# Coherent and semiclassical states of a free particle 

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#### Abstract

Coherent states (CSs) were first introduced and $\overline{\text { studied in detail for bound motion and discrete-spectrum sys- }}$ tems like harmonic oscillators and similar systems with a quadratic Hamiltonian. However, the problem of constructing CSs has still not been investigated in detail for the simplest and physically important case of a free particle, for which, besides being physically important, the CS problem is of didactic value in teaching quantum mechanics, with the CSs regarded as examples of wave packets representing semiclassical motion. In this paper, we essentially follow the Malkin-DodonovMan'ko method to construct the CSs of a free nonrelativistic particle. We give a detailed discussion of the properties of the CSs obtained, in particular, the completeness relations, the minimization of uncertainty relations, and the evolution of the corresponding probability density. We describe the physical conditions under which free-particle CSs can be considered semiclassical states.


## 1. Introduction

Coherent states (CSs) play an important role in modern quantum theory as states that provide a natural relation between quantum mechanical and classical descriptions.

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They have a number of useful properties and, as a consequence, a wide range of applications, e.g., in semiclassical descriptions of quantum systems, in quantization theory, in condensed matter physics, in radiation theory, and in quantum computations (see, e.g., Refs [1-7]). Although there are numerous publications devoted to constructing CSs of different systems, a universal definition of a CS and a workable scheme to construct them for an arbitrary physical system is not known. However, we believe that the problem of constructing CSs for systems with quadratic Hamiltonians of the general form was completely solved by Dodonov and Man'ko, using Malkin's and Man'ko's integral of motion method (see [6-8]). It should be noted that extracting concrete sets of CSs and their properties (for a given quadratic system) from their general results sometimes requires additional technical efforts. In this article, we turn our attention to the CSs of a free particle. Besides their physical importance, there is a didactic advantage of using free-particle CSs in teaching quantum mechanics, regarding them as examples of exact wave packages representing semiclassical particle motion. In this relation, we note that CSs were first introduced and studied in detail for systems with bounded motion and a discrete spectrum, like a harmonic oscillator or a charged particle in a magnetic field. However, for such a simple and physically important system as a free particle, the problem of CS construction was not solved at that time. We believe that this situation is explained by the fact that the free particle represents unbounded motion with a continuous energy spectrum, and a generalization of the initial (Glauber) scheme of constructing the CSs of a harmonic oscillator was not so obvious in this case. Although CSs of a free particle, in principle, could be extracted from the abovementioned general results of Dodonov and Man'ko, many authors (ignoring or simply unaware of their results) keep trying to construct CSs of a free particle, inventing their own ways. Describing these attempts, we have to cite Refs [9-12] devoted to this problem. In our opinion, no single one of these studies completely solves the problem under consideration. The authors of Ref. [10] have quite closely approached the goal, choosing a particular case of initial states for their CSs. But even for such initial states, they did not derive an explicit form of time-dependent free particle CSs and did not study their
properties. In fact, their program was realized in [9], but the author did not identify his states with some kind of CSs. In [11], the authors consider the limit of zero frequency in the CSs of a harmonic oscillator, deriving a sort of CSs for a free particle. Their CSs are expressed in terms of sums of Hermite polynomials, and the complicated forms of the CSs hampers their interpretation, study, and application. Another study [12] treats free-particle CSs in the framework of a general approach to constructing CSs for systems with a continuous spectrum. The approach is based on using nonnormalizable fiducial states and involves quite complicated techniques. The authors do not present free-particle CSs well defined for any time instant.

In this article, in fact following the Dodonov-Man'ko method, we construct different families of generalized CSs of a free massive nonrelativistic particle. We discuss the properties of the constructed CSs in detail, including completeness relations, minimization of uncertainty relations, and evolution of the corresponding probability density in time. We describe physical conditions under which free-particle CSs can be considered semiclassical states.

