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Gravitational mass of the photons

R I Khrapko

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Contents

Abstract. It is shown that the gravitational and inertial masses of a photon or, indeed, of any body are equal. Motion past and free fall onto a center of gravitational attraction are considered.

Keywords: equivalence principle, mass

1. Introduction

As is well known, the trajectory of a pulse of radiation passing near the Sun bends from a straight line toward the Sun. This indicates the presence of acceleration of a photon, directed perpendicular to its velocity. In turn, the acceleration means that

(1) the photon experiences an attraction force **F** and exhibits inertia, i.e., it has an inertial mass m_{i} in accordance with the law

$$\mathbf{F} = m_{\mathrm{i}}\mathbf{a}\,, \tag{1}$$

(2) the force acting on the photon is the gravity force **F** obeying the law

$$\mathbf{F} = \gamma\,\frac{Mm_{\mathrm{g}}\mathbf{r}}{r^3} = m_{\mathrm{g}}\,g\,\frac{\mathbf{r}}{r}\,, \tag{2}$$

and, accordingly, the photon has a gravitational mass, which is denoted by m_{g}.

As shown in Section 2, the photon acceleration in this case equals the free fall acceleration $a = g \equiv \gamma M/r^2$, and hence the equality $m_{\mathrm{i}}a = m_{\mathrm{g}}g$ implies the equality of the gravitational and inertial photon masses, $m_{\mathrm{i}} = m_{\mathrm{g}}$.

It is essential that inertial properties of any object (a body or a photon) are correctly represented by its acceleration according to formula (1) only in two cases:

(1) when the velocity of the object is perpendicular to its acceleration, as in the case where it passes by the Sun,

(2) when the velocity is absent altogether, i.e., the body is at rest.

In general, the proportionality coefficient in formula (1) becomes a tensor, $\mathbf{F} = \hat{m}\mathbf{a}$, because the acceleration does not coincide with the force in direction. Accordingly, the formula $\mathbf{F} = \hat{m}\mathbf{a}$ does not provide a correct value of the object's inertia, i.e., its inertial mass m_{i}. This is explicitly seen in the example of a radially falling photon: gravity force (2) acts on the photon, whereas its acceleration is equal to zero. Therefore, the desire to use the formula $\mathbf{F} = \hat{m}\mathbf{a}$ for measuring inertia results in an infinitely large inertia of a photon. And the inertia of ultrarelativistic particles in such measurements turns out to be exceedingly large. This follows from the fact that the particle acceleration under the action of a longitudinal force decreases too fast with the increase in velocity, because the coefficient of proportionality between the force and acceleration, in accordance with the formula

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}\,, \tag{3}$$

increases faster than the inertial mass does [1, 2],

$$m_{\mathrm{i}} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{4}$$

(here, m_0 denotes the invariant mass, i.e., the mass of a particle at rest).

The determination of the inertial mass of a radially falling object is discussed in Section 3. It is shown there that the acceleration of vertical fall depends on the velocity as $a = g(1 - v^2/c^2)$, i.e., force (3), being in this case the gravitational force, depends on the velocity as

$$F = \frac{m_0 g}{\sqrt{1 - v^2/c^2}} = m_{\mathrm{i}} g \tag{5}$$

[here, relation (4) is used]. By virtue of Eqn (2), $F = m_{\mathrm{g}}g$, and equality (5), the gravitational mass is once again equal to the inertial mass.

We note that in order to consistently determine the inertial properties of an object, instead of the formula $\mathbf{F} = \hat{m}\mathbf{a}$, the formula $\mathbf{p} = m_{\mathrm{i}}\mathbf{v}$ has to be used as the definition of the inertial mass because there are indubitable operational definitions for the velocity and momentum [3]. In particular, to determine

R I Khrapko Moscow Aviation Institute,
Volokolamskoe shosse 4, 125993 Moscow, Russian Federation
E-mail: khrapko_ri@hotmail.com, http://khrapkori.wmsite.ru

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the momentum, one has to stop the moving object and measure the force impulse $\int \mathbf{F}\,\mathrm{d}t = \mathbf{p}$ transferred to the obstacle stopping the object. When derived from the formula $\mathbf{p} = m_{\mathrm{i}}\mathbf{v}$, the inertial mass m_{i} satisfies the conservation law.

