

FROM THE HISTORY OF PHYSICS

PACS numbers: **01.60.+q, 01.65.+g, 03.65.–w**

"The host of heaven is far..." (my recollections of Ya B Zeldovich)

V S Popov

DOI: 10.3367/UFNe.0184.201404i.0451

Abstract. **Recollections about one of the great Soviet theoretical physicist Academician Ya B Zeldovich and his influence on the scientific work of the author and the author's colleagues.**

It was ... a time which called for giants and produced giants –giants in power of thought, passion, and character, in universality and learning. The men who founded the modern rule of the bourgeoisie had anything but bourgeois limitations.
Friedrich Engels. *Dialectics of Nature*
(Moscow: Politizdat, 1957), p. 7

Yakov Borisovich Zeldovich (1914–1987) made major contributions to many fields of theoretical and applied physics: the theory of combustion and detonation, the physics of explosions and shock waves, atomic and nuclear physics, the theory of the weak interaction (conserved vector current), relativistic astrophysics and cosmology... and this list could well be continued [1–3]. In our epoch of narrow specialization, he was perhaps one of the last (together with Fermi, Feynman, Gamow, and Landau) universalists in the entire area of physics and astrophysics.

Beginning, I should say at once that I was neither a disciple nor a close colleague of Yakov Borisovich — when a young researcher I had no chance to listen to his lectures or work under his immediate guidance, and we only had a single joint article [4]. Our close scientific contacts refer to 1970–1973; they concerned special issues of theoretical physics, and first and foremost quantum electrodynamics of superstrong fields. Yakov Borisovich was a bright and many-sided person, and my reminiscences are of an inevitably fragmentary and somewhat one-sided character. I nevertheless dare present them in the hope that they may add some details to the image of this prominent scientist.

My direct acquaintance with Yakov Borisovich goes back to the time soon after the appearance of the paper by Gershtein and Zeldovich [5] (see also Ref. [6]), where they considered the properties of the atomic spectrum depending on the nuclear charge Z in superheavy ($Z > \alpha^{-1} = 137$) atoms and spontaneous positron production from a vacuum upon approaching the heavy nuclei within a distance $R < R_{\rm cr} \ll l_{\rm C}$. They supposed that when the energy ε_0 of a discrete electron level approaches the boundary of the lower continuum of solutions to the Dirac equation ($\varepsilon_0 \to -mc^2$): "the behavior of the wave function of the bound electron becomes pathological and the vacuum polarization gets delocalized" [5] — that is, the vacuum charge density moves far away from the nucleus. The authors therewith proceeded from the principal term of the asymptotics of the electron wave function[1]

$$\psi_0(r) \propto \exp(-\lambda r)\,, \quad \lambda = \sqrt{m^2 - \varepsilon_0^2}\,, \quad r \to \infty\,. \tag{1}$$

When I got interested in this issue and thoroughly studied papers [5–7], I noticed that it follows from the Dirac equation in the Coulomb field $V(r) = -Z\alpha/r$ that, along with the exponent (1), the wave function also has a substantial pre-exponential factor:

$$\psi_0(r) \approx r^{\nu} \exp(-\lambda r)\,, \quad \nu = \frac{Z\alpha\varepsilon_0}{\lambda}\,, \tag{2}$$

which in the limit $\varepsilon_0 \to -m$ decreases at infinity faster than any finite power of r and leads to the following asymptotic dependence

$$\psi_0(r) \propto \exp\left(-\sqrt{8Z_{\rm cr}\alpha\frac{r}{l_{\rm C}}}\right), \quad r \to \infty \tag{3}$$

(for $\varepsilon_0 = -m$ and $Z = Z_{\rm cr}$), and so the wave function of the level at the edge of lower continuum remains localized (although $\lambda \to 0$ in this case), and the mean radius of the electron cloud will be $\langle r \rangle \ll l_{\rm C}$. Thus, the arguments presented in Ref. [5] in favor of bound state delocalization and vacuum polarization rearrangement as $Z \to Z_{\rm cr}$ are no longer pertinent.

We shall briefly explain the derivation of these formulas. The level energy ε_0 includes the electron rest energy m and depends on Z, with the inequality $-m \leqslant \varepsilon_0(Z) < m$ met for the states of the discrete spectrum. The value of $\varepsilon_0 = m$ corresponds to the free electron at rest, $\varepsilon_0 > m$ to the upper continuum, $\varepsilon_0 = -m$ to the critical nuclear charge,[2] and

[1] Hereinafter, m and $l_{\rm C} = \hbar/mc$ are the electron mass and Compton wave length, respectively, $a_{\rm B} = \hbar^2/me^2 = \alpha^{-1} l_{\rm C}$ is the Bohr radius, r is the distance between the electron and the nucleus, λ is the dimensionless momentum of the bound state, $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant, $Z_{\rm cr}$ and $R_{\rm cr}$ are the critical charge of the nucleus and the appropriate distance between colliding nuclei at which the electron level descends to the lower continuum boundary $\varepsilon_0 = -mc^2$. We make use of the relativistic units: $m_{\rm e} = \hbar = c = 1$.

[2] The exact value of the critical charge $Z_{\rm cr} > 137$ depends on the nucleus radius $r_{\rm N}$ and, to a lesser extent, on the bulk distribution of the electric charge in a nucleus, the degree of ionization of the outer electron shells in the atom, the nucleus edge diffusivity, and some other factors [8–18]. The first estimate of $Z_{\rm cr}$ belongs to Pomeranchuk and Smorodinsky [7], who allowed for the Coulomb potential cut-off at small distances in the Dirac equation, but this $Z_{\rm cr}$ value turned out to be overestimated. The calculations done in Refs [8–11] showed that $Z_{\rm cr} \approx 170$ for the ground level $1{\rm s}_{1/2}$, $Z_{\rm cr} = 185$ and 230 for the excited states $2{\rm p}_{1/2}$ and $2{\rm s}_{1/2}$, respectively, etc. The evaluation of $R_{\rm cr}$, which requires a numerical solution of the 'two-centers problem' for the Dirac equation, was performed in Refs [14–20].