## 2. Constructing time-dependent CSs of a free particle

### 2.1 Basic equations

For simplicity, we consider one-dimensional quantum motion of a free nonrelativistic particle of mass $m$ on the whole real axis $\mathbb{R}=(-\infty, \infty)$. This motion is described by the Schrödinger equation

$$
\begin{equation*}
\mathrm{i} \hbar \partial_{t} \Psi(x, t)=\hat{H}_{x} \Psi(x, t), \quad x \in \mathbb{R}, \tag{1}
\end{equation*}
$$

where the Hamiltonian $\hat{H}_{x}$ and the momentum operator $\hat{p}_{x}$,

$$
\begin{equation*}
\hat{H}_{x}=-\frac{\hbar^{2}}{2 m} \partial_{x}^{2}=\frac{\hat{p}_{x}^{2}}{2 m}, \quad \hat{p}_{x}=-\mathrm{i} \hbar \partial_{x} \tag{2}
\end{equation*}
$$

are self-adjoint on their natural domains [13].
It is useful to introduce the dimensionless variables

$$
\begin{equation*}
q=x l^{-1}, \quad \tau=\frac{\hbar}{m l^{2}} t \tag{3}
\end{equation*}
$$

Then Eqn (1) takes the form

$$
\begin{align*}
& \hat{S} \psi(q, \tau)=0, \quad \hat{S}=\mathrm{i} \partial_{\tau}-\hat{H}, \quad \hat{H}_{x}=\frac{\hbar^{2}}{m l^{2}} \hat{H}, \\
& \hat{H}=\frac{\hat{p}^{2}}{2}, \quad \hat{p}=-\mathrm{i}_{q}, \quad \psi(q, \tau)=\sqrt{l} \Psi\left(l q, \frac{m l^{2}}{\hbar} \tau\right), \tag{4}
\end{align*}
$$

with $|\Psi(x, t)|^{2} \mathrm{~d} x=|\psi(q, \tau)|^{2} \mathrm{~d} q$. We call the operator $\hat{S}$ the equation operator.

In terms of creation and annihilation operators

$$
\hat{a}=\frac{\hat{q}+\mathrm{i} \hat{p}}{\sqrt{2}}, \quad \hat{a}^{\dagger}=\frac{\hat{q}-i \hat{p}}{\sqrt{2}}, \quad\left[\hat{a}, \hat{a}^{\dagger}\right]=1,
$$

the Hamiltonian $\hat{H}$ is a quadratic form:

$$
\begin{equation*}
\hat{H}=\frac{1}{4}\left[\hat{a}^{\dagger} \hat{a}+\hat{a} \hat{a}^{\dagger}-\left(\hat{a}^{\dagger}\right)^{2}-\hat{a}^{2}\right] . \tag{5}
\end{equation*}
$$

It cannot be reduced to the first canonical form for a quadratic combination of creation and annihilation operators, which is the oscillator-like form, by any canonical transformation; this indicates that the spectrum of $\hat{H}$ is continuous (see, e.g., [14]).

### 2.2 Integrals of motion linear

## in canonical operators $\hat{q}$ and $\hat{p}$

We construct an integral of motion $\hat{A}(\tau)$ linear in $\hat{q}$ and $\hat{p}$. The general form of such an integral of motion is

$$
\begin{equation*}
\hat{A}(\tau)=f(\tau) \hat{q}+\mathrm{i} g(\tau) \hat{p}+\varphi(\tau), \tag{6}
\end{equation*}
$$

where $f(\tau), g(\tau)$, and $\varphi(\tau)$ are some complex functions of the time $\tau$. For the operator $\hat{A}(\tau)$ to be an integral of motion, it has to commute with the equation operator in (4);

$$
\begin{equation*}
[\hat{S}, \hat{A}(\tau)]=0 \tag{7}
\end{equation*}
$$

If the Hamiltonian is self-adjoint, the adjoint operator $\hat{A}^{\dagger}(\tau)$ is also an integral of motion, $\left[\hat{S}, \hat{A}^{\dagger}(\tau)\right]=0$.

