METHODOLOGICAL NOTES

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Lagrangian equations of motion of particles and photons in a Schwarzschild field*

V I Ritus

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Contents

Abstract. The equations of motion of a particle in the gravitational field of a black hole are considered in a formulation that uses generalized coordinates, velocities, and accelerations and is convenient for finding the integrals of motion. The equations are rewritten in terms of the physical velocities and accelerations measured in the Schwarzschild frame by a stationary observer using proper local length and time standards. The attractive force due to the field and the centripetal acceleration of a particle is proportional to the particle kinetic energy $m/\sqrt{1 - v^2}$, consistently with the fact that the particle kinetic energy and the photon energy $\hbar\omega$ in the field increase by the same factor compared with their values without a field. The attraction exerted on particles and photons by a gravitational field source is proportional to their kinetic energies. The particle trajectory in the ultrarelativistic limit $v \to 1$ coincides with the photon trajectory.

Keywords: gravitational field, Schwarzschild geometry, mass and energy in gravitation

1. Equation of particle motion along a geodesic trajectory

The motion of a material particle with mass m in a gravity field is defined by the least action principle $\delta S = 0$, stating that the trajectory of a particle between points a and b in the 4-space x^α, $\alpha = 0, 1, 2, 3$, is an extremum of the action S considered as a functional of the trajectory. The action S can be taken in the covariant form [1]

[1] We use the system of units where $c = G = 1$.

V I Ritus Lebedev Physical Institute, Russian Academy of Sciences,
Leninskii prosp. 53, 119991 Moscow, Russian Federation
Tel. +7 (499) 132 64 26. Fax +7 (499) 135 85 33
E-mail: ritus@lpi.ru

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$$S = \int_a^b L\,\mathrm{d}s\,, \qquad L = \frac{1}{2}\,m g_{\alpha\beta}(x)\,\frac{\mathrm{d}x^\alpha}{\mathrm{d}s}\,\frac{\mathrm{d}x^\beta}{\mathrm{d}s}\,, \tag{1}$$

where s is a *scalar* variable, independent of the form of the trajectory $x^\alpha(s)$ being varied, running through the same range of values as for the extremal trajectory. It can be interpreted as the particle proper time if the proper time interval between two close points is defined by the metric tensor $g_{\alpha\beta}$,

$$\mathrm{d}s^2 = -g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\,,$$

with the signature $(-,+,+,+)$.

The Lagrangian function L is a *scalar* that depends on the generalized 4-coordinates $x^\alpha(s)$ and the 4-velocities

$$\frac{\mathrm{d}x^\alpha}{\mathrm{d}s} \equiv \dot{x}^\alpha \equiv u^\alpha\,, \qquad \alpha = 0, 1, 2, 3\,.$$

The Lagrangian equations have the standard form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0\,. \tag{2}$$

Because L does not explicitly depend on s, these equations admit the integral

$$u_\alpha u^\alpha = \mathrm{const}\,. \tag{3}$$

Choosing the constant equal to -1 (i.e., measuring the physical velocity components in the units of the speed of light), we obtain the condition under which s is the particle proper time. In this case, the generalized 4-momentum of a particle in a gravitational field and its square are given by the

* The incentive to write this article was the article by R I Khrapko, submitted to *Physics–Uspekhi* (see p. 1115 of this issue) and his correspondence with the Editorial Board, which contains critical comments on formula (8.1) given in a review by L B Okun' (*Usp. Fiz. Nauk* **158** 511 (1989) [*Phys. Usp.* **32** 543 (1989)]) for the force of attraction of a relativistic particle by a gravity center. This paper shows, in particular, how such a formula could have emerged.

formulas

$$p^{\alpha} = mu^{\alpha}\,, \quad p_{\alpha} p^{\alpha} = -m^2\,, \quad p_{\alpha} = mu_{\alpha} = mg_{\alpha\beta} u^{\beta}\,, \qquad (4)$$

which are analogous to the formulas for the Minkowski space, where, specifically, $g_{\alpha\beta} = \mathrm{diag}\,(-1, 1, 1, 1)$.

Using Lagrangian (1), we can represent equations of motion (2) as

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} + \Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma} = 0\,, \qquad (5)$$

where

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\, g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\delta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\delta}} \right). \qquad (6)$$

are the Christoffel symbols. We draw attention to the fact that the relation of the Christoffel symbols to the metric tensor arose as a consequence of the least action principle with Lagrangian function (1).

