PHYSICS OF OUR DAYS

On the possible turbulence mechanism in accretion disks in nonmagnetic binary stars

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Contents

1. Introduction	787
2. Precessional density wave	788
3. Linear analysis of perturbations in a disk with a precessional wave	790
3.1 Approach; 3.2 Computation	
4. Physical analysis of the results	793
5. Conclusion	795
Appendix A	796
Appendix B	796
References	797

Abstract. One of the major challenges in modern astrophysics is the unexplained turbulence of gas-dynamic (nonmagnetic) accretion disks. Since they are stable, such disks should not theoretically be turbulent, but observations show they are. The search for instabilities that can develop into turbulence is one of the most intriguing problems in modern astrophysics. In 2004, we pointed to the formation of the so-called 'precessional' density wave in accretion disks of binary stars, which produces additional density and velocity gradients in the disk. A linear hydrodynamics stability analysis of an accretion disk in a binary shows that the presence in the disk of a precessional wave produced by the tidal influence of the second binary component gives rise to the instability of radial modes, whose characteristic growth times are about one tenth or one hundredth of the binary's orbital period. The immediate reason for the instability is the radial velocity gradient in the precessional wave, the destabilizing perturbations being those in which the radial velocity variation on the wavelength scale is near or greater than the speed of sound. Unstable perturbations occur in the interior of the disk and make the gas turbulent as they propagate outward. The characteristic turbulence parameters are in agreement with observations (the Shakura-Sunyaev parameter $a \leq 0.01$).

1. Introduction

High accretion rates observed in accretion disks in binary stars can be explained only by the presence of turbulent

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Received 23 September 2013, revised 18 March 2014 Uspekhi Fizicheskikh Nauk **184** (8) 851–863 (2014) DOI: 10.3367/UFNr.0184.201408c.0851 Translated by K A Postnov; edited by A M Semikhatov viscosity [1–3] (see also historical review [4]). The turbulence itself should arise due to some instability [1, 3]. For a long time, attempts have been made to search for fluid instabilities in Keplerian disks (see, e.g., the references in [5]). But it can be shown that small radial perturbations in a Keplerian disk are stable according to the Rayleigh criterion [6]. In addition, numerical simulations [6] suggest that azimuthal shortwavelength modes also do not display instability. The numerical study of long-wavelength perturbations in a thin Keplerian disk [5] revealed that such perturbations grow to become nonlinear and then decay without quenching the disk turbulence.

Many authors have applied magneto-rotational instability (MRI) [7, 8] to accretion disks [9]. However, this type of instability as the reason for disk turbulence meets with some difficulties: a) in the majority of close binary stars, there is no observational evidence for the magnetic field; b) the disk turbulence due to MRI requires the presence of a seed magnetic field; c) the magnetic field growth stabilizes perturbations, i.e., suppresses the instability [8]; d) at the nonlinear stage, MRI saturates and hence the angular momentum transfer through the disk significantly decreases [10]. In addition, the very existence of MRI in thin disks was recently questioned [11].

There have been several papers that further examined fluid turbulence in disks. For example, in [12], turbulence was proposed to arise due to a super-reflection instability. It was argued in [13, 14] that disks with a negative entropy gradient (in the presence of radiative cooling) can be subjected to baroclinic instability for axially nonsymmetric modes. Among the recent papers on gas-dynamic turbulence in astrophysical disks, we note [15], where a statistical approach to turbulence modeling was used. In such models, the field of pressure fluctuations is represented by a stochastic force in the equations of motion. Due to its simplicity and generality, such a source is usually modeled as a Gaussian random process that is delta-correlated in time. In our opinion, this approach may help in studying only well-developed turbulence, but not the process of how it arises, as claimed by the authors of [15]. The reason is that this random force is not consistent with gas-dynamic equations; therefore, the growth of perturbations due to this force cannot be viewed as the appearance of fluid instability.

Instabilities are usually studied under certain a priori assumptions on the structure and parameters of the accretion disk. In particular, a near-Keplerian velocity distribution in the disk, its homogeneity in the equatorial plane, and a circular form of the stream lines or their weak eccentricity are usually assumed. But such idealized assumptions can already be invalid within the 'pure' gas-dynamic framework. For example, it was shown in [16] that in a viscous accretion disk, orbits of particles are unstable with respect to the eccentricity growth, and hence the disk ellipticity increases. In accretion disks in binary stars, specific physical conditions can occur, including shocks, tidal interaction, and resonances. These features can significantly affect the gas flow, instability growth, turbulence, and angular momentum transfer. In accretion disks in semi-detached binaries, steady shocks arise due to the tidal interaction with the secondary component [17-19] and the interaction of the gas stream from the inner Lagrangian point with the circumdisk halo [20–33].

Numerical simulations have also revealed that the tidal interaction with the secondary component leads to the appearance of a specific type of waves in accretion disks, the precessional density waves [34]. Waves of this kind have a spiral form and occur in almost the entire disk. Qualitatively, such a wave can be represented as the envelope of a family of elliptical orbits precessing in a nonsymmetric gravitational field [34]. The ellipticity of orbits can result from the eccentric instability that arises due to either viscous forces [16] or resonances [35]. In the latter case, linear modes are excited in the disk, and the single-arm spiral with the azimuthal wave number m = 1 has the maximum increment [35].

The stability of the disk in which the m = 1 mode is present have been studied previously. For example, the interaction of linear perturbations with an originally specified linear mode in the approximation of large radial wave numbers was considered in [36]. It was shown there that perturbations propagating within the disk plane can be unstable, and the increment, according to the authors' opinion, can exceed the Keplerian frequency in the disk.

This paper is devoted to the study of the instability of radial (axially symmetric) perturbations that are formed in the presence of a precessional density wave, and the wave itself here is given as a numerical solution of gas-dynamic equations. That the wave is tightly wound allows us to neglect the angular dependence of all considered quantities for each radial direction chosen. As a result, we restrict ourself to the analysis of the radial perturbation modes only. An important feature of the precessional wave is that it represents a smooth solution, which provides fast convergence of spectral methods used in the stability analysis. The presence of gradients in the wave significantly changes the dispersion relation for linear perturbations and leads to conditions facilitating the instability growth.

In Section 2, a scenario of the density wave arising is presented. In Section 3, a linear analysis of perturbations in a thin isothermal Keplerian disk is introduced and applied to the numerical model of an accretion disk with a precessional wave. In Section 4, a physical analysis of the results is performed. A discussion of results and a conclusion are given in Section 5.

