Physics – Uspekhi **58** (12) 1206 – 1214 (2015)

PHYSICS OF OUR DAYS

PACS numbers: **98.65.–r**, 98.65.Cw

Galaxy clusters, similarity parameters, and ratios between measurable characteristics

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DOI: 10.3367/UFNe.0185.201512c.1323

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Abstract. The study of galaxy clusters provides insights into the different stages of the evolution of the Universe. Cluster observations measure luminosity, size, temperature, and mass. What binds a cluster into a single entity is gravity, its force being proportional to the Newtonian constant of gravitation *G*. Because all five of these quantities are measured in units of mass, length, and time, two nondimensional parameters, commonly known as similarity parameters, can be argued to characterize the system. One of these is the well-known virial ratio of kinetic to potential energies. The velocities of galaxy clusters are not measured, however. The luminosity *L* and the constant *G* can be combined to introduce the dynamic velocity scale $U_\alpha = (LG)^{1/5}$. The ratio of this scale to the particle thermal velocity gives the similarity parameter Π_1, which is constant to within about 10% for all 30 objects studied, allowing the virial similarity parameter Π_2 to be evaluated for 31 objects. For nearby objects with a red shift of $z \leqslant 0.2$, the parameter Π_2 is of order 10 and decreases with increasing *z*, i.e., with decreasing age. To test the quality of the data, the value of *G* was determined using other measured quantities and found to be equal to its true value to within $\leqslant 6\%$ and 28% for close and distant objects, respectively. A number of other ratios between measured quantities have been proposed and checked, showing a scatter of 10 – 20% from constancy on the linear scale in the numerical coefficients involved. Older clusters are, on average, larger in mass and size, implying that smaller clusters can be absorbed by large ones. The results obtained can be valid for clusters with a temperature of $T > 1$ keV, i.e., in the X-ray range of the spectrum. The cluster mass reduction with increasing *z*, i.e., with decreasing age, is also traced, on the average, in other spectral regions. It is shown that by knowing the temperature and the received X-ray intensity, the possibility arises to estimate the distance to the cluster.

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Received 25 June 2015, revised 10 August 2015
Uspekhi Fizicheskikh Nauk **185** (12) 1323 – 1332 (2015)
DOI: 10.3367/UFNr.0185.201512c.1323
Translated by K A Postnov; edited by A Radzig

Keywords: clusters of galaxies, similarity parameters, numerical estimates of virial, data quality test, distances to clusters

1. Introduction

The study of galaxies and their clusters is a very topical line of inquiry in modern astrophysics. Our Local Group includes several dozen galaxies, and galaxy clusters consist of hundreds and thousands of objects (see, e.g., reviews [1–4]). Galaxy clusters have first been studied in X-rays [5, 6] and, recently, at millimeter wavelengths [4]. They are observed up to redshifts $z \sim 1.5$, i.e., the age of distant objects is as low as 1.5 or 2 bln years (at $z = 1.5$, the age is about 3.5 bln years). Whereas, star formation began 0.5 bln years after the Big Bang, which occurred 13.8 bln years ago, and galaxies started forming 1.5–2 bln years after star formation began [7].

Modern X-ray observations enable determining the temperature T_e, luminosity L, mass M, and redshift of clusters [5, 6]. These parameters can be improved by taking new measurements, for example, by millimeter observations [4] from the thermal Sunyaev–Zeldovich effect [8] of inverse Compton scattering of cosmic microwave background (CMB) photons on hot electrons in galaxy cluster halos. These rich observational data are available only in the X-ray band, i.e., for temperatures $T_e > 1$ keV, which allows the determination of similarity parameter ratios for clusters at these energies.

If the velocities of objects (orbital, thermal, or turbulent) U are known, the following relation with the luminosity can be used:

$$L = KU^n \,, \quad (1)$$

where K is a numerical factor derived from measurements of some sample of objects. Such a relation was first published by Tully and Fisher (TF) [9] in 1977. For several nearby radio galaxies, Tully and Fisher found $n = 2.5 \pm 0.3$ and determined the value of the dimensional factor K. For objects with known velocities, formula (1) could be used to estimate the true luminosities, giving a tool to measure the object distance from measured velocity and luminosity. Later on, relations like (1) were proposed for use in the optical and near infrared (IR) bands as well. In the last case, the recent review [10] gives $n = 3.5 \pm 0.3$ for the relation between the luminosity and the spectral line width. However, the simplest form of relation (1), at least physically motivated from the point of view of dimensional analysis, is obtained for $n = 5$, when the factor K has the inverse dimension of the Newtonian constant of gravitation $G = 0.6672 \times 10^{-10}$ m^3 kg^{-1} s^{-2}.

To obviate difficulties related to dimensional factors like K in formula (1), the physical quantities under consideration are usually related to some well-known value, i.e., instead of Eqn (1), one uses the relation

$$\frac{L}{L_{\rm r}} = \left(\frac{U}{U_{\rm r}}\right)^m , \quad (1')$$

where the subscript r denotes the known fiducial value. Relation (1′) can be called self-similar. It should be borne in mind that self-similarity can be observed only in some restricted domain of existence of self-similarity parameters. Even in such well-known case as Kolmogorov–Obukhov small-scale turbulence, there are some small but crucial improvements, such as the dependence of the prefactor on the Reynolds number logarithm [11].

The size of galaxy clusters is about several megaparsecs (1 Mpc = 3.0856×10^{22} m), their luminosities exceed 10^{37} W (the solar luminosity is 4×10^{26} W), and the mass is around $10^{14}\, m_{\odot}$ (the solar mass is $m_{\odot} = 2 \times 10^{30}$ kg). The cluster temperature [5] ranges from ~ 2 keV to ~ 14 keV, i.e., in the X-ray band. This is the temperature of intergalactic gas consisting of protons, electrons, and about 10% helium atoms. The molecular weight of such a gas is $\mu = 0.6$, close to the solar value. The intergalactic gas is observed by X-ray satellites and from high-altitude telescopes.