Because the derivative $\mathrm{d}u^{\alpha}/\mathrm{d}s$ is the generalized 4-acceleration of the particle, it is natural to call the quantity $-m\Gamma^{\alpha}_{\beta\gamma} u^{\beta} u^{\gamma}$ the '4-force' acting on the particle in a gravitational field, and refer to the $\Gamma^{\alpha}_{\beta\gamma}$ as the strength of this field (see Ref. [1] §§ 85, 87 and [2] § 1, Chapter 4).

The curves $x^{\alpha}(s)$ satisfying Eqn (5) are called geodesics.

We note that integral of motion (3) arising because L has no explicit dependence on s essentially coincides with the Hamiltonian function related to L by the Legendre transformation

$$H = \dot{x}^{\alpha}\, \frac{\partial L}{\partial \dot{x}^{\alpha}} - L\,. \qquad (7)$$

Moreover,

$$H = \frac{1}{2m}\, p^{\alpha} p_{\alpha} = \frac{1}{2}\, m \dot{x}^{\alpha} \dot{x}_{\alpha} = L\,. \qquad (8)$$

This implies that neither function contains the potential energy — an *additive term* dependent solely on 4-coordinates. The kinetic 'energy', however, depends on 4-coordinates because of their presence in the metric tensor $g_{\alpha\beta}(x)$. The Hamiltonian and Lagrangian functions are *scalars*, and this is why the word 'energy' is enclosed in quotation marks.

The use of the proper time and the related scalar Lagrangian function in relativistic classical and quantum mechanics was proposed by Fock [3]. It was subsequently elaborated by Schwinger [4].

An actual gravitational field is created by material bodies and decays far from them owing to the island-like distribution of matter. The metric of such a field transforms into the Minkowski metric $g_{\alpha\beta}(x) \to \eta_{\alpha\beta} = \mathrm{diag}\,(-1, 1, 1, 1)$ at large distances from the bodies that create it. This implies that the space–time curvature is maintained by material bodies in a finite spatial domain, while far from it the 4-space remains flat.

2. Particle motion in the field of a black hole

We consider equations of motion of a particle with the mass m in a centrally symmetric gravitational field of a black hole with a mass M and the gravitational radius $r_{\mathrm{g}} = 2GM/c^2$. Such a field is described by the Schwarzschild metric

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 - \mathrm{d}l^2 = -g_{\alpha\beta}\,\mathrm{d}x^{\alpha}\,\mathrm{d}x^{\beta} = \left(1 - \frac{r_{\mathrm{g}}}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_{\mathrm{g}}}{r}\right)^{-1}\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2 \sin^2\theta\,\mathrm{d}\varphi^2 \qquad (9)$$

with the nonzero components

$$g_{00} = -\left(1 - \frac{r_{\mathrm{g}}}{r}\right), \qquad g_{11} = \left(1 - \frac{r_{\mathrm{g}}}{r}\right)^{-1}, \qquad g_{22} = r^2\,, \qquad g_{33} = r^2 \sin^2\theta\,, \qquad (10)$$

and $\mathrm{d}s$ and $\mathrm{d}\tau$ have the meaning of the respective proper time intervals for a moving observer and an observer at rest in the Schwarzschild frame of reference with the generalized coordinates t, r, θ, φ. The physical meaning of the intervals $\mathrm{d}s$ and $\mathrm{d}\tau$ and the notation agree with those used by Landau and Lifshitz [1].

We note that the gravitational field of a spherical star is described by the Schwarzschild metric down to its surface, where it smoothly matches the internal metric of the star.

A local Cartesian frame of reference can be introduced in the vicinity of each spatial point r, θ, φ, with a triple of unit vectors $\mathbf{e}_r$, $\mathbf{e}_{\theta}$, $\mathbf{e}_{\varphi}$ along the radial, meridional, and azimuthal directions. Changes in actual physical distances along these directions are related to the changes in respective coordinates r, θ, φ as

$$\mathrm{d}x = \sqrt{g_{11}}\;\mathrm{d}r = \left(1 - \frac{r_{\mathrm{g}}}{r}\right)^{-1/2}\mathrm{d}r\,, \qquad (11)$$

$$\mathrm{d}y = \sqrt{g_{22}}\;\mathrm{d}\theta = r\,\mathrm{d}\theta\,, \qquad (12)$$

$$\mathrm{d}z = \sqrt{g_{33}}\;\mathrm{d}\varphi = r \sin\theta\,\mathrm{d}\varphi\,. \qquad (13)$$