Substituting representation (6) in Eqn (7), we obtain the following equations for the functions $f(\tau), g(\tau)$, and $\varphi(\tau)$ :

$$
\begin{equation*}
\dot{f}(\tau)=0, \quad \dot{g}(\tau)-\mathrm{i} f(\tau)=0, \quad \dot{\varphi}(\tau)=0, \tag{8}
\end{equation*}
$$

where the dots denote derivatives with respect to $\tau$. The general solution of Eqns (8) is

$$
\begin{equation*}
f(\tau)=c_{1}, \quad g(\tau)=c_{2}+\mathrm{i} c_{1} \tau, \quad \varphi(\tau)=\mathrm{const}, \tag{9}
\end{equation*}
$$

where $c_{1}$ and $c_{2}$ are arbitrary constants. Without loss of generality, we can set $\varphi(\tau)=0$. Thus,

$$
\begin{equation*}
\hat{A}(\tau)=c_{1} \hat{q}+\mathrm{i} g(\tau) \hat{p}, \quad g(\tau)=c_{2}+\mathrm{i} c_{1} \tau . \tag{10}
\end{equation*}
$$

The commutator $\left[\hat{A}(\tau), \hat{A}^{\dagger}(\tau)\right]$ is given by

$$
\begin{equation*}
\left[\hat{A}(\tau), \hat{A}^{\dagger}(\tau)\right]=2 \operatorname{Re}\left(g^{*}(\tau) f(\tau)\right)=2 \operatorname{Re}\left(c_{1}^{*} c_{2}\right)=\delta \tag{11}
\end{equation*}
$$

Equations (9) imply that $\delta$ is a real-valued integral of motion, $\delta=$ const. In what follows, we set $\delta=1$,

$$
\begin{equation*}
\delta=2 \operatorname{Re}\left(c_{1}^{*} c_{2}\right)=1 \tag{12}
\end{equation*}
$$

Let $c_{1}=\left|c_{1}\right| \exp \left(\mathrm{i} \mu_{1}\right)$ and $c_{2}=\left|c_{2}\right| \exp \left(\mathrm{i} \mu_{2}\right)$. Condition (12) then implies that

$$
\begin{equation*}
\left|c_{2}\right|\left|c_{1}\right| \cos \left(\mu_{2}-\mu_{1}\right)=\frac{1}{2} . \tag{13}
\end{equation*}
$$

Choosing $\delta=1$, we set $\hat{A}(\tau)$ and $\hat{A}^{\dagger}(\tau)$ to be annihilation and creation operators,

$$
\begin{equation*}
\left[\hat{A}(\tau), \hat{A}^{\dagger}(\tau)\right]=1 \tag{14}
\end{equation*}
$$

It follows from Eqns (10) and (12) that

$$
\begin{align*}
& \hat{q}=g^{*}(\tau) \hat{A}(\tau)+g(\tau) \hat{A}^{\dagger}(\tau), \quad g(\tau)=c_{2}+\mathrm{i} c_{1} \tau \\
& \mathrm{i} \hat{p}=c_{1}^{*} \hat{A}(\tau)-c_{1} \hat{A}^{\dagger}(\tau) \tag{15}
\end{align*}
$$

We note that the operators $\hat{q}$ and $\hat{p}$ cannot depend on the constants $c_{1}, c_{2}$ or time $\tau$. Indeed, using Eqns (6) and (12), it can be verified that the relations $\partial_{\tau} \hat{q}=\partial_{\tau} \hat{p}=\partial_{c_{1}} \hat{p}=\partial_{c_{1}} \hat{q}=$ $\partial_{c_{2}} \hat{p}=\partial_{c_{2}} \hat{q}=0$ hold.

### 2.3 Time-dependent generalized CSs

We consider eigenvectors $|z, \tau\rangle$ of the annihilation operator $\hat{A}(\tau)$ corresponding to an eigenvalue $z$ :

$$
\begin{equation*}
\hat{A}(\tau)|z, \tau\rangle=z|z, \tau\rangle . \tag{16}
\end{equation*}
$$

In general, $z$ is a complex number. It follows from Eqns (15) and (16) that

$$
\begin{align*}
& q(\tau) \equiv\langle z, \tau| \hat{q}|z, \tau\rangle=q_{0}+p \tau, \quad q_{0}=c_{2}^{*} z+c_{2} z^{*}, \\
& p(\tau) \equiv\langle z, \tau| \hat{p}|z, \tau\rangle=\mathrm{i}\left(c_{1} z^{*}-c_{1}^{*} z\right)=p, \\
& z=c_{1} q(\tau)+\mathrm{i} g(\tau) p=c_{1} q_{0}+\mathrm{i} c_{2} p . \tag{17}
\end{align*}
$$