V S Popov RF State Scientific Center 'Alikhanov Institute for Theoretical and Experimental Physics',
ul. B. Cheremushkinskaya 25, 117218 Moscow, Russian Federation
E-mail: markina@itep.ru

Received 29 April 2013
Uspekhi Fizicheskikh Nauk **184** (4) 451 – 456 (2014)
DOI: 10.3367/UFNr.0184.201404i.0451
Translated by M V Tsaplina; edited by A Radzig

$\varepsilon_0 < -m$ to the states of the lower continuum (the so-called Dirac sea filled with unobservable vacuum electrons; the hole, i.e., the unfilled state in this sea, corresponds to the positron [21, 22]).

Analogously, if the condition $\varepsilon_0(R; Z_1, Z_2) = -m$ (where R is the distance between the nuclei) is met for the energy ε_0 of the ground term of the set of two nuclei with charges Z_1 and Z_2, this determines the 'critical' internuclear distance $R_{\rm cr}(Z_1, Z_2)$. When these nuclei approach so that $R < R_{\rm cr}$, the bound level vanishes from the discrete spectrum, becoming quasistationary and going to the lower continuum, and the spontaneous production of e^+e^- pairs from the vacuum begins with electrons 'alighting' into the K-orbit (if for $R > R_{\rm cr}$ it was not occupied with electrons), while positrons, penetrating through the Coulomb barrier, tend to infinity, where they can be registered. In the threshold region ($Z - Z_{\rm cr} \ll Z_{\rm cr}$), the probability of spontaneous positron production is exponentially low [9–11], because the barrier in the effective potential $U \approx \varepsilon V - V^2/2$ is wide and has a low penetrability.

After these explanations, let us go over to formulas (1)–(3). The first two follow from the quasiclassical expression for the electron wave function $\psi_0(r) \propto \exp\{{\rm i}\int^r p(r')\,{\rm d}r'\}$, where the momentum under the barrier is expressed for the case of vector coupling [3] as

$$p(r) = \sqrt{\big(\varepsilon_0 - V(r)\big)^2 - m^2}$$

(we consider the potential $V(r) \to 0$ at infinity). For the Coulomb field, when $V(r) = -Z\alpha/r$ for $r > r_{\rm N}$, we arrive at

$$ {\rm i}p(r) = -\lambda + \frac{Z\alpha\varepsilon_0}{\lambda r} + O\left(\frac{1}{r^2}\right), \qquad (4)$$

which immediately gives formula (2), while for the short-range ($Z = 0$) potential it leads to expression (1). In general, one has

$$\psi_0(r) \propto \exp(-\lambda r)\exp\left(-\frac{\varepsilon_0}{\lambda}\int^r V(r')\,{\rm d}r' + \ldots\right), \quad r \to \infty, \qquad (5)$$

from which the presence of the power function r^ν in the pre-exponential factor of formula (2) is seen to be connected with precisely the Coulomb 'tail' of the atomic potential $V(r)$.

As concerns the asymptotics (3) at the edge of the lower continuum, its derivation is more complicated and cannot be obtained directly from formula (2) but needs an exact solution of the Dirac equation at an energy $\varepsilon_0 = -m$, which, in the case of the Coulomb attractive field, is found analytically and has the form [4]

$$\psi_0(r) = {\rm const}\; K_{{\rm i}\nu}(\sqrt{8Z\alpha r}), \quad \nu = 2\sqrt{(Z\alpha)^2 - 1}, \quad Z\alpha > 1, \qquad (6)$$

where $K_{{\rm i}\nu}(x)$ is the Macdonald function known from the theory of special functions. Since $K_{{\rm i}\nu}(x) \propto \exp(-x)$ for $x \gg 1$, this immediately implies asymptotics (3). Note the specific character of the decrease in the level wave function: $\psi_0(r) \propto \exp(-c_1\sqrt{r})$ for $\varepsilon_0 = -m$ and $r \to \infty$, different from the typical behavior $\psi_0(r) \propto \exp(-\lambda r)$ in short-range potentials.

The behavior of atomic wave functions for $\varepsilon_0 \to m$ and $\varepsilon_0 \to -m$ is substantially different, although in both cases $\lambda \to 0$. In the former case (a hydrogen-like atom), the level energy is $\varepsilon_n = m(1 - (Z\alpha)^2/2n^2)$, with $n = 1, 2, 3, \ldots$ being the principal quantum number of the level, and

$$\lambda = \frac{Z}{n}, \quad \nu = n, \quad \psi_n(r) \propto r^n \exp\left(-\frac{Z}{n}r\right) \qquad (7)$$

(in Hartree atomic units $\hbar = m = e = 1$). The electron density reaches its maximum at distances $\langle r\rangle \sim n^2 Z^{-1} a_{\rm B}$ from the nucleus and, as $n \to \infty$, i.e., $\varepsilon_n \to m$, the bound state becomes delocalized.[5] On the contrary, as $Z \to Z_{\rm cr}$ and $\varepsilon_0 \to -m$, the atomic state is compressed, and its mean radius $\langle r\rangle$ decreases with increasing Z. At $Z = Z_{\rm cr}$, we have $\langle r\rangle \sim \alpha^{1/2} l_{\rm C} \approx 0.001 a_{\rm B}$, with the numerical $\langle r\rangle$ value depending on the nucleus radius $r_{\rm N}$ [9–11]. Thus, solutions of the Dirac equation in the presence of a Coulomb field are no longer symmetric in the sign of the energy ε.