2. Gravitational force is perpendicular to the velocity

Strictly speaking, it was rigorously shown by Einstein that radiation carries inertia between emitting and absorbing bodies [4]. However, it was understood even earlier that radiation is attracted to Earth and, consequently, has gravitational and inertial masses. According to this understanding, in 1801 Soldner used Newton's laws

$$F = \frac{kmM}{r^2}\,, \qquad F = ma\,, \tag{6}$$

assuming that the gravitational mass equals the inertial one (cited from Ref. [5]). He argued in the same way as did Rutherford 110 years later on observing the deflection of alpha particles. Soldner found that the light flux passing a mass M must deflect by the angle α,

$$\tan\frac{\alpha}{2} = \frac{kM}{c^2 R}\,, \qquad \alpha \approx \frac{2kM}{c^2 R} \tag{7}$$

(here, R is the impact parameter). But in 1919, Eddington's expedition observing a solar eclipse found the deflection to be twice as large, in agreement with Einstein's theory.

Ginzburg [5] explained this actual doubling of deflection with respect to formula (7) by the non-Euclidean character of space in a gravitational field; the attraction of the Sun deflects radiation or a moving body from motion along a geodesic in three-dimensional space. But the geodesic line in the vicinity of the Sun deviates from the geodesic passing far from the Sun by the same angle α. This is why the trajectory is deflected twice as strongly in the vicinity of the Sun compared with the trajectory passing far from the Sun. Only half of this double deflection is due to the attraction to the Sun, the other half coming as a consequence of the three-dimensional space being curved by the Sun. This phenomenon is illustrated in Fig. 1.

For the actual computation of this phenomenon in three-dimensional space, we use a spacetime with Schwarzschild coordinates t, r, θ, φ and the metric

$$\mathrm{d}s^2 = \frac{r-1}{r}\,\mathrm{d}t^2 - \frac{r}{r-1}\,\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\varphi^2 \tag{8}$$

(we set $c = r_{\mathrm{g}} = 2M = 1$), and the equation for geodesics with t as a parameter ($\sin^2\theta = 1$):

$$\begin{gathered}\frac{D}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} \equiv \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = \alpha\frac{\mathrm{d}x^i}{\mathrm{d}t}\,,\\ \Gamma^r_{tt} = \frac{r-1}{2r^3}\,, \quad \Gamma^r_{rr} = -\Gamma^t_{tr} = \frac{-1}{2r(r-1)}\,, \quad \Gamma^r_{\varphi\varphi} = -(r-1)\,.\end{gathered} \tag{9}$$

Here, the right-hand side of the equation for the geodesic is not equal to zero but proportional to the tangent vector, because t is not a canonical parameter.

To compute the acceleration experienced by an object flying past the Sun, it suffices to consider only circular orbits [6, 7]. For such orbits, we set $\mathrm{d}r/\mathrm{d}t \equiv 0$ in Eqn (9), and are then left with only the equation with $i = r$,

$$\begin{gathered}\Gamma^r_{tt} + \Gamma^r_{\varphi\varphi}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = 0\,,\\ \frac{r-1}{2r^3} = (r-1)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2, \qquad \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 = \frac{1}{2r^3}\,.\end{gathered} \tag{10}$$

It can be easily found that for $r = 3/2$, geodesic (10) becomes isotropic, i.e., it represents an orbit of a photon. Indeed, from Eqn (8), with due regard for Eqn (10), it follows that

$$\begin{gathered}\mathrm{d}s^2 \equiv \frac{r-1}{r}\,\mathrm{d}t^2 - r^2\,\mathrm{d}\varphi^2 = \left(\frac{r-1}{r} - \frac{r^2}{2r^3}\right)\mathrm{d}t^2 = 0\,,\\ r = \frac{3}{2}\,.\end{gathered} \tag{11}$$

World line (10) is a spacetime geodesic that represents motion along a circle in space. The centripetal acceleration of such motion is determined by the second time derivative of the radial deviation of this circle, $r = \mathrm{const}$, from the *tangent geodesic* line of the two-dimensional space r, φ with the metric

$$\mathrm{d}l^2 = \frac{r}{r-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2\,. \tag{12}$$

We now find the equation of such a geodesic line $r(\varphi)$. Considering φ in Eqn (9) as a parameter and taking into account that for the tangent line at $\varphi = 0$ we have $\mathrm{d}r/\mathrm{d}\varphi = 0$, for $\varphi = 0$ and $i = r$ we find

$$\frac{\mathrm{d}^2r}{\mathrm{d}\varphi^2} + \Gamma^r_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \Gamma^r_{\varphi\varphi} = \alpha\frac{\mathrm{d}r}{\mathrm{d}\varphi}\,, \qquad \frac{\mathrm{d}^2r}{\mathrm{d}\varphi^2} = r-1\,. \tag{13}$$

This equation gives the sought second derivative of the radial displacement of a circular orbit from the geodesic. The second derivative with respect to the time coordinate follows after applying Eqn (10):