2. Precessional density wave

Accretion disks in semi-detached binaries have a sufficiently complicated structure because the tidal forces and the disk interaction with the inter-component envelope and the gas stream from the inner Lagrange point L_1 deform the disk shape and lead to the formation of different shocks. Tidal shocks and the 'hot line'-the region of impact of the gas stream from the inner Lagrange point L_1 with the circumdisk halo-mostly affect the flow. Three-dimensional hydrodynamic simulations demonstrate that these waves do not penetrate deep inside the cold accretion disk (with a temperature $\sim 10^4$ K) and leave most of the disk weakly perturbed, which creates conditions for the third type of wave, the precessional density wave, to appear [34]. The form of stream lines in the accretion disk is close to the corresponding elliptical Keplerian orbits with the accreting star residing in one of the focuses. This can be explained by the fact that in the region free from strong gas-dynamic perturbations due to steady shocks, the disk is almost homogeneous and gravitational forces dominate over forces due to gas pressure gradients in the equatorial plane of the system. The tidal interaction increases the stream line eccentricity in the disk and forces their apse line to counterrotate the disk. The tidal forces act on the stream lines nonuniformly: the outer stream lines tend to rotate faster than the inner ones. But because there can be no intersecting stream lines in a gas disk (at least if the Knudsen number¹ is much less than unity), some mean precessional velocity is established, which is the same for all stream lines. This velocity can be approximately estimated using the formula [37, 38]

$$\frac{P_{\rm pr}}{P_{\rm orb}} \approx \frac{4}{3} \frac{(1+q)^{1/2}}{q} \left(\frac{r}{A}\right)^{-3/2},\tag{1}$$

where P_{pr} is the precessional period, P_{orb} is the system orbital period, q is the binary component mass ratio, r is the characteristic size of the orbit, and A is the distance between the binary components.

The convergence of stream lines that move with different velocities leads to the formation of a spiral pattern shown schematically in Fig. 1. The matter velocity along the stream line, whose shape is primarily determined by gravity, is close to the local Keplerian velocity for the corresponding (elliptical) orbit. Accordingly, the velocity is minimal at the stream line apastron. But because the flux should be conserved along the stream line, the matter density should also change along the stream line and reach a maximum at the apastron. Therefore, the spiral arm formed by the stream line apastrons look likes a density wave, as was shown in [34].

In the observer's reference frame, the precessional wave is almost steady: its shifts by $1^{\circ}-3^{\circ}$ in the retrograde direction in one orbital period [34]. Hence, the density and velocity distribution in the wave can be considered stationary on time scales of the order of several dozen characteristic disk periods.

Figure 2 shows distributions of the surface density and the radial and angular velocity in numerical simulations of the disk in a close binary system [34]. The calculations were

¹ The Knudsen number here is the ratio of the particle mean free path to the characteristic scale of the problem, which here can be the disk thickness. For a typical number density in the accretion disk of the order of 10^{11} cm⁻³, the free path length is 10^{-10} a.u., while the disk thickness is $\leq 10^{-2}$ a.u.



Figure 1. Schematics of the spiral pattern formation in the inner unperturbed parts of cold gas disks. The accretor is at the center O, A_1, \ldots, A_5 denote the stream line apastrons.

carried out for the binary with the following parameters: the accretor mass $M_1 = 1M_{\odot}$, the donor mass $M_2 = 0.05M_{\odot}$, the binary separation $A = 0.625R_{\odot}$, and the binary orbital period $P_{\rm orb} = 4830$ s. The precessional wave is distinctly seen as a spiral-like overdensity on the surface density map. On the radial velocity map, the region with the density wave is bounded by the zero radial velocity lines. The zero tangential velocity lines coincide with radial velocity extrema.

The surface density and radial and tangential velocity distributions are shown in Fig. 3 along four radial directions in the disk: 180°, 225°, 270°, and 315° (0° corresponds to the direction from the secondary component along the line connecting the binary component centers). The density peaks observed at phases 0.15-0.20 correspond to intersections of the profiles with the precessional wave and correlate well with the radial velocity minima. The radial and tangential velocity profiles are significantly different along different directions. Nevertheless, these profiles share common features determined by the properties of elliptical orbits. For example, for all profiles, the tangential velocity is sub-Keplerian on average (because accretion occurs through the disk); but in the outer parts of the disk, the tangential velocity is much smaller than in the inner parts, because the velocity of matter moving along a Keplerian orbit decreases with the distance from the star. The radial velocity distribution significantly depends on the direction: three profiles presented in Fig. 3a demonstrate a positive velocity (directed away from the accretor) in the inner parts of the disk; at the same time, the radial velocity in the fourth profile is negative in the same region. This behavior is clear because matter moving along an eccentric orbit can more towards or away from the star at different parts of the trajectory. Nevertheless, it should be noted that in the outer parts of the disk, the radial velocity is positive for all profiles, because the angular



Figure 2. Maps of the surface density (upper panel), radial velocity (middle panel), and deviations of the angular velocity from the Keplerian profile (bottom panel) according to numerical model [34]. Velocities are in units of the sound speed $c_{\rm T}$. The thick lines show zero velocity levels.

momentum of the disk decreases in the outer parts due to the decretion of matter. We stress that the radial velocity difference over the disk is fairly large and along some directions can be as high as several dozen times the speed of sound.

As follows from the distributions presented in Fig. 2, the precessional wave is tightly wound. This allows approximately treating perturbations caused by the wave as axially symmetric. This approximation cannot be valid in the central parts of the disk, where the axial symmetry is significantly violated (which can be seen most clearly in the velocity



Figure 3. Plots of (a) the surface density, (b) radial velocity, (c) angular velocity, and (d) rotational curve slope in the numerical accretion disk model [34]. The plots correspond to the radial disk cuts in the directions 180° (solid curves), 225° (dashed curves), 270° (dashed-dotted curves), and 315° (dashed-double-dotted curves).

distribution). In what follows, we analyze perturbations excluding the central part of the disk with a radius smaller than 0.08A from calculations. The results of calculations suggest that the outer parts of the disk are subjected to strong gas-dynamic perturbations. In shocks located close to the disk edge, the stream lines are broken, and correspondingly the methods we use in this paper cannot be applied to study instabilities in these regions, because the calculation domain is limited by the radius 0.37A.

3. Linear analysis of perturbations in a disk with a precessional wave

The density and radial and tangential velocity distributions obtained in numerical simulations [34] described in Section 2 were taken as the background distributions for the analysis of perturbations. Perturbations propagating in the direction perpendicular to the disk are sound perturbations, i.e., they do not show instability. Therefore, it seems plausible to assume that accounting for vertical perturbations can lead to the appearance of additional sound modes and would only insignificantly change the frequencies and increments of radial perturbations. These considerations allow significantly simplifying the calculations by excluding the vertical degree of freedom. All the subsequent analysis is carried out in two dimensions. Thus the results of numerical simulations were reduced to two dimensions by integrating distributions along the direction perpendicular to the disk [39].

