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Ya B Zeldovich's ideas and modern Brans–Dicke cosmology

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Abstract. We compare Ya B Zeldovich's ideas about Brans–Dicke theory and Mach's principles, contained in now-classical books, with modern results in this field. Our recent results on the Brans–Dicke cosmology with the cosmological term are presented. The Friedmann–Brans–Dicke equations are written for the flat Universe. The initial conditions for the model are provided by the presently observed Hubble constant, its first time derivative (the deceleration parameter), and matter density. The cosmological scenario with the scale factor not vanishing in the past, corresponding to the absence of the cosmological singularity, is analytically calculated. Instead of the singularity, the scale factor experiences a 'bounce' and demonstrates regular behavior at all times. Some ideas related to Mach's principle are also discussed.

1. Introduction

Yakov Borisovich Zeldovich was as outstanding physicist of the 20th century. He always paid special attention to new physical theories and their astronomical applications. In monographs [1–3], Yakov Borisovich discussed cosmological consequences of the Brans–Dicke theory and ideas related to Mach's principle. One of the present authors, I D N, discussed these topics with Yakov Borisovich a great deal. Results of these discussions can be found in the monographs mentioned above. As far as we know, Yakov Borisovich did not publish dedicated papers on these issues (see, however, paper [4]).

Let us recall Yakov Borisovich's ideas about the Brans–Dicke (BD) theory and Mach's principle. In paper [1], Mach's principle is expounded as follows: "In the literature, there continuously appear papers discussing Mach's principle from different viewpoints.... This principle essentially states that the inertia of a body is determined by its interaction (gravitational and inertial) with other bodies in the Universe. This principle played a large heuristic role in Einstein's construction of GR. But after the development of general relativity, it became clear that Mach's principle does not enter the theory.... From this point of view, every confirmation of general relativity strikes Mach's principle."

And further: "The direct application of Mach's principle may lead to the conjecture that the reference frame connected with receding galaxies is inertial. But then, a motion with constant velocity (and not necessarily with acceleration relative to this frame) must lead to physical distinctions. However, this is not the case."

In paper [2], Zeldovich wrote: "The scalar theory of gravitation gives results which are significantly different from the tensor theory, and hence, from experiments — for rapidly moving bodies.... Predictions for light rays grazing the Sun vary. In recent years, the coincidence of experimental results with GR has systematically improved, leaving less and less room for possible admixtures of scalar interaction.... Presently, it can be stated that the scalar theory contribution to gravitation is less than 10% of that of GR."

Notice that, according to modern data, the accuracy of this statement is much higher (see below). In monograph [3], Zeldovich wrote: " ...there is another hypothesis as well — the conjecture about a certain role of the Universe in local laws. This hypothesis is called Mach's principle.... Thus, the reference frame connected with the cosmic microwave background radiation, with the total mass of remote matter, is indeed physically preferred, and it is inertial at each of the points. Perhaps, this could somehow be treated in the spirit of

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Mach's principle? We believe this should not be done. The direct application of Mach's principle in this context leads to the following. Once a preferential reference frame is chosen, even inertial motion with respect to it (and not necessarily with acceleration or rotation) must lead to differences in physical laws in the new frame relative to those in the preferential frame. But this is not the case....''

Zeldovich next discusses the BD theory, which is closely related to Mach's principle: '' ...along with known particles and fields, a hypothetical... massless scalar field, the so-called φ field, is considered.... 'The correspondence' to Mach's principle, according to authors of the BD theory, is the advantage of the theory. However, Mach's principle is not proved... and the 'correspondence' to it is not proof that the BD theory is correct...

Note that in the BD theory, in addition to the 'ordinary' solutions with the initial condition $t = 0$, $\varphi = 0$, one may also consider the condition $\varphi_0 \neq 0$ or $\varphi a^3 = \text{const} \neq 0$, i.e., with the 'free' φ-field. These solutions exhibit interesting features. Under certain conditions, the field behaves like maximally rigid... matter. With a certain choice of parameters... there can be a smooth transition from contraction to expansion in the isotropic Friedmann model....

...However, after eliminating the singularity, new questions arise: How does the nucleosynthesis occur? How was the equilibrium Planck radiation formed? Infinite density, perhaps, is not a prerequisite, but a density of about $10^6 - 10^8$ g cm^{-3} and a temperature above 10^{10} K are needed to understand the observed Universe.

...Let us conclude the BD theory discussion. The diversity of solutions turned out to be much larger than the authors of this theory had assumed. New variants which are not connected with Mach's principle appeared: free fields are involved. Many papers devoted to the BD theory are issuing.''

In the present paper, we consider the developments of ideas discussed by Ya B Zeldovich in light of advances in modern cosmology [5].

