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METHODOLOGICAL NOTES

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Analytical mechanics and field theory: derivation of equations from energy conservation

N A Vinokurov

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Contents

Abstract. Equations of motion in mechanics and field equations in field theory are conventionally derived using the least action principle. This paper presents a nonvariational derivation of Hamilton's and Lagrange's equations. The derivation starts by specifying the system energy as a function of generalized coordinates and velocities and then introduces generalized momenta in such a way that the energy remains unchanged under variations of any degree of freedom. This immediately leads to Hamilton's equations with an as yet undefined Hamiltonian. The explicit dependence of generalized momenta on the coordinates and velocities is determined by first finding the Lagrangian from the known energy function. We discuss electrodynamics as an illustrative example. The proposed approach provides new insight into the nature of canonical momenta and offers a way to find the Lagrangian from the known energy of the system.

1. Introduction

Equations of motion for a broad class of mechanical systems can be derived from variational principles (see, e.g., the courses [1–5]). On the one hand, it can be argued that this fact reflects some special properties of the corresponding differential equations and can be regarded as a more compact form of writing these equations. On the other hand, variational principles and related differential equations form the basis of quantum mechanics. In this approach, the conservation of energy follows from the absence of explicit dependence of the Lagrangian function on time. Below we discuss an alternative approach. Relying on the energy conservation law, we introduce the generalized momenta and then find the Lagrangian function and explicit expressions for momenta in terms of velocities and coordinates.

N A Vinokurov Budker Institute of Nuclear Physics, Siberian Branch of the Russian Academy of Sciences, prosp. Akademika Lavrent'eva 14, 630090 Novosibirsk, Russian Federation
E-mail: vinokurov@inp.nsk.su;
Korea Atomic Energy Research Institute, 1045 Daedeok-Daero, Yuseong-gu, Daejeon 305-353, Republic of Korea

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2. Energy conservation and Hamilton's equations

Let $E(q, v)$ be the system energy that depends on the generalized coordinates q^i, $i = 1, \ldots, N$, and generalized velocities $v^i = \mathrm{d}q^i/\mathrm{d}t$. The state of the system is represented by a point with coordinates (q, v) in a $2N$-dimensional space. The conservation of energy along the trajectory can be written as

$$0 = \mathrm{d}E(q, v) = \sum_{i=1}^{N} \left(\frac{\partial E}{\partial q^i}\, \mathrm{d}q^i + \frac{\partial E}{\partial v^i}\, \mathrm{d}v^i \right). \tag{1}$$

The sum in the right-hand side is equal to zero, but each of its contributing terms differs from zero in general. In order to provide a detailed energy conservation, i.e., the conservation under independent changes in each degree of freedom, we replace velocities by different variables p_i such that $E(q, v) = H(q, p)$ and

$$0 = \frac{\partial H}{\partial q^1}\, \mathrm{d}q^1 + \frac{\partial H}{\partial p_1}\, \mathrm{d}p_1 = \frac{\partial H}{\partial q^2}\, \mathrm{d}q^2 + \frac{\partial H}{\partial p_2}\, \mathrm{d}p_2 = \ldots$$

$$= \frac{\partial H}{\partial q^N}\, \mathrm{d}q^N + \frac{\partial H}{\partial p_N}\, \mathrm{d}p_N\,. \tag{2}$$

If the system is composed of independent (noninteracting) parts, the energy of each of them is conserved. It can therefore be said that we have defined the $2N$-dimensional space (q, p) as the space in which the degrees of freedom are 'most independent'. Equations (2) can be rewritten as N proportions:

$$\frac{\mathrm{d}q^1}{\partial H/\partial p_1} = -\frac{\mathrm{d}p_1}{\partial H/\partial q^1} = \mathrm{d}t_1, \ \ldots, \ \frac{\mathrm{d}q^N}{\partial H/\partial p_N} = -\frac{\mathrm{d}p_N}{\partial H/\partial q^N} = \mathrm{d}t_N\,. \tag{3}$$