Written in the $q$-representation, Eqn (16) becomes

$$
\begin{align*}
& {\left[c_{1} q+g(\tau) \partial_{q}\right] \Phi_{z}^{c_{1,2}}(q, \tau)=z \Phi_{z}^{c_{1,2}}(q, \tau),} \\
& \Phi_{z}^{c_{1,2}}(q, \tau) \equiv\langle q \mid z, \tau\rangle . \tag{18}
\end{align*}
$$

The general solution of this equation has the form

$$
\begin{equation*}
\langle q \mid z, \tau\rangle=\Phi_{z}^{c_{1,2}}(q, \tau)=\exp \left[-\frac{c_{1}}{g(\tau)} \frac{q^{2}}{2}+\frac{z q}{g(\tau)}+\chi(\tau, z)\right], \tag{19}
\end{equation*}
$$

where $\chi(\tau, z)$ is an arbitrary function of $\tau$ and $z$.
We can see that the functions $\Phi_{z}^{c_{1,2}}(q, \tau)$ can be written in terms of the mean values $q(\tau)$ and $p(\tau)$,

$$
\begin{equation*}
\Phi_{z}^{c_{1,2}}(q, \tau)=\exp \left\{\mathrm{i} p q-\frac{c_{1}}{2 g(\tau)}[q-q(\tau)]^{2}+\phi(\tau, z)\right\} \tag{20}
\end{equation*}
$$

where $\phi(\tau, z)$ is again an arbitrary function of $\tau$ and $z$.
The functions $\Phi_{z}$ satisfy the equation

$$
\begin{equation*}
\hat{S} \Phi_{z}^{c_{1,2}}(q, \tau)=\lambda(\tau, z) \Phi_{z}^{c_{1,2}}(q, \tau) \tag{21}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda(\tau, z)=\mathrm{i} \dot{\phi}(\tau, z)-\frac{1}{2}\left[p^{2}+\frac{c_{1}}{g(\tau)}\right] . \tag{22}
\end{equation*}
$$

For functions (20) to satisfy Schrödinger equation (4), we have to fix $\phi(\tau, z)$ from the condition $\lambda(\tau, z)=0$. Thus, for the function $\phi(\tau, z)$, we obtain

$$
\begin{equation*}
\phi(\tau, z)=-\frac{\mathrm{i}}{2} p^{2} \tau-\frac{1}{2} \ln g(\tau)+\ln N \tag{23}
\end{equation*}
$$

where $N$ is a normalization constant, which we suppose to be real.

The density probability generated by function (20) is given by

$$
\begin{equation*}
\rho(q, \tau)=\left|\Phi_{z}^{c_{1,2}}(q, \tau)\right|^{2}=\frac{N^{2}}{|g(\tau)|} \exp \left\{-\frac{[q-q(\tau)]^{2}}{2|g(\tau)|^{2}}\right\} . \tag{24}
\end{equation*}
$$

By considering the normalization integral, we find the constant $N$ :

$$
\begin{equation*}
\int_{-\infty}^{\infty} \rho(q, \tau) \mathrm{d} q=1 \Rightarrow N=(2 \pi)^{-1 / 4} . \tag{25}
\end{equation*}
$$

Thus, normalized solutions of the Schrödinger equation that are eigenfunctions of the annihilation operator $\hat{A}(\tau)$ have the
form

$$
\begin{align*}
& \Phi_{z}^{c_{1,2}}(q, \tau)=\frac{1}{\sqrt{\sqrt{2 \pi} g(\tau)}} \\
& \quad \times \exp \left\{\mathrm{i}\left(p q-\frac{1}{2} p^{2} \tau\right)-\frac{c_{1}}{g(\tau)} \frac{[q-q(\tau)]^{2}}{2}\right\}, \tag{26}
\end{align*}
$$

and the corresponding probability density is

$$
\begin{align*}
\rho_{z}^{c_{1,2}}(q, \tau) & =\left|\Phi_{z}^{c_{1,2}}(q, \tau)\right|^{2} \\
& =\frac{1}{\sqrt{2 \pi}|g(\tau)|} \exp \left\{-\frac{[q-q(\tau)]^{2}}{2|g(\tau)|^{2}}\right\} . \tag{27}
\end{align*}
$$

In what follows, we call solutions (26) the time-dependent generalized CSs. In fact, we have a family of states parameterized by two complex constants $c_{1}$ and $c_{2}$ that satisfy constraint (12). As we see in what follows, each family of generalized CSs represents so-called squeezed states. Additional constraints on the constants $c_{1}$ and $c_{2}$ transform these states into CSs of the free particle (see below).