Having ascertained all this, I called Yakov Borisovich with certain agitation and set forth my considerations. His first reaction was negative. "You have made some mistake." However, having listened to my arguments, he said: "I should think it over." And in no more than two or three days he himself called me: "Yes, it seems likely that you are right. Come to my place and we shall discuss everything in more detail." And, at the end of our conversation, he said: "You should meet Migdal, for he has ideas concerning positron levels emanating from the lower continuum. Try to look into it".

This was the beginning of my acquaintance and collaboration with Yakov Borisovich Zeldovich and Arkady Benediktovich Migdal (YaB and AB, as their disciples and colleagues usually called them among themselves). Looking back, I see that those were the most splendid events in my scientific biography.

Although our close collaboration did not take so much time and encompassed the years of 1970–1977, it was of invaluable benefit for me. I learnt much from these remarkable physicists, including striving for a visual qualitative explanation (in lay terms) of physical effects and the results of cumbersome calculations. Work on the quantum electrodynamics of superstrong Coulomb fields, the critical charge of a nucleus, the properties of the electron shell in a supercritical atom, spontaneous positron production from a vacuum for $Z > Z_{\rm cr}$ or $R < R_{\rm cr}$ in collisions of heavy nuclei,[6] the theory of anomalous and supercharged nuclei was discussed in detail with YaB and (or) AB, which was of great importance for us. Essentially, all my work in the range of $Z > 137$ began upon studying papers [5–7], which proposed the statement of the problem. It should be noted

[3] That is, for the electromagnetic (Coulomb) electron–nucleus interaction.
[4] The solution decreasing at infinity. The details of calculations can be found in Refs [8] (numerical calculations) and [9–11] (analytical solution) (see also papers [4, 14–17]). Note that expression (6) refers to the upper component of the Dirac bispinor; the lower component has a somewhat different form, but asymptotics (3) remains unchanged for it, too.

[5] The authors of paper [5] evidently meant precisely this case.
[6] For a supercritical resultant nuclear charge $Z_1 + Z_2 > Z_{\rm cr} \approx 170$ [8–11], which actually occurs upon the collision of two uranium nuclei — YaB's and Gershtein's remark [5] which gave start to the numerous studies (of both our group [9–13] and Greiner and colleagues [14–16]) on the calculation of the critical distance $R_{\rm cr}(Z_1, Z_2)$ in the two-nucleus system, the cross sections of spontaneous and stimulated generation of positrons for $R < R_{\rm cr}$, their momentum spectrum, etc. For bare (fully stripped) uranium nuclei, the critical distance is $R_{\rm cr} \approx 0.1\, l_{\rm C} \approx 35$ fm. The dependences of $Z_{\rm cr}$ and $R_{\rm cr}$ on the degree of filling of outer atomic electron shells was discussed in detail in Refs [12–20].

that the problem of vacuum polarization and superbound electrons in the lower continuum for $Z > Z_{cr}$ and $\varepsilon < -m$ was first formulated clearly and consistently in papers [4, 14–16, 23].

I would also like to dwell on the discussion of the problem of positron levels which, in the opinion of AB, could appear from the lower continuum with increasing nuclear charge $Z > 137$. On YaB's advice, I investigated this issue on the basis of the relativistic Dirac equation (describing both electrons and positrons) and found no such levels.[7] This was the subject of the discussion that took place in the early 1970; participants in the discussion were YaB, AB, and the present author. At that time, AB was absolutely positive about the existence of such states, while YaB and I objected. Obviously, AB gradually got more and more irritated by our objections; the discussion became hot and let itself go, throwing aside all restraint. Finally, YaB said: "Kadya! You have forgotten about the Pauli principle!" and AB's boisterous reaction immediately followed, after which YaB cut short the discussion, saying, "Let it be all for today, but don't you think, Kadya, that I cannot have answered you adequately and in the same style. The point is that I am a bit hindered by the presence of Vladimir Stepanovich." I hastened to retire from YaB's place where this discussion had taken place. Although more than 40 years have passed since then, I perfectly well remember the scene and guarantee the exact, if not literal, meaning of what I have narrated.

This episode found certain reflection in the paper "Electronic structure of superheavy atoms" [4] in which YaB inserted 'the distich of Velimir Khlebnikov':[8]

"The host of heaven is far, powerful, and mighty,

but if you crave for understanding, know the atom-house." *

This is precisely what was written in the manuscript and even in the proof to the paper, but at the last moment YaB replaced 'crave for' by a more neutral 'you seek', and this last version appeared in the journal. I was bewildered by the verse and tried to convince YaB to remove it, but to try to dissuade him was useless (an analogous episode was also described in L B Okun's reminiscences [24]). I should confess that I grasped the undertone only later.[9] This was how YaB replied to the rudeness permitted by Migdal in the course of the above-described discussion. It should be emphasized that this episode was not representative of the warm, friendly relations between these two outstanding physicists.

Further on, YaB showed immutable interest in M S Marinov's and my calculations of the critical internuclear distance

The above-described hot discussion took place in Zeldovich's flat in this house, in which not only Zeldovich, but also other prominent scientists, notably N N Semenov, the Nobel Prize Winner in Chemistry 1956, and N M Emanuel, lived for many years. It is now seen to be in a sad state (although it was planned to make a museum in this house: perhaps these plans will someday be realized).

R_{cr} [19, 20] and spontaneous positron production in slow (adiabatic) collisions of two heavy nuclei and in the problem of e^+e^--pair production from a vacuum by applying an alternating electric field or tapping intense laser radiation [25–27] (the so-called Schwinger effect).