Equations (2) and (3) describe N independent displacements $\mathrm{d}q^i$. Because $\mathrm{d}q^i = v^i\,\mathrm{d}t$ in motion, we can set $v^i = \partial H/\partial p_i$ in order to make all the terms in proportions (3) equal, i.e., to obtain $\mathrm{d}t_1 = \ldots = \mathrm{d}t_N = \mathrm{d}t$. In this way, we arrive at Hamilton's equations,

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}\,. \tag{4}$$

We note that in writing Eqns (2), we assumed the existence of as yet undefined variables p_i and a function H satisfying these equations and the conditions $v^i = \partial H/\partial p_i$. In other words, we seek differential equations (equations of motion) in form (4) that would satisfy the energy conservation law (1). As is shown below, the sought equations do exist, but there is a wider class of equations that preserve energy.

Using a more formal line of reasoning, in order to obtain energy conservation in the new variables $X = (q_1, p_1, \ldots, q_N, p_N)$, $0 = \mathrm{d}H(X)/\mathrm{d}t = (\partial H/\partial X_\alpha)\mathrm{d}X_\alpha/\mathrm{d}t$ ($\alpha = 1, \ldots, 2N$; here and below, summation over repeated indices is assumed), it suffices to require that $\mathrm{d}X_\alpha/\mathrm{d}t = J_{\alpha\beta}\,\partial H/\partial X_\beta$ for any skew-symmetric matrix $J_{\alpha\beta}$. In matrix form, these $2N$ equalities can be written as $\dot{\mathbf{X}} = \mathbf{J}\nabla H$. Any real-valued skew-symmetric matrix can be reduced by an orthogonal transformation to a block-diagonal form with N diagonal blocks,

$$\begin{pmatrix} 0 & \mu_k \\ -\mu_k & 0 \end{pmatrix},$$

where $\pm\mathrm{i}\mu_k$ are the eigenvalues of the matrix [6]. By selecting scaling factors at the variables p, all the eigenvalues can be made equal to $\pm\mathrm{i}$, and the matrix $\mathbf{J}$ can be transformed into a unit symplectic matrix with N unit symplectic 2×2 blocks on the diagonal:

$$\mathbf{J} = \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \ldots, \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \tag{5}$$

We thus introduce the variables p in terms of which, as follows from quasidiagonal form (5) of the matrix $\mathbf{J}$, the degrees of freedom are 'maximally independent' and a symplectic structure arises (hence, the Poisson brackets). We note that the appearance of such a mathematically rich system in phase space is, at least partially, a consequence of our selection of very convenient, but to a certain degree artificial variables, the generalized momenta.

Equation (3) also leads to an alternative explanation of the derivation of Hamilton's equations (4). Let each degree of freedom depend on its time t_i. We require the conservation of $H[q^1(t_1), p_1(t_1), \ldots, q^N(t_N), p_N(t_N)]$ under independent changes of each t_i: $\mathrm{d}H/\mathrm{d}t_i = 0$. After this step, we carry out the 'synchronization' $t_1 = \ldots = t_N = t$. We recall that in the relativity theory, a clock at each point in three-dimensional space is considered; the clock synchronization can be carried out in different ways (in particular, depending on the reference system used). For this reason, for a system with spatially separated parts, such an explanation acquires a simple sense. It also clarifies the statement on 'maximally independent' degrees of freedom.

3. Finding momenta and the Lagrangian function

Now we need to find explicit expressions for the momenta p in terms of velocities v. Because $v^i = \partial H/\partial p_i$, according to Donkin's theorem (see, e.g., [2]) there exists a function of velocities $L(q, v) = p_i v^i - H(p, q)$ that is dual to the Hamiltonian function, with $p_i = \partial L/\partial v^i$. Using the last expression, we can obtain a partial differential equation for the Lagrange function L,

$$v^i\,\frac{\partial L}{\partial v^i} = L + E(q, v)\,. \tag{6}$$

In courses of analytic mechanics (see, e.g., [5]), Eqn (6) is used to determine the energy E given the Lagrangian function L. Here, we have to find L by solving partial differential equation (6). We also note that although the Legendre transformation has been used to derive Eqn (6), this equation on its own does not fall into this class, because it relates two functions of the same variables, the velocities.