Galaxy clustering is due to gravitation, which brings about gas compression and leads to increasing gas temperature to about several billion kelvins. The simultaneous action of gravitational forces, thermodynamic and other physical processes results in an interrelation of different physical quantities that can be measured. The factor K on the right-hand side of formula (1) at $n = 5$ has the inverse dimension of the Newtonian constant of gravitation, i.e., the empirical relation (1) can be written out in the form

$$GL = U^5 \,. \quad (2)$$

According to Albert Einstein [12], numerical coefficients in correct dimensional relations should not differ greatly from unity, although this statement, strictly speaking, is not proven. The first monograph devoted to dimensional analysis [13] describes this paper by Einstein [12], in which the solid body compressibility was derived from dimensional considerations, and the numerical coefficient turned out to be 0.14.

In Section 3, we will show that observations [5, 6] of galaxy clusters can be used to derive the numerical coefficient in formula (2), which turns out to be equal to unity with a 30% uncertainty. In Section 7, we demonstrate that the power of gravitational wave emission during coalescence of binary black holes is also bounded from above by formula (2) with the substitution of the speed of light c for U. A similar formula is valid on Planckian scales determined by the Planck constant, Newtonian constant of gravitation, and speed of light.

Equation (2) permits introducing the velocity scale

$$U = (LG)^{1/5} \,. \quad (3)$$

In the units we use here, the luminosity $L = 10^{37}$ W (which corresponds approximately to the optical luminosity of our Galaxy) implies

$$U = 232 L^{1/5} \ [\text{km s}^{-1}] \,. \quad (4)$$

This is to be compared with the orbital velocity of the Sun at a distance of 8.5 kpc from the galactic center, which, according to different estimates, is 230 ± 30 km s^{-1}. According to book [14], the luminosities and the mean velocities measured for the Milky Way and Andromeda (M31) galaxies are (230 ± 10) km s^{-1} and 260 km s^{-1}, respectively, for the corresponding luminosities $L = 0.80 \times 10^{37}$ and 0.94×10^{37} W. Such luminosities yield velocities of 222 and 229 km s^{-1}, i.e., only 10% below the observed values. Good correspondence can also be obtained for convective velocities in the Sun. Thus, scale (3) can be referred to as the dynamical velocity $U_{\rm d}$.

In the present paper, in addition to four measured quantities, namely temperature $T_{\rm e}$, luminosity L, mass M, and size R, we add the Newtonian constant of gravitation $G = 6.672 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, since gravitational interaction in galaxy clusters combines these values into one self-consistent set. In addition, we add two relations that are widely used in astrophysics, which will be needed below. The first is the time scale of galaxy formation or their clusters [7] for a given mass:

$$t_{\rm d} = (G\rho)^{-1/2} = R^{3/2}(GM)^{-1/2} \,, \quad (5)$$

where ρ is the mean density of matter, and the thermal gas velocity is

$$U_T \,[\text{km s}^{-1}] = \left(\frac{3kT}{\mu m_{\rm p}}\right)^{1/2} = 693\, T_{\rm e}^{1/2} \,, \quad (6)$$

where $T_{\rm e}$ is the temperature in kiloelectron-volts (1 keV = 1.161×10^7 K), $\mu = 0.6$ is the molecular weight of the gas mixture, and $m_{\rm p} = 1.67 \times 10^{-27}$ kg is the mass of a proton. For monoatomic gas, thermal velocities exceed the velocity of sound $c_{\rm s}$ by about 30%.

2. Similarity parameters

All five physical quantities discussed in Section 1 are defined by three units of measure for: mass M, length L, and time T. Therefore, there must be two similarity parameters, and the possibility arises to obtain different relations between measurands (see monographs [13, 15, 16]).

With two velocity scales, the first similarity parameter can be suggested in the form

$$\Pi_1 = \frac{U_\mathrm{d}}{U_T} = \frac{(LG)^{1/5}}{(3kT/\mu m_\mathrm{p})^{1/2}} = 0.334\, L^{1/5} T^{-1/2}\,, \qquad (7)$$

where the luminosity is expressed in the units of 10^{37} W, and T in keV. The quantity Π_1 can be expressed through the Mach number

$$\mathrm{Ma} = \frac{U_\mathrm{d}}{c_\mathrm{s}} = \frac{U_\mathrm{d}}{0.7\, U_T} \approx 1.3\, \Pi_1\,, \qquad (8)$$

where the numerical coefficients are calculated from the adiabatic index and are presented for monoatomic gas.

The second similarity parameter used in astrophysics concerns the virial relation: $\Pi_2 = MG/U^2R$. Using formulas (7) and (8), the parameter Π_2 can be given in the form

$$\Pi_2 = \frac{MG}{U^2R} = \frac{MG}{\Pi_1^2 U_T^2 R}\,, \qquad (9)$$

and may be calculated for all galaxy clusters described in Refs [5, 6]. Both similarity parameters are listed in Tables 1 and 2. In other words, the numerical value of the virial parameter Π_2 indicates the relative depth of the potential well [1, 2], i.e., measures the ratio of kinetic to potential energies. Note that in the literature on galaxy clusters the numerical values of the virial parameter are not presented. Nor can one find them for individual galaxies [14]. Then, nearby clusters are assumed already to have reached relaxation, while remote clusters are not. In Section 6, we will assign the numerical values to those notions that fit concrete objects.

It should be noted that in the system with four measurands and constant G, we can relate according to the rules of operation with dimensional variables one of the variables with any other variable, using two remaining variables, with G being possibly one of them. The dimensionless numerical coefficients appear in such relations from comparisons with observations. They can also be used as the similarity criteria. But these coefficients must be functions of already determined parameters Π_1 and Π_2, which will be illustrated by examples in Sections 4–6.