Gradually, the main theoretical problems concerning superstrong Coulomb fields were solved, and my contacts with YaB weakened. In 1975–1976, Dima Voskresenskii (at that time AB's postgraduate student) and I developed, under the guidance of AB, the theory of supercharged nuclei (for which $Ze^3 = Z/1600 \gtrsim 1$), whose electrodynamics would possess peculiar features [28–30]. For instance, the electric charge of such a supernucleus is concentrated near its surface,[10] and the electroneutral plasma of p, e, and n

[7] The characteristic property of the Dirac equation is the fact [4, 14–16] that any level of the discrete spectrum, $\varepsilon_0(Z)$, descends monotonously with increasing nuclear charge Z until it reaches the boundary of lower continuum at $Z = Z_{cr}$. For other relativistic wave equations, the situation is already not so simple (in this connection, see, e.g., review [4, Section 5]).

[8] "The poet — word-creator refers to the microworld theory as the atom's constitution" [4].

[9] I was, however, surprised that no reference to V Khlebnikov, including the exact title, publishing house, and year (which in those times was obviously a necessary condition for publication in Soviet journals) was demanded by the editors of *UFN,* who accepted the reference to "Ya B Zeldovich's searches." On other occasions, this requirement was strictly met (see, for instance, Refs [37], [54], and [57] in paper [4] concerning Valery Bryusov's verses, the expression 'in the Pickwick sense', and the opinion, imputed to Bohr, of what is a 'real specialist in a given area' (needless to say that all these 'lyrical digressions' were included in the paper by Yakov Borisovich).

* *Editor's note*: This is a poetic joke wherein the first letters of Russian words are summed to compose Zeldovich adequate answer to Migdal.

[10] According to papers [28–30], a noncompensated electric charge is located near the supernucleus surface in a narrow (compared to the radius of this nucleus) layer $\sim 12-15$ fm thick, and a huge electric field is induced exceeding the critical, or the Schwinger, field $\mathcal{E}_{cr} = m_e^2 c^3/e\hbar$ in quantum electrodynamics. An additional attractive internucleon interaction is needed for securing the stability of supercharged nuclei, which, according to Migdal, can emerge upon compaction of nuclear matter and the formation of pion condensate. The idea of π-condensation suggested by Migdal was then developed by many authors. Now it is conventionally considered (the remark by D N Voskresenskii) that π-condensate can exist in dense cores of neutron stars. The possibility of π-condensation in quark matter is being discussed in the literature as well.

particles is formed in the interior of the nucleus, which lowers substantially the Coulomb energy of the nucleus obstructing its stability. YaB was skeptical in respect of this activity, and as time showed, he was right, for such nuclei are unstable. In those years, I often used to meet AB and could see the difference between his and YaB's style of work, but this is yet another story [31]....

YaB's influence also substantially manifested itself in other work by our group having nothing in common with the '$Z > 137$ problem'. For example, in the paper "The energy levels in a distorted Coulomb field" [32] (see also book [2, p. 83]), he pointed out a beautiful physical effect of atomic spectrum restructuring by a close 'quasinuclear' level (with an example of the electron levels in an impurity semiconductor). This effect was further rediscovered in the relativistic Coulomb problem with $Z > 137$ [33] and in the theory of very light hadron atoms [34, 35], was studied in detail [36–38], and was called the 'Zeldovich effect' [36, 38]. This effect can show up in the spectra of quantum systems for which the interaction potential consists of two parts with strongly incommensurate radii, for example, the short-range ('strong') potential V_s with radius r_0 and the long-range Coulomb potential $U = -Ze^2/r$, with $r_0 \ll a_B = \hbar^2/Zme^2$. Under this condition, one can obtain a model-free equation [35] defining the s-level energies in an atomic spectrum in terms of the s-scattering length a_s on the strong potential. This allows the whole spectrum of atomic states with $l = 0$ to be reconstructed from the energy of one of the levels (taken from either experiment or numerical calculations) without solving the Schrödinger equation. The specific features of the Zeldovich effect for states with angular momentum $l \neq 0$ were considered in papers [37, 38]. It has recently been noted [39, 40] that, in the spectrum of the hydrogen atom placed in a superstrong magnetic field $B \sim 10^{12} - 10^{15}$ Gs [such fields exist on pulsar (neutron star) surfaces and in so-called magnetars], the Zeldovich effect manifests itself for even (under the reflection $z \to -z$ along the magnetic field) levels.

YaB was interested in the theory of unstable (quasi-stationary) states and proposed a way of Gamow's wave function normalization [11] for them and an original form of the perturbation theory (PT) [41, 42]. This approach (Zeldovich's regularization method) was applied in problems of quantum mechanics and atomic physics, including calculations of negative-ion (H^-, Na^-, etc.) photoionization probability in a laser field with circular polarization [43, 44]. In the simplest (discrete) version, the Zeldovich method consists in the fact that the generalized sum S is assigned to the divergent series:

$$S \equiv \sum_n^{\infty} a_n = \lim \sum_n^{\infty} a_n \exp(-\alpha n^2), \quad \alpha \to +0, \qquad (8)$$

and the same procedure is applied to integrals. This method is more powerful than the Cesare, Abel, and Borel methods of divergent series summation, known from mathematics. It is, as a rule, sufficient for attaching physical meaning to divergences appearing in the theory of quasistationary states [42–45].

YaB obtained [46] a peculiar second-order PT formula which, as distinct from the standard formula, does not need knowledge of the entire energy spectrum of the system. The technique [12] proposed in Refs [46–49] turned out to be convenient and was applied to calculations of PT higher-order terms for anharmonic oscillators, the funnel potential, and the Stark effect in a hydrogen atom [47–49]. The corresponding PT series diverge factorially and can be used directly in the domain of weak fields only. Applying divergent series summation techniques, the Zeldovich method among them, the authors of Refs [47–49] managed to calculate the Stark shifts and the widths of hydrogen atomic levels in a strong electric field far beyond the applicability limits of a conventional PT.