We note that the generalized CSs can be constructed in the Glauber manner, by acting with the displacement operator $\mathcal{D}(z, \tau)=\exp \left[z \hat{A}^{\dagger}(\tau)-z^{*} \hat{A}(\tau)\right]$ on the vacuum vector $|0, \tau\rangle$ defined by $\hat{A}(\tau)|0, \tau\rangle=0$ :

$$
\begin{align*}
& \langle q \mid z, \tau\rangle=\mathcal{D}(z, \tau)\langle q \mid 0, \tau\rangle=\exp \left(-\frac{|z|^{2}}{2}\right) \sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{n!}}\langle q \mid n, \tau\rangle, \\
& |n, \tau\rangle=\frac{\left[\hat{A}^{\dagger}(\tau)\right]^{n}}{\sqrt{n!}}|0, \tau\rangle \\
& |0, \tau\rangle=\frac{1}{\sqrt{\sqrt{2 \pi} g(\tau)}} \exp \left(-\frac{c_{1}}{g(\tau)} \frac{q^{2}}{2}\right) . \tag{28}
\end{align*}
$$

Functions (28) differ from those in (26) by a constant phase factor only.

Using the completeness property of the states $|n, \tau\rangle$,

$$
\begin{equation*}
\sum_{n=0}^{\infty}|n, \tau\rangle\langle n, \tau|=1, \quad \forall \tau \tag{29}
\end{equation*}
$$

we can find the overlapping and prove the completeness relations for the generalized CS of the free particle:

$$
\begin{align*}
& \left\langle z^{\prime}, \tau \mid z, \tau\right\rangle=\exp \left(z^{\prime *} z-\frac{\left|z^{\prime}\right|^{2}+|z|^{2}}{2}\right), \quad \forall \tau ; \\
& \iint\langle q \mid z, \tau\rangle\left\langle z, \tau \mid q^{\prime}\right\rangle \mathrm{d}^{2} z=\pi \delta\left(q-q^{\prime}\right), \\
& \mathrm{d}^{2} z=\operatorname{dRe} z \operatorname{dIm} z, \quad \forall \tau . \tag{30}
\end{align*}
$$

## 3. Standard deviations, uncertainty relations, and CSs of a free particle

Calculating standard deviations $\sigma_{q}(\tau), \sigma_{p}$, and the quantity $\sigma_{q p}(\tau)$ in a generalized CS, we obtain

$$
\begin{align*}
\sigma_{q}(\tau) & =\sqrt{\left\langle(\hat{q}-\langle q\rangle)^{2}\right\rangle}=\sqrt{\left\langle\hat{q}^{2}\right\rangle-\langle q\rangle^{2}}=|g(\tau)| \\
\sigma_{p}(\tau) & =\sqrt{\left\langle(\hat{p}-\langle p\rangle)^{2}\right\rangle}=\sqrt{\left\langle\hat{p}^{2}\right\rangle-\langle p\rangle^{2}}=|f(\tau)|=\left|c_{1}\right| \\
\sigma_{q p}(\tau) & =\frac{1}{2}\langle(\hat{q}-\langle q\rangle)(\hat{p}-\langle p\rangle)+(\hat{p}-\langle p\rangle)(\hat{q}-\langle q\rangle)\rangle \\
& =\mathrm{i}\left[\frac{1}{2}-g(\tau) f^{*}(\tau)\right] \tag{31}
\end{align*}
$$

It is easy to see that the generalized CSs minimize the Robertson-Schrödinger uncertainty relation [15, 16]:

$$
\begin{equation*}
\sigma_{q}^{2}(\tau) \sigma_{p}^{2}-\sigma_{q p}^{2}(\tau)=\frac{1}{4} . \tag{32}
\end{equation*}
$$

This means that these states are squeezed states for any time instant.