Analogously, the change in the actual (physical or proper) time τ measured by a clock resting near a point (r, θ, φ) is related to the change in the coordinate time t measured by a clock resting at infinity by

$$\mathrm{d}\tau = \sqrt{-g_{00}}\;\mathrm{d}t = \left(1 - \frac{r_{\mathrm{g}}}{r}\right)^{1/2}\mathrm{d}t\,. \qquad (14)$$

The angular momentum $\mathbf{L} = [\mathbf{rp}]$ of a particle in a centrally symmetric field is orthogonal to the plane containing the radius vector of the particle $\mathbf{r}$ and its 3-velocity $\mathbf{v}$ at any time instant. Because the angular momentum is preserved in both amplitude and direction, the particle orbit lies in the same plane, which can be treated as the equatorial plane of the Schwarzschild frame of reference by selecting $\theta = \pi/2$.

Among first integrals of motion, there is the square of 4-velocity (3), equal to -1,

$$g^{00} u_0^2 + g_{rr} u^{r\,2} + g_{00} u^{\theta\,2} + g^{\varphi\varphi} u_{\varphi}^2 = -1\,. \qquad (15)$$

Because $\theta = \pi/2$, it follows that $u^{\theta} = 0$. Two more conserved quantities follow from the manifest independence of the Lagrangian function with metric (9) of the angle φ and time t. They are the angular momentum p_{φ} and the energy p_0:

$$p_{\varphi} = \frac{\partial L}{\partial \dot{x}^{\varphi}} = mg_{\varphi\varphi} u^{\varphi} = mu_{\varphi} = m\tilde{L}\,, \qquad (16)$$

$$p_0 = \frac{\partial L}{\partial \dot{x}^0} = mg_{00} u^0 = mu_0 = -m\tilde{E}\,. \qquad (17)$$

Here, we use the established notation for the angular momentum $\tilde{L}$ in units of mc and for the energy $\tilde{E}$ in units of mc^2 (see Refs [5–7]). In this case, we find the contravariant 4-velocity components

$$u^\varphi = g^{\varphi\varphi} u_\varphi = \frac{1}{r^2}\,\tilde{L}\,, \tag{18}$$

$$u^0 = g^{00} u_0 = \frac{\tilde{E}}{1 - r_{\mathrm{g}}/r}\,, \tag{19}$$

$$u^r = \mp\sqrt{\tilde{E}^2 - \left(1 - \frac{r_{\mathrm{g}}}{r}\right)\left(1 + \frac{1}{r^2}\,\tilde{L}^2\right)}\,. \tag{20}$$

The expression for u^r follows from Eqn (15) with Eqns (18) and (19) taken into account. The values $u^r \lessgtr 0$ correspond to the particle motion towards and away from the center. It is noteworthy that the components of the 4-velocity change only if the radius r changes.

We write the universal expressions for the constants of motion in terms of the radius and the particle velocity:

$$\tilde{L} = \frac{r v_{\hat{\varphi}}}{\sqrt{1 - v^2}}\,, \qquad \tilde{E} = \sqrt{\frac{1 - r_{\mathrm{g}}/r}{1 - v^2}}\,. \tag{21}$$

The hat on an index is used to denote the physical components of a vector in a local Lorentz reference system.

Now, using the 4-velocity components u^α and Christoffel symbols for the Schwarzschild metric (with $\theta = \pi/2$),

$$\Gamma^t_{\ tr} = \Gamma^t_{\ rt} = -\Gamma^r_{\ rr} = \frac{r_{\mathrm{g}}}{2r^2(1 - r_{\mathrm{g}}/r)}\,,$$

$$\Gamma^r_{\ tt} = \frac{r_{\mathrm{g}}}{2r^2}\left(1 - \frac{r_{\mathrm{g}}}{r}\right), \quad \Gamma^r_{\ \theta\theta} = \Gamma^r_{\ \varphi\varphi} = -r\left(1 - \frac{r_{\mathrm{g}}}{r}\right),$$

$$\Gamma^\varphi_{\ \varphi r} = \Gamma^\varphi_{\ r\varphi} = \Gamma^\theta_{\ r\theta} = \Gamma^\theta_{\ \theta r} = \frac{1}{r}\,,$$

$$\Gamma^\varphi_{\ \varphi\theta} = \Gamma^\varphi_{\ \theta\varphi} = \Gamma^\theta_{\ \varphi\varphi} = 0\,,$$

we can obtain equations of motion (5):