The calculations were performed in the inertial reference frame where the precessional wave is almost at rest. The presence of the time-dependent gravitational field of the secondary component in this frame, generally speaking, can give rise to an additional spiral pattern comoving with the rotation [40]. The amplitude of deviations in the density and radial and tangential velocity distributions due to this effect are scaled with the component mass ratio [40], which in our case is very small. Hence, we assume that the gas distribution in the inertial reference frame is steady.

3.1 Approach

We use the isothermal thin-disk approximation in two dimensions. This approximation is quite suitable for accretion disks in binary systems because the effective temperature is of the order of 10⁴ K and the characteristic ratio of the disk thickness to its radius for this temperature is ≤ 0.01 . The twodimensional disk flow in the (r, ϕ) plane is obtained by integrating the complete system of gas-dynamic equations along the vertical coordinate z. The parameters of this system are the surface density $\sigma = \int dz \rho$, the radial and angular velocities u and $v = \Omega r$, and the flat pressure $p = \int dz P$, where ρ and P are the volume density and pressure. For a perfect gas with the adiabatic index γ , the flat pressure has a power-law dependence on the surface density with the flat 'adiabatic index' $\gamma_{\rm S} = 1 + 2(\gamma - 1)/(\gamma + 1)$ [39, 41, 42]. But in the isothermal case, $\gamma_{\rm S} = 1$ [39, 42], and the flat pressure has the same form as the volume pressure, $p = c_T^2 \sigma$. The correct reduction of the system of three-dimensional equations to two dimensions, generally speaking, gives rise to additional terms in the equations compared to the three-dimensional form [39, 42]. The initial two-dimensional system is

$$\frac{\partial\sigma}{\partial t} + \nabla(\sigma \mathbf{V}) = 0, \qquad (2)$$

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla) \mathbf{V} = -\nabla \Phi_1 - \nabla \Phi_2 - c_{\mathrm{T}}^2 \nabla \ln\left(\Omega_{\mathrm{K}}\sigma\right).$$
(3)

where σ is the surface density, $\mathbf{V} = \mathbf{e}_r u + \mathbf{e}_{\phi} v$, $\Phi_1 = -GM_1/r$ and $\Phi_2 = -GM_2/|\mathbf{R}_1 - \mathbf{R}_2 + \mathbf{r}|$ are the respective gravitational potentials of the accretor and donor, c_T is the speed of sound, \mathbf{R}_1 and \mathbf{R}_2 are the respective radius vectors from the barycenter to the accretor and donor, $\Omega_{\rm K} = (GM_1/r^3)^{1/2}$ is the Keplerian angular velocity, and G is the gravitational constant.

Because the adopted mass ratio M_2/M_1 is small, the binary system barycenter lies close to the accretor. We assume the accretor center to be at the barycenter: $R_1 \approx 0$, $R_2 \approx A$. We also set $M \equiv M_1$, $q \equiv M_2/M_1$. In cylindrical coordinates, we then obtain

$$\frac{\partial\sigma}{\partial t} + \frac{1}{r}\frac{\partial(r\sigma u)}{\partial r} + \frac{1}{r}\frac{\partial(\sigma v)}{\partial\phi} = 0, \qquad (4)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \phi} - \frac{v^2}{r} = -\frac{GM}{r^2} - \frac{qGM(A\cos\phi + r)}{(A^2 + r^2 + 2Ar\cos\phi)^{3/2}} - c_{\rm T}^2 \frac{\partial \ln\left(\Omega_{\rm K}\sigma\right)}{\partial r}, \qquad (5)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \phi} + \frac{uv}{r}$$
$$= -\frac{c_{\rm T}^2}{r} \frac{\partial \ln \sigma}{\partial \phi} + \frac{qGMA\sin \phi}{\left(A^2 + r^2 + 2Ar\cos \phi\right)^{3/2}}.$$
(6)

where ϕ is the azimuthal angle in the disk; $\phi = 0$ corresponds to the direction toward the secondary companion along the line connecting the centers of the binary components.

We superimpose small perturbations and linearize the equations. We take the disk model with a precessional wave (see Section 2) as the unperturbed solution. The perturbations are written in the form $\sigma \to \sigma_0(1 + \delta)$, $u \to u_0 + u$, $v \to \Omega_0 r + v$, where $|\delta| \leq 1$, $|u| \leq |u_0|$ and $|v| \leq |\Omega_0 r|$; the quantities σ_0 , u_0 , and Ω_0 correspond to the unperturbed solution. Linearizing the equations results in the vanishing of terms responsible for gravitational interaction with the donor. Equations for the perturbations take the form

$$\frac{\partial \delta}{\partial t} + u_0 \frac{\partial \delta}{\partial r} + \Omega_0 \frac{\partial \delta}{\partial \phi} + \frac{\partial u}{\partial r} + \left(\frac{\partial \ln \sigma_0}{\partial r} + \frac{1}{r}\right) u + \frac{1}{r} \frac{\partial v}{\partial \phi} + \frac{1}{r} \frac{\partial \ln \sigma_0}{\partial \phi} v = 0, \qquad (7)$$

$$\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial r} + \Omega_0 \frac{\partial u}{\partial \phi} + \frac{\partial u_0}{\partial r} u + \left(\frac{1}{r} \frac{\partial u_0}{\partial \phi} - 2\Omega_0\right) v + c_{\rm T}^2 \frac{\partial \delta}{\partial r} = 0, \qquad (8)$$

$$\frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial r} + \Omega_0 \frac{\partial v}{\partial \phi} + \left(\frac{u_0}{r} + \frac{\partial \Omega_0}{\partial \phi}\right) v + \frac{\varkappa_0^2}{2\Omega_0} u + \frac{c_T^2}{r} \frac{\partial \delta}{\partial \phi} = 0, \qquad (9)$$

where

$$\varkappa_0^2 = 2\Omega_0 \left(2\Omega_0 + r \, \frac{\mathrm{d}\Omega_0}{\mathrm{d}r} \right) \tag{10}$$

is the square of the epicyclic frequency. We express the perturbations in terms of harmonics of a kind, $f_{\alpha}(t, r, \phi) \rightarrow \exp(-i\omega t)f_{\alpha}(r, \phi)$, where $(f_{\alpha}) = [\delta, u, v]$. Then the system of equations (7)–(9) can be represented in the matrix form

$$\sum_{\beta=1}^{3} A_{\alpha\beta}(r,\phi) f_{\beta}(r,\phi) = \mathrm{i}\omega f_{\alpha}(r,\phi) \,. \tag{11}$$

The characteristic scale of variations of the background variables along the radius is about 0.1A (see Fig. 3), and along the angular coordinate, about $2\pi r$. Hence, at $r \ge 0.016A$, the angular dependence of the background variables can be neglected. Then the problem splits into independent one-dimensional problems, and the density and velocity along the corresponding radial direction in the disk can be taken as unperturbed solutions of each of them. Assuming that the perturbations are also independent of the angle, we obtain the system of equations for radial perturbations (see Appendix A):

$$\sum_{\beta=1}^{3} A_{\alpha\beta}(r) f_{\beta}(r) = i\omega f_{\alpha}(r) , \qquad (12)$$

where the matrix of the operator $A_{\alpha\beta}$ has the form

$$\begin{bmatrix} u_0 \frac{d}{dr} & \frac{d \ln \sigma_0}{dr} + \frac{1}{r} + \frac{d}{dr} & 0\\ c_T^2 \frac{d}{dr} & \frac{d u_0}{dr} + u_0 \frac{d}{dr} & -2\Omega_0\\ 0 & \frac{\varkappa_0^2}{2\Omega_0} & \frac{u_0}{r} + u_0 \frac{d}{dr} \end{bmatrix}.$$
 (13)

The i ω are eigenvalues of the operator $A_{\alpha\beta}$, and its eigenfunctions are the solution f_{α} of the system. Therefore, Eqn (12) is a Sturm–Liouville problem.