3. Data review and similarity criteria for galaxy clusters

Let us review observational data mainly obtained by A A Vikhlinin [5] for 21 distant galaxy clusters with $z \geqslant 0.4$, and for 10 nearby clusters [6] with $z \leqslant 0.23$ (in paper [6], data

Table 1. Parameters* of distant galaxy clusters taken from paper [5] with the similarity parameters Π_1 and Π_2 depending on the age of the objects

No.	z	T_e, keV	L, 10^{37} W	M_{200}*, $10^{14}\, M_\odot$	R, Mpc	Π_1	Π_2	t_a, 10^9 y	t_d, 10^9 y	$t_\mathrm{a}/t_\mathrm{d}$
1	0.394	4.8	9.2	1.24	0.5	0.238	7.0	8.12	0.48	16.9
2	0.400	3.7	8.9	1.42	0.7	0.269	7.4	8.09	0.49	16.8
3	0.424	3.6	10.6	1.07	0.5	0.282	8.1	7.92	0.61	13.0
4	0.426	7.6	27.0	2.89	0.9	0.234	5.3	7.91	0.50	15.8
5	0.451	14.1	260.4	8.77	0.9	0.270	9.4	7.75	0.43	18.0
6	0.453	5.8	15.9	1.81	0.7	0.241	6.0	7.73	0.65	11.9
7	0.460	5.3	16.3	1.57	0.5	0.254	8.0	7.68	0.55	14.0
8	0.516	5.1	15.7	1.67	0.6	0.252	7.4	7.34	0.54	13.6
9	0.537	8.1	91.7	3.68	1.0	0.290	6.2	7.21	0.78	9.2
10	0.541	9.9	113.3	6.43	1.0	0.274	8.8	7.19	0.59	12.2
11	0.562	4.8	12.5	1.19	0.5	0.253	6.7	7.07	0.49	14.4
12	0.574	2.7	38.8	0.36	0.5	0.265	3.6	7.00	0.88	8.0
13	0.583	5.2	10.8	0.95	0.5	0.236	5.0	6.95	0.54	12.9
14	0.700	7.2	28.7	2.01	0.7	0.244	5.4	6.36	0.62	10.2
15	0.782	6.3	32.4	1.41	0.7	0.266	4.3	5.99	0.74	8.1
16	0.805	2.2	2.0	0.21	0.5	0.258	2.6	5.89	1.16	5.1
17	0.805	4.3	13.2	1.04	0.8	0.269	4.1	5.89	1.05	5.6
18	0.813	6.6	28.8	1.25	0.7	0.255	3.7	5.86	0.79	7.4
19	0.823	7.8	70.9	2.58	1.0	0.280	4.5	5.81	0.93	6.2
20	1.100	3.5	5.9	0.26	0.5	0.255	2.0	4.82	1.04	4.6
21	1.261	4.7	6.0	0.20	0.5	0.220	1.2	4.36	1.19	3.7

* M_{200} is the total mass enclosed in a volume at the outer boundary of which the density is 200 times as high as the critical density in the Universe (where the Universe is flat at the present time).

Table 2. Parameters of nearby galaxy clusters taken from paper [6].

No.	z	R_{500}, Mpc	T, keV	M_{500}, $10^{14}\,M_\odot$	Π_2	t_a, 10^9 y	t_d, 10^9 y	t_a/t_d
22	0.0569	1.007	4.14	3.17	10.3	11.3	0.844	13.4
23	0.0162	0.650	2.08	1.0	10.0	11.8	0.791	14.9
24	0.1883	0.944	4.81	3.06	9.1	9.8	0.781	12.5
25	0.0881	1.337	7.94	7.68	9.8	10.9	0.831	13.1
26	0.1603	1.096	5.96	4.56	9.5	10.1	0.799	12.6
27	0.1429	1.299	7.38	7.57	10.7	10.3	0.800	12.5
28	0.0622	1.235	6.12	6.03	10.8	11.2	0.813	13.8
29	0.0592	1.362	8.47	8.01	9.4	11.2	0.835	13.4
30	0.2302	1.416	8.89	10.74	11.5	9.4	0.765	12.3
31	0.0199	0.634	1.64	0.77	10.0	11.7	0.855	13.7

on three clusters are insufficient in volume for the present analysis). These data, obtained with both specialized satellites and large ground-based telescopes, provide the most complete sets of simultaneously measured quantities, which makes them very convenient for the comprehensive analysis. The quality of these measurements will be tested in Sections 4–6 by determining from them the constant G: the accuracy of the value of G derived in this way can serve as a measure of the accuracy of the original data.

Table 1 presents redshifts z, temperatures, luminosities, masses, and sizes of the clusters, as well as the similarity parameters Π_1 and Π_2 derived from these data using formulas (7) and (9), respectively [5]. Table 2 lists the same quantities as in Table 1 with the exception of luminosity for 10 nearby clusters. The cluster age t_a is defined as $t_a = [t_0(1+z)^{-1} - 1.8]$ in bln years, $t_0 = 13.8$ bln years is the time from the Big Bang, 1.8 bln years is the conventional time of the beginning of cluster formation [7], and t_d is the duration of cluster formation according to formula (5). The ratio t_a/t_d characterizes the relative time of existence of galaxy clusters that are more or less formed. Reviews [1, 2] consider relaxed clusters in the sense that the kinetic energies of the clusters are much less than their potential energies. In our terms, for such clusters the value of the virial, i.e., the similarity parameter Π_1, changes insignificantly.

Figure 1 presents the similarity criteria Π_1 for remote clusters with known luminosities [5], which ranges from 0.22 for the most distant cluster with $z = 1.26$ to 0.29 for cluster No. 9. As seen from Fig. 1, with an accuracy of about 80% the statistical mean is 0.26 ± 0.02. Unfortunately, despite our detailed knowledge of our Galaxy, it is difficult to precisely measure the mean temperature of intergalactic gas, and we cannot reliably estimate the galactic luminosity. These values are too averaged, and can be measured only from outside and from sufficiently large distances. But we can estimate the similarity parameter Π_1 for the Sun with the radiation temperature $T = 5750$ K and luminosity $L = 4 \times 10^{26}$ W. The dynamical velocity scale is then $U_d = 2$ km s^{-1}, which is the scale of convective motions and, hence, $\Pi_1 = 0.125$. This value is only half the corresponding similarity criteria for clusters whose parameters are 10–12 orders of magnitude larger. With account for reasonable Mach number estimates for remote clusters, with a variance of about 10%, we assume that the value of $\Pi_1 = 0.26$ is the same for 10 nearby clusters. The data for the closest clusters are presented in Table 2, which differs from Table 1 by the absence of luminosities of the clusters. Note that the virial value is close to 10, namely, the mean value is $\Pi_2 = 10.1 \pm 0.6$.