The general solution of Eqn (6) is the sum of a particular solution L_1 and the complete solution of the corresponding homogeneous equation,

$$L = L_1(q, v) + a_i(q)v^i\,. \tag{7}$$

The second term in the right-hand side of Eqn (7), containing N arbitrary constants a_i, describes hyroscopic forces that do not contribute to the energy $E(q, v)$. The particular solution of Eqn (6) can be found using the Taylor series expansion. If

$$E(q, v) = \sum_{n=0}^{\infty} A^{(n)}_{ik\ldots l}\, v^i v^k \ldots v^l\,, \tag{8}$$

then

$$\begin{aligned} L_1(v) &= -A^{(0)} + \sum_{n=2}^{\infty} \frac{1}{n-1}\, A^{(n)}_{ik\ldots l}\, v^i v^k \ldots v^l = \\ &= -E(q, 0) + \int_1^{\infty} \left[E\left(q, \frac{v}{x}\right) - E(q, 0) \right] \mathrm{d}x\,. \end{aligned} \tag{9}$$

Solution (9) exists only for $A^{(1)}_i = 0$. If series (8) is terminated after $n = 2$, the energy $E = A^{(0)} + A^{(2)}_{ik} v^i v^k = U(q) + T(q, v)$ is the sum of potential U and kinetic T energies and Eqn (9) leads to the usual formula $L_1 = T - U$. It is easy to verify that for a free relativistic particle, Eqn (9) also gives the correct answer.

Using Eqns (7) and (9), we can determine the generalized momenta

$$p_i = a_i + \int_1^{\infty} \frac{\partial}{\partial v^i}\, E\left(\frac{v}{x}\right) \mathrm{d}x \tag{10}$$

and the Hamiltonian function $H = E[q, v(p)]$. The second part of Hamilton's equations (4) can be transformed into the Lagrange equations. From Eqn (7), we find an explicit expression for the forces related to a 'covector field' a_i [2],

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial L_1}{\partial v^i} = \frac{\partial L_1}{\partial q^i} + v^j \left(\frac{\partial a_j}{\partial q^i} - \frac{\partial a_i}{\partial q^j} \right). \tag{11}$$

We see from Eqn (11) that only the nonintegrable (vortex) part of a_i contributes to the forces. Moreover, these additional forces do not change the system energy.

It is worth mentioning that instead of the energy $E(q, v)$, we can use its arbitrary function $F[E(q, v)]$ in the equations above. Then, with the help of Eqns (7) and (9), we can find other Lagrangian functions. A simple example is $E = A^{(0)} + A^{(2)}_{ik} v^i v^k = U(q) + T(q, v)$ and $F(E) = E^2$. In this case, Eqn (9) gives $L_1 = T^2/3 + 2UT - U^2$.

Thus, from the constancy of the known function $E(q, v)$ of generalized coordinates and velocities, we can derive Lagrange's and Hamilton's equations (up to a certain nonuniqueness related to the existence of gyroscopic forces). This derivation relies on the assumption of detailed energy conservation, which is expressed by Eqns (2)–(4). If we use only the energy conservation law, then, having determined the Lagrangian function as a solution of Eqn (6), it is easy to obtain the generalization of Lagrange's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i} - \frac{\partial L}{\partial q^i} = G_{ij}(q, v, \dot{v})v^j\,, \tag{11a}$$

where $G_{ij} = -G_{ij}$ is an arbitrary antisymmetric tensor. In the special case in (11), it can be reduced to zero by adding the term $a_i(q)v^i$ linear in velocities to the Lagrangian function, in agreement with formula (7).