Because the unperturbed solution is sufficiently smooth, and in order to avoid differentiation with respect to the radius, we solve Eqn (12) by the Galerkin method in some functional basis representation. The problem geometry suggests that the zeroth-order Bessel functions of the first kind can be conveniently chosen as the new functional basis. In this paper, the unperturbed disk state is specified by the numerical solution on a discrete spatial grid, and we therefore use the discrete Hankel transform for the Bessel functions in the form [43]

$$\hat{f}_{\alpha}(k_p) = \sum_{q=1}^{N} \frac{2J_0(k_p r_q)}{\mu_{N+1}^2 J_1^2(\mu_q)} f_{\alpha}(r_q) , \qquad (14)$$

$$f_{\alpha}(r_q) = \sum_{p=1}^{N} \frac{2J_0(k_p r_q)}{\mu_{N+1}^2 J_1^2(\mu_p)} \hat{f}_{\alpha}(k_p) , \qquad (15)$$

where μ_q is the *q*th root of J_0 and *N* is the dimension of the spatial grid. This transform assumes that the function f(r) is defined on the finite interval $0 \le r \le R$ and vanishes at the interval boundaries, and its values should be calculated at the points $r_q = R\mu_q/\mu_{N+1}$. The value of the image $\hat{f}(k_p)$ should be calculated at the points $k_p = \mu_p/R$.

Equation (12) can be rewritten as

$$\sum_{s=1}^{N} \sum_{\beta=1}^{3} \hat{A}_{\alpha\beta}(k_p, k_s) \hat{f}_{\beta}(k_s) = \mathrm{i}\omega \hat{f}_{\alpha}(k_p) , \qquad (16)$$

where

$$\hat{A}_{\alpha\beta}(k_p,k_s) = \frac{4}{\mu_{N+1}^4} \sum_{q=1}^N \frac{J_0(k_p r_q)}{J_1^2(\mu_q)} A_{\alpha\beta}(r_q) \frac{J_0(k_s r_q)}{J_1^2(\mu_q)} .$$
(17)

We note that the differential operator of the form g(r) d/drfor an arbitrary function must transform as

$$g(r) \frac{\mathrm{d}}{\mathrm{d}r} \to \frac{4}{\mu_{N+1}^4} \sum_{q=1}^N \frac{J_0(k_p r_q)}{J_1^2(\mu_q)} \frac{k_s}{2} \frac{J_{-1}(k_s r_q) - J_1(k_s r_q)}{J_1^2(\mu_q)} g(r_q) \,.$$
(18)

Finally, to reduce the system of algebraic equations to a suitable form allowing the application of known methods for solving eigenvalue problems, we linearly collect the elements of the vector $\hat{f}_{\alpha}(k_p)$ as follows:

$$(\hat{f}_{I}) = [\hat{f}_{1}(k_{1}), \hat{f}_{2}(k_{1}), \hat{f}_{3}(k_{1}), \hat{f}_{1}(k_{2}), \hat{f}_{2}(k_{2}), \hat{f}_{3}(k_{2}), \dots].$$
 (19)

In a similar way, we compose the matrix \hat{A}_{IJ} to finally obtain

$$\sum_{J=1}^{3N} \hat{A}_{IJ} \hat{f}_J = i\omega \hat{f}_I \,. \tag{20}$$

Eigenvalues of the matrix \hat{A}_{IJ} give the spectrum of angular frequencies of possible solutions, and eigenvectors give the solution in terms of the Bessel functions. For a given spatial grid dimension N, we have 3N complex eigenfrequencies and N eigenvectors for each field (δ , u, and v).

3.2 Computation

The method described in Section 3.1 can be used to compute linear perturbations on axially symmetric backgrounds. In the framework of the numerical model of an accretion disk considered in Section 2, this assumption, strictly speaking, is invalid. But we can state that for each radial direction, the angular dependence of all variables is insignificant, and therefore each radial distribution can be considered axially symmetric. In our problem, this method was independently applied to each radial cut of the disk. The parameters of the method are the computational domain size $0.08A \le r \le R = 0.37A$ and the grid dimension N = 439. Equation (20) was solved using the LAPACK library [44].

Maps of complex frequencies for all perturbation modes are presented in Fig. 4. The maximum absolute values of angular frequencies in calculations reach $1500\Omega_{orb}$ and the increments lie in the range from $-70 P_{orb}$ to $50 P_{orb}$. The minimal perturbation length for a fixed grid dimension N is around $\lambda_N \equiv 2R/N \approx 3 \times 10^{-3} A$. This wavelength can be compared to the Keplerian angular frequency $(GM/\lambda_N^3)^{1/2} \approx$ $6 \times 10^3 \Omega_{orb}$, whereas the corresponding sound frequency is $2\pi c_T/\lambda_N \approx 40\Omega_{orb}$. We note that these estimates depend on the spatial discretization scale. This scale can be related to the maximum and minimum angular frequencies obtained in calculations.

Because the original unperturbed solution is nonuniform, the perturbation amplitude is different at different points of the disk. The rate of growth or decay of perturbations is determined both by the imaginary part of the eigenfrequency and by its local amplitude. Figure 5 shows real parts of eigenvectors for some modes in two cuts of the disk, corresponding to 180° and 270° (imaginary parts of the solutions are different from the real parts only in the spatial phase).



Figure 4. Maps of complex frequencies of eigensolutions in the disk cuts from Fig. 3.

Eigenvectors can have a nonuniform spectral composition: the local wavelength, defined as the distance between maxima in different parts of the disk, can differ by many times (see Fig. 4). In studying the spectral composition of the solutions in different parts of the disk, the wavelet analysis may be helpful. For each mode, we calculate the convolution²

$$w(r,\lambda) = \int \mathrm{d}r' \left| W(r-r',\lambda) \,\delta(r') \right|,\tag{21}$$

where W is the Morlet wavelet [46] of order 5,

$$W(r,\lambda) = \exp\left[-\frac{1}{2}\left(\frac{2\pi}{5}\frac{r}{\lambda} - 2\pi\right)^2\right] \exp\left(i2\pi\frac{r}{\lambda}\right).$$
 (22)

The distribution of $w(r, \lambda)$ over wavelengths shows the characteristic wavelength scale λ dominating around a given *r*.