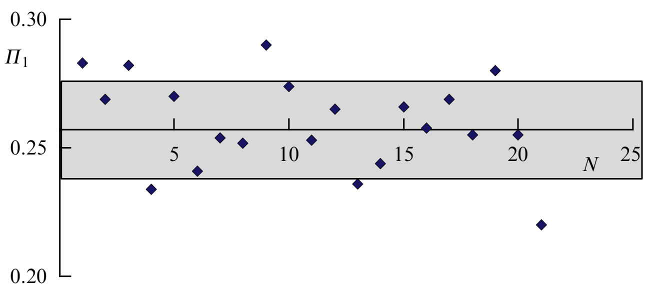

Figure 1. Similarity parameter Π_1 for 21 objects from Table 1 (N is the number of the cluster). The shaded belt contains 2/3 of the objects.

4. Cluster evolution with age

Now, using Tables 1 and 2, we are in the position to determine the cluster evolution with age. First of all, it is manifested in the change in the virial parameter Π_2 with the relative age of the cluster. Velocities in the galactic clusters are not measured directly, but this problem can be circumvented by using the approximate constancy of the similarity parameter Π_1 or the Mach number. Then, one obtains

$$\Pi_2 = \frac{MG}{\Pi_1^2 aTR}\,, \tag{10}$$

where $a = 3kA/(\mu m_p) = 4.8 \times 10^{11}$ in SI units. In our special units of measure we have

$$\Pi_2 = 3.3\,\frac{M}{T_e R}\,, \tag{10'}$$

which was actually used to calculate the virial Π_2 in Tables 1 and 2 (note that in the literature cited here no numerical values of the virial can be found). Values of Π_2 as a function of the relative age t_a/t_d are shown in Fig. 2. For the 10 nearby clusters from Table 2 (white dots), which are quite relaxed in the virial sense, as noted in Refs [2, 6], the similarity parameter is $\Pi_2 \approx 10$. Only the giant cluster No. 5, with luminosity and mass one order of magnitude larger than for other clusters,

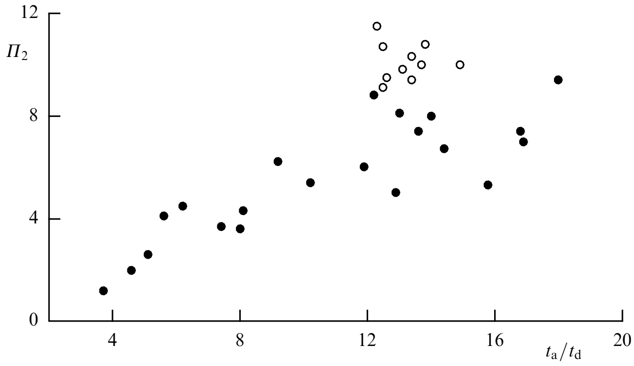

Figure 2. Similarity parameter Π_2 as a function of the relative age of the cluster. Black and white dots correspond to objects from Table 1 and Table 2, respectively.

has $\Pi_2 = 9.4$, which is close to 10. The general tendency is a decrease in the parameter Π_2 with decreasing the relative cluster age t_a/t_d. Generally, masses of distant clusters with $z \geqslant 0.4$ are significantly smaller than those from Table 2, where the mean cluster age is $t_a/t_d \approx 13.2 \pm 0.6$, although masses in both samples differ by more than an order of magnitude. However, the value of Π_2 remains close to 10. With a small difference in the relative age, the virial value of about 10 can probably signal the sufficiently complete relaxation of the object. The smaller masses in Table 1 and their increase, on the average, with the relative age t_a/t_d imply that during evolution smaller clusters are frequently swallowed by larger ones.

Figure 3 plots the dependence of the cluster masses on their relative age. The general tendency of cluster mass increasing with age is seen. Tables 1 and 2 show that for $0.57 \leqslant z \leqslant 1.26$ the mean mass is $\bar{M} = 1.03 \pm 0.63$, while for 11 clusters with $0.4 \leqslant z \leqslant 0.56$ the mass is $\bar{M} = 2.9 \pm 1.7$, and for the 10 nearby clusters from Table 2, $\bar{M} = 5.3 \pm 2.3$, i.e., the mass increases, on the average, by 5 times when passing from distant to nearby galaxy clusters. The mean radii $\bar{R}$ of clusters increase to less extent, but significantly, with age: for the 10 distant objects, $\bar{R} = 0.67 \pm 0.14$, for the first 11 objects from Table 1, $\bar{R} = 0.71 \pm 0.17$, and for the 10 nearby clusters from Table 2, $\bar{R} = 1.03 \pm 0.26$. Here, the ratio M/R increases approximately threefold. All these effects are explained by neighbor merging.

Notice that the dynamical age t_d is estimated from the present-day observations of masses and radii which relate to the distant past. During the merging of a smaller cluster with a larger one, the resulting density increases, and then, according to formula (5), the time t_d decreases, i.e., the relative age t_a/t_d should increase during neighbor mergings. The nearby clusters (white dots in Fig. 3) are grouped, clearly, around relative ages 10–14, as in Fig. 2 for virial similarity coefficients. Although dispersion in the nearby cluster masses is much larger than in their Π_2 values, this merely confirms that they are relaxed to the virial equilibrium. Cluster No. 5 represents a special case: it has a very high mass, huge luminosity, and relatively small size, which determine its large relative age. According to the similarity parameter Π_2, this cluster also approaches the already very relaxed clusters from Table 2. Of course, the set of 31 clusters is too small and should be increased, but this requires complex cluster parameter measurements, as in Ref. [5], which were not carried out in Ref. [6].