4. Classical electrodynamics

The method described in Section 3 can be applied to a field. In the Coulomb gauge ($\operatorname{div}\mathbf{A} = 0$), the energy of charged particles and an electromagnetic field can be written as [7, 8]

$$E = \frac{1}{8\pi}\iiint\left[\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)^2 + (\operatorname{rot}\mathbf{A})^2\right]\mathrm{d}^3\mathbf{r} + \sum_\alpha \frac{m_\alpha c^2}{\sqrt{1-\mathbf{v}_\alpha^2/c^2}} + \sum_{\alpha<\beta}\frac{Q_\alpha Q_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|}\,, \tag{12}$$

where $\mathbf{A}$ is the vector potential, m_α, Q_α, $\mathbf{r}_\alpha$, and $\mathbf{v}_\alpha$ are respectively the mass, charge, coordinate, and velocity of a particle with number α, and c is the speed of light. For the field, the number of a degree of freedom i is replaced by the coordinate $\mathbf{r}$ of the point of observations. For this reason, the derivatives with respect to $\mathbf{A}$ and $\partial\mathbf{A}/\partial t$ must be variational. With the help of Eqns (7), (9), and (12), we obtain

$$L = \frac{1}{8\pi}\iiint\left[\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)^2 - (\operatorname{rot}\mathbf{A})^2\right]\mathrm{d}^3\mathbf{r} - \sum_\alpha m_\alpha c^2\sqrt{1-\frac{\mathbf{v}_\alpha^2}{c^2}} - \sum_{\alpha<\beta}\frac{Q_\alpha Q_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|} + \sum_\alpha \mathbf{a}_\alpha\mathbf{v}_\alpha + \iiint\mathbf{a}\,\frac{\partial\mathbf{A}}{\partial t}\,\mathrm{d}^3\mathbf{r}\,. \tag{13}$$

Then

$$p_i(\mathbf{r}, t) = \frac{\delta L}{\delta(\partial A_i/\partial t)} = \frac{1}{4\pi c^2}\frac{\partial A_i}{\partial t} + a_i\big(\{\mathbf{A}\}, \{\mathbf{r}_\alpha\}, \mathbf{r}\big)\,, \tag{14}$$

where $i = 1, 2, 3$, and

$$\mathbf{p}_\alpha(t) = \frac{\partial L}{\partial\mathbf{v}_\alpha} = \frac{m_\alpha\mathbf{v}_\alpha}{\sqrt{1-\mathbf{v}_\alpha^2/c^2}} + \mathbf{a}_\alpha\big(\{\mathbf{A}\}, \{\mathbf{r}_\beta\}\big)\,. \tag{15}$$

The braces in Eqn (14) point to the fact that the as yet undefined vector field $\mathbf{a}$ and vectors $\mathbf{a}_\alpha$ may depend on the vector potential at all points of three-dimensional space and on the coordinates of all particles (i.e., they are functionals of $\mathbf{A}$ and functions of coordinates of all particles). It can be readily seen that for $\mathbf{a} = \mathbf{a}_\alpha = 0$, Lagrangian function (13) leads to equations with the transverse field component not coupled to the particles. It is therefore natural to select appropriate $\mathbf{a}$ in order to couple them. Namely, the vector field $\mathbf{a}$ should depend on the particle coordinates $\mathbf{r}_\alpha$, and the vectors $\mathbf{a}_\alpha$ should depend on the vector potential $\mathbf{A}$. The simplest choice for $\mathbf{a}$ is the longitudinal electric field with some coefficient $-\kappa$: $\mathbf{a} = \kappa\operatorname{grad}\varphi$, $\varphi = \sum_\alpha\varphi_\alpha = \sum_\alpha Q_\alpha/|\mathbf{r} - \mathbf{r}_\alpha|$. The simplest choice for $\mathbf{a}_\alpha$ is the vector potential at the point where the particle is located, $\mathbf{a}_\alpha = \kappa_\alpha\mathbf{A}(\mathbf{r}_\alpha, t)$. Under these assumptions, expression (13) leads to Lagrange's equations