Observational data collected by the beginning of the 21st century evidence that the Universe expands with acceleration [6–11], although the physical reasons behind this phenomenon are not completely clear. Thus, the construction of a self-consistent cosmological model with a minimal number of assumptions and adjustable parameters, which can explain observational facts, seems to be a relevant task. Presently, the simplest (and thus the most elaborated) of the proposed models is the cosmological model with cold dark matter and the cosmological constant (ΛCDM theory). This model, while providing good quantitative agreement with observations, does not explain the physical nature of dark matter and dark energy. Another shortcoming of the ΛCDM model consists in the fact that it does not offer an explanation of the observed smallness of the cosmological constant (by assuming that it characterizes the so-called vacuum energy). All these facts call for the construction of a broader dynamical theory of dark energy (see, for example, review [12]). The candidates most discussed at present in the literature include the models of gravitation with higher-order curvature corrections (for example, $f(R)$-gravity first proposed in Ref. [13]) and quintessence (slowly changing scalar field) [14]. In addition, $f(R)$ gravitation theories are closely related to the Brans–Dicke theory, which is the main topic of the present paper (see, for example, Ref. [15]).

The model constructed by Brans and Dicke (BD) is the first extended version of gravitation theory with a scalar field [16], proposed in 1961. The BD model also includes additional arbitrary parameter w to be fixed from observations. The larger this parameter, the stronger the contribution from the tensor part (the scalar curvature), and vice versa, the smaller this parameter, the larger the contribution from the scalar field. The BD theory converts into general relativity (GR) in the limit of $|w| \to \infty$. In this model, the gravitational constant is inversely proportional to the scalar field value ($G \approx 1/\Phi$), i.e., an additional link between the parameters exists. Presently, the most precise estimate of the parameter w, which follows from observations by the *Cassini–Huigens* spacecraft for the post-Newtonian parameter γ, is $|w| > 50{,}000$ (see Ref. [17]).

The BD theory is one of the most natural extensions of GR. The interest in this theory has not decreased due to, first of all, the fact that the model can represent the effective low-energy limit of Grand Unification and Supersymmetry theories (which, according to the latest data from the Large Hadron Collider [18], are not rejected). Moreover, the scalar field in the BD model can be identified with dilaton in string theory. Second, the BD model is the simplest GR extension. Therefore, when studying properties of the general theory, this model seems to be the candidate of choice to search for deviations from GR (see Ref. [19]).

Furthermore, the BD model is actively invoked in inflationary cosmology which requires the presence of a scalar field in the early Universe, and in the model discussed this scalar field arises in the most natural way. Many inflationary models [20–22] are based on both the BD theory and more general scalar–tensor theories.

In 1973, Gurevich, Finkelstein, and Ruban [23] analytically calculated cosmology with bounce using the standard BD theory. At that time, the accelerating expansion of the Universe was not yet recognized. In the standard form, the BD theory does not lead to accelerated expansion of the Universe, so it is necessary to investigate extensions of the BD theory. One of the most widespread extensions includes the BD model with a scalar field potential (see Ref. [24]). As the precise form of this potential is still unknown, the effective contribution can be substituted by the Λ-term (BDΛ below). The BDΛ theory offers an explanation of the smallness of the cosmological constant, as was suggested in paper [25]. Using the scalar field of the BDΛ model, a dark matter halo around galaxies can also be modelled [26].

The BDΛ theory is mathematically much more complicated than the BD theory. The exact solution of the Einstein–Friedmann equations in the BDΛ theory was obtained for the first time by Uehara and Kim [27]. The authors of this work assumed a positive value of w and zero initial condition for the scale factor, $a(t_\mathrm{m}) = 0$, where t_m is the Big Bang instant of time. Particular solutions, by assuming a power-law dependence of the scalar field on the scale factor, were obtained in Refs [28–31]. Vacuum solutions are presented in papers [32–34]. Some papers have considered the model with the Λ-term depending on the scalar field (see, for example, Ref. [35]). Numerical integration and the stability analysis of the extensive family of BDΛ solutions with matter were performed in Ref. [36]. The BDΛ solution leading to the so-called Big Rip is presented in paper [37].

In our work, we examined the BDΛ solution of the Friedmann–Einstein equations with parameter $w < 0$ and the scale factor $a(t)$, which takes the nonzero minimal value of $a_\mathrm{m} > 0$. The Friedmann equations are written down for a flat universe, with the present-day value of the Hubble

constant and its time derivative (the acceleration parameter) being taken as initial conditions. When solving the field equations, we apply the approach elaborated in Ref. [27]. However, the authors of this work considered only positive values of w, so our solution represents a new branch, which has not been described in Ref. [27]. The scale factor in our model, unlike the standard ΛCDM model, does not vanish in the past. This situation corresponds to the so-called bounce of the scale factor from the minimal value a_{m}. Formula (24) (see Section 4), leading to the bounce, is obtained for a 'cold' universe, $p = 0$, and it cannot be applied in a 'hot' universe. Therefore, near the bounce the results obtained are only qualitative. They can be used to obtain the initial values of all functions before the transition of a universe to the hot stage (when considering evolution back in time) (see Section 6 for more detail).