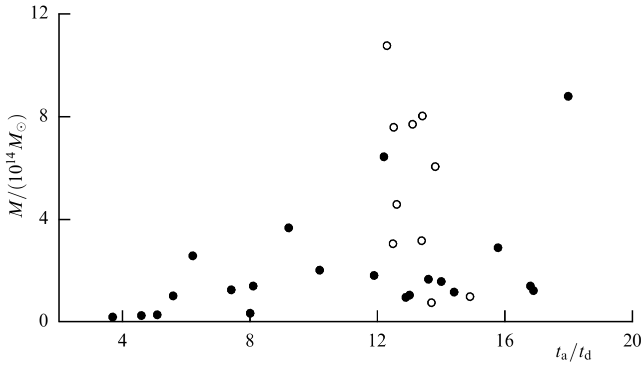

Figure 3. Cluster mass as a function of the cluster age. Black and white dots correspond to clusters with $z \geqslant 0.4$ and $z \leqslant 0.23$, respectively.

The relative age is directly related to the redshift z. In review [2], there is a remarkable Fig. 6 showing the cluster mass dependence on z, which clearly demonstrates the mass increase, on the average, with decreasing z, i.e., when their relative age increases. In paper [4], the cluster mass was estimated from the thermal Sunyaev–Zeldovich (SZ) effect [8], related to inverse Compton scattering of CMB photons on the cluster's halo electrons. These measurements were carried out by the 10-m South Pole Telescope running since 2008. About 800 SZ-clusters and more than 1000 objects observed in other spectral ranges (IR, visible, and X-ray) point to a statistical increase in the cluster masses with age.

5. Check of the data quality and relations among different cluster characteristics. The luminosity-temperature relation

We now give several examples testing the formulas obtained from the dimensional analysis and estimating the dispersion of parameters presented in the above tables. We start with a dependence similar to the TF relation, equation (2), which we have not so far applied for galaxy clusters. Let us estimate the numerical coefficient expressed in our units of measure:

$$c_g = LGU_d^{-5} = LG(\Pi_1 U_T)^{-5} = 2.94\, LT_e^{-5/2}\,. \qquad (11)$$

Using the sample of 20 clusters from Table 1 but object No. 12, which has a very unusual mass–luminosity ratio, we find $c_g = 1.00 \pm 0.28$. The dispersion falls within 67% with the minimal value of 0.47 for the most distant object No. 21, and maximal value of 1.82 for object No. 9, which also has some peculiar properties, thus dropping out of the ensemble. The closeness of the experimentally derived numerical coefficient c_g in formula (11) to unity suggests the very deep nature of this dependence, and the 30% dispersion is evidence of the good quality of experimental data, i.e., the consistency of the luminosity and temperature measurements.

Data presented in Table 2 seem to have a higher accuracy than in Table 1. This can be shown by calculating the constant G_d from formulas like (10) or (10′). It should be borne in mind that Table 2 presents masses M_{500}, i.e., as derived from densities 500 times as high as the critical density in the Universe. At the same time, Table 1 presents masses M_{200}, i.e., those within the volume on the surface of which the density is only 200 times as high as the critical one. According to paper [6], $M_{500} \approx 0.7\, M_{200}$. Below, we shall use the object's size estimate from the dimensional analysis, $R = MG/aT$, for which we find

$$G = \frac{aTR}{M} = \frac{5.184\,RT}{M} \equiv G_{\rm d}\,, \tag{12}$$

where the coefficient a was determined immediately after formula (10).

According to data from Table 2, the thus determined value of the gravity constant is $G_{\rm d} = (6.99 \pm 0.39) \times 10^{-11}$ m^3 kg^{-1} s^{-2}. Its ratio to the universal (Newtonian) constant of gravitation is 1.048 ± 0.053. The 5% coincidence should be considered the measurement accuracy estimate for nearby objects. Notice that $G_{\rm d}$ takes the maximum value of 7.69×10^{-11} m^3 kg^{-1} s^{-2} for object No. 25, and the minimal value, $G_{\rm d} = 6.10 \times 10^{-11}$ m^3 kg^{-1} s^{-2}, for object No. 31. Note also that Table 2 presents the central values of the measured quantities from paper [6], with a dispersion of 4%–6%.

The luminosity–temperature dependence for clusters is a topical study (see, for example, paper [6]). The experimental data (taken from review [2]) are shown in Fig. 4. Review [1] presents a very informative Fig. 14, which depicts the results of many dozens of observational studies and modeling. For $T > 1.5$ keV, the very clear dependence $L \propto T^{5/2}$ is seen, and for smaller temperatures the dispersion of points increases, and the power-law index increases as well. The dependence $L \sim T^{5/2}$ justifies the dynamical velocity scale $U_{\rm d} = (GL)^{1/5}$ introduced above [see formula (11)] and confirms the good constancy of the first similarity parameter $\Pi_1 = U_{\rm d}/U_T$.

From the approximate constancy of the Mach number, i.e., the parameter Π_1, and formula (2), one can obtain

$$L \approx G^{-1} \Pi_1^5 a^{5/2} T^{5/2} = 0.286\, T_{\rm e}^{5/2} \tag{13}$$

in the units of measure adopted here. The same dependence can be obtained from direct calculations using data from Table 1 with the numerical coefficient 0.29 ± 0.08. This holds for $T_{\rm e} \geqslant 2$ keV. At lower temperatures, numerical calculations are required. Apparently, the entire theory with constant pre-power numerical factors presented here can be applied to objects with temperatures $T_{\rm e} > 1$ keV, i.e., higher than 10^7 K.

Relation (13) was checked by means of its comparison with data given in Table 1 from the point of view of steadiness of the numerical coefficient 0.286. All 20 clusters but one (No. 12, in which the temperature is inconsistent with luminosity) yielded the value of this coefficient 0.286 ± 0.08, i.e., with a 28% error. With increasing distance r from the object, the energy flux decreases as $L/(4\pi r^2) = q_0$, i.e. the luminosity level measured on Earth's surface. From here and the luminosity–temperature relation (13), we can estimate the distance to the object:

$$r = 0.73 q_0^{-1/2} T^{5/4}\,, \tag{14}$$

where, we recall, the distance r is measured in Mpc, and temperature in keV. The error in the distance estimate will already be around 14%. In the X-ray band of $T > 1$ keV, this relation could play the same role for distance estimates as the TF relation [9] at much lower energies.