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{r-1}{2r^3}\,. \tag{14}$$

We can now find the centripetal acceleration on an arbitrary circular orbit. It turns out to be equal to the usual g, as mentioned in the Introduction:

$$a = \frac{\sqrt{g_{rr}}}{g_{tt}}\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{1}{2r^2}\sqrt{\frac{r}{r-1}} = g\,. \tag{15}$$

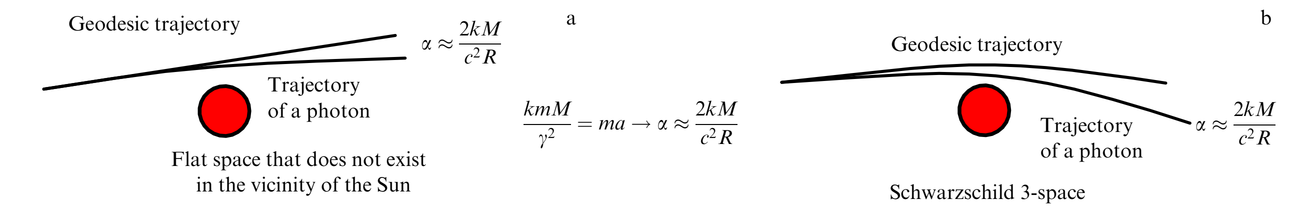

Figure 1. (a) Trajectory of a photon deviates from the geodesic of the space because of the Sun's attraction. (b) However, the geodesic proper is also curved because the Sun makes the space curved.

The curvature of the world line of a body at rest (for which $r = \text{const}$ and $\varphi = \text{const}$) is $D^2r/\mathrm{d}t^2 = \Gamma^r_{tt} = (r-1)/(2r^3)$. And in the General Relativity framework, a body at rest has the acceleration of free fall, which is given exactly by the formula

$$g = \frac{\sqrt{g_{rr}}}{g_{tt}} \frac{D^2 r}{\mathrm{d}t^2} = \frac{1}{2r^2}\sqrt{\frac{r}{r-1}}\,. \tag{16}$$

3. Gravitational force aligned with the velocity

For a vertically falling object, Eqn (9) gives [8]

$$\text{for} \quad i = t: \quad \Gamma^t_{tr} 2 \frac{\mathrm{d}r}{\mathrm{d}t} \equiv \frac{1}{r(r-1)} \frac{\mathrm{d}r}{\mathrm{d}t} = \alpha\,,$$

$$\text{for} \quad i = r: \quad \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \Gamma^r_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \Gamma^r_{tt}$$

$$\equiv \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{1}{2r(r-1)}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{r-1}{2r^3} = \alpha \frac{\mathrm{d}r}{\mathrm{d}t}\,.$$

Eliminating α, we arrive at the equation for a geodesic:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{3}{2r(r-1)}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{r-1}{2r^3} = 0\,. \tag{17}$$

The derivative $\mathrm{d}r/\mathrm{d}t$ is related to the velocity,

$$v = -\frac{\mathrm{d}r}{\mathrm{d}t}\sqrt{\frac{g_{rr}}{g_{tt}}} = -\frac{\mathrm{d}r}{\mathrm{d}t}\frac{r}{r-1}\,,$$
$$g_{tt} = \frac{r-1}{r}\,, \quad g_{rr} = \frac{r}{r-1}\,, \tag{18}$$

and the second derivative $\mathrm{d}^2r/\mathrm{d}t^2$ is related to the acceleration,

$$a = \frac{1}{\sqrt{g_{tt}}}\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{1}{\sqrt{g_{tt}}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\frac{r}{r-1}\right)$$
$$= -\sqrt{\frac{r}{r-1}}\left(\frac{\mathrm{d}^2 r}{\mathrm{d}t^2}\frac{r}{r-1} - \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 \frac{1}{(r-1)^2}\right). \tag{19}$$

Substituting the second derivative from Eqn (17) and the first derivative from Eqn (18), we obtain

$$a = \frac{1}{2r^2}\sqrt{\frac{r}{r-1}}\,(1 - v^2) = g(1 - v^2)\,. \tag{20}$$

Hence, $a = g(1 - v^2/c^2)$, as mentioned in the Introduction.

4. Conclusions

We consider the motion of bodies and radiation past a center of gravitational attraction and their free fall on this center in the framework of General Relativity in order to calculate the three-dimensional acceleration that accompanies such motion and is observed in three-dimensional space around the attracting center. As a result, it is shown that the gravitational force exerted on the gravitational mass of the object causes acceleration that corresponds to the inertial mass of the object equal to its gravitational mass. This is also valid for photons. The double deflection of photons passing in the vicinity of an attracting center is explained not by their double gravitational mass but by the curvature of the geodesic in the vicinity of an attracting center compared with the remote geodesic [9].

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