We evaluate the Heisenberg uncertainty relation in a generalized CS taking constraint (12) into account:

$$
\begin{align*}
& \left.\sigma_{q}(\tau) \sigma_{p}(\tau)\right|_{2 \operatorname{Re}\left(c_{1}^{*} c_{2}\right)=1} \\
& =\sqrt{\frac{1}{4}+\left[\left|c_{2}\right|\left|c_{1}\right| \sin \left(\mu_{2}-\mu_{1}\right)+\left|c_{1}\right|^{2} \tau\right]^{2}} \geqslant \frac{1}{2} . \tag{33}
\end{align*}
$$

Using (31), we then find $\sigma_{q}(0)=\sigma_{q}=\left|c_{2}\right|$ and $\sigma_{p}(0)=$ $\sigma_{p}=\left|c_{1}\right|$, and hence at $\tau=0$ this relation becomes

$$
\begin{equation*}
\left.\sigma_{q} \sigma_{p}\right|_{2 \operatorname{Re}\left(c_{1}^{*} c_{2}\right)=1}=\left|c_{2}\right|\left|c_{1}\right|=\sqrt{\frac{1}{4}+\left[\left|c_{2}\right|\left|c_{1}\right| \sin \left(\mu_{2}-\mu_{1}\right)\right]^{2}} \tag{34}
\end{equation*}
$$

We see that $\left|c_{i}\right| \neq 0, i=1,2$, and the left-hand side of (34) is minimal if $\mu_{1}=\mu_{2}=\mu$, which provides the minimization of the Heisenberg uncertainty relation in the generalized CS at the initial time instant:

$$
\begin{equation*}
\sigma_{q} \sigma_{p}=\frac{1}{2} . \tag{35}
\end{equation*}
$$

In what follows, we consider the free-particle generalized CS with the restriction $\mu_{1}=\mu_{2}$. Such states are simply called CSs of a free particle.

Now, constraint (12) takes the form

$$
\begin{equation*}
\left|c_{2}\right|\left|c_{1}\right|=\frac{1}{2} \Longrightarrow c_{2}^{*}=\frac{c_{1}^{-1}}{2} . \tag{36}
\end{equation*}
$$

We see that the constant $\mu$ does not enter CS (26). We therefore set $\mu=0$ in what follows. Then

$$
\begin{align*}
& c_{2}=\left|c_{2}\right|=\sigma_{q}, \quad c_{1}=\left|c_{1}\right|=\sigma_{p}=\frac{1}{2 \sigma_{q}}, \\
& g(\tau)=\left(\sigma_{q}+\frac{\mathrm{i} \tau}{2 \sigma_{q}}\right), \sigma_{q}(\tau)=|g(\tau)|=\sqrt{\sigma_{q}^{2}+\frac{\tau^{2}}{4 \sigma_{q}^{2}}} . \tag{37}
\end{align*}
$$

From Eqn (37), we conclude that for any $\tau$, the Heisenberg uncertainty relation in the CS takes the form

$$
\begin{equation*}
\sigma_{q}(\tau) \sigma_{p}=\frac{1}{2} \sqrt{1+\frac{\tau^{2}}{4 \sigma_{q}^{4}}} \geqslant \frac{1}{2}, \tag{38}
\end{equation*}
$$

and the CSs of a free particle are given by

$$
\begin{align*}
& \Phi_{z}^{\sigma_{q}}(q, \tau) \\
& =\frac{\exp \left\{\mathrm{i}\left[p q-\left(p^{2} / 2\right) \tau\right]-[q-q(\tau)]^{2} /\left[4\left(\sigma_{q}^{2}+\mathrm{i} \tau / 2\right)\right]\right\}}{\sqrt{\left[\sigma_{q}+\mathrm{i} \tau /\left(2 \sigma_{q}\right)\right] \sqrt{2 \pi}}} . \tag{39}
\end{align*}
$$

In fact, we have a family of CSs parameterized by one real parameter $\sigma_{q}$. Each set of CSs in the family has its specific initial standard deviations $\sigma_{q}>0$. Coherent states from a