The cluster size is determined by the balance between the gas pressure, $p = nkT$, and gravitation. A decrease in temperature in this balance means a number particle density increase, which increases the optical depth. This might explain the dispersion in the power-law index m in relations $L \sim T^m$ at temperatures much below 2 keV. The range of temperatures is usually resided within one order of magnitude, while luminosities can vary by several orders. For example, if we directly calculate the correlation for all objects from Table 1, with a 99% probability we get $m = 2.4 \pm 0.3$ with the correlation coefficient $R^2 = 0.74$, which is in agreement with $m = 2.5$, obtained from the dimensional analysis and following from formulas (2)–(4).

The mass–temperature $M(T_{\rm e})$ dependence is also widely used in galaxy cluster studies. For example, by assuming isothermal gas in hydrostatic equilibrium, the relation $M \sim T^{1.5}$ is obtained [2, 16–18]. Many authors have found the power-law index in this relation to fall within the range 1.45–1.85, depending on the mass and temperature intervals

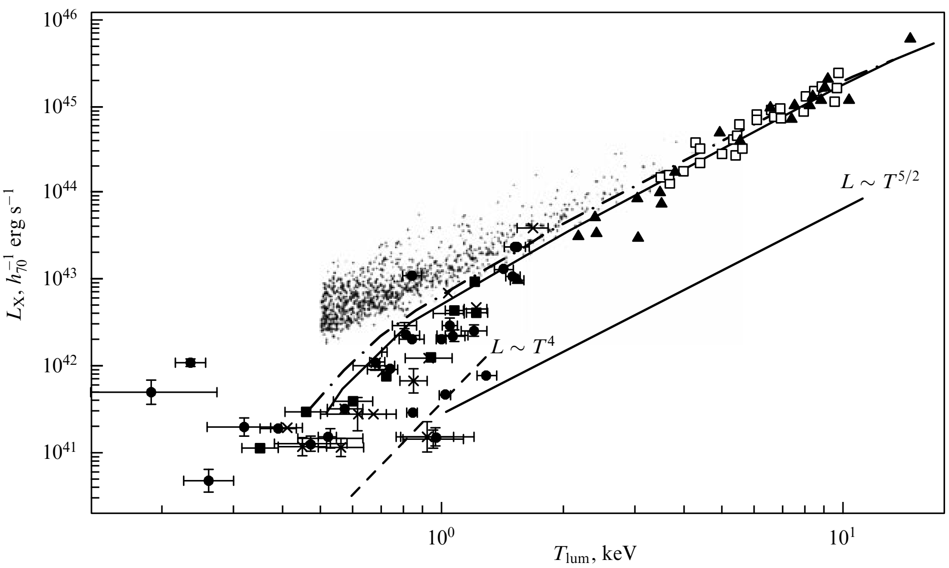

Figure 4. Cluster X-ray luminosity as a function of cluster temperature (figure adapted from review [2]). The 'cloud' of points in the center of the figure shows the result of numerical simulations (h_{70} is the normalized Hubble constant). Different symbols mark the results of different experiments (see review [2] for details). The dashed line $L \sim T^4$ and the solid line $L \sim T^{5/2}$ are also presented.

used [2, 6], with a low temperature inclusion increasing this index. A dimensional analysis shows that, to estimate the mass, in addition to the temperature and gravity constant, another measurable parameter should be utilized, for example, luminosity or size. We have checked the dimensional dependence $M_d = RT_e G^{-1}$, which transforms to $M_d = 1.11\, a_m R T_e$ in our units of measure. The numerical coefficient a_m arises when comparing this value with observational data. Such a comparison with the first 11 objects from Table 1, which are presumably more or less relaxed, according to Fig. 2, yields $a_m = 0.49 \pm 0.06$. Ten well-relaxed objects from Table 2 yield $a_m = 0.67 \pm 0.04$, which reflects their higher virial value than for the distant objects from Table 1.

The mass can be estimated with larger dispersion using the cluster luminosity rather than temperature. The dimensional analysis then suggests $M = a'_m R L^{2/5} G^{-3/5}$. The first 11 objects from Table 1 yield $a'_m = 7.46 \pm 0.93$.

One more dependence to be checked is

$$L = a_e \left(\frac{M}{R}\right)^{5/2} G^{3/2}\,, \tag{15}$$

where the numerical factor a_e, as obtained from a comparison with data from Table 1 (except for cluster No. 12 with anomalous luminosity and mass), turns out to be 1.00 ± 0.28. Formula (15) suggests a high luminosity for massive compact objects, for example, for quasars.

Formulas made up to estimate the cluster size (which itself cannot be determined with high accuracy) using their other parameters provide another good example. In SI units and our units of measurements we get

$$R_1 = \frac{MG}{U_T^2} = \frac{MG}{aT}\,, \quad a = \frac{3kA}{\mu m_p} = 4.81 \times 10^{11}\,. \tag{16}$$

A comparison of R_1 with observational data collected in Table 2 yields the numerical factor $c_1 = R_{obs}/R_1 = 1.49 \pm 0.08$ with the correlation coefficient $R^2 = 0.89$ (Fig. 5).

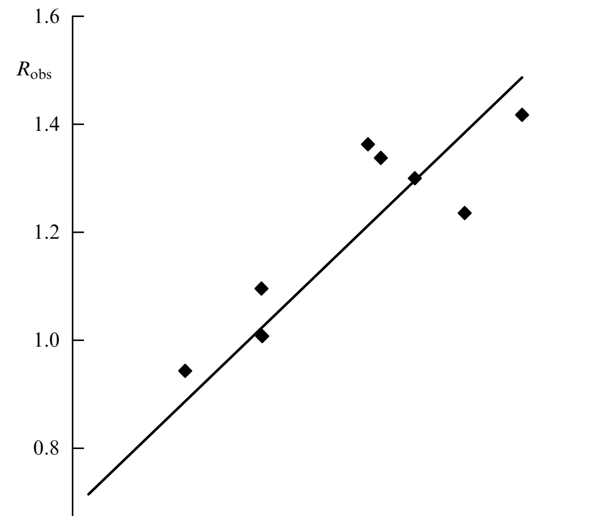

Figure 5. Dispersion of the observed cluster sizes as a function of their values derived from the dimensional analysis according to formula (16) for objects from Table 2. The correlation coefficient is $R^2 = 0.89$.