The turbulence viscosity coefficient v_{turb} , or the Shakura– Sunyaev parameter $\alpha = v_{turb}/(c_T h)$ related to it, where $h = c_T/\Omega_K$ is the half-thickness of a Keplerian disk [2], is the commonly accepted characteristic of the efficiency of the angular momentum transfer in accretion disks. Although v_{turb} pertains to well-developed turbulence, its relation to the characteristics of unstable liner perturbations was obtained in [47] in the form

$$v_{\rm turb} = \frac{\gamma \lambda^2}{4\pi^2} \,, \tag{23}$$

where λ and γ are the maximum wavelength and maximum increment for all growing modes for a given type (given branch) of perturbations. A similar approach to the turbulent viscosity estimate was proposed in [12] based on plasma theory results [48, p. 299], where the instability was regarded as a consequence of the background nonuniformity only.

On the eigenfrequency maps (see Fig. 4), it is difficult to uniquely identify the perturbation branches. The estimate of v_{turb} using maximum increments for all modes appears to be senseless because the modes with maximum increments and frequencies may reflect the spatial discretization and boundary effects. Therefore, the values of the coefficient v_{turb} defined by formula (23) should characterize not the whole set of eigenvectors but an individual mode at each point of the disk. This approach corresponds to the general equation (7) in [47]. In our setting, for the mode with the increment γ and the local wavelength λ , we have

$$\alpha = 0.21 P_{\text{orb}} \gamma \, \frac{(\lambda/A)^2}{h/A} \,. \tag{24}$$

To describe a reasonably realistic accretion disk, we should specify the power density of the family of modes and calculate the Shalura–Sunyaev parameter as the average over the mode ensemble. For example, in [6], the initial conditions for the evolution of plane waves in the outer part of the accretion disk were chosen as a power-law power density spectrum (quadratic in the wave number) exponentially decaying at short wavelengths. In the present problem, the plane-wavelength approximation is invalid because the

² In cylindrical coordinates, the Morlet wavelet and this type of transform are, strictly speaking, inapplicable [45], but for approximate estimates this transform turns out to be sufficient.



Figure 5. Real parts of the complex solutions for density and radial and angular velocity perturbations for some modes in the cuts (see Fig. 3) corresponding to 180° (left panels) and 270° (right panels). Left plots correspond to $\omega = 379.6\Omega_{orb}$, $\gamma = 22.3/P_{orb}$ (solid curves) and $\omega = 35.2\Omega_{orb}$, $\gamma = 22.6/P_{orb}$ (dashed curves). Right plots correspond to $\omega = 158.0\Omega_{orb}$, $\gamma = 25.7/P_{orb}$ (solid curves) and $\omega = 10.4\Omega_{orb}$, $\gamma = 48.8/P_{orb}$ (dashed curves).

wavelength is a local variable for one mode. The statistical weight in the mode ensemble can be specified from the following considerations. Expression (23) is applicable to accretion disks under two physical conditions: a) only threedimensional turbulence can be induced; b) the characteristic growth time of a perturbation cannot exceed one disk revolution period. These conditions suggest two local restrictions, on the wavelength and on the increment of a perturbation:

$$\lambda \leqslant h \,, \qquad \gamma \geqslant \frac{\Omega_0}{2\pi} \,. \tag{25}$$

The final expression for the Shalura–Sunyaev parameter can be written in the form

$$\alpha_k(r) = \frac{\int' d\lambda \, w_k(r,\lambda) \, \alpha_k(r,\lambda)}{\int' d\lambda \, w_k(r,\lambda)} \tag{26}$$

for the mode with a number k at the point r, and

$$\alpha(r) = \frac{\sum_{k}^{\prime} w_{k}(r) \,\alpha_{k}(r)}{\sum_{k}^{\prime} w_{k}(r)} \tag{27}$$

for all modes at the point r. The local amplitude of perturbations for all modes is

$$w(r) = \sum_{k} \int d\lambda \, w_k(r, \lambda) \,. \tag{28}$$

Primes over the integral and the sum mean that the summation is performed over the modes and wavelengths for which conditions (25) are satisfied. The results of calculations are presented in Fig. 6.

4. Physical analysis of the results

A comparison of the unperturbed distributions (see Figs 2 and 3) with perturbation profiles (see Fig. 4) reveals several features. First, near zeros of the function u_0 , the local wave of perturbations decreases. Second, immediately near these points, the local amplitude of perturbations tends to zero. The first effect can easily be explained by the following considerations. We simplify the problem as much as possible and consider the advection part of linearized equation (7):

$$\frac{\partial\delta}{\partial t} + u_0 \frac{\partial\delta}{\partial r} = 0.$$
⁽²⁹⁾

Direct substitution shows that near a root r_* of u_0 , the solution has the form

$$\delta = \exp\left(\mathrm{i}\omega\int\frac{\mathrm{d}r}{u_0} - \mathrm{i}\omega t\right). \tag{30}$$

Specifying the velocity variation law as $u_0 \propto r - r_*$, we obtain the local perturbation wavelength (defined as λ in the expression $\omega \int_r^{r+\lambda} dr/u_0 = 2\pi$) in the vicinity of r_* behaving as $O(r - r_*)$.

In the immediate vicinity of a zero of u_0 in Eqn (7), the divergence term becomes important. We write Eqns (7) and (8) ignoring geometrical terms and the tangential velocity:

$$\frac{\partial\delta}{\partial t} + u_0 \frac{\partial\delta}{\partial r} + \frac{\partial u}{\partial r} = 0, \qquad (31)$$

$$\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial r} + \frac{\partial u_0}{\partial r} u + c_{\rm T}^2 \frac{\partial \delta}{\partial r} = 0.$$
(32)

For the background velocity variation law $u_0 \propto r - r_*$, in the limit $u_0 \ll c_T$, it is easy to show (by differentiating the Euler



Figure 6. Perturbation amplitudes (upper panel) and the Shakura–Sunyaev parameter α (bottom panel) in a disk.

equation with respect to the radius and by eliminating velocity perturbations) that solutions take the form $\delta \propto \exp(-i\omega t)(r-r_*)$ and $u \propto \exp(-i\omega t)(r-r_*)^2$ such that in the immediate vicinity of the point r_* , the perturbation amplitude decreases and the spatial oscillations become 'frozen'.