Figure 1. Evolution of probability densities.
family with a given $\sigma_{q}$ are labeled by quantum numbers

$$
\begin{equation*}
z=\frac{q_{0}}{2 \sigma_{q}}+\mathrm{i} \sigma_{q} p, \tag{40}
\end{equation*}
$$

which are in a one-to-one correspondence with the trajectory initial data $q_{0}$ and $p$ :

$$
\begin{equation*}
q_{0}=2 \sigma_{q} \operatorname{Re} z, \quad p=\frac{\operatorname{Im} z}{\sigma_{q}} . \tag{41}
\end{equation*}
$$

The probability densities that correspond to the CSs are

$$
\begin{align*}
\rho_{z}^{\sigma_{q}}(q, \tau) & =\frac{1}{\sqrt{\left[\sigma_{q}^{2}+\tau^{2} /\left(4 \sigma_{q}^{2}\right)\right] 2 \pi}} \\
& \times \exp \left\{-\frac{1}{2} \frac{[q-q(\tau)]^{2}}{\sigma_{q}^{2}+\tau^{2} /\left(4 \sigma_{q}^{2}\right)}\right\} . \tag{42}
\end{align*}
$$

It can be seen that at any time instant $\tau$, probability densities (42) are given by Gaussian distributions with standard deviations $\sigma_{q}(\tau)$. The means $\langle q\rangle=q(\tau)=q_{0}+p \tau$ move along the classical trajectory with the particle velocity $p$. The maxima of the probability densities move with the same velocity (42).

Figure 1 plots function (42) with $\sigma=2^{-1 / 2}, p=2$, and $q_{0}=0$ for two time instants $\tau=0$ and $\tau=1$.

We compare CSs (39) with the plane waves

$$
\begin{equation*}
\Phi_{p}(q, \tau)=\frac{1}{\sqrt{2 \pi}} \exp \left[\mathrm{i}\left(p q-\frac{p^{2}}{2} \tau\right)\right] \tag{43}
\end{equation*}
$$

Both sets of functions are solutions of the Schrödinger equation for the free particle. The CSs belong to $L^{2}(\mathbb{R})$, whereas a plane waves do not. A plane wave propagates with the phase velocity, that is, $p / 2$. The CS set generates the probability density that propagates exactly with the particle velocity $p$. We can say that CSs (39) represent wave packets that allow establishing a natural connection between the classical and quantum description of free particles. Depending on the parameters of the CSs, some of them can be treated as semiclassical states of free particles, while some cannot, because they describe pure quantum states (see below).

## 4. Semiclassical CSs of a free particle

To discuss the question of which CSs can be treated as representing semiclassical particle motion, we have to return to the initial dimensional variables $x$ and $t$ in (3) and to the
initial wave function $\Psi(x, t)$ written in these variables, Eqn (4). Taking into account that

$$
\begin{align*}
& x(t)=l q(\tau)=x_{0}+\frac{p_{x}}{m} t, \quad p=\frac{l}{\hbar} p_{x} \\
& \sigma_{x}(0)=l \sigma_{q}(0)=l \sigma=\sigma_{x}, \quad \sigma_{x}^{2}(t)=\sigma_{x}^{2}+\frac{\hbar^{2}}{4 m^{2} \sigma_{x}^{2}} t^{2} \tag{44}
\end{align*}
$$

we obtain

$$
\begin{align*}
\Psi(x, t) & =\frac{1}{\sqrt{\left[\sigma_{x}+\mathrm{i} \hbar t /\left(2 m \sigma_{x}\right)\right] \sqrt{2 \pi}}} \\
& \times \exp \left\{\frac{\mathrm{i}}{\hbar}\left(p_{x} x-\frac{p_{x}^{2}}{2 m} t\right)-\frac{[x-x(t)]^{2}}{4\left[\sigma_{x}^{2}+\mathrm{i} \hbar t /(2 m)\right]}\right\} \\
\rho(x, t) & =|\Psi(x, t)|^{2}=\frac{1}{\sqrt{\left[\sigma_{x}^{2}+\hbar^{2} t^{2} /\left(4 m^{2} \sigma_{x}^{2}\right)\right] 2 \pi}} \\
& \times \exp \left\{-\frac{1}{2} \frac{[x-x(t)]^{2}}{\sigma_{x}^{2}+\hbar^{2} t^{2} /\left(4 m^{2} \sigma_{x}^{2}\right)}\right\} \tag{45}
\end{align*}
$$