$$\frac{1}{4\pi c^2}\frac{\partial^2 A_i}{\partial t^2} - \kappa\sum_\alpha v_{\alpha j}\partial_j\partial_i\varphi_\alpha = \frac{1}{4\pi}\Delta A_i + \sum_\alpha\kappa_\alpha v_{\alpha i}\delta(\mathbf{r} - \mathbf{r}_\alpha)\,, \tag{16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_\alpha\mathbf{v}_\alpha}{\sqrt{1-\mathbf{v}_\alpha^2/c^2}} + \kappa_\alpha\left(\frac{\partial\mathbf{A}}{\partial t} - \mathbf{v}_\alpha\times\operatorname{rot}\mathbf{A}\right) = Q_\alpha\sum_{\beta\neq\alpha}\frac{Q_\beta(\mathbf{r}_\alpha - \mathbf{r}_\beta)}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|^3}\,. \tag{17}$$

In order to satisfy the Coulomb gauge condition, the current in Eqn (16) should be divergence-free,

$$\partial_i\sum_\alpha\big(\kappa_\alpha v_{\alpha i}\delta(\mathbf{r} - \mathbf{r}_\alpha) + \kappa v_{\alpha j}\partial_j\partial_i\varphi_\alpha\big) = 0\,. \tag{18}$$

Hence, $\kappa_\alpha/\kappa = 4\pi Q_\alpha$. For $1/\kappa = 4\pi c$, Eqns (16) and (17) take their usual form:

$$\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \frac{4\pi}{c}\sum_\alpha Q_\alpha\mathbf{v}_\alpha\delta(\mathbf{r} - \mathbf{r}_\alpha) - \frac{1}{c}\operatorname{grad}\frac{\partial\varphi}{\partial t}\,, \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{m_\alpha\mathbf{v}_\alpha}{\sqrt{1-\mathbf{v}_\alpha^2/c^2}} = Q_\alpha\left[\sum_{\beta\neq\alpha}\frac{Q_\beta(\mathbf{r}_\alpha - \mathbf{r}_\beta)}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|^3} - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right] + \frac{Q_\alpha}{c}\,\mathbf{v}_\alpha\times\operatorname{rot}\mathbf{A}\,. \tag{20}$$

Such a choice of the constant κ cannot be substantiated in the framework of this approach. The reason is the difference in the description of transverse and longitudinal components of the electromagnetic field. The former is described by the respective degrees of freedom, and the latter is not. The coupling between the transverse and longitudinal fields cannot be established until we require the equations to be relativistically invariant.

5. Conclusions

Hamilton's equations are equivalent to the Hamilton–Jacobi partial differential equations [9]

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0\,, \tag{21}$$

$$H\left(q, \frac{\partial W}{\partial q}\right) = E_0\,, \tag{22}$$

for $S = W - E_0 t$, since Hamilton's equations describe characteristic bands of the Hamilton–Jacobi equations. Additionally, they are equivalent to variational principles for the actions S and W. The variational principles show that W and S have the geometrical meaning of natural symplectic parameters (the 'length' of curves) of trajectories in phase space (q, p) and in the extended phase space $(q, t, p, -E)$.

The particular choice of momenta p, via 'the principle of detailed energy conservation' (2), ensures the preservation of phase volume and phase density (Liouville's theorem). The latter is used in statistical mechanics.

The definition of generalized momenta in Eqn (2) resembles that of entropy S in the fundamental thermodynamic equation $\mathrm{d}E = -P\,\mathrm{d}V + T\,\mathrm{d}S$ (traditional notation is used for the pressure P, volume V, and temperature T). The first term in the right-hand side of the last equation is the work, and the second describes the motion (of gas molecules) with the help of the measurable parameter $T = \partial E/\partial S$ [instead of $v^i = \partial H/\partial p_i$ in Eqn (2)] and the variation of the specially introduced parameter S (instead of p_i).