YaB [50, 51] introduced the notion (simultaneously with V I Ritus [52]) of quasienergy E_α for an atom residing in the light-wave field:

$$\psi_\alpha(\mathbf{r}, t + T) = \exp\left(-i\frac{E_\alpha T}{\hbar}\right)\psi_\alpha(\mathbf{r}, t), \quad -\infty < t < \infty \qquad (9)$$

($T = 2\pi/\omega$ is the wave period, and ψ_α is the periodic time function) and considered the problem of stimulated photon emission from the quasienergy state.[13] Quasienergy is similar to quasimomentum of an electron in a periodic potential of a crystal lattice (the Bloch wave). At the present time, the approach using quasienergies is a standard in atomic and laser physics. To determine the quasienergy spectrum for a particular atomic system is a fairly complicated problem. The quasienergy spectrum for the model problem of a nonstationary harmonic oscillator permitting an exact solution was found in Ref. [54] (see also book [42, p. 268]). This example demonstrates that even in the case of the 'cut-off' oscillator potential $V(x) = (1/2)\,\omega^2(t)\,x^2$, the quasienergy spectrum can be both discrete and continuous, depending on the particular form of the frequency $\omega(t)$.

Notice that YaB called our attention to the 'paradox of the harmonic oscillator' in the quantum radiation theory. It is a known fact that the dipole radiation probability in the transition between two energy levels E_i and E_f has the form

$$w_{if} = \frac{4\omega_{if}^3}{3\hbar c^3}\left|\langle f|e\mathbf{r}|i\rangle\right|^2, \quad \omega_{if} = \frac{E_i - E_f}{\hbar}. \qquad (10)$$

In the case of a harmonic oscillator with frequency ω, all the differences $\omega_n - \omega_{n-1} = \omega$ are independent of the level number n, and the transition matrix element is $\langle n|\hat{x}|n-1\rangle \propto \sqrt{n}$. This should seem to imply that the $|n\rangle \to |n-1\rangle$ transition probability is proportional to n, and the spectral line width must decrease $\propto 1/n$. Meanwhile, the quasiclassical approximation holds true for $n \gg 1$, and, therefore, the line width is determined by the oscillator damping constant and does not depend on n. This (seeming) contradiction constitutes the paradox of a harmonic oscillator, which was made clear in papers [55, 56] where quantum kinetic equations for the density matrix were derived for

[11] Gamow's wave functions increase exponentially at infinity, and, therefore, the usual normalization condition cannot be met and the perturbation theory formulas need modification.

[12] Replacement of the Schrödinger equation by a Riccati type nonlinear equation for a logarithmic derivative of the wave function allows effective calculations of PT higher-order terms in the case of simple analytic potentials. Using this method in the problem of the Stark effect in a hydrogen atom, 160 PT orders for the ground-state energy and 100 orders for excited states were calculated in Refs [47–49].

[13] It should be noted that in the consideration of a relativistic electron in the field of a strong electromagnetic wave, Nikishov and Ritus [53] introduced a four-dimensional quasimomentum whose time component coincides with quasienergy.

systems with an equidistant spectrum. In this special case, the amplitudes of all transitions correspond to one and the same energy $\hbar\omega$ and interfere among themselves. As a result, the same expressions (for any n) for the transition probabilities are obtained as those in classical electrodynamics, and the paradox of the harmonic oscillator is resolved on its own. In this connection, YaB once told me (perhaps in jest): "Paradoxes are the driving force of science." It is pertinent to recall here the note "How quantum mechanics helps understand the conclusions of classical mechanics," published under the name of P Paradoxov [57], but written undoubtedly by YaB himself.

The work by Yakov Borisovich described above makes up a small, and not at all essential, part of his scientific heritage [1–3]. However, it already shows how wide the range of his scientific interests was and how significant the results he obtained were (see, in particular, papers [58–82]). It should be stressed that the characteristic feature of YaB, and also of AB, was democracy: each of them was ready to discuss scientific problems equally with a student and with an academician, and disagreement with an opposite point of view, which was sometimes expressed in a rather sharp form, never extended to private relations. To discuss with them different aspects of quantum electrodynamics of strong fields and a wider range of physical, but not only physical, issues was always interesting and instructive for me [31].

It is worth noting that Yakov Borisovich found time to delve into the gist of mathematical education necessary for future physicists and engineers [82, 83]. He believed that many textbooks in higher mathematics exploited the formal axiomatic method based on the 'neat $\varepsilon - \delta$ technique' [83], which was inconvenient for applications and failed to guarantee an active mastering of this science by students. YaB expressed his viewpoint clearly in books [82–84] (see also [42]).

I am grateful to destiny for the opportunity to have met and worked in those already distant 1980s with such coryphaei of theoretical physics as Yakov Borisovich Zeldovich and Arkady Benediktovich Migdal, my impressions of the meetings with whom I have tried to expound on above.

Acknowledgment
The present paper is based on the talk by the author at the International Conference 'The Sun, the Stars, the Universe and General Relativity', devoted to the 95th anniversary of the birth of Ya B Zeldovich (Zeldovich Meeting, Minsk, 20–23 April 2009). I am grateful to R Ruffini for the offer to write these recollections, and also to S I Blinnikov, D N Voskresenskii, M I Vysotskii, S I Godunov, V S Imshennik, V D Mur, L B Okun, and V A Rubakov, who read the manuscript and made useful remarks. My thanks are also due to V M Weinberg and D V Popov for assistance in the preparation of the manuscript for publication.