The layout of the paper is as follows:

— in Section 2 we write down the metric of spacetime and field equations;

— in Section 3 we determine cosmological parameters at the present time for the solution of field equations;

— in Section 4 we present an analytical solution with bounce for a 'cold' universe (with $p = 0$);

— in Section 5 we discuss the preliminary results for a cold universe;

— in Section 6 we investigate the solution for a hot universe (with an ultrarelativistic equation of state of matter);

— in Section 7 we formulate our conclusions and compare Ya B Zeldovich's ideas with modern cosmological models.

2. Friedmann–Einstein equations

We assume our metric equal to the Friedmann metric [1]

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right), \tag{1}$$

where $\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. The flatness parameter in cosmology is $k = 0, \pm 1$.

The action in the BDΛ theory is written out as

$$S = \frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g}\Big[\Phi(R+2\Lambda) - \frac{w}{\Phi}\,g^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi + 16\pi L_{\mathrm{matter}}\Big]. \tag{2}$$

Here, w is the BD theory parameter, $\Phi(t)$ is the scalar field, and Λ is the Λ-term (constant) [2].

Variation of the action with respect to $g_{\mu\nu}$ and Φ yields the Friedmann–Einstein equations and Klein–Gordon equations, respectively:

$$G_{\mu\nu} = \frac{8\pi}{\Phi}\,T_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{w}{\Phi^2}\left(\partial_\mu\Phi\,\partial_\nu\Phi - \frac{1}{2}\,g_{\mu\nu}g^{\sigma\lambda}\,\partial_\sigma\Phi\,\partial_\lambda\Phi\right) + \frac{1}{\Phi}\left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla_\lambda\nabla^\lambda\Phi\right), \tag{3}$$

$$\frac{8\pi}{\Phi}\,T^\mu_\mu + 2\Lambda = \frac{3+2w}{\Phi}\,\nabla_\lambda\nabla^\lambda\Phi, \tag{4}$$

where ∇_μ is the covariant derivative, and

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,Rg_{\mu\nu}, \qquad T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu - pg_{\mu\nu}, \quad \partial_\mu\Phi = \delta^t_\mu\,\partial_t\Phi. \tag{5}$$

Here, $\rho(t)$ and $p(t)$ stand for the matter density and pressure, respectively, and the energy–momentum tensor corresponds to a perfect fluid.

Let us introduce new dimensionless variables: [3]

$$\Phi(t) \equiv \frac{\phi(t)}{G_0}, \quad \epsilon(t) \equiv \frac{\partial_t\phi}{\sqrt{\Lambda}\phi}, \tag{6}$$

$$\tilde{H}(t) \equiv \frac{H(t)}{\sqrt{\Lambda}} = \frac{\partial_t a}{\sqrt{\Lambda}a}, \quad \tilde{\rho}(t) = \frac{4\pi G_0\rho}{\Lambda}, \quad \tilde{p}(t) = \frac{4\pi G_0 p}{\Lambda}. \tag{7}$$

Here, H is the Hubble function, and $\tilde{H}$ is the dimensionless Hubble function.

Then, the dimensionless Friedmann equations for a flat universe (with $k = 0$) in the matter-comoving frame of reference ($u_\mu = [1, 0, 0, 0]$) take the form

$$\frac{G^t_t}{\Lambda} = 3\tilde{H}^2 = \frac{2\tilde{\rho}}{\phi} + 1 + \frac{w}{2}\,\epsilon^2 - 3\tilde{H}\epsilon, \tag{8}$$

$$\frac{G^r_r}{\Lambda} = 2\dot{\tilde{H}} + 3\tilde{H}^2 = -\frac{2\tilde{p}}{\phi} + 1 - \frac{w}{2}\,\epsilon^2 - \frac{\ddot{\phi}}{\phi} - 2\tilde{H}\epsilon, \tag{9}$$

and the Klein–Gordon equation (4) is transformed to the following equation

$$\frac{2\tilde{\rho} - 6\tilde{p}}{\phi} + 2 = (3+2w)\left(\frac{\ddot{\phi}}{\phi} + 3\tilde{H}\epsilon\right). \tag{10}$$

Hereinafter, the dot above the quantity denotes the derivative with respect to the dimensionless time $\tilde{t} \equiv \sqrt{\Lambda}t$.

Equations (8)–(10) lead to the continuity equation (which include the equivalence principle)

$$\frac{\dot{\tilde{\rho}}}{\tilde{\rho} + \tilde{p}} + 3\tilde{H} = 0. \tag{11}$$

3. Determination of model parameters

Let us introduce the deceleration parameter of a universe, q, and the parameter β corresponding to the present-day dimensionless matter density:

$$\dot{\tilde{H}} \equiv -(1+q)\,\tilde{H}^2, \quad \beta \equiv \frac{4\pi G_0(\rho_0 - p_0)}{H_0^2} = \frac{\tilde{\rho}_0 - \tilde{p}_0}{\tilde{H}_0^2}. \tag{12}$$

After excluding the quantities ϵ and $\ddot{\phi}/\phi$ from equations (8)–(10), at the present time we obtain for zero pressure ($p = 0$):

$$w\left[\tilde{H}_0^2(2 - q_0 - \beta z) - z\right]^2 - 2\tilde{H}_0^2(3z - 1) + \tilde{H}_0^4(6 - 6q_0 - 6\beta z + 4\beta) = 0, \quad z \equiv \frac{2+2w}{3+2w}. \tag{13}$$

[1] Below we set the velocity of light equal to $c = 1$.

[2] Here, the value of Λ can be different from its value in the ΛCDM theory.

[3] Here inafter, the present-day values are marked with index 0, so G_0 is the present-day value of the gravitational constant. The preset-day time is set to 0. In the new notation, $\phi_0 = 1$.

In the approximation[4] for $|w| \gg 1$, equation (13) in the principal order in $1/w$ yields

$$\frac{1}{\tilde{H}_0^2} \to (2 - q_0 - \beta) \pm \sqrt{\frac{2(1 + q_0 - \beta)}{w}}. \tag{14}$$

In the leading approximation, the second term in formula (14) can be neglected.

Observational data [38, 39] give the following values of cosmological parameters: $H_0 \approx 2.3 \times 10^{-18}$ s^{-1}, $\rho_0 \approx 0.27 \times 10^{-29}$ g cm^{-3} (with account of both baryonic and dark matter), and $q_0 \approx -0.6$; the present-day pressure can be neglected by assuming a universe is filled with dust-like matter (a cold universe).

Hence follows the cosmological constant:

$$\Lambda \to (2 - q_0)\, H_0^2 - 4\pi G_0(\rho_0 - p_0) \approx 11.3 \times 10^{-36}\ \mathrm{s}^{-2}. \tag{15}$$

From the lunar laser ranging experiments [40], one can infer $|\partial_t G/G|_{(0)} \leqslant 4 \times 10^{-20}$ s^{-1}, which leads to the following bound on $|\epsilon_0|$ (in our notations): $|\epsilon_0| < 0.01$.

For $|w| \gg 1$, we arrive at the following cosmological parameters:

$$\tilde{H}_0 \approx 0.68\,, \quad \tilde{\rho}_0 \approx 0.2\,, \quad \beta \approx 0.4\,. \tag{16}$$

We add expressions (8) and (9), multiply the result by $1/\tilde{H}_0^2$, and, by expressing the quantity $\ddot{\phi}/\phi$ from equation (10), get the following relationship for the present-day time:

$$\frac{\epsilon_0}{\tilde{H}_0} = \frac{1}{\tilde{H}_0^2} - (2 - q_0 - \beta) + \frac{\beta + 1/\tilde{H}_0^2}{3 + 2w}\,. \tag{17}$$

By substituting expression (14) into equation (17), in the leading approximation in $1/w$ we obtain[5]

$$\epsilon_0 \to \pm\sqrt{\frac{2(1 + q_0 - \beta)}{w(2 - q_0 - \beta)}}. \tag{18}$$

4. Solution for a cold universe

Let us consider the problem for a cold universe, i.e., for $p = 0$. By introducing the notation $f \equiv \phi a^3$, from equation (11) we find $\tilde{\rho}/\phi = \tilde{\rho}_0 f_0/f$.

Taking into account the relationship $\ddot{f}/f = \ddot{\phi}/\phi + 6\tilde{H}\epsilon + 3\dot{\tilde{H}} + 9\tilde{H}^2$ and by writing down the following combination of equations (8)–(10): $(3/2)[(8) + (9)] + (10)/(6 + 4w)$, we get

$$\ddot{f} - \eta^2(f + \tilde{\rho}_0 f_0) = 0\,, \quad \eta^2 \equiv \frac{8 + 6w}{3 + 2w}\,. \tag{19}$$

The solution to this equation reads

$$\frac{f(\tilde{t})}{f_0} = c^+ E + \frac{c^-}{E} - \tilde{\rho}_0\,, \quad E(\tilde{t}) \equiv \exp(\eta\tilde{t})\,. \tag{20}$$

Here, c^+ and c^- are constants determined via parameters of our model.

Equation (10) can be rewritten in the form

$$2f + 2\tilde{\rho}_0 f_0 = (3 + 2w)(\dot{\phi} a^3)^{\cdot}. \tag{21}$$

From equation (21) with regard to Eqn (20), one can find the expression for the Hubble function:[6]

$$\begin{aligned} 3\tilde{H} &= \frac{\dot{f}}{f} - \frac{\dot{\phi}}{\phi} = \frac{\dot{f}}{f} - \frac{2f_0}{f(3 + 2w)} \int_{\mathrm{const}}^{\tilde{t}} \left(\frac{f}{f_0} + \tilde{\rho}_0\right) \mathrm{d}\tilde{t} \\ &= \frac{\dot{f}}{f} - \frac{2(c^+E - c^-/E + c_H)}{\eta(3 + 2w)(c^+E + c^-/E - \tilde{\rho}_0)} \\ &= \frac{6(1 + w)(c^+E - c^-/E) - 2c_H}{\eta(3 + 2w)(c^+E + c^-/E - \tilde{\rho}_0)}\,. \end{aligned} \tag{22}$$

Here, c_H is an additional constant determined through our model parameters.

From expressions (20) and (22), we can find the coefficients

$$\begin{aligned} c^+ &= \frac{1 + \tilde{\rho}_0}{2} + \frac{\epsilon_0 + 3\tilde{H}_0}{2\eta}\,, \quad c^- = \frac{1 + \tilde{\rho}_0}{2} - \frac{\epsilon_0 + 3\tilde{H}_0}{2\eta}\,, \\ c_H &= \frac{\eta\epsilon_0(3 + 2w)}{2} - \frac{\epsilon_0 + 3\tilde{H}_0}{\eta}\,. \end{aligned} \tag{23}$$

Having integrated equation (22), we get the following expression for the scale factor (see Fig. 1):

$$\begin{aligned} \frac{a}{a_0} &= \left(c^+E + \frac{c^-}{E} - \tilde{\rho}_0\right)^{1/3} \\ &\quad \times \exp\left[\frac{-1}{3(4 + 3w)} \int_1^E \left(\frac{c^+E^2 + c_H E - c^-}{c^+E^2 - \tilde{\rho}_0 E + c^-}\right) \frac{\mathrm{d}E}{E}\right] \\ &= \left(c^+E + \frac{c^-}{E} - \tilde{\rho}_0\right)^{(1+w)/(4+3w)} \exp\frac{-2c_H(A - A_0)}{3(4 + 3w)\sqrt{\Delta}}\,, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \Delta &\equiv 4c^+c^- - \tilde{\rho}_0^2 = 1 + 2\tilde{\rho}_0 - \frac{(3\tilde{H}_0 + \epsilon_0)^2}{\eta^2} \\ &= -\frac{3}{8 + 6w}\left[\tilde{H}_0 - \epsilon_0(1 + w)\right]^2, \\ A(E) &\equiv \arctan\frac{2c^+E - \tilde{\rho}_0}{\sqrt{\Delta}}\,. \end{aligned} \tag{25}$$

It is important to note that the solution for $|w| \gg 1$ exists only as $w < 0$.[7]

For the field ϕ, we obtain from equations (20) and (24):

$$\phi = \left(c^+E + \frac{c^-}{E} - \tilde{\rho}_0\right)^{1/(4+3w)} \exp\frac{2c_H(A - A_0)}{(4 + 3w)\sqrt{\Delta}}\,. \tag{26}$$

In the limit $|w| \to \infty$, equation (24) tends to the standard Friedmann solution:

[4] Hereinafter (unless stated otherwise), the arrow specifies the approximation for $|w| \gg 1$.

[5] When deriving this expression, in the first term on the right-hand side of equation (17) we have taken into account terms on the order of $1/\sqrt{w}$ from equation (14); the last term in equation (17) was omitted by virtue of the accuracy demanded, i.e., the terms on the order of $1/\sqrt{w}$ inclusive.

[6] We have taken into account that $\mathrm{d}\tilde{t} = \mathrm{d}E/(\eta E)$.

[7] Interestingly, for $w < 0$ the BD theory allows the existence of wormholes without violation of energy conditions (see Refs [41, 42]).

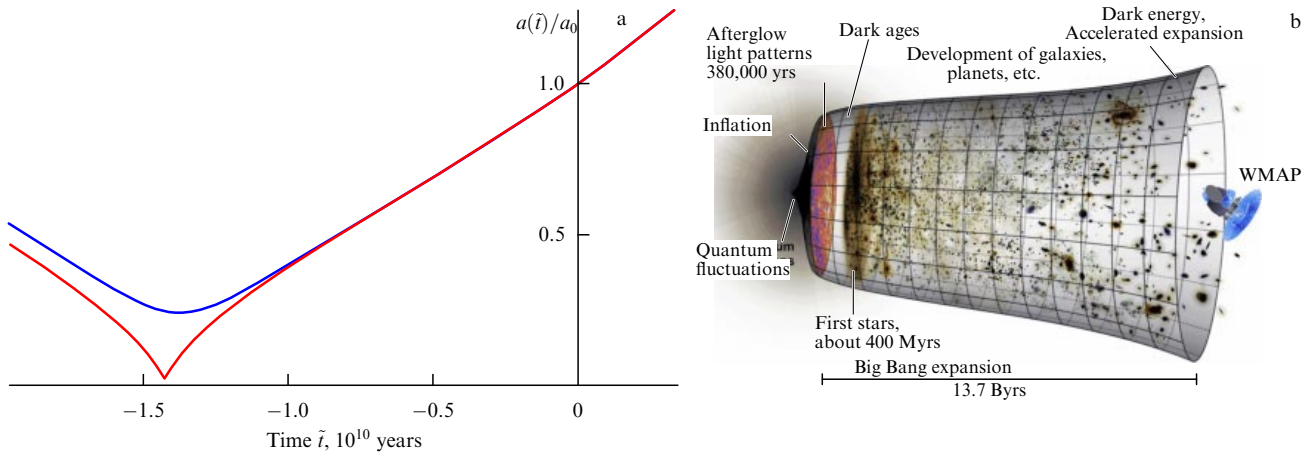

Figure. 1 (a) The scale factor $a(\tilde{t})/a_0$ (24) in the model of a cold universe with bounce for parameters $w = -1000$, $q = -0.6$, and $\beta = 0.45$ (for the upper curve), $\beta = 0.43653$ (for the bottom curve). (b) Illustration (taken from NASA site http://map.gsfc.nasa.gov) of the scale factor evolution in the ΛCDM model, corresponding to the Friedmann solution. (WMAP is the NASA Wilkinson Microwave Anisotropy Probe mission.)

$$H_{\rm Fr} = \frac{1}{\sqrt{3}}\frac{E+E_{\rm cr}}{E-E_{\rm cr}}\,, \quad E_{\rm cr} \equiv \frac{\sqrt{3}\tilde{H}_0 - 1}{\sqrt{3}\tilde{H}_0 + 1}\,, \quad \eta_{\rm Fr} = \sqrt{3}\,, \tag{27}$$

$$\frac{a_{\rm Fr}}{a_0} = \frac{(\sqrt{3}\tilde{H}_0 + 1)^{2/3}(E - E_{\rm cr})^{2/3}}{(4E)^{1/3}}\,; \tag{28}$$

note that in the Friedmann model at the point $E = E_{\rm cr}$ the values $\Delta = 0$ and $a = 0$ and the function $a(t)$ demonstrate a bend (which is absent for $\Delta > 0$). The Big Bang instant of time corresponds to $t_1 \approx -1.46\Lambda^{-1/2}$, with $\Lambda^{-1/2} \approx 10^{10}$ years.

5. Singularity-free cosmology

The scale factor in the BDΛ theory, unlike the scale factor in the standard ΛCDM model, can remain nonzero all the time in the past evolution. This situation corresponds to a bounce from the minimal value $a_{\rm m}$ of the scale factor.

The parameter region, starting from which the bounce becomes possible, corresponds to the scale factor local minimum vanishing: $a_{\rm m}(E_{\rm m}) = 0$ (i.e., when the scale factor minimum 'touches' the abscissa).

We can obtain from formula (24) the following condition for the bounce to exist in the past evolution of the scale factor:

$$\Delta > 0\,, \tag{29}$$

while the value of E in the local minimum (at the bounce) is estimated to be

$$E_{\rm m} \approx \sqrt{\frac{c^-}{c^+}}\,. \tag{30}$$

The precise equality $E_{\rm m} = \sqrt{c^-/c^+}$ holds for $\Delta = 0$, i.e., when the scale factor at the local minimum point is zero, $a_{\rm m}(E_{\rm m}) = 0$. In the scenario with $\Delta > 0$, a universe does not reach singularity in the past evolution, with the scale factor remaining a smooth function of time at all times (including at the bounce: see formula (22)).

Observations suggest a hot (radiation-dominated) stage in the past evolution of the Universe, the direct evidence being provided by the existence of the cosmic microwave background with the mean temperature $T_0 \approx 2.7$ K (see monograph [43]). Using the relation[8] $a_{\rm hot}/a_0 = 4 \times 10^{-5}$ (corresponding to the adiabatic expansion of the Universe: see monograph [3]), for $|w| \gg 1$ we can obtain the upper limit on Δ:

$$\Delta \approx \frac{2\tilde{\rho}_0 a_{\rm m}^3}{a_0^3} < \frac{2\tilde{\rho}_0 a_{\rm hot}^3}{a_0^3} \approx 2.6 \times 10^{-14}\,. \tag{31}$$

This anomalously small value for $|w| \gg 1$ can be reached only due to the flatness of a universe, i.e., at $1 + q_0 - \beta \approx 0$. Thus, the very small values of $|\dot{G}_0/G_0|$ correspond to very large values of $|w|$, and, as a consequence, to an almost flat universe. Therefore, bounds on $|\dot{G}_0/G_0|$ from Solar System observations are in agreement with the cosmological data for our model.

The model considered above, as well the bounce-free cosmology[9] in Ref. [27], ignores the pressure, and, consequently, cannot be applied to a hot universe. Therefore, the above results near the bounce should be considered only as a qualitative description; they can be utilized to obtain the boundary values of all functions before the Universe's transition to the hot stage (when considering evolution back in time).

6. Solution for a hot universe

Quantitative results near the bounce can be obtained only by taking into account the ultrarelativistic equation of state of matter. For a hot universe with pressure $p = \rho/3$, equations (8)–(10) yield

$$\dot{\tilde{H}} + 2\tilde{H}^2 = \frac{1}{6}\left(-w\epsilon^2 + \frac{6+8w}{3+2w}\right) \equiv Q(\tilde{t})\,. \tag{32}$$

As $w < -1.5$, the value of Q is positive.

In the case of a Friedmann universe (i.e., at $\epsilon = 0$ and $|w| \to \infty$), we can easily derive from equation (32) analogs of

[8] Here, $a_{\rm hot}$ is the scale factor at the instant of time of transition of the Universe from radiation-dominated to the cold stage.

[9] In paper [27], the analytical solution for the case of $\Delta < 0$ and $w > 0$ was obtained.

equations (27), (28) for a hot universe:

$$H_{\mathrm{Fr}} = \frac{1}{\sqrt{3}}\frac{U+1}{U-1}\,, \quad U(\tilde{\tau}) \equiv \exp\left(\frac{4\tilde{\tau}}{\sqrt{3}}\right),$$

$$\frac{a_{\mathrm{Fr}}}{a_{\mathrm{hot}}} = \left[\frac{(U-1)^2 U_{\mathrm{hot}}}{(U_{\mathrm{hot}}-1)^2 U}\right]^{1/4}. \tag{33}$$

Here, the subscript 'hot' corresponds to the time of transition to the cold stage from the hot stage of evolution of a universe, and the dimensionless time $\tilde{\tau}$ has been introduced, which is reckoned from the instant of time of the scale factor minimum; the relation between $\tilde{\tau}$ and $\tilde{t}$ is determined by the expression $\tilde{\tau} - \tilde{t} =$ age of the Universe $\times\sqrt{\Lambda}$.

Expression (33) for the scale factor has singular derivatives [as well as in the case of a cold Friedmann universe: see Eqns (27), (28)]. It should be noted also that in a cold universe, as $\Delta \to 0$, the second derivative $\ddot{a}$ at the bounce point tends to $+\infty$, and in the close vicinity to the bounce point already tends to $-\infty$. This means that the Hubble function in a cold universe case near the bounce turns out to be rapidly oscillating as $\Delta \to 0$.

A similar situation near the bounce must take place in a hot universe. As a hot universe epoch corresponds to very small (relative to a_0) values of the scale factor and sufficiently large values of $\tilde{H}$, this region spans a short interval of the dimensionless time $\tilde{\tau}$. The above consideration implies that, for $|w| \gg 1$ and $\epsilon \ll 1$, the solution in the BDΛ theory is almost identical to the Friedmann solution (excluding the region near the bounce).

Let us find the solution for a hot universe by expanding the scale factor $a(\tilde{\tau})$ near its local minimum (bounce) in power series in time $\tilde{\tau}$. To an accuracy up to the fourth order, we have

$$a = a_{\mathrm{m}} + \frac{1}{2}\, a_{\mathrm{m}}\dot{\tilde{H}}_{\mathrm{m}}\tilde{\tau}^2 - \frac{1}{12}\, a_{\mathrm{m}} b^2 \dot{\tilde{H}}_{\mathrm{m}}^2 \tilde{\tau}^4 + \ldots\,. \tag{34}$$

Here, $\dot{\tilde{H}}_{\mathrm{m}}$ and b are constants, with the quantity $\dot{\tilde{H}}_{\mathrm{m}}$ corresponding to the second derivative of the scale factor at the bounce time; thus, it must be positive, $\dot{\tilde{H}}_{\mathrm{m}} > 0$, as $a_{\mathrm{m}} > 0$. The third-order term can be neglected, since it does not significantly affect the results.

Then, the Hubble function and its derivative to the second order in $\tilde{\tau}$ take the form

$$\tilde{H}^2 = \dot{\tilde{H}}_{\mathrm{m}}^2\tilde{\tau}^2\,, \quad \dot{\tilde{H}} = \frac{\dot{\tilde{H}}_{\mathrm{m}}(1 - b^2\dot{\tilde{H}}_{\mathrm{m}}\tilde{\tau}^2)}{1 + \dot{\tilde{H}}_{\mathrm{m}}\tilde{\tau}^2/2} - \dot{\tilde{H}}_{\mathrm{m}}^2\tilde{\tau}^2\,. \tag{35}$$

By substituting these expressions into equation (32), we obtain

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \frac{\dot{\tilde{H}}_{\mathrm{m}}\left[1 + \tilde{\tau}^2\dot{\tilde{H}}_{\mathrm{m}}(3/2 - b^2)\right]}{(1 + \dot{\tilde{H}}_{\mathrm{m}}\tilde{\tau}^2/2)^2} = Q > 0\,. \tag{36}$$

This necessary inequality [see Eqn (32)] remains valid at any time only if $0 < b^2 < 3/2$.

From expansion (34) it is seen that at the instant of time $\tilde{\tau}_1 = 1/(\dot{\tilde{H}}_{\mathrm{m}} b^2)^{1/2}$ the second derivative of the scale factor changes sign ($\tilde{\tau}_1$ is the inflection point), i.e., there is one more additional inflection point of the scale factor time evolution curve , which is absent in the Friedmann solution. At the time $\tilde{\tau}_2 = \sqrt{3}/(\dot{\tilde{H}}_{\mathrm{m}} b^2)^{1/2} = \sqrt{3}\tilde{\tau}_1$, the first derivative of the scale factor changes its sign. Thus, sewing solutions for hot and cold universes must take place between points $\tilde{\tau}_1$ and $\tilde{\tau}_2$.

After the time $\tilde{\tau}_1$, at large[10] $\dot{\tilde{H}}_{\mathrm{m}}$ and small b, the second derivative of the scale factor takes a negative value with large module, while the Hubble function is still positive (until the time $\tilde{\tau}_2$). So, in the time interval from $\tilde{\tau}_1$ to $\tilde{\tau}_2$ the solution for the radiation-dominated universe can be coupled with the solution for a cold universe. By adjusting parameters $a_{\mathrm{m}}, \tilde{H}_{\mathrm{m}}$, and b, it is possible to reach a smooth joint of these functions at the boundary between the hot and cold stages of the Universe's evolution.

7. Conclusion

Thus, we have shown that the Friedmann solution with the cosmological term is the degenerate case of a more general cosmological model (for example, in the framework of the BDΛ theory).

The introduction of a cosmological constant into the theory allows us not only to reach agreement with observational data, but also to explain the need for this constant and to estimate its value; it is fundamentally impossible to precisely measure a dimensional quantity (for example, distance). It is only possible to count 'units' (in this case, 'units' of parsecs in a distance). However, determination of the size of the 'units' is fraught with additional errors. The introduction of a cosmological term allows us to make distances (and not only distances) dimensionless. Thus, in units of $\Lambda^{-1/2}$, the size of the Metagalaxy turns out to be of order unity. Therefore, any speculation as to why the cosmological constant in the theory is so large (or so small) immediately becomes senseless! This is why we are persuaded that the true fundamental physical theory must operate only with dimensionless quantities!

In the Friedmann cosmology, the scale factor function $a(t)$ approaches zero vertically (after which a bounce is possible), while in the framework of the BDΛ theory with $w > 0$ the function $a(t)$ passes through zero with a finite slope (the value of the first derivative at the point $a = 0$ is finite), and for $w < 0$ the scale factor does not vanish and experiences a smooth bounce, with all functions at the bounce remaining regular.

The quantitative theory with bounce can be considered only numerically due to the inapplicability of the cold Universe model near the bounce, as well as because near the bounce the parameter k (which is responsible for the nonplain geometry of the Universe; it was assumed to be zero in our consideration) can play a big role. Therefore, for quantitative estimates near the bounce, it is necessary to analyze equations with different values of $k = 0, \pm 1$, as well as with $p \neq 0$, which is associated with major mathematical difficulties and can be performed only numerically.

The cosmological solution with bounce enables many (yet unsolved) problems of quantum gravity and quantum singularity to be avoided.

The theory parameter $|w| > 10^{40}$ — is it large or small?

The opponents of the Brans–Dicke theory argue that even 10^4 is too much for any parameter in theory, but we think that in modern cosmology the number 10^{40} is quite normal. The corresponding inverse value $|1/w| \ll 1$ just suggests an extremely small difference between BD theory and GR. But this is good! As a result, we avoid the cosmological singularity! But the bounce in the Universe (if it actually took place) must have occurred at very small values (relative

[10] Here, the notion of 'large' and 'small' assumes a comparison with unity.

to the present-day value) of the scale factor — in order to allow the primordial nucleosynthesis and the hot radiation-dominated stages of the Universe. It is this requirement that leads to the bound $|w| > 10^{40}$.

The Brans–Dicke gravitation theory is closely related to Mach's principle. According to one possible formulation of Mach's principle, the value of the inertia (mass) of any physical body depends on the mass distribution of all bodies in the Universe. Ideas of Mach's principle have many aspects. These many aspects were called by Ya B Zeldovich in books [1–3] as "the many faces of Mach". In the above formulation, the connection of the Brans–Dicke theory with Mach's principle is determined by the gravitational 'constant' $G = 1/\Phi(t)$; here this quantity is a function of time.

The condition $|w| > 10^{40}$ also implies that $\epsilon_0 \sim 10^{40}$, or $|\partial_t G/G|_0 \sim 10^{-56}\ \mathrm{s}^{-1}$. This extremely small value suggests that it is experimentally impossible to observe such a relationship (with Mach's principle)!

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