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}s^2} = \pm\frac{2\tilde{L}}{r^3}\sqrt{\tilde{E}^2 - \left(1 - \frac{r_{\mathrm{g}}}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)}\,, \tag{22}$$

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} = \pm\frac{r_{\mathrm{g}}\tilde{E}}{r^2(1 - r_{\mathrm{g}}/r)^2}\sqrt{\tilde{E}^2 - \left(1 - \frac{r_{\mathrm{g}}}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)}\,, \tag{23}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}s^2} = -\frac{r_{\mathrm{g}}}{2r^2} + \frac{\tilde{L}^2}{r^3}\left(1 - \frac{3r_{\mathrm{g}}}{2r}\right). \tag{24}$$

Thus, three components of the 4-force entering the particle equations of motion are different from zero. This means that, in general, the azimuthal u^φ, radial u^r, and null u^0 components of the generalized 4-velocity u^α change as the particle moves. However, the component u^θ does not enter the equations because the motion stays in the same plane due to the angular momentum conservation.

3. Velocity and acceleration of a particle measured by a Schwarzschild observer

We now adapt the above equations to an observer located at a certain point (r, θ, φ) in the Schwarzschild frame of reference and measuring the three-dimensional physical velocity $\mathbf{v}$ and the acceleration $\mathbf{w}$ based on readings of the local proper time τ of the observer's clock.

The absolute value of the physical velocity is

$$v = \frac{\mathrm{d}l}{\mathrm{d}\tau} = \sqrt{g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + g_{\varphi\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2}\,, \tag{25}$$

and its radial and azimuthal components

$$v_{\hat{r}} = \sqrt{g_{rr}}\,\frac{\mathrm{d}r}{\mathrm{d}\tau}\,, \qquad v_{\hat{\varphi}} = \sqrt{g_{\varphi\varphi}}\,\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \tag{26}$$

define the velocity 3-vector

$$\mathbf{v} = v_{\hat{r}}\mathbf{e}_r + v_{\hat{\varphi}}\mathbf{e}_\varphi\,. \tag{27}$$

Because $\mathrm{d}s = \mathrm{d}\tau\sqrt{1 - v^2}$, the Lorentzian physical components of the 4-velocity are related to its generalized contravariant components as

$$u_{\hat{r}} \equiv \frac{v_{\hat{r}}}{\sqrt{1 - v^2}} = \sqrt{g_{rr}}\,u^r\,, \quad u_{\hat{\varphi}} \equiv \frac{v_{\hat{\varphi}}}{\sqrt{1 - v^2}} = \sqrt{g_{\varphi\varphi}}\,u^\varphi\,,$$
$$u^{\hat{0}} \equiv \frac{1}{\sqrt{1 - v^2}} = \frac{\mathrm{d}\tau}{\mathrm{d}s} = \sqrt{-g_{00}}\,\frac{\mathrm{d}t}{\mathrm{d}s} = \sqrt{-g_{00}}\,u^0\,. \tag{28}$$

The physical momentum $\mathbf{p}$ and the energy ε of the particle form the physical 4-momentum $p^{\hat{\alpha}}$ and are expressed in terms of the physical velocity as

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2}}\,, \qquad \varepsilon = \frac{m}{\sqrt{1 - v^2}}\,. \tag{29}$$

With these relations between the physical and generalized components of the 4-velocity, system of equations (22)–(24) can be easily transformed into the equivalent system

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \frac{m}{\sqrt{1 - v^2}\sqrt{1 - r_{\mathrm{g}}/r}}\left\{-\frac{M}{r^2}\,\mathbf{e}_r + \left(1 - \frac{r_{\mathrm{g}}}{r}\right)\frac{v_{\hat{\varphi}}}{r}\,\mathbf{v}_R\right\}, \tag{30}$$

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} = -\frac{m}{\sqrt{1 - v^2}\sqrt{1 - r_{\mathrm{g}}/r}}\,\frac{M}{r^2}\,v_{\hat{r}}\,, \tag{31}$$

where the 3-vector

$$\mathbf{v}_R = v_{\hat{\varphi}}\mathbf{e}_r - v_{\hat{r}}\mathbf{e}_\varphi \tag{32}$$

is orthogonal to the velocity $\mathbf{v}$, equal to it in absolute value, and obtained by a *right* turn of $\mathbf{v}$ through the angle $\pi/2$.