References

1. Zeldovich Ya B *Chemical Physics and Hydrodynamics*: *Selected Works of Yakov Borisovich Zeldovich* (Eds J P Ostriker, G I Barenblatt, R A Sunyaev) (Princeton, N.J.: Princeton Univ. Press, 1993); Translated from Russian: *Khimicheskaya Fizika i Gidrodinamika*: *Izbrannye Trudy* (Ed. Yu B Khariton) (Moscow: Nauka, 1984)
2. Zeldovich Ya B *Particles, Nuclei, and the Universe*: *Selected Works of Yakov Borisovich Zeldovich* (Eds J P Ostriker, G I Barenblatt, R A Sunyaev) (Princeton, N.J.: Princeton Univ. Press, 1993); Translated from Russian *Chastitsy, Yadra, Vselennaya*: *Izbrannye Trudy* (Ed. Yu B Khariton) (Moscow: Nauka, 1985)
3. Aleksandrov A P et al. *Sov. Phys. Usp.* **31** 778 (1988); *Usp. Fiz. Nauk* **155** 729 (1988)
4. Zeldovich Ya B, Popov V S *Sov. Phys. Usp.* **14** 673 (1972); *Usp. Fiz. Nauk* **105** 403 (1971)
5. Gershtein S S, Zel'dovich Ya B *Sov. Phys. JETP* **30** 358 (1969); *Zh. Eksp. Teor. Fiz.* **57** 654 (1969)
6. Gershtein S S, Zel'dovich Ya B *Nuovo Cimento Lett.* **1** 835 (1969)
7. Pomeranchuk I Ya, Smorodinsky Ya A *J. Phys. USSR* **9** 97 (1945)
8. Pieper W, Greiner W *Z. Phys.* **218** 327 (1969)
9. Popov V S *JETP Lett.* **11** 162 (1970); *Pis'ma Zh. Eksp. Teor. Fiz.* **11** 254 (1970)
10. Popov V S *Sov. J. Nucl. Phys.* **12** 235 (1971); *Yad. Fiz.* **12** 429 (1970)
11. Popov V S *Sov. Phys. JETP* **32** 526 (1971); *Zh. Eksp. Teor. Fiz.* **59** 956 (1970)
12. Gershtein S S, Popov V S *Lett. Nuovo Cimento* **6** 593 (1973)
13. Popov V S *Sov. Phys. JETP* **38** 18 (1974); *Zh. Eksp. Teor. Fiz.* **65** 35 (1973)
14. Greiner W, Müller B, Rafelski J *Quantum Electrodynamics of Strong Fields* (Berlin: Springer-Verlag, 1985)
15. Rafelski J, Müller B, Greiner W *Nucl. Phys. B* **68** 585 (1974)
16. Rafelski J, Müller B *Phys. Lett. B* **65** 205 (1976)
17. Popov V S *Priroda* (10) 14 (1981)
18. Popov V S *Phys. Atom. Nucl.* **64** 367 (2001); *Yad. Fiz.* **64** 421 (2001)
19. Lisin V I, Marinov M S, Popov V S *Phys. Lett. B* **69** 141 (1977)
20. Lisin V I, Marinov M S, Popov V S *Phys. Lett. B* **91** 20 (1980)
21. Dirac P A M *Proc. R. Soc. London A* **117** 610 (1928)
22. Dirac P A M *The Principles of Quantum Mechanics* (Oxford: Clarendon Press, 1958)
23. Okun L *Comm. Nucl. Part. Phys.* **6** 25 (1974)
24. Okun L B, in *Znakomyi i Neznakomyi Zeldovich v Vospominaniyakh Druzei, Kolleg, Uchenikov* (Familiar and Unfamiliar Zeldovich in Recollections of Friends, Colleagues, Disciples) (Exec. Eds S S Gershtein, R A Sunyaev; compiler N D Morozova) (Moscow: Nauka, 1993) p. 182
25. Popov V S *Sov. Phys. JETP* **34** 709 (1972); *Zh. Eksp. Teor. Fiz.* **61** 1334 (1971)
26. Popov V S *Sov. J. Nucl. Phys.* **19** 584 (1974); *Yad. Fiz.* **19** 1140 (1974)
27. Marinov M S, Popov V S *Fortschr. Phys.* **25** 373 (1977)
28. Migdal A B, Voskresenskii D N, Popov V S *JETP Lett.* **24** 163 (1976); *Pis'ma Zh. Eksp. Teor. Fiz.* **24** 186 (1976)
29. Migdal A B, Popov V S, Voskresenskii D N *Sov. Phys. JETP* **45** 436 (1977); *Zh. Eksp. Teor. Fiz.* **72** 834 (1977)
30. Popov V *AIP Conf. Proc.* **1205** 127 (2010)
31. Popov V S, Preprint No. 34-01 (Moscow: ITEP, 2001)
32. Zel'dovich Ya B *Sov. Phys. Solid State* **1** 1497 (1959); *Fiz. Tverd. Tela* **1** 1637 (1959)
33. Popov V S *Sov. Phys. JETP* **33** 665 (1971); *Zh. Eksp. Teor. Fiz.* **60** 1228 (1971)
34. Kudryavtsev A E, Markushin V E, Shapiro I S *Sov. Phys. JETP* **47** 225 (1978); *Zh. Eksp. Teor. Fiz.* **74** 432 (1978)
35. Kudryavtsev A E, Popov V S *JETP Lett.* **29** 280 (1979); *Pis'ma Zh. Eksp. Teor. Fiz.* **29** 311 (1979)
36. Badalyan A M, Kok L P, Polikarpov M I, Simonov Yu A *Phys. Rep.* **82** 31 (1982)
37. Karnakov B M, Mur V D, Popov V S *Sov. Phys. Dokl.* **29** 938 (1984); *Dokl. Akad. Nauk SSSR* **279** 345 (1984)
38. Karnakov B M, Kudryavtsev A E, Mur V D, Popov V S *Sov. Phys. JETP* **67** 1333 (1988); *Zh. Eksp. Teor. Fiz.* **94** 65 (1988)
39. Karnakov B M, Popov V S *JETP* **97** 890 (2003); *Zh. Eksp. Teor. Fiz.* **124** 996 (2003)
40. Popov V S, Karnakov B M *JETP* **114** 1 (2012); *Zh. Eksp. Teor. Fiz.* **141** 5 (2012)
41. Zel'dovich Ya B *Sov. Phys. JETP* **12** 542 (1961); *Zh. Eksp. Teor. Fiz.* **39** 776 (1960)
42. Baz' A I, Zel'dovich Ya B, Perelomov A M *Scattering, Reactions and Decay in Nonrelativistic Quantum Mechanics* 1st ed. (Jerusalem: Israel Program for Scientific Translations, 1969); Translated from Russian: *Rasseyanie, Reaktsii i Raspady v Nerelyativistskoi Kvantovoi Mekhanike* 2nd ed. (Moscow: Nauka, 1971)
43. Mur V D, Pozdnyakov S G, Popov V S, Popruzhenko S V *JETP Lett.* **75** 249 (2002); *Pis'ma Zh. Eksp. Teor. Fiz.* **75** 294 (2002)
44. Mur V D, Popruzhenko S V, Pozdnyakov S G, Popov V S *Phys. Lett. A* **316** 226 (2003)