To summarize, this article formulates the principle of detailed energy conservation. It defines the generalized momenta as special variables that provide 'maximum independence' between the degrees of freedom. Given the dependence of energy on generalized coordinates and velocities, the Lagrangian function is found with the help of formulas (7) and (9). In turn, differentiating it with respect to velocities gives generalized momenta. Their substitution into the energy gives the Hamiltonian function. The Hamilton–Jacobi equations and variational principles arise as a consequence of the equations of motion (Lagrange's and Hamilton's equations).

The viewpoint proposed here elucidates the meaning of generalized momenta. These quantities are introduced for obtaining (Hamilton's) equations with the 'most independent' degrees of freedom. We note that because the approach proposed above assumes that the state of the system is specified by generalized coordinates and related velocities, the phase space has an even dimension. Dynamical systems with odd-dimensional spaces require separate consideration.

We answer a naturally arising question as to why classical equations have to be derived in one more way. Any mathematical model hinges on some initial assumptions and notions (axiomatics). Discussing different axiomatics, which lead to the same results, might be helpful for understanding the model applicability limits and their possible extension. A well-known example is the comparison of different axiomatics in thermodynamics. Mathematical models in physics rely on experimental results. For this reason, the closer the notions used in axioms are to quantities dealt with in experiments, the more 'transparent' the model seems. In this sense, the conservation laws and their further development — the principle of detailed energy conservation — look at least no worse than integral variational principles with a physically obscure and ambiguously defined (and hence nonmeasurable) Lagrangian function and generalized momenta. It is anything but simple to explain why $L = T - U$ (once again, using the notion of energy) and $\mathbf{P} = \gamma m\mathbf{v} + (Q/c)\mathbf{A}$ (see, however, Ref. [8] and a reference therein). Formulas (6)–(9) allow finding the Lagrangian function (and, based on it, the generalized momenta) from the experimentally known energy function. Additionally, they help in obtaining the general form of the equations of motion that preserve energy, the generalized Lagrangian equations (11a). In any case, it is enlightening to examine a task from various angles.

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References

1. Landau L D, Lifshitz E M *Mechanics* 3rd ed. (Oxford: Pergamon, 1974); Translated from Russian: *Mekhanika* 4th ed. (Moscow: Nauka, 1988)
2. Gantmacher F *Lectures in Analytical Mechanics* (Moscow: Mir, 1975); Translated from Russian: *Lektsii po Analiticheskoi Mekhanike* 3rd ed. (Moscow: Fizmatlit, 2002)
3. Sommerfeld A *Lectures on Theoretical Physics* Vol. 1 *Mechanics* 4th ed. (New York: Academic Press, 1964); Translated into Russian: *Mekhanika* (Izhevsk: RKhD, 2001)
4. Lanczos C *The Variational Principles of Mechanics* 4th ed. (Toronto: Univ. of Toronto Press, 1970; New York: Dover Publ., 1986); Translated into Russian: *Variatsionnye Printsipy Mekhaniki* (Moscow: Mir, 1965)
5. José J V, Saletan E J *Classical Dynamics: a Contemporary Approach* (Cambridge: Cambridge Univ. Press, 1998)
6. Gantmacher F R *Applications of the Theory of Matrices* Vols 1, 2 (New York: Interscience Publ., 1959); Translated from Russian: *Teoriya Matrits* 5th ed. (Moscow: Fizmalit, 2004)
7. Heitler W *The Quantum Theory of Radiation* 3rd ed. (Oxford: Clarendon Press, 1954); Translated into Russian: *Kvantovaya Teoriya Izlucheniya* (Moscow: IL, 1956)
8. Ginzburg V L *Theoretical Physics and Astrophysics* (Oxford: Pergamon Press, 1979); Translated from Russian: *Teoreticheskaya Fizika i Astrofizika: Dopolnitel'nye Glavy* 2nd ed. (Moscow: Nauka, 1981)
9. Courant R, Hilbert D *Methods of Mathematical Physics* Vol. 2 *Partial Differential Equations* (New York: Interscience Publ., 1989); Translated into Russian: *Uravneniya s Chastnymi Proizvodnymi* (Moscow: Mir, 1964)