Using the physical 3-vector of acceleration

$$\mathbf{w} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 - r_{\mathrm{g}}/r}}\left\{-\frac{1 - v^2}{r^2}\,M\mathbf{e}_r + \left(1 - \frac{3r_{\mathrm{g}}}{2r}\right)\frac{v_{\hat{\varphi}}}{r}\,\mathbf{v}_R\right\}, \tag{33}$$

we rewrite Eqns (30)–(32) in the form of spatial and temporal components of the 4-acceleration $a^{\hat{\alpha}} \equiv \mathrm{d}u^{\hat{\alpha}}/\mathrm{d}s$:

$$\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau}, \frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau}\right) = m\sqrt{1 - v^2}(\mathbf{a}, a^{\hat{0}})\,, \tag{34}$$

$$(\mathbf{a}, a^{\hat{0}}) = \frac{\mathrm{d}u^{\hat{\alpha}}}{\mathrm{d}s} = \left(\frac{\mathbf{w}}{1 - v^2} + \mathbf{v}\,\frac{(\mathbf{v}\mathbf{w})}{(1 - v^2)^2}\,, \frac{(\mathbf{v}\mathbf{w})}{(1 - v^2)^2}\right). \tag{35}$$

The appearance of the factor $\sqrt{1-v^2}$ in the right-hand side of Eqn (34) is related to the use of the derivative with respect to the local time τ in the left-hand side, in contrast to the derivative with respect to the proper time s of a moving particle in the definition of $a^{\hat{\alpha}}$ (see Ref. [8], formula (193), or [1] §§ 7, 9). Formula (34) should be compared with formula (7.3) in the review by Okun' [9].

The metric manifests itself in the expressions for the velocity $\mathbf{v}$ and acceleration $\mathbf{w}$ [see Eqns (26) and (33)].

In representation (30), the first term comes from the force of attraction to the center, and the second term corresponds to the inertia force; the latter owes its existence to the nonzero angular momentum and the related azimuthal velocity $v_{\hat{\varphi}}$ [see Eqn (21)]. We stress that the attraction force is always directed to the center of attraction, whereas the inertia force is aligned with the vector $\mathbf{v}_R$. The proportionality of the 4-force to the kinetic energy ε in equations of motion (30) and (31) is obvious.

Although the radial velocity $v_{\hat{r}}$ can have any sign (negative when moving toward the center and positive otherwise), the vector $\mathbf{v}_R$ is always directed to the convex side of the trajectory. For circular motion, as well as for the apastron and periastron where the radial velocity vanishes, the vector $\mathbf{v}_R$ is directed along the radius, $\mathbf{v}_R = v\mathbf{e}_r$. In these cases, the acceleration vanishes, $\mathbf{w} = 0$, connecting the velocity and radius by formula (43).

We draw attention to the fact that the particle kinetic energy varies only in the case of a nonzero radial velocity [see Eqn (31)].

Formula (30) can also be written as

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \frac{m}{\sqrt{1-v^2}\sqrt{1-r_{\mathrm{g}}/r}} \times \left\{ -\frac{M}{r^2}(\mathbf{e}_r + v_{\hat{\varphi}}\mathbf{v}_R) + \left(1-\frac{r_{\mathrm{g}}}{2r}\right)\frac{v_{\hat{\varphi}}}{r}\,\mathbf{v}_R \right\}. \quad (36)$$

In this representation, the attraction term coincides with that given in the review by Okun' [9]. This term acquires half the Schwarzschild contribution from the orbital term, such that the vector $\mathbf{e}_r$ is replaced by the vector

$$\mathbf{e}_r + v_{\hat{\varphi}}\mathbf{v}_R = \frac{1}{r}\left[(1+v^2)\,\mathbf{r} - (\mathbf{r}\mathbf{v})\mathbf{v}\right] \quad (37)$$

given by Okun'. The orbital term changes accordingly. However, the attraction term has lost its invariable orientation to the center of attraction.