Note that all these numerical estimates were obtained in linear, not logarithmic, coordinates, whereas in astrophysics the relations are commonly sought in logarithmic units.

6. Cluster energy and the energy similarity parameter

The total energy of the cluster in dimensional units reads as follows

$$E_d = M^2 G R^{-1}\,. \tag{17}$$

Using expressions (7)–(10), formula (17) can be transformed into

$$E_d = a_T \Pi_1^2 \Pi_2 M c_p T\,, \tag{17'}$$

where $c_p = 3k/\mu$ is the specific heat capacity at constant pressure. We will see that the total energy of a cluster is proportional to its enthalpy, which can be estimated from our data upon determining the numerical coefficient a_T. For first 11 clusters from Table 1 we get $a_T = 2.25 \pm 0.29$, and for 10 nearby clusters from Table 2 we obtain $a_T = 1.80 \pm 0.10$. Clearly, the difference is due to different degrees of virialization of distant and nearby objects because of their different relative ages.

If the total enthalpy of a galaxy cluster is known, we can estimate its expected lifetime as $t = E_d/L$ for the present-day values of the mass, temperature, luminosity, and gravity constant. Using expression (17′) at $a_T \approx 2$, we obtain the result for the lifetime t (in billions of years):

$$t = \frac{548 M^2}{RL}\,, \tag{18}$$

which, with account for expression (3) for the formation time and formula (12) for the luminosity, can be transformed into dimensionless time $\tau = t/t_b$. Using Eqns (5)–(8), the last formula may be presented in the form

$$\tau \approx 2\Pi_2^{3/2}\,. \tag{19}$$

This is a more exact characteristic of the cluster than the virial value Π_2. For example, the low-mass but high-luminous galaxy cluster No. 12 has a relative lifetime two orders of magnitude shorter than closer clusters from Table 1. According to formula (18), with given X-ray luminosity, the relative age τ of the latter is many hundred billion years. This time is unrealistically long, and estimate (18) is purely formal. However, it clearly demonstrates the peculiar character of cluster No. 12. It cannot be ruled out that some parameters of this cluster were erroneously determined.

The similarity criterion

$$\Pi_3 = \frac{MG}{R c_p T}\,, \tag{20}$$

which is the ratio of the cluster's potential energy to its enthalpy, can also be useful. Criterion (20) contains neither luminosity nor some velocity, both of which are frequently unknown in the available literature. The physical sense of criterion (20) is rather simple: if the potential energy is much larger than the kinetic energy, the thermal dissipation of gas is small. The mass of gas in galaxy clusters, which is a small fraction f of their total mass (galaxies, dark matter), is about

0.1, but with large uncertainties [2, 6]. The heat capacity at constant pressure for gas with $\mu = 0.6$ is $c_p = 48 \times 10^3$ J kg K^{-1}. Then, in our units of measure formula (2) gives

$$f\Pi_3 \approx \frac{0.9M}{RT} .$$

This quantity can be calculated for all objects from Tables 1 and 2. For the first 11 clusters from Table 1, the maximum and minimum values of $f\Pi_3$ are 0.62 and 0.38 for objects No. 5 and No. 4, respectively. For the other, more distant, 10 clusters from Table 1, the corresponding values are 0.38 and 0.08, respectively. For clusters from Table 2, the value of Π_3 ranges from 0.61 to 0.77 with the mean value of 0.67 ± 0.04. If $f = 0.1$, then Π_3 is of order unity for most of our objects. It can be shown that this new parameter is related to the similarity parameters Π_1 and Π_2 by the relation

$$\Pi_3 \approx \Pi_2 \Pi_1^2 . \qquad (22)$$

7. Possible relations between emission power and velocities in relativistic gravitationally coupled systems

Let us consider Planckian scales determined by the Planck constant h, speed of light c, and Newtonian constant of gravitation G. From the dimensional analysis, we find, as expected, that

$$L = c^5 G^{-1} , \qquad (23)$$

which equals 4×10^{52} W in SI units. Thus, such a huge power is not surprising, considering the Planckian time scale of 5×10^{-44} s. By multiplying these two values, we obtain the Planckian energy scale of 2×10^9 J $\approx 10^{19}$ GeV.

Another example is provided by emission of gravitational waves by a binary system of two black holes. This problem is considered in monograph [19], where it is shown that the emission in this case is quadruple and its power is as follows:

$$\frac{\mathrm{d}E_{\mathrm{g}}}{\mathrm{d}t} = \frac{c^5}{G}\left(\frac{r_{\mathrm{S}}}{R}\right)^5 , \quad r_{\mathrm{S}} = \frac{GM}{c^2} , \qquad (24)$$

where r_{S} is the Schwarzschild gravitational radius characterizing the size of a black hole with mass M, and R is the size of the system. As R is always larger than r_{S}, relation (23) is an upper limit on the power of gravitational wave emission.

8. Conclusions

An analysis of observed characteristics for 31 X-ray galaxy clusters with different ages and redshifts from 0.02 to 1.26 has been carried out. Four measurable quantities, viz. mass, luminosity, temperature, and size, related by gravitation and thermodynamics, are determined by three dimensional measurement units. This enables us to introduce two dimensionless similarity parameters. One of them is well known in mechanics — the virial relation between the potential and kinetic energies. In the present paper, this is the parameter Π_2. To estimate it for specific objects considered here, the velocity scale should be introduced.

The main results are as follows.

(1) From dimensional considerations, the velocity scale $U_{\mathrm{d}} = (GL)^{1/5}$ is introduced, which seems to be reasonable not only for our Galaxy and the Andromeda galaxy, but also for the Sun. In the last case, it corresponds to the velocity of convective motions in the photosphere. If the temperature is known, the thermal velocity U_{T} can be introduced, which is somewhat larger than the velocity of sound in gas.