Figure 6 demonstrates that the perturbation amplitude reaches a maximum in the inner part of the disk restricted by the precessional wave, while the turbulence (understood as large values of α) is strongest in the outer part of the disk. This is most clearly seen in the second and third quadrants of the disk, where the location of the amplitude maximum coincides with the radial velocity maximum (see Fig. 2). The perturbation amplitude decreases with decreasing the velocity. In the region bounded by the zero radial velocity, some amplitude growth is observed, especially close to the outer edge of the precessional wave. In the fourth quadrant, where the radial velocity in the inner part of the disk is negative (see also Fig. 3), perturbations and turbulence are almost absent, but there is an amplitude peak at the outer edge of the wave.

This behavior of the amplitude can be explained as follows. We use Eqns (7)–(9) in the approximation of a simulation box whose sizes are much smaller than the distance from the box to the center, as in [6]. We represent perturbations in the form $\exp(-i\omega t + ikr)$ and assume that the perturbation length is sufficiently small such that all unperturbed variables can be considered constant and

geometrical terms can be neglected. We obtain the equations

$$(-i\omega + iku_0)\,\delta + \left(\frac{\partial \ln \sigma_0}{\partial r} + ik\right)u = 0\,,\tag{33}$$

$$ic_{\rm T}^2 k\delta + \left(-i\omega + iku_0 + \frac{\partial u_0}{\partial r}\right)u - 2\Omega_0 v = 0, \qquad (34)$$

$$(2-q)\,\Omega_0 u + (-i\omega + iku_0)\,v = 0\,. \tag{35}$$

The epicyclic term in the angular momentum conservation law is written as $\varkappa_0^2/(2\Omega_0) \approx (2-q)\Omega_0$, where $q \equiv -\partial \ln \Omega_0/\partial \ln r$ [6]. The dispersion relation for this system is

$$\omega_{\pm} = u_0 k - \frac{\mathrm{i}}{2} \frac{\partial u_0}{\partial r} \pm \left[c_{\mathrm{T}}^2 k^2 - \frac{1}{4} \left(\frac{\partial u_0}{\partial r} \right)^2 + 2(2-q) \Omega_0^2 - \mathrm{i} c_{\mathrm{T}}^2 \frac{\partial \ln \sigma_0}{\partial r} k \right]^{1/2}.$$
(36)

The second term in the right-hand side of (36) is a divergence term that stabilizes or destabilizes perturbations depending on whether the flow is divergent $(\partial u_0/\partial r > 0)$, as in the outer part of the disk, or convergent $(\partial u_0/\partial r < 0)$, as in the inner part of the disk. The second term in the radicand helps destabilize the flow: if the radial velocity gradient is strong enough, the perturbation phases over the one-wavelength interval start surpassing each other. The third term in the radicand, in the case of zero gradients, would correspond to the Rayleigh stability criterion [6]: a flow with the angular velocity profile q > 2 is unstable. Finally, the last term contributes to the instability increment at all wave numbers.

We evaluate each term in dispersion equation (36) using conditions (25) and Fig. 3. The minimum local wave number at a given radius must be determined by the disk halfthickness: $|Ak_{\min}|^2 \equiv (2\pi A \Omega_{\rm K}/c_{\rm T})^2 \approx 10^5 (A/r)^3$; everywhere except in the vicinity of roots of the radial velocity, we then have $|c_{\rm T}k_{\rm min}| \ll |u_0k_{\rm min}|$. The term with the velocity gradient turns out to be of the same order: $|\partial(u_0/c_T)/\partial(r/A)|^2 \leq 10^5$. The term corresponding to the Rayleigh criterion can be estimated as follows. The slope of the rotational curve can be assumed to be Kepelerian everywhere (see Fig. 3), q = 3/2, whence $2(2-q)(A\Omega_0/c_T)^2 \approx 10^3 (A/r)^3$. We see that this term introduces only a small stabilizing effect. The density gradient can be estimated as $|A^2 k_{\min} \partial \ln \sigma_0 / \partial r| \approx 5 \times$ $10^3 (A/r)^{3/2}$. Hence, the instability, at least in the simulation box approximation, can only be due to the radial velocity gradient. Dispersion equation (36) must therefore have the form

$$\omega_{\pm} = u_0 k \pm \left[c_{\rm T}^2 k^2 - \frac{1}{4} \left(\frac{\partial u_0}{\partial r} \right)^2 + 2(2-q) \,\Omega_0^2 \right]^{1/2}.$$
 (37)

The above estimates strongly suggest that the disk is in a border-line state between stability and instability. To clarify this point, we use the general integral method of stability analysis described in [49]. In this method, gas-dynamic equations are written in terms of the displacement vector of a gas element, and the analysis assumes this displacement to be small. In this approach, the necessary and sufficient condition for the instability in our problem can be presented as (see Appendix B)

$$\left(\int dE\right)^{-1} \left[\int \frac{dE}{\lambda} \frac{u_0}{c_{\rm T}}\right]^2 + \int \frac{dE}{\lambda^2} \left(1 - \frac{u_0^2}{c_{\rm T}^2} + \frac{\lambda^2}{h^2}\right) < 0, \quad (38)$$

where $dE = dr r\sigma_0 |\xi|^2$, ξ is the radial gas displacement, and λ is the characteristic wavelength scale of perturbation. For

inequality (38) to be satisfied, it is necessary that the first term, giving a positive contribution, be small enough and the second term be negative. The second condition is satisfied if the wavelength scale of perturbations is smaller than the disk thickness (this is one of the conditions of developed turbulence we adopted) and if the radial velocity in a sufficiently large region of the disk is supersonic. The first condition is satisfied if the radial velocity over the disk is close to zero, i.e., the velocity changes sign. In other words, the velocity should take higher values and should have a large gradient over a sufficiently extended part of the disk. Both these conditions are satisfied in the present problem (see Appendix B).

The behavior of modes near the zero radial velocity points and dispersion equation (37) are sufficient, in principle, to explain many properties of unstable perturbations presented in Fig. 6. If the radial velocity has a large gradient in the inner part of the disk bounded by the precessional wave, then local unstable modes arise. In the fourth quadrant of the disk, the velocity gradient is not too high, and additionally the character of the velocity in which it alternates in sign is less pronounced, and hence unstable modes are suppressed. The sub-Keplerian gas rotation in this region (see Fig. 3) also has a stabilizing effect. At the precessional wave boundary, as shown above, the perturbation amplitude vanishes. The amplitude increase in the outer parts of the disk, in the first and forth quadrants, can be due to boundary effects.³

The physical meaning of this instability can be explained as follows. In the radial Euler equation, the term with the background velocity gradient acts on the gas element as an external force, in addition to the pressure gradient force and the centrifugal force. In our setting, the rotational flow has a stabilizing effect and together with pressure prevents the instability development [both corresponding terms make a positive contribution to the radicand in (37)]. But if the velocity gradient is sufficiently high, the momentum flux transmitted to a perturbation due to this term can exceed the counter-acting contribution from stabilizing terms. According to (37), this condition occurs when the background radial velocity change on the perturbation wavelength scale is greater than or approximately equal to the speed of sound. In this case, it is possible to argue that the rear phase of the perturbation catches up with its front phase in one wave period. Here, the amplitude of perturbation maxima increases by the mass conservation. In terms of the dispersion relation, this signals the appearance of nonzero values of the perturbation growth increment.