Because $\mathbf{e}_r$ and $\mathbf{v}_R$ are directed to the convex side of the trajectory, and $v_{\hat{\varphi}} \geqslant 0$, the attraction term in Eqn (36) is always directed to where the trajectory is concave, and the orbital term is directed to the convex side. Clearly, here too, these terms compensate each other when $v_{\hat{r}} = 0$ and $v_{\hat{\varphi}} = v$.

Decompositions (27) and (32) of the vectors $\mathbf{v}$ and $\mathbf{v}_R$ with respect to the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ allow obtaining other representations of formula (30) as well.

4. Finite and infinite orbits

When moving along a circle or in the vicinity of the apastron and periastron, the radial velocity u^r vanishes. This implies that

$$\tilde{E} = \sqrt{\left(1-\frac{r_{\mathrm{g}}}{r}\right)\left(1+\frac{\tilde{L}^2}{r^2}\right)}\,. \quad (38)$$

The function in the right-hand side is called the effective potential,

$$U(x,L) = \sqrt{\left(1-\frac{1}{x}\right)\left(1+\frac{L^2}{x^2}\right)}\,. \quad (39)$$

Considered as a function of the dimensionless radius $x = r/r_{\mathrm{g}}$ and the dimensionless angular momentum $L = \tilde{L}/r_{\mathrm{g}}$, the potential is positive in the physical range $1 < x < \infty$, is equal to 0 at $x = 1$ and 1 at $x = \infty$, and attains the respective maximum and minimum at the points

$$x_{1,2} = L^2 \mp L\sqrt{L^2-3}\,. \quad (40)$$

For $L < \sqrt{3}$, the potential monotonically increases with x from 0 to 1.

For $L = \sqrt{3}$, the potential has an inflection point at $x = x_0 = 3$, attaining the value $\sqrt{8/9}$ there.

For $L > \sqrt{3}$, the maximum $U(x_1, L)$ and minimum $U(x_2, L)$ values of the potential increase monotonically with L in the intervals

$$\sqrt{\frac{8}{9}} < U(x_1, L) < \infty \quad \text{and} \quad \sqrt{\frac{8}{9}} < U(x_2, L) < 1\,,$$

taking the values 1 and $\sqrt{25/27}$ at $L = 2$ (see Fig. 1).

Thus, for $L > \sqrt{3}$, the potential takes the form of a scoop within which are the finite Kepler orbits with the energy $\tilde{E}$ lying between the minimum and maximum of the potential if $L < 2$, or between the minimum and 1 if $L > 2$.

We are also interested in infinite orbits that begin and end at infinity or end at $r = r_{\mathrm{g}}$, i.e., in the black hole.

For circular motion and at the apastron and periastron, the velocity is linked to the radius and angular momentum via the definition of the latter,

$$L = x\,\frac{v}{\sqrt{1-v^2}}\,, \qquad x = \frac{r}{r_{\mathrm{g}}}\,, \quad (41)$$

because in these cases, the radial velocity vanishes and $v_{\hat{\varphi}} = v$. On the other hand, from the radial acceleration being zero at these points, a relation between the angular momentum and

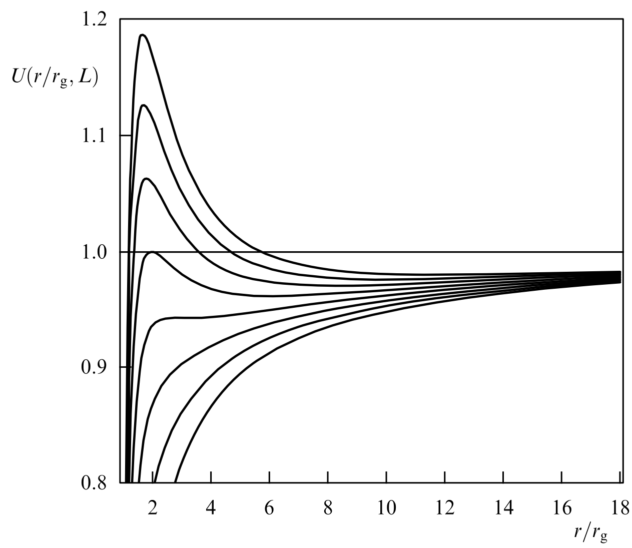

Figure 1. Effective potential $U(x, L)$ for $L = \sqrt{n}$, $n = 0, 1, 2, 3, 4, 5, 6, 7$.

the radius follows:

$$L = \frac{x}{\sqrt{2x-3}}\,, \qquad x = \frac{r}{r_{\mathrm{g}}} \qquad (42)$$

[see Eqn (24)]. According to Eqns (41) and (42), the velocity and radius are then related by

$$v = \frac{1}{\sqrt{2(x-1)}}\,, \qquad x = \frac{r}{r_{\mathrm{g}}}\,. \qquad (43)$$

Thus, the last two formulas follow from the equality of acceleration to zero.