45. Mur V D, Pozdnyakov S G, Popruzhenko S V, Popov V S *Phys. Atom. Nucl.* **68** 677 (2005); *Yad. Fiz.* **68** 708 (2005)
46. Zel'dovich Ya B *Sov. Phys. JETP* **4** 142 (1956); *Zh. Eksp. Teor. Fiz.* **31** 1101 (1956)
47. Popov V S, Mur V D, Shcheblykin A V, Weinberg V M *Phys. Lett. A* **124** 77 (1987)
48. Popov V S, Mur V D, Sergeev A V, Weinberg V M *Phys. Lett. A* **149** 418 (1990); Popov V S, Mur V D, Sergeev A V *Phys. Lett. A* **149** 425 (1990)
49. Mur V D, Popov V S *Laser Phys.* **3** 462 (1993)
50. Zel'dovich Ya B *Sov. Phys. JETP* **24** 1006 (1967); *Zh. Eksp. Teor. Fiz.* **51** 1492 (1966)
51. Zel'dovich Ya B *Sov. Phys. Usp.* **16** 427 (1973); *Usp. Fiz. Nauk* **110** 139 (1973)
52. Ritus V I *Sov. Phys. JETP* **24** 1041 (1967); *Zh. Eksp. Teor. Fiz.* **51** 1544 (1966)
53. Nikishov A I, Ritus V I *Sov. Phys. JETP* **19** 529 (1964); *Zh. Eksp. Teor. Fiz.* **46** 776 (1964)
54. Perelomov A M, Popov V S *Theor. Math. Phys.* **3** 582 (1970); *Teor. Mat. Fiz.* **3** 377 (1970)
55. Zel'dovich Ya B, Perelomov A M, Popov V S *Sov. Phys. JETP* **28** 308 (1969); *Zh. Eksp. Teor. Fiz.* **55** 589 (1968)
56. Belavin A A, Zel'dovich Ya B, Perelomov A M, Popov V S *Sov. Phys. JETP* **29** 145 (1969); *Zh. Eksp. Teor. Fiz.* **56** 264 (1969)
57. Paradoksov P *Sov. Phys. Usp.* **9** 618 (1967); *Usp. Fiz. Nauk* **89** 707 (1966)
58. Al'tshuler L V, Zel'dovich Ya B, Styazhkin Yu M "Investigation of isentropic compression and equations of state of fissionable materials" *Phys. Usp.* **40** 101 (1997); "Issledovanie izentropicheskoi szhimaemosti i uravnenii sostoyaniya delyashchikhsya materialov" *Usp. Fiz. Nauk* **167** 107 (1997)
59. Zeldovich Ya B, Khariton Yu B "Fission and chain decay of uranium" *Phys. Usp.* **36** 311 (1993); "Delenie i tsepnoi raspad urana" *Usp. Fiz. Nauk* **163** (4) 107 (1993)
60. Gurevich I I, Zeldovich Ya B, Pomeranchuk I Ya, Khariton Yu B "Utilization of the nuclear energy of the light elements" *Sov. Phys. Usp.* **34** 445 (1991); "Ispol'zovanie yadernoi energii legkikh elementov" *Usp. Fiz. Nauk* **161** (5) 171 (1991)
61. Zel'dovich Ya B, Buchachenko A L, Frankevich E L "Magnetic-spin effects in chemistry and molecular physics" *Sov. Phys. Usp.* **31** 385 (1988); "Magnitno-spinovye effekty v khimii i molekulyarnoi fizike" *Usp. Fiz. Nauk* **155** 3 (1988)
62. Zel'dovich Ya B, Mikhailov A S "Fluctuation kinetics of reactions"*Sov. Phys. Usp.* **30** 977 (1987); "Fluktuatsionnaya kinetika reaktsii"*Usp. Fiz. Nauk* **153** 469 (1987)
63. Zeldovich Ya B, Ruzmaikin A A "The hydromagnetic dynamo as the source of planetary, solar, and galactic magnetism" *Sov. Phys. Usp.* **30** 494 (1987); "Gidromagnitnoe dinamo kak istochnik planetarnogo, solnechnogo i galakticheskogo magnetizma"*Usp. Fiz. Nauk* **152** 263 (1987)
64. Zel'dovich Ya B, Molchanov S A, Ruzmaikin A A, Sokolov D D "Intermittency in random media" *Sov. Phys. Usp.* **30** 353 (1987); "Peremezhaemost' v sluchainoi srede" *Usp. Fiz. Nauk* **152** 3 (1987)
65. Zel'dovich Ya B, Grishchuk L P "Gravitation, the general theory of relativity, and alternative theories" *Sov. Phys. Usp.* **29** 780 (1986); "Tyagotenie, obshchaya teoriya otnositel'nosti i al'ternativnye teorii" *Usp. Fiz. Nauk* **149** 695 (1986)
66. Zel'dovich Ya B, Sokolov D D "Fractals, similarity, intermediate asymptotics" *Sov. Phys. Usp.* **28** 608 (1985); "Fraktaly, podobie, promezhutochnaya asimptotika" *Usp. Fiz. Nauk* **146** 493 (1985)
67. Shandarin S F, Doroshkevich A G, Zel'dovich Ya B "The large-scale structure of the Universe" *Sov. Phys. Usp.* **26** 46 (1983); "Krupnomasshtabnaya struktura Vselennoi" *Usp. Fiz. Nauk* **139** 83 (1983)
68. Zel'dovich Ya B, Khlopov M Yu "The neutrino mass in elementary-particle physics and in Big Bang cosmology" *Sov. Phys. Usp.* **24** 755 (1981); "Massa neitrino v fizike elementarnykh chastits i kosmologii rannei Vselennoi" *Usp. Fiz. Nauk* **135** 45 (1981)
69. Zel'dovich Ya B "Vacuum theory: a possible solution to the singularity problem of cosmology" *Sov. Phys. Usp.* **24** 216 (1981); "Teoriya vakuuma, byt' mozhet, reshaet zagadku kosmologii" *Usp. Fiz. Nauk* **133** 479 (1981)
70. Vysotskii M I, Dolgov A D, Zel'dovich Ya B "Cosmological limitations on the mass of neutral leptons" *Sov. Phys. Usp.* **20** 1031 (1977); "Kosmicheskie ogranicheniya na massy neitral'nykh leptonov" *Usp. Fiz. Nauk* **123** 703 (1977)
71. Zel'dovich Ya B, Manakov N L, Rapoport L P "Quasienergy of a system subjected to a periodic external disturbance" *Sov. Phys. Usp.* **18** 920 (1975); "Kvazienergiya sistemy, podvergayushcheisya periodicheskomu vneshnemu vozdeistviyu" *Usp. Fiz. Nauk* **117** 563 (1975)
72. Zel'dovich Ya B "Interaction of free electrons with electromagnetic radiation" *Sov. Phys. Usp.* **18** 79 (1975); "Vzaimodeistvie svobodnykh elektronov s elektromagnitnym izlucheniem" *Usp. Fiz. Nauk* **115** 161 (1975)
73. Zel'dovich Ya B "The cosmological constant and the theory of elementary particles" *Sov. Phys. Usp.* **11** 381 (1968); "Kosmologicheskaya postoyannaya i teoriya elementarnykh chastits" *Usp. Fiz. Nauk* **95** 209 (1968)
74. Zel'dovich Ya B "The 'hot' model of the Universe" *Sov. Phys. Usp.* **9** 602 (1967); "'Goryachaya' model' Vselennoi" *Usp. Fiz. Nauk* **89** 647 (1966)
75. Zel'dovich Ya B, Okun L B, Pikel'ner S B "Quarks: astrophysical and physico-chemical aspects" *Sov. Phys. Usp.* **8** 702 (1966); "Kvarki: astrofizicheskii i fiziko-khimicheskii aspekty" *Usp. Fiz. Nauk* **87** 113 (1965)
76. Zel'dovich Ya B, Novikov I D "Relativistic astrophysics. I" *Sov. Phys. Usp.* **7** 763 (1965); "Relyativistskaya astrofizika. I" *Usp. Fiz. Nauk* **84** 377 (1964)
77. Zel'dovich Ya B, Novikov I D "Relativistic astrophysics. II" *Sov. Phys. Usp.* **8** 522 (1966); "Relyativistskaya astrofizika. II" *Usp. Fiz. Nauk* **86** 447 (1965)
78. Baz' A I, Gol'danskii V I, Zel'dovich Ya B "Systematics of the lightest nuclei" *Sov. Phys. Usp.* **8** 177 (1965); "Sistematika legchaishikh yader" *Usp. Fiz. Nauk* **85** 445 (1965)
79. Zel'dovich Ya B, Gershtein S S "Nuclear reactions in cold hydrogen I. Mesonic catalysis" *Sov. Phys. Usp.* **3** 593 (1961); "Yadernye reaktsii v kholodnom vodorode. I. Mezonnyi kataliz" *Usp. Fiz. Nauk* **71** 581 (1960)
80. Zeldovich Ya B, Raizer Yu P "Udarnye volny bol'shoi amplitudy v gazakh" ("Shock waves of large amplitude in gases") *Usp. Fiz. Nauk* **63** 613 (1957)
81. Zeldovich Ya B, Luk'yanov S Yu, Smorodinskii Ya A "Svoistva neitrino i dvoinoi β-raspad" ("Properties of neutrinos and double β-decay") *Usp. Fiz. Nauk* **54** 361 (1954)
82. Zeldovich Ya B *Vysshaya Matematika dlya Nachinayushchikh i ee Prilozheniya k Fizike* (Higher Mathematics for Beginners and Its Applications to Physics) 5th ed. (Moscow: Nauka, 1970)
83. Zeldovich Ya B, Yaglom I M *Vysshaya Matematika dlya Nachinayushchikh Fizikov i Tekhnikov* (Higher Mathematics for Beginners in Physics and Engineering) (Moscow: Nauka, 1982)
84. Zeldovich Ya B, Myshkis A D *Elementy Prikladnoi Matematiki* (Elements of Applied Mathematics) 3d ed. (Moscow: Nauka, 1972)