(2) The new similarity parameter $\Pi_1 = U_{\mathrm{d}}/U_T$ is introduced. The analysis of distant galaxy clusters with known luminosity shows that, for them, $\Pi_1 = 0.26 \pm 0.02$, i.e., it is constant to within 10% (see Fig. 1). This allows us to interpret the similarity parameter as the Mach number, since the Mach number for monoatomic intergalactic gas is Ma $\approx 1.3\,\Pi_1$.

(3) The second similarity parameter, Π_2, is known as virial ratio, but in the present paper Π_2 is first proposed in the form which allows it to be numerically found through Π_1. The parameter Π_2 increases with age of the object by approaching saturation at $\Pi_2 \approx 10$, as seen from Fig. 2 and Tables 1, 2.

(4) The proposed dimensional relationships among different parameters derived for 30 clusters from Refs [5, 6] are consistent in linear, but not in logarithmic, scales with an accuracy of better than 30%. Such a massive check of data quality using dimensional relations is methodically new in astrophysics, allowing us to estimate some parameters from others.

(5) A special test of the quality of data considered is that any three measured parameters can be used to estimate the universal (Newtonian) constant of gravitation G. The ratio of thus determined G_{exp} to its true value of $G = 6.672 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is found to be 1.00 ± 0.28 and 1.05 ± 0.05 for one data set and for nearby clusters from Table 2.

(6) For $T > 1$ keV, the luminosity is proportional to $T^{5/2}$, which confirms the good constancy of the similarity parameter Π_1 and thus the dependence $L \propto U^5$, defining limits of its applicability.

(7) The X-ray luminosity dependence on temperature can be used to determine the distance to the cluster according to formula (14). This formula plays the same role as the Tully–Fisher relation at larger wavelengths.

(8) In Section 7, the universal relation $L = c^5/G$ is discussed, where c is the speed of light. It gives the upper limit on the power of gravitational wave emission in a binary system of two black holes [19]. This universal relation holds on Planckian scales as well, although there it bears a purely formal sense.

Acknowledgments

In conclusion, I would like to thank Professor R A Sunyaev for introducing me the problem, and A A Vikhlinin for providing observational data. I am especially grateful to the referees, whose critical remarks helped to significantly improve the presentation of the results.

The work was partially supported by the Program of Fundamental Research of the Presidium of the RAS itemized under No. 19.

Notes added in proofs. After this paper was submitted, I obtained relation (13) $L \propto T^{5/2}$ directly from the dimensional analysis of the cluster mass $M = N\mu m$, where N is the total number of particles in an object, its thermal energy NkT_{e}, and the gravity constant G in the form

$$L = c_{\mathrm{e}} \left(\frac{kT_{\mathrm{e}}}{\mu m}\right)^{5/2} G^{-1} ,$$

where $c_{\mathrm{e}} = 0.286$ is the numerical factor which is constant to within 28% for 20 objects from Table 1. Combining all constants and conversion factors from keV to kelvins, we arrive at the working relation

$$L = \frac{T_e^{5/2}}{150} \, ,$$

where the luminosity is in units of 10^{37} W, and T_e in keV. This leads to formula (14). Such simple formulas, which are confirmed numerically and supported by observations, demand a detailed theoretical derivation. The commonly accepted explanation of the X-ray luminosity of galaxy clusters by bremsstrahlung radiation needs further development. Of course, these conclusions require better statistical analysis and further complex observations.

The specific per unit mass luminosity of galaxy clusters in units of [$m^2 s^{-3}$] was also calculated. According to data from Table 1, this value, on the average, decreases severalfold with increasing cluster age. This can be explained by the cluster mass increase due to adding low-luminosity (dark?) matter.

References

1. Voit G M *Rev. Mod. Phys.* **77** 207 (2005)
2. Kravtsov A V, Borgani S *Annu. Rev. Astron. Astrophys.* **50** 353 (2012)
3. Vikhlinin A A et al. *Phys. Usp.* **57** 317 (2014); *Usp. Fiz. Nauk* **184** 339 (2014)
4. Bleem L E et al. *Astrophys. J. Suppl.* **216** 27 (2015)
5. Vikhlinin A A, Doctor of Phys.-Math. Sciences Thesis (Moscow: Space Research Institute RAS, 2002)
6. Vikhlinin A et al. *Astrophys. J.* **640** 691 (2006)
7. Rees M *New Perspectives in Astrophysical Cosmology* 2nd ed. (Cambridge: Cambridge Univ. Press, 2000)
8. Sunyaev R A, Zel'dovich Ya B *Annu. Rev. Astron. Astrophys.* **18** 537 (1980)
9. Tully R B, Fisher J R *Astron. Astrophys.* **54** 661 (1977)
10. Zaritsky D et al. *Astron. J.* **147** 134 (2014)
11. Barenblatt G I, Chorin A J, Prostokishin V M *Phys. Usp.* **57** 250 (2014); *Usp. Fiz. Nauk* **184** 265 (2014)
12. Einstein A *Ann. Physik* **35** 679 (1911)
13. Bridgman P W *Dimensional Analysis* (New Haven: Yale Univ. Press, 1931); Translated into Russian: *Analiz Razmernostei* (Moscow – Izhevsk: RKhD, 2001)
14. Surdin V G (Ed.-Compil.) *Galaktiki* (Galaxies) (Moscow:Fizmatlit, 2013)
15. Sedov L I *Similarity and Dimensional Methods in Mechanics* (Moscow: Mir Publ., 1982); Translated from Russian: *Metody Podobiya i Razmernosti v Mekhanike* (Moscow: Nauka, 1972)
16. Barenblatt G I *Scaling* (Cambridge: Cambridge Univ. Press, 2003); *Avtomodel'nye Yavleniya — Analiz Razmernostei i Skeiling* (Automodel Phenomena — Dimensional Analysis and Scaling) (Dolgoprudnyi: Intellekt, 2009)
17. Dibai E A, Kaplan S A *Razmernosti i Podobie Astrofizicheskikh Velichin* (Dimensionality and Similarity of Astrophysical Quantities) (Moscow: Nauka, 1976)
18. Kaizer N *Mon. Not. R. Astron. Soc.* **222** 323 (1986)
19. Peacock J A *Cosmological Physics* (Cambridge: Cambridge Univ. Press, 1999)