The velocity of a particle moving along a circle increases if the radius decreases, and attains the speed of light at $r = 1.5 r_{\mathrm{g}}$.

Weinberg derived formula (42) and called it the equilibrium condition, considering the circle as a limit of an elliptic orbit (see (8.4.24) in Ref. [2]). This is valid only for circles with the radius $r_2(L)$ corresponding to the potential minimum $U(x, L)$, because for circles with the radius $r_1(L)$ that corresponds to the potential maximum, the radii close to r_1 belong to infinite orbits. At the same time, the formulas above are valid for any finite orbit at points where the radial velocity and acceleration are equal to zero.

Useful information is contained in the longitudinal and transverse components of the acceleration $\mathbf{w}$:

$$\mathbf{w} = w_{\parallel}\,\frac{\mathbf{v}}{v} + w_{\perp}\,\frac{\mathbf{v}_R}{v}\,,$$
$$w_{\parallel} = -\frac{v_{\hat r}(1-v^2)}{v\sqrt{1-r_{\mathrm{g}}/r}}\,\frac{M}{r^2}\,, \qquad (44)$$
$$w_{\perp} = -\frac{v_{\hat\varphi}}{v\sqrt{1-r_{\mathrm{g}}/r}}\left[(1-v^2)\,\frac{M}{r^2} - \left(1-\frac{3r_{\mathrm{g}}}{2r}\right)\frac{v^2}{r}\right].$$

In particular, for a particle falling radially with a small velocity ($-v_{\hat r} = v \ll 1$), the acceleration is $w_{\parallel} = g$, the local free-fall acceleration measured by Galilei.

If $w_{\perp} = 0$, then we have relation (43) between the velocity and radius of the circular orbit, whence, in particular, $v = 1$ on the orbit with the radius $r = (3/2) r_{\mathrm{g}}$. For the radius $r \lessgtr (3/2) r_{\mathrm{g}}$, a particle with $v = 1$ acquires a nonzero acceleration $w_{\perp} \lessgtr 0$, which expels the particle from its circular orbit toward the black hole or infinity.

A present-day Galilei measuring the speed of a satellite on a circular orbit would be able to verify that $w_{\parallel}$ and $w_{\perp}$ are equal to zero because $v_{\hat r} = 0$ and $v_{\hat\varphi} = v$ satisfies Eqn (43).

5. Motion of an ultrarelativistic particle and a photon

The trajectory of a photon follows from the particle equation of motion

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{u^r}{u^{\varphi}} = \mp r^2 \sqrt{\gamma v^2 - \left(1-\frac{r_{\mathrm{g}}}{r}\right)\frac{1}{r^2}}\,\Bigg|_{v\to 1}$$
$$\to \mp r^2 \sqrt{\gamma - \left(1-\frac{r_{\mathrm{g}}}{r}\right)\frac{1}{r^2}}\,, \qquad (45)$$

if we let γ denote the constant of motion $(\tilde E/\tilde L)^2$ and let the particle velocity v tend to the speed of light. Only one constant of motion γ remains in the resultant equation for a photon, instead of the two featuring in Eqn (21).

It is convenient to make a transformation from $\mathrm{d}r/\mathrm{d}\varphi$ to $\mathrm{d}u/\mathrm{d}\varphi$, where $u = r_{\mathrm{g}}/r \leqslant 1$. Then

$$u' \equiv \frac{\mathrm{d}u}{\mathrm{d}\varphi} = \pm\sqrt{\gamma r_{\mathrm{g}}^2 - u^2 + u^3}\,. \qquad (46)$$

Hence, it follows that

$$u'' + u = \frac{3}{2}\,u^2\,. \qquad (47)$$

For infinite trajectories, the constant of motion is $\gamma r_{\mathrm{g}}^2 = (r_{\mathrm{g}}/b)^2$, where b is the impact parameter. We assume that r_{g}/b is small.