We note once again that in our method of instability analysis, we assume axially symmetric perturbations, while the background density and velocity distributions do not have axial symmetry and weakly depend on the angle with the characteristic angular scale 2π (see Fig. 2). In a real disk, perturbations with this or smaller angular scale would shift in the tangential direction due to the background rotation. This could weaken the growth of perturbations due to the radial velocity gradient. However, the necessary condition adopted here for the turbulence to appear, Eqn (25), requires that the characteristic growth time of perturbations be shorter than the Keplerian time. Thus, a perturbation in its growth time cannot leave the region of the precessional wave in the tangential direction and we therefore observe the instability growth.

In Section 3, we applied the necessary conditions of threedimensional turbulence development, Eqn (25), to unstable modes. In the inner part of the disk, where the angular frequency of the gas is high, the conditions for turbulence development are more stringent. They become favorable only in the outer part of the disk and near the zero radial velocity line, where the local wavelength of perturbations decreases.

Thus, the turbulence arising in the disk can be described as follows. Perturbations that have time to grow into the nonlinear stage in less than one disk revolution and whose wavelength does not exceed the disk half-thickness serve as sources for three-dimensional turbulence. The turbulence arises predominantly along the precessional wave boundary and beyond its outer edge and is then taken by the gas rotation and accretion flow over the entire disk.

5. Conclusion

Thin Keplerian disks are known to be stable under fluid perturbations [5, 6]. For a hydrodynamic instability to arise, it is necessary that the density and velocity distributions in the disk differ from Keplerian ones [6]. The accretion disk in the axially asymmetric gravitational field of a binary stellar system provides an obvious example of such a configuration. In [34], we have shown that the gravitational field of the secondary binary component excites the precessional wave in the disk. The wave significantly changes the flow in the disk, causing the appearance of regions with large density and velocity gradients. In the presented solution, the radial velocity gradient can be as high as Mach 40.

In this paper, we performed a linear analysis of perturbations to study the stability of an isothermal accretion disk with a precessional wave. The numerical model of the accretion disk in a binary system obtained in [34] was taken as the unperturbed background solution. The problem of linear perturbations growth was formulated in two dimensions in the inertial frame where the precessional wave can be considered stationary. The gas flow perturbations due to the time-dependent gravitational field of the secondary component can be considered small. The strong twisting of the precessional wave allowed a linear analysis of radial perturbations only for each radial cut of the disk.

We have shown that the presence of the precessional wave gives rise to unstable radial modes with increments up to $\sim 50/P_{\rm orb}$. However, actually important for turbulence development is the presence of background regions with large radial velocity gradients. The instability arises if the radial velocity change in the unperturbed flow on the wavelength of a perturbation is of the order of or greater than the speed of sound. The physical reason for the instability is that the rear phase of a perturbation starts catching up with the frontal phase, which, by the mass conservation, causes the growth of the perturbation amplitude at maxima. The necessary conditions for turbulence (a wavelength shorter than the disk thickness, the increment exceeding the rotation frequency) are fulfilled only at the precessional wave boundaries and in the outer part of the disk.

The results of our analysis suggest that a precessional wave in the disk can lead to turbulence with the characteristic Shakura–Sunyaev α parameter about 0.01.

³ Perturbations vanish at the calculation domain boundary, but at the same time the eigenvectors of problem (20) have a fixed norm; therefore, in narrow regions where there are favorable conditions for the instability development, the perturbation amplitude can be relatively high.

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Appendix A

The method for solving linearized equations we use in this paper ignores the angular dependence of the background variables. This approach is different from the ordinarily used spectral method (see, e.g., [36]), and its formulation is not fully rigorous. To show this, we expand system (11) in the basis of some functions of the angular coordinate. We conventionally define functional transformations of the form

$$\tilde{f}(m) = \sum_{\phi} \Phi(\phi, m) f(\phi) , \qquad (39)$$

$$f(\phi) = \sum_{m} \Psi(\phi, m) \tilde{f}(m) , \qquad (40)$$

where $\{\Phi(\phi, m)\}$ and $\{\Psi(\phi, m)\}$ are mutually dual sets of functions depending on the angle ϕ and the parameter *m*. Then expanded system (11) takes the form

$$\sum_{\beta=1}^{3} \sum_{\phi',m'} \Phi(\phi',m) A_{\alpha\beta}(r,\phi') \Psi(\phi',m') \tilde{f}_{\beta}(r,m')$$
$$= i\omega \sum_{\phi',m'} \Phi(\phi',m) \Psi(\phi',m') \tilde{f}_{\alpha}(r,m').$$
(41)

In the 'standard' approach, the expansion is done with respect to an orthogonal set of functions,

$$\int \mathrm{d}\phi' \, \Phi(\phi',m) \, \Psi(\phi',m') = 0$$

for $m \neq m'$. In particular, this is valid for $\Phi(\phi',m) \propto \exp(-im\phi')$ and $\Psi(\phi',m') \propto \exp(im'\phi')$. The approach proposed in this paper relies on the expansion in two harmonics only, denoted as $m \equiv M$ and $m' \equiv 0$; here, the orthogonality of functions is not assumed. Expansion (12) can be obtained if we set

$$\Phi(\phi', M) = \delta_{\rm D}(\phi' - \phi), \qquad (42)$$

$$\Psi(\phi',0) \equiv 1\,,\tag{43}$$

where δ_D is the Dirac delta function. In this case, Eqns (39) and (40) take the form $\tilde{f}(0) = f(\phi)$.

It can be seen that $\Psi(\phi', 0)$ represents an axially symmetric harmonic and the operator $\int d\phi' \Phi(\phi', M)$ 'cuts' the given direction in the disk. One of the assumptions of this approach is as follows. Although the expansion of system (11) in the orthogonal set of functions is quite admissible, we cannot be sure that functions (42) and (43) belong to two dual sets in the sense of definitions (39) and (40) for all ϕ from 0 to 2π . Despite this fact, we prefer this method because it offers a more clear physical interpretation of the perturbation growth as a function of the background variable distribution, ensuring the angular coordinate locality. Another assumption is that the proposed approach assumes the use of axially symmetric modes, whereas the background solution depends on the angle. The distributions shown in Fig. 2 demonstrate that the main angular scale of changes is 2π . Therefore, this scale should mainly contribute to perturbations, and in the standard approach, this scale would correspond to the azimuthal number m = 1. However, for simplicity, we assume that the perturbation is axially symmetric. Thus, the problem can be formulated independently for each radial direction in the disk.