Semiclassical motion implies that the form of distribution (45) changes slowly with time $t$ in a certain sense. This form changes due to the change in the quantity $\hbar^{2} t^{2} /\left(4 m^{2} \sigma_{x}^{2}\right)$ with time, which is responsible for the change in $\sigma_{x}^{2}(t)$ [see Eqn (44)]. We suppose that in semiclassical motion, this quantity is much less than the square of the distance that the particle travels in the same time. We then have the inequality

$$
\begin{equation*}
\frac{\hbar^{2}}{4 m^{2} \sigma_{x}^{2}} t^{2} \ll\left(\frac{p_{x}}{m} t\right)^{2} \Longrightarrow p_{x} \gg \frac{\hbar}{2 \sigma_{x}} \sim v \gg \frac{\hbar}{2 m \sigma_{x}} \tag{46}
\end{equation*}
$$

which can be rewritten in another form:

$$
\begin{equation*}
\lambda \ll 4 \pi \sigma_{x}, \quad \lambda=\frac{2 \pi \hbar}{p_{x}}, \tag{47}
\end{equation*}
$$

where $\lambda$ is the Compton wavelength of the particle. Hence, the CSs of a free particle can be considered semiclassical states if the Compton wavelength of the particle is much less than the coordinate standard deviation $\sigma_{x}$ at the initial instant. It is known that in a cyclotron, nonrelativistic electrons move with velocities $v \simeq 10^{3} \mathrm{~m} \mathrm{~s}^{-1}$. Then, according to (46), the CSs of such electrons with $2 \sigma_{x} \simeq 10^{-7}$ can be treated as semiclassical states.

We note that similar criteria of semiclassicality were used in the theory of potential scattering [17] and in classifying CSs in a magnetic-solenoid field [18].

## 5. Some concluding remarks

In this article, we have studied different types of generalized CSs of a free massive nonrelativistic particle and established properties of these states such as the completeness relations, the minimization of uncertainty relations, and the evolution of the corresponding probability densities in time. Among all these types of generalized CSs, families of states are naturally distinguished which we suggest identifying with the CSs of a free massive nonrelativistic particle. These CS families are parameterized by one real-valued parameter, the coordinate standard deviation $\sigma_{q}$ at the initial time instant. The CSs from a family with a given $\sigma_{q}$ form a complete system of functions and are labeled by a complex-valued quantum number $z$,
which is in a one-to-one correspondence with the initial data of the corresponding trajectory of the coordinate mean value. CSs minimize the Robertson-Schrödinger uncertainty relation at all time instants and the Heisenberg uncertainty relations at the initial instant. The smaller the coordinate standard deviation $\sigma_{q}(\tau)$ at the initial instant is, the faster $\tau$ grows with time at an arbitrary instant. At any time instant $\tau$, the probability density corresponding to free-particle CSs is given by Gaussian distributions with standard deviations $\sigma_{q}(\tau)$. The coordinate mean value propagates along the classical trajectory with the mean particle velocity. The probability density maximum propagates with the same velocity. The constructed CSs are wave packets that are solutions of the Schrödinger equation for a free particle. They belong to the Hilbert space $L^{2}(\mathbb{R})$, whereas plain waves do not belong to this space. The CSs allow establishing a natural relation between the classical and quantum descriptions of free particles. Depending on the parameters of the CSs, some of them can be considered semiclassical states of free particles, and some of them cannot, inasmuch as the latter are purely quantum states. We provide arguments in favor of the fact that free-particle CSs can be considered semiclassical states when the Compton wavelength is much less than the standard coordinate deviation at the initial instant. The suggested CSs can be apparently identified with the asymptotic free states in nonrelativistic quantum scattering theory.

We believe it is useful for a lecture course in quantum mechanics to complete the description of quantum motion of a free particle with the free-particle CSs as an example of exact wave packets, which, under certain conditions, admit a semiclassical description of such a particle, and which allow illustrating a large number of general principles of quantum mechanics, such as the minimization of uncertainty relations. The acquaintance of an audience with free-particle CSs would naturally make it easier to understand the CSs of an oscillator and other quantum systems.

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