In the zeroth-order approximation, setting the right-hand side of Eqn (47) to zero, we find the solution

$$u_0 = \frac{r_{\mathrm{g}}}{b}\,\sin\varphi\,,$$

which corresponds to a straight trajectory, unperturbed by the field, with the impact parameter $b = r\sin\varphi$.

In the next approximation, we write $u = u_0 + u_1$. The equation for u_1,

$$u_1'' + u_1 = \frac{3}{2}\,u_0^2\,,$$

admits the solution

$$u_1 = \frac{1}{2}\left(\frac{r_{\mathrm{g}}}{b}\right)^2 (1+\cos^2\varphi)\,.$$

The solution

$$u = u_0 + u_1 = \frac{r_{\mathrm{g}}}{b}\,\sin\varphi + \frac{1}{2}\left(\frac{r_{\mathrm{g}}}{b}\right)^2 (1+\cos^2\varphi)$$

should tend to zero for $r = \infty$. This condition is satisfied for the angles

$$\varphi_1 = -\frac{r_{\mathrm{g}}}{b} \quad \text{and} \quad \varphi_2 = \pi + \frac{r_{\mathrm{g}}}{b}\,. \qquad (48)$$

Then the angle a photon deflected from unperturbed straight motion is

$$\Delta\varphi = \varphi_2 - \varphi_1 - \pi = \frac{2r_{\mathrm{g}}}{b}\,. \qquad (49)$$

This result was obtained by Einstein in 1915 [10]. Clearly, it is also valid for an ultrarelativistic particle.

Equation (46) leads to real-valued solutions if the function

$$f(x) = \left(\frac{r_{\mathrm{g}}}{b}\right)^2 - \frac{1}{x^2} + \frac{1}{x^3}\,, \qquad x = \frac{r}{r_{\mathrm{g}}} \qquad (50)$$

in the radicand is positive. In the interval $r_{\mathrm{g}} < r < \infty$, this function has a minimum at the point $r = 1.5 r_{\mathrm{g}}$, where

$$f(x_0) = \left(\frac{r_{\mathrm{g}}}{b}\right)^2 - \frac{4}{27}\,, \qquad x_0 = \frac{3}{2}\,,$$

and takes the value $(r_{\mathrm{g}}/b)^2$ at the ends of the interval. Hence, for the impact parameter $b < b_{\min} = \sqrt{27/4}\, r_{\mathrm{g}}$, the function $f(x)$ is positive in the entire physical range of distances and the fall of a photon with such an impact parameter ends in a black hole.

For $b = b_{\min}$, the orbit of a photon falling from infinity reaches the radius $r_0 = 1.5 r_g$, makes many turns there, and eventually ends at the black hole or infinity.

For $b > b_{\min}$, the orbit of a photon reaches a radius larger than r_0 and then continues to infinity.

It follows that the method of solving Eqn (47) considered above is appropriate if $(r_g/b) \ll \sqrt{4/27} \approx 0.38$.

6. Conclusions

A gravitational field equally increases the *kinetic* energy of a particle and the energy (frequency) of a photon falling in this field from infinity. If their energies are $\varepsilon_\infty = m/\sqrt{1 - v_\infty^2}$ and $\hbar\omega_\infty$ outside the field, then in the field they increase in the same way:

$$\varepsilon = \frac{m}{\sqrt{1 - v^2}} = \frac{\varepsilon_\infty}{\sqrt{-g_{00}}}\,, \qquad \hbar\omega = \frac{\hbar\omega_\infty}{\sqrt{-g_{00}}}\,, \tag{51}$$

(cf. (88.9) and (88.6) in Ref. [1]). One can assert the identical attraction of energies ε and $\hbar\omega$ by the field. And yet, the change in ε entails a change in velocity and the appearance of particle acceleration, whereas the change in $\hbar\omega$ does not affect the velocity of the photon, and there is no acceleration. However, the conservation of momentum leads to a change in the direction of the particle and the photon, and in the ultrarelativistic case, to the coincidence of their trajectories.

To conclude, we note that according to a proposal by H Weyl, the mass of a particle is identified with the relativistic invariant $m = \sqrt{\varepsilon^2 - \mathbf{p}^2}$ (see Ref. [11], § 27). The mass of a photon is zero.

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