Appendix **B**

Below, we briefly describe the instability analysis in a disk following the Lynden-Bell–Ostriker method. A detailed presentation of the method and examples for axially symmetric flows can be found in [49].

The gas-dynamic equations can be written in terms of the displacement vector of gas ξ relative to its equilibrium position (marked with index '0'):

$$\left(\frac{\partial}{\partial t} + \mathbf{V}_0 \nabla\right)^2 \boldsymbol{\xi} = -\Delta (c_{\mathrm{T}}^2 \nabla \ln \sigma + \nabla \Phi), \qquad (44)$$

where Δ is the Lagrangian difference operator:

$$\Delta f = f\left(t, \mathbf{r} + \boldsymbol{\xi}(t, \mathbf{r})\right) - f_0(t, \mathbf{r})$$
(45)

$$= f(t, \mathbf{r}) - f_0(t, \mathbf{r}) + \boldsymbol{\xi}(t, \mathbf{r}) \,\nabla f_0(t, \mathbf{r}) + \mathcal{O}(|\boldsymbol{\xi}|^2) \,. \tag{46}$$

Equation (44) should be supplemented with the continuity equation

$$\Delta \sigma + \sigma_0 \nabla \xi = 0. \tag{47}$$

Further analysis is performed by assuming that the displacement is small and the unperturbed disk is steady. We redefine the displacement vector as $\xi \rightarrow \exp(i\omega t)\xi$. The linearized dynamic equations have the form

$$-\omega^2 \mathbf{A}\boldsymbol{\xi} + \omega \mathbf{B}\boldsymbol{\xi} + \mathbf{C}\boldsymbol{\xi} = 0, \qquad (48)$$

where **A**, **B**, and **C** are matrices composed of the unperturbed variables and their derivatives. After multiplying this equation by the vector ξ^+ and integrating over the volume, we finally obtain a quadratic equation for ω :

$$-\omega^2 \int d^2 r \,\xi^+ \mathbf{A} \xi + \omega \int d^2 r \,\xi^+ \mathbf{B} \xi + \int d^2 r \,\xi^+ \mathbf{C} \xi = 0 \qquad (49)$$

or

$$-\omega^2 a + \omega b + c = 0. \tag{50}$$

The necessary and sufficient condition for a linear perturbation to grow is the condition for the discriminant of this equation:

$$b^2 + 4ac < 0. (51)$$

In our problem, the displacement vector ξ contains only the radial component, and the coefficients *a*, *b*, and *c* have the form [49]

$$a = \int \mathrm{d}r \, r\sigma_0 |\xi|^2 \,, \tag{52}$$

$$b = i \int dr \, r \sigma_0 \left(\xi^* \, \frac{d\xi}{dr} - \xi \, \frac{d\xi^*}{dr} \right) u_0 \,, \tag{53}$$

c = t + v + p,

$$t = \int \mathrm{d}r \, r\sigma_0 \left(-u_0^2 \left| \frac{\mathrm{d}\xi}{\mathrm{d}r} \right|^2 + 3\Omega_0 |\xi|^2 \right),\tag{54}$$

$$v = \int \mathrm{d}r \, r\sigma_0 |\xi|^2 \, \frac{\mathrm{d}(r\Omega_{\mathrm{K}}^2)}{\mathrm{d}r} \,, \tag{55}$$

$$p = c_{\rm T}^2 \int dr \, r\sigma_0 \left| \frac{d\xi}{dr} \right|^2.$$
(56)

Using the equilibrium flow equation

$$c_{\rm T}^2 \, \frac{\mathrm{d} \ln \sigma_0}{\mathrm{d} r} + r \Omega_{\rm K}^2 - r \Omega_0^2 = 0 \tag{57}$$

and the disk half-thickness $h = c_T/\Omega_K$, we can rewrite the instability growth condition as

$$\left(\int \mathrm{d}r \, r\sigma_0 |\xi|^2\right)^{-1} \left[\int \mathrm{d}r \, r\sigma_0 \operatorname{Im}\left(\xi \, \frac{\mathrm{d}\xi^*}{\mathrm{d}r}\right) \frac{u_0}{c_{\mathrm{T}}}\right]^2 + \int \mathrm{d}r \, r\sigma_0 \left|\frac{\mathrm{d}\xi}{\mathrm{d}r}\right|^2 \left(1 - \frac{u_0^2}{c_{\mathrm{T}}^2}\right) + \int \mathrm{d}r \, r\sigma_0 |\xi|^2 \left(\frac{1}{h^2} + \frac{1}{r} \, \frac{\mathrm{d}\ln\sigma_0}{\mathrm{d}r}\right) < 0.$$
(58)

Using (57), it is easy to show that in the particular case where the fluid is incompressible and the radial flow is absent, expression (58) becomes the classical Rayleigh criterion [6]:

$$\int \mathrm{d}r \, r\sigma_0 |\xi|^2 \varkappa_0^2 < 0 \,. \tag{59}$$

The integrand in the first term in (58) is the product of the radial velocity and a function whose scale of change corresponds to the given mode change scale. Indeed, Eqn (47) for the displacement ξ can be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} + \frac{\mathrm{d}\ln(r\sigma_0)}{\mathrm{d}r}\,\xi + \delta = 0\,,\tag{60}$$

whence it follows that Im $(\xi d\xi^*/dr) = -\text{Im}(\xi \delta^*)$. The plot of this function for two modes is presented in Fig. 7. By comparing these modes with those shown in Fig. 5, we conclude that the characteristic scale of change of these functions is the same, although the functions Im $(\xi d\xi^*/dr)$ apparently do not change sign. On the contrary, the radial velocity distribution does change sign. We define the local scale of perturbation change λ as Im $(\xi d\xi^*/dr) \equiv \lambda^{-1} |\xi|^2$. Because the background distribution change scale is larger than the scale of perturbations of interest here, the estimate



Figure 7. The value of Im $(\xi \delta^*)$ as a function of the radial disk coordinate for the two modes shown in the left panel of Fig. 5.

 $|d\xi/dr|^2 \simeq \lambda^{-2}|\xi|^2$ approximately holds. For the same reason, we can neglect the density gradient in the last term in the left-hand side of (58). Then the inequality takes the form

$$\left(\int dE\right)^{-1} \left[\int \frac{dE}{\lambda} \frac{u_0}{c_{\rm T}}\right]^2 + \int \frac{dE}{\lambda^2} \left(1 - \frac{u_0^2}{c_{\rm T}^2} + \frac{\lambda^2}{h^2}\right) < 0, \quad (61)$$

where $dE = dr r\sigma_0 |\xi|^2$.

The numerical check of inequality (61) that we did also confirms the presence of instability. For example, for the modes shown in Fig. 5, the typical value of the discriminant $1 + 4ac/b^2$ is of the order of -1.

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