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# Geometrodynamics: the nonlinear dynamics of curved spacetime

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**Contents** 

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1.	Introduction	342
2.	Gravitational collapse: phase transition and critical behavior	342
3.	Generic spacetime singularities	343
	3.1 BKL singularity; 3.2 Singularities inside a black hole	
4.	Black string in five spacetime dimensions	345
5.	Black-hole collisions	346
	5.1 Vortex-tendex tools; 5.2 Black-hole collisions	
6.	Gravitational-wave observations	349
7.	Conclusions	350
	References	350

**Abstract.** We review discoveries in the nonlinear dynamics of curved spacetime, largely made possible by numerical solutions of Einstein's equations. We discuss critical phenomena and self-similarity in gravitational collapse, the behavior of spacetime curvature near singularities, the instability of black strings in five spacetime dimensions, and the collision of four-dimensional black holes. We also discuss the prospects for further discoveries in geometrodynamics via observations of gravitational waves.

# 1. Introduction

In the 1950s and 60s, John Archibald Wheeler [1] speculated that curved, empty spacetime could exhibit rich, nonlinear dynamics — which he called *geometrodynamics* — analogous to the writhing surface of the ocean in a storm. Wheeler exhorted his students and colleagues to explore geometrodynamics by solving Einstein's general relativistic field equations.

In 1965, Yakov Borisovich Zeldovich, with his young protégés Andrei Doroshkevich and Igor Novikov [2], gave strong evidence that when a highly deformed star collapses to form what would later be called a black hole, the hole's curved spacetime, by its nonlinear dynamics, will somehow shake off all the deformations, thereby becoming a completely smooth, axially symmetric hole.

The Wheeler–Zeldovich challenge of exploring geometrodynamics in general, and for black holes in particular, has largely resisted analytic techniques. In the 1980s and 90s, that

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Received 9 January 2014 Uspekhi Fizicheskikh Nauk **184** (4) 367–378 (2014) DOI: 10.3367/UFNr.0184.201404b.0367 Edited by A M Semikhatov resistance motivated the authors and our colleagues to formulate a two-pronged attack on geometrodynamics: *numerical solutions of Einstein's equations* to discover general relativity's predictions, and *observations of gravitational waves* from black-hole births and black-hole collisions to test those predictions. The numerical simulations have now reached fruition, and the observations will do so in the near future.

In this article, dedicated to the memory of Zeldovich and Wheeler (who deeply respected each other, despite cold-war barriers, and who were the primary mentors for one of us, Kip Thorne), we present an overview of some of the most interesting things we have learned about geometrodynamics from numerical simulations, and a preview of what gravitational-wave observations may bring.

More specifically, we describe geometrodynamic discoveries in four venues: *gravitational implosion*, where a phase transition, discrete self-similarity, and critical behavior have been observed (Section 2); the dynamics of spacetime near *singularities*, where richly chaotic behavior has been observed (Section 3); the unstable evolution of a *black string* in five spacetime dimensions, where a dynamical sequence of strings linking black holes has been observed (Section 4); and *collisions of black holes* in four spacetime dimensions, where dynamical interactions of tidal tendexes and frame-drag vortexes have been observed (Section 5). Then, we briefly describe the prospects for observing some of these phenomena via *gravitational waves* (Section 6).

# **2.** Gravitational collapse: phase transition and critical behavior

The first numerical simulations to exhibit rich geometrodynamics were done in 1993 by Matthew Choptuik [3], who was then a postdoc at the University of Texas. Choptuik simulated the spherical implosion (Fig. 1) of a linear, massless scalar field (satisfying  $\Box \Psi = 0$ ). The field energy, momentum, and stress (which are quadratic in the field) generated spacetime curvature, with which the field inter-



**Figure 1.** The implosion of a scalar wave  $\Psi$  with the amplitude *p* and a particular waveform.

acted via the curvature's influence on its wave operator  $\Box$ . That interaction produced surprising nonlinear dynamics.

If the wave amplitude p for some chosen ingoing waveform was larger than some critical value  $p_*$ , the implosion produced a black hole. If p was smaller than  $p_*$ , the imploding waves passed through themselves, traveled back outward, and dispersed. For  $p = p_*$ , the imploding waves interacted with themselves nonlinearly (via their spacetime curvature), producing a sequence of frequency doublings and wavelength halvings, with an intriguing, discretely self-similar pattern that was independent of the initial, ingoing waveform. Waves with ever decreasing wavelength emerged from the 'nonlinearly boiling' field, carrying away its energy and ultimately leaving behind what appeared to be an infinitesimal naked singularity (a region with infinite spacetime curvature, not hidden by a black-hole horizon). One year after Choptuik's simulations, the mathematician Demetrios Christodoulou [4] developed a rigorous proof that the endpoint for  $p = p_*$  was, indeed, a naked singularity.

The transition of the implosion's endpoint, from black hole for  $p > p_*$ , to naked singularity for  $p = p_*$ , to wave dispersal for  $p < p_*$ , was a phase transition analogous to those that occur in condensed-matter physics. And, as in condensed matter, so also here, the phase transition exhibited scaling: for *p* slightly larger than  $p_*$ , the mass of the final black hole scaled as  $M_{\rm BH} \propto (p - p_*)^{\beta}$ . For *p* slightly below  $p_*$ , the curvature radius of spacetime at the center of the 'boiling region' reached a minimum value, before field dispersal, that scaled as  $\mathcal{R}_{\rm min} \equiv (R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho})_{\rm max}^{-1/4} \propto (p_* - p)^{\beta}$ , with the same numerically measured exponent  $\beta = 0.374$ . Here,  $R_{\mu\nu\sigma\rho}$  is the Riemann curvature tensor.

Choptuik's discovery triggered many follow-on simulations. Most interesting to us was one by Andrew Abrahams and Charles Evans [5], later repeated with higher resolution by Evgeny Sorkin [6]. In their simulations, the imploding, spherically symmetric scalar field was replaced by an imploding, axially symmetric (quadrupolar) gravitational wave, such that they were dealing with pure, vacuum spacetime as envisioned by Wheeler. Once again, there was a critical wave amplitude  $p_*$ : for p near  $p_*$ , the behavior was similar to the scalar-wave case, to within numerical error: for  $p = p_*$ , strong evidence for discretely self-similar evolution leading to an infinitesimal final singularity; for  $p > p_*$ , the same black-hole mass scaling  $M_{\rm BH} \propto (p - p_*)^{\beta}$ ; for  $p < p_*$ , the same spacetime-curvature scaling  $\mathcal{R} \propto (p_* - p)^{\beta}$ ; and to within numerical accuracy, the scaling exponent was the same,  $\beta = 0.38$  for quadrupolar gravitational waves and  $\beta = 0.374$  for the spherical scalar wave. This is reminiscent, of course, of the universality encountered in condensedmatter phase transitions.

For a detailed review of these and many other studies of critical gravitational implosion, see the article by Carsten Gundlach [7].

## 3. Generic spacetime singularities

#### 3.1 BKL singularity

The singular endpoint of the implosions described above is *nongeneric*, in the sense that no singularity occurs if  $p_*$  is only infinitesimally different from p.

However, there are other situations that lead to *generic* singularities.<sup>1</sup> This was demonstrated analytically in the 1960s by Roger Penrose, Stephen Hawking, and others, using tools from differential topology [10]. In 1969–1970, Vladimir Belinsky, Isaac Khalatnikov, and Evgeny Lifshitz [11] used approximate differential geometry techniques to deduce the geometrodynamical behavior of spacetime as it nears one generic type of singularity, a type now called BKL.

In the 1970s, 80s, and 90s, there was much skepticism in the US and UK about this BKL analysis, because its rigor was much lower than that of the Penrose–Hawking singularity theorems. (This lower rigor was inevitable because the geometrodynamical approach to a singularity is very complex (see below) and deducing its details is much more difficult than proving that a singularity occurs.) As a result, the BKL geometrodynamics came to be called, in the West, the *BKL conjecture*.

There was little hope of proving or disproving this 'conjecture' by analytic techniques, and the skeptics turned to numerical simulations to probe it. After nearly a decade of code development, David Garfinkle in 2003 [12] carried out simulations that demonstrated that Belinsky, Khalatnikov, and Lifshitz had been correct. The geometrodynamical evolution, approaching a BKL singularity, is as they predicted, although with one additional feature that they had missed: a set of nonlocal spikes in the spacetime curvature [13].

The BKL geometrodynamics can be described in terms of tidal-gravity observations by observers who fall into the BKL singularity on timelike geodesics (Fig. 2). As two observers, A and B, approach the singularity, they lose causal contact, in the sense that after passing through A's particle horizon (at point P in the diagram), B can no longer influence A. This causal decoupling is so strong in the BKL spacetime that spatial derivatives cease having significant influence on the geometrodynamics, as the singularity is approached: a *spatial decoupling* occurs and, as a result, it turns out that there is no correlation between the late-time observations of adjacent observers.

Each observer's experience, when approaching the singularity, can be described in terms of the tidal gravitational field  $\mathcal{E}_{jk}$  that they feel. This tidal gravitational field has components, in the observer's local Lorentz frame, that are equal to

<sup>&</sup>lt;sup>1</sup> Perhaps the most important generic singularity for astrophysics is one that arises when matter with negligible pressure is present. Leonid Petrovich Grishchuk in 1967 [8] showed that matter, generically, undergoes gravitational collapse to form flat pancakes with infinite density and curvature; and in 1970, Zeldovich [9] showed that, astrophysically, pressure halts the collapse before the infinities but after the pancake structure has been strongly established. A few years later, Zeldovich realized that these pancakes, seen edge on, are filamentary structures that astronomers observe in the distribution of galaxies in the sky.



Figure 2. The world lines of two observers, A and B, falling into a BKL singularity (solid lines), and the *particle horizon* of observer A (dashed lines; a past light cone). Events outside the particle horizon can never influence observer A.

the space-time-space-time part of the Riemann curvature tensor

$$\mathcal{E}_{jk} = R_{j\,0k0} \,. \tag{1}$$

The physical manifestation of the tidal field is the relative gravitational acceleration

$$\Delta a_j = -\mathcal{E}_{jk} \,\Delta x_k \tag{2}$$

of particles with a vector separation  $\Delta x_k$ . (The tidal field acquires its name from the fact that the Sun's and Moon's tidal field produces the tides on Earth's oceans. In the Newtonian limit, the tidal field is  $\mathcal{E}_{jk} = \partial^2 \Phi / \partial x_j \partial x_k$ , where  $\Phi$  is the Newtonian gravitational potential.)

Being a symmetric tensor, the tidal field can be described by three orthogonal principal axes (unit vectors  $\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}}$ ) and their eigenvalues,  $\mathcal{E}_{\hat{j}\hat{j}} \equiv \mathbf{e}_{\hat{j}} \mathcal{E} \mathbf{e}_{\hat{j}}$ . If  $\mathcal{E}_{\hat{1}\hat{1}} < 0$ , then an object is tidally stretched along its  $\mathbf{e}_{\hat{1}}$  principal axis, and similarly for the other principal axes. If  $\mathcal{E}_{\hat{1}\hat{1}} > 0$ , the object is tidally squeezed along  $\mathbf{e}_{\hat{1}}$ . The tidal field in vacuum is traceless, and hence its eigenvalues must sum to zero, which means that there must be a squeeze along at least one principal axis and a stretch along at least one.

Figure 3 shows the pattern of stretches and squeezes experienced by an observer falling into the BKL singularity. The pattern is divided, in time, into *cycles* and *eras*. During a single cycle, there is a stretch along one axis and a squeeze along the other two. Between cycles, the stretch axis switches to squeeze and the more strongly squeezing axis switches to stretch. At the end of each era, the axes rotate in some manner, relative to the observer's local Lorentz frame, and the pattern begins over. The number of cycles in each era and the details of their dynamics are governed by a continuedfraction map that is chaotic, in the technical sense of chaos (extreme sensitivity to initial conditions). This chaos plays a key role in destroying correlations between the observations of adjacent observers as they approach the singularity.

The full details of this, as worked out by Belinsky, Khalatnikov, and Lifshitz [11], are occasionally violated (as numerical simulations reveal): a cycle can be skipped, and during that skip there is an extreme spike in the tidal field that is more sensitive to spatial derivatives than expected and whose details are not yet fully understood (see [13] and the references therein).

#### 3.2 Singularities inside a black hole

This BKL behavior is speculated to occur in the core of a young black hole. However, numerical simulations are needed to confirm or refute this speculation.



**Figure 3.** The qualitative pattern of tidal stretches and squeezes experienced by an observer falling into a BKL singularity. The three eigenvalues of the tidal field are plotted vertically as functions of time, with one axis shown solid, another dashed, and the third dotted. Adapted from [14].

As the black hole ages, the singularity in its core is speculated to break up into three singularities, one of the BKL type and two that are much more gentle than the BKL singularity. This speculation arises from the expectation that, just as the hole's exterior spacetime geometry settles down into the quiescent, axially symmetric state described by the Kerr metric, its interior settles down into the Kerr-metric state, except near two special regions called *Cauchy horizons*, where the Kerr metric is dynamically unstable. Nonlinear geometrodynamics is thought to convert those Cauchy horizons into *null, generic singularities* (Fig. 4).

These singularities are null in the sense that ingoing or outgoing photons, in principle, could skim along them without being captured. The ingoing singularity (called a *mass inflation singularity*) is generated, according to approximate analytic analyses [16, 17], by radiation and matter that fall into the black hole and pile up along the ingoing Cauchy horizon, where they gravitate intensely. The outgoing singularity (called a *shock singularity* [15]) is generated by ingoing radiation that backscatters off the hole's spacetime curvature, and then travels outward, piling up along the outgoing Cauchy horizon, where it gravitates intensely. In both cases, what piles up could be gravitational waves rather than material or nongravitational radiation, in which case we are dealing with pure vacuum spacetime curvature—pure geometrodynamics.

Neither of these null singularities is oscillatory, but the spacetime curvature goes to infinity at both of them (the radius of curvature  $\mathcal{R}$  of spacetime goes to zero). This divergence of curvature happens so quickly at the shock singularity that objects *might* be able to pass through it, although with a destructive net compression along two dimensions and a net stretch along the third. If so, they will likely then be destroyed in the BKL singularity.

The mass-inflation singularity is also expected to produce only a finite, not infinite, net compression and stretch of objects falling through. If anything survives, its subsequent fate is totally unknown.



**Figure 4.** Penrose diagram depicting the causal structure outside and inside an old black hole, as we best understand it today. Ingoing and outgoing null rays (hypothetical photons) travel along 45 degree lines, and a conformal transformation has been used to compress our Universe into a finite diamond in the diagram. The Kerr spacetime is shaded. The true spacetime is superposed on the Kerr spacetime; it is the region bounded by the center of the imploding star (thin left line), the BKL singularity (thick horizontal line), the mass-inflation singularity (dashed line), and the infinities of our Universe: future timelike infinity labeled  $I_+$ , future null infinity labeled  $\mathcal{I}_+$ , spacelike infinity labeled  $I_0$ , past null infinity labeled  $\mathcal{I}_-$ , and past timelike infinity labeled  $I_-$ . It might well be that the true spacetime ends at the shock singularity, and there is no BKL singularity beyond [15].

These (highly informed) speculations, which arise from extensive, approximate, analytic analyses, mostly perturbation theory, will be tested in the coming few years by numerical simulations. And just as the BKL conjecture missed an important phenomenon (curvature spikes), these speculations about the geometrodynamics of black-hole interiors may fail in some modest, or even spectacular, way.

For far greater detail on what we now know and speculate about the interiors of black holes and the bases for that knowledge and speculation, see [15] and the references therein.

# 4. Black string in five spacetime dimensions

A remarkable example of geometrodynamics has been discovered by Luis Lehner and Frans Pretorius [18], in numerical simulations carried out in five spacetime dimensions.

Lehner and Pretorius began with a *black string* in its equilibrium state. This black string is a vacuum solution of Einstein's equations in five spacetime dimensions, with the metric

$$ds^{2} = -\left(1 - \frac{2M}{r}\right)dt^{2} + \frac{dr^{2}}{1 - 2M/r} + r^{2}(d\theta^{2} + \sin^{2}\theta \,d\phi^{2}) + dz^{2}.$$
 (3)

This is precisely a four-spacetime-dimensional Schwarzschild black hole translated along the *z* axis in the fifth (spatial) dimension. The event horizon is at r = 2M; at a fixed time *t*, it is a cylinder with a spherical cross section,  $R \times S^2$ .

In 1993, Ruth Gregory and Raymond LaFlamme [19] proved analytically that such a black string is unstable against linear, axisymmetric perturbations with wavelengths longer than about 1.2 times the string circumference. But little was definitively known about the nonlinear, geometrodynamical evolution of the instability until Lehner's and Pretorius's 2010 simulations [18]. They revealed that the string horizon evolves as depicted in Fig. 5.

The string develops a sausage instability — analogous to that of a magnetically confined plasma in a Z-pinch, and the Rayleigh–Plateau instability for a cylinder of fluid confined by its surface tension — but with outgoing gravitational waves carrying off energy. This instability gives rise to a chain of five-spacetime-dimensional black holes linked by segments of shrunken black string — segments whose circumferences are far smaller than that of the original string. The instability then repeats on each shrunken string, producing smaller black holes linked by segments of more extremely shrunken string. Each successive sausage instability produces its smaller black holes on an evolution timescale proportional to the string circumference. These successively shorter time-



**Figure 5.** (Color online.) A sequence of snapshots from a simulation of the geometrodynamical evolution of a black string in five spacetime dimensions, by Lehner and Pretorius [18]. Each snapshot is an embedding diagram of the black string's event horizon: the horizon's intrinsic geometry is the same as the intrinsic geometry of the depicted surface in flat space. Colors (or shades of gray) labeled 'AH radius' denote the real radius of the apparent horizon  $\sqrt{A(z)/4\pi}$ , where A(z) is the surface area of the horizon at a fixed value of z.

scales converge: an infinite sequence of instabilities presumably ends in a naked singularity in finite time.

Because these simulations assumed two-sphere symmetry from the outset, we cannot be certain that their predictions are the black string's true geometrodynamical evolution. To learn the true evolution, we need simulations in five spacetime dimensions that do not assume any symmetry. Such simulations are beyond current capabilities, but may be possible in a decade or so. In the meantime, it is conjectured that the Lehner–Pretorius evolution (see Fig. 5) is the true evolution, because black strings have been proved stable against all *linear* nonspherical perturbations.

# 5. Black-hole collisions

Recent advances in numerical simulations have enabled the study of geometrodynamics in the violent collisions of two black holes, including the generation of gravitational waves and the relaxation of the remnant to a single, quiescent, spinning black hole (as predicted by Zeldovich, Doroshkevich, and Novikov [2]). We and collaborators have developed a new set of so-called *vortex/tendex tools* for visualizing this black-hole geometrodynamics [20]. We first describe these tools and then use them to visualize the geometrodynamics of black-hole collisions.

#### 5.1 Vortex-tendex tools

The gravitational field felt by a local observer is described by the Riemann curvature tensor  $R_{\mu\nu\sigma\rho}$ . Any such observer, freely falling or accelerated, can decompose the Riemann tensor into an 'electric' part  $\mathcal{E}_{jk}$  defined by Eqn (1) and a 'magnetic' part  $\mathcal{B}_{jk}$ , defined by

$$\mathcal{B}_{jk} = \epsilon_{jpq} R_{pqk0} \,. \tag{4}$$

Here, the indices are components on the observer's local, orthonormal basis, the index 0 refers to the time component (i.e., the component along the observer's world line), Latin indices refer to the observer's three spatial components, and  $\epsilon_{jpq}$  is the spatial Levi-Civita tensor. Both  $\mathcal{E}_{jk}$  and  $\mathcal{B}_{jk}$  are symmetric and traceless in vacuum (the situation of interest to us).

As discussed in Section 3.1,  $\mathcal{E}_{jk}$  is called the *tidal field*, and describes the tidal stretching and squeezing experienced by objects in the observer's local reference frame, according to Eqn (2). The 'magnetic' quantity  $\mathcal{B}_{jk}$  is called the *frame-drag field*. The physical manifestation of this field is a relative precession, or dragging of inertial frames: two gyroscopes with a vector separation  $\Delta x_k$  precess relative to each other with the angular velocity

$$\Delta\Omega_j = \mathcal{B}_{jk} \,\Delta x_k \,. \tag{5}$$

This differential frame dragging is a general relativistic effect with no analogue in Newtonian gravity. Its global (nondifferential) analog is one of the two relativistic effects recently measured by Gravity Probe B [21].

We note that the decomposition of the Riemann tensor into  $\mathcal{E}_{jk}$  and  $\mathcal{B}_{jk}$  depends on the observer's reference frame. Different observers at the same location, moving relative to each other, would measure different tidal fields and framedrag fields. This is the same situation as in classical electromagnetism, where the electromagnetic tensor  $F_{\mu\nu}$  can be decomposed into the familiar electric and magnetic field



**Figure 6.** (Color online.) A spinning black hole. (a) Colors represent the horizon tendicity  $\mathcal{E}_{nn}$ . There is a positive (blue or light gray) tendex on each of the poles, and a negative (red or dark gray) tendex on the equator. Other (green or medium gray) regions have small tendicity. (b) Colors represent the horizon vorticity  $\mathcal{B}_{nn}$ . There is a negative (red or dark gray) vortex on the north pole, and a positive (blue or light gray) vortex on the south pole. Other (green or medium gray) regions have small vorticity. The spin vector points out of the north pole.

vectors  $E_j = F_{j0}$  and  $B_j = \epsilon_{jpq}F_{pq}$  in the same observerdependent manner. This correspondence between gravitation and electromagnetism motivates the use of the names 'electric' and 'magnetic' for  $\mathcal{E}_{jk}$  and  $\mathcal{B}_{jk}$ .

The frame-drag field  $\mathcal{B}_{jk}$  and the tidal field  $\mathcal{E}_{jk}$  are useful in describing geometrodynamics on the surface of a black hole (its event horizon). If **n** is a unit vector normal to the horizon with spatial components  $n^i$ , then we define the *horizon tendicity*  $\mathcal{E}_{nn} \equiv n^j n^k \mathcal{E}_{jk}$  as the normal–normal component of the tidal field. For a positive horizon tendicity, an object is tidally squeezed normal to the horizon, and for a negative horizon tendicity, an object is tidally stretched normal to the horizon. This is illustrated in the left panel of Fig. 6. We call a region on the horizon with large tendicity a *horizon tendex*.

We can similarly define the *horizon vorticity*  $\mathcal{B}_{nn} \equiv n^j n^k \mathcal{B}_{jk}$  as the normal–normal component of the frame-drag field on the surface of a black hole. For a positive horizon vorticity, an object experiences a clockwise twist; for a negative horizon vorticity, the twist is counterclockwise. We call a region on the horizon with large vorticity a *horizon vortex*. The horizon vorticity of a spinning black hole is illustrated in the right panel of Fig. 6.

We now turn to vortex/tendex tools in regions away from the horizon. In Section 3.1, we discussed how, at any point,  $\mathcal{E}_{ik}$ can be described by three orthogonal eigenvectors (unit vectors  $\mathbf{e}_{\hat{1}}, \mathbf{e}_{\hat{2}}, \mathbf{e}_{\hat{3}}$ ), and their eigenvalues,  $\mathcal{E}_{\hat{i}\hat{i}} \equiv \mathbf{e}_{\hat{i}} \mathcal{E} \mathbf{e}_{\hat{i}}$ . We call the eigenvalue  $\mathcal{E}_{\hat{j}\hat{j}}$  the *tendicity* associated with the corresponding eigenvector  $\mathbf{e}_{i}$ ; the tendicity measures the tidal stretching or squeezing of an object oriented along the eigenvector. In analogy with electric field lines, we define tendex lines as the integral curves along each of the three eigenvectors  $\mathbf{e}_{\hat{i}}$ . In electromagnetism, there is a single electric field line passing through each point, but in geometrodynamics, there are generically three tendex lines passing through each point: one tendex line for each of the three eigenvectors of  $\mathcal{E}_{jk}$ . Because (in vacuum)  $\mathcal{E}_{jk}$  has a vanishing trace, the eigenvalues must sum to zero, and hence generically both positive and negative tendex lines exist at each point.

Figure 7a shows the tendex lines outside a rapidly rotating black hole. A collection of tendex lines with particularly large tendicity is called a *tendex*. The rotating hole has a fan-



**Figure 7.** (Color online.) (a) Tendex lines and (b) vortex lines near a spinning black hole. Lines with positive eigenvalues (tidal squeeze or clockwise twist) are shown blue or light gray, and lines with negative eigenvalues (tidal squeeze or counterclockwise twist) are shown red or dark gray. At each point in space, there are three intersecting tendex lines and three intersecting vortex lines.

shaped, stretching (red or dark gray) tendex sticking out of its equatorial horizon tendex, and a poloidal squeezing (blue or light gray) tendex that emerges from its north polar horizon tendex and reaches around the hole to its south polar horizon tendex.

As with  $\mathcal{E}_{jk}$ , the frame-drag field  $\mathcal{B}_{jk}$  can be described by three orthogonal eigenvectors and their eigenvalues. An integral curve of one of these eigenvectors is called a *vortex line*, and the corresponding eigenvalue is the *vorticity* associated with that vortex line. The vorticity of a vortex line describes the twist, or differential frame dragging, experienced by an object oriented along that vortex line: positive vorticity corresponds to a clockwise twist, and negative vorticity corresponds to a counterclockwise twist.

The vortex lines associated with a spinning black hole are shown in Fig. 7b. A counterclockwise (red or dark gray) *vortex* (collection of large-negative-vorticity lines) emerges from the horizon's north polar vortex, reaches around the south polar region, and descends back into the north polar vortex. Similarly, a clockwise (blue or light gray) vortex emerges from the south polar horizon vortex, reaches around the north polar region, and descends back into the south horizon vortex.

The black holes in Figs 6 and 7 have stationary (timeindependent) vortex and tendex structures. Vortex and tendex lines, and their associated vortexes and tendexes, can also behave dynamically. The equations of motion for  $\mathcal{E}_{jk}$  and  $\mathcal{B}_{jk}$ are similar to Maxwell's equations and, like Maxwell's equations, they have wavelike solutions—gravitational waves—in which energy is fed back and forth between  $\mathcal{E}_{jk}$ and  $\mathcal{B}_{jk}$ . Figure 8 shows vortex and tendex lines for a plane gravitational wave without any nearby sources. As the wave passes an observer, the tendicities and vorticities oscillate in sign with one oscillation per gravitational wave period, leading to an oscillatory stretch and squeeze in horizontal and vertical directions, and an oscillatory twist in diagonal directions, out of phase with the stretch and squeeze.

#### 5.2 Black-hole collisions

We illustrate the geometrodynamical richness of black-hole collisions by describing the vortex/tendex behaviors in several numerical simulations. All these simulations were performed by members of the Collaboration to Simulate Extreme Spacetimes (SXS), which included numerical relativists from Caltech, Cornell, the Canadian Institute for Theoretical

![](_page_5_Figure_10.jpeg)

**Figure 8.** (Color online.) Snapshot of vortex and tendex lines for a gravitational plane wave traveling into the page. Two orthogonal sets of tendex lines (a) are oriented 45 degrees with respect to two orthogonal sets of vortex lines (b). The third set of vortex (and tendex) lines is normal to the page, with zero vorticity (tendicity).

Astrophysics, and Washington State University at the time of these simulations, and has since been expanded. We are members of this collaboration, and one of us (Scheel) played a significant role in most of these simulations.

**5.2.1 Head-on collision of two black holes with transverse spins.** Our first example is a simulation [22] of the head-on collision of two transversely spinning black holes, depicted in Fig. 9.

As the black holes merge, the vortexes retain their individuality. When the four vortexes (one pair from each hole) discover each other, all attached to the same horizon, they begin to interact: each one tries to convert those adjacent to it into a replica of itself. As a result, they exchange vorticities; each oscillates back and forth between clockwise and counterclockwise. At the moment when all are switching direction, such that the horizon momentarily has zero

![](_page_5_Figure_15.jpeg)

**Figure 9.** (Color online). Event horizons and vortex lines of spinning black holes, colliding head-on with transverse spins; from the simulation in [22]. (a) Horizon vortexes and spin directions just before merger. (b) Horizon vortexes just after merger, which retain their individuality. (c) Vortex lines linking horizon vortexes of the same polarity (red to red or dark gray to dark gray; blue to blue or light gray to light gray). Lines are color coded by vorticity (different scale from the horizon). (d) Sloshing of near-zone vortexes generates toroidal vortex loops (two shown here) composed of orthogonal vortex lines, traveling outward as gravitational waves; these are accompanied by intertwined tendex lines (not shown). The horizon, with attached vortex lines, is visible in the center. Figure adapted from [20]. See [23] for a movie of horizon vortexes for this simulation.

![](_page_6_Figure_2.jpeg)

**Figure 10.** (Color online.) (a) Schematic of vortex lines and horizon vortexes for two orbiting, spinning black holes about to merge. (b) Schematic of vortex lines of the remnant black hole just after merger, showing vortex lines extending to large distances; the entire pattern is rotating counterclockwise.

vorticity, the vortex lines pop off the horizon and join onto each other, creating a toroidal structure, much like a smoke ring, that begins to travel outward. Simultaneously, adjacent to the horizon much of the oscillation energy is stored in tendexes, which then regenerate the horizon vortexes, but with the twist directions reversed. As the toroidal bundle of vortex lines travels outward, its motion generates toroidal tendex lines, intertwined with the vortex lines in just such a manner that they become, locally, the gravitational-wave structure described in Fig. 8 above.

The process repeats over and over, with successive, toroidal, tendex/vortex structures being ejected and morphing into gravitational waves. The waves carry away some oscillation energy, and some oscillation energy goes down the hole, so the oscillations die out, with an exponentiation time of the order of the oscillation period.

**5.2.2 Collision of identical, spinning black holes in an inspiraling circular orbit.** For collisions of orbiting (i.e., non-head-on) black holes, the vortex and tendex lines similarly travel to the wave zone and become gravitational waves.

Figure 10a shows a schematic diagram of the horizon vortexes and the vortex lines for the collision of two orbiting, spinning black holes that are about to merge. Just after merger, the horizon vortexes retain their individuality and travel around the horizon of the merged black hole, trailing their vortex lines outward and backward in a pattern like water from a spinning sprinkler head (shown schematically in Fig. 10b). In the wave zone, the vortex lines acquire tendex lines and become gravitational waves.

Similarly, the near-field tendex lines attached to the merged hole horizon tendexes trail backward and outward in a spiral pattern, acquiring accompanying vortex lines as they travel and becoming gravitational waves. These horizon-tendex-generated gravitational waves have the opposite parity from the horizon-vortex-generated waves, and there is a remarkable duality between the two sets of waves [24].

Figure 10 is a schematic. For a more precise depiction, we focus on the merged hole at late times, when it is weakly perturbed from its final, Kerr-metric state, and its perturbations are predominantly  $\ell = 2$ , m = 2 quasinormal modes [24]. Then the perturbations of the frame-drag field  $\delta B_{ij}$  generated by the horizon vortexes have the vortex lines and vorticities shown in Fig. 11a, and the perturbations of the tendex field  $\delta \mathcal{E}_{ij}$  generated by the horizon tendexes have the tendex lines and tendicities shown in Fig. 11b. We note that the two figures are very nearly identical, aside from sign

![](_page_6_Figure_10.jpeg)

**Figure 11.** (Color online.) Vortex and tendex structures for deviations from the final Kerr black hole at late times after the merger of a black-hole binary. These are the structures in the final hole's equatorial plane, and the final hole has the dimensionless spin  $S/M^2 = a/M = 0.945$ . (a) Perturbations generated by horizon vortexes — vortex lines (solid black for clockwise, dashed for counterclockwise) and vorticity of the dominant vortex at each point (colored blue or light gray for clockwise and red or dark gray for counterclockwise). (b) Perturbations generated by horizon tendexes — tendex lines (solid black for squeeze, dashed for stretch) and tendicity of the dominant tendex (blue or light gray for squeeze and red or dark gray for stretch). Adapted from Fig. 12 in [24].

(interchange of red and blue, or dark and light gray). This is due to the (near) duality between the two sets of perturbations.

**5.2.3 Extreme-kick black-hole collision.** An interesting example of geometrodynamics and of the interplay between vortexes and tendexes is the 'extreme-kick' black-hole collision first simulated not by our SXS collaboration but by Campanelli et al. [25] and others [26, 27]. Our collaboration has repeated their simulations in order to extract the vortex and tendex structures.

In these simulations, two identical black holes merge from an initially circular orbit, with oppositely directed spins lying in the orbital (x, y) plane. The name 'extreme-kick' arises because gravitational waves generated during the merger carry linear momentum preferentially in either the +z or -zdirection, resulting in a gravitational recoil of the remnant black hole with speeds as high as thousands of km s<sup>-1</sup>. The magnitude and direction of the recoil depend on the angle between the holes' identical spin axes and the distance between the holes at the moment they merge. This angle can be fine-tuned (for instance, to produce the maximum recoil in the +z direction) by adjusting the initial conditions in the simulation.

To understand the mechanism of the recoil, we consider the remnant black hole just after merger. Figure 12a shows the horizon tendicity and the tendex lines emerging from the remnant black hole at some particular time. The tendex structure rotates counterclockwise around the hole's horizon. The rotating tendex lines acquire accompanying vortex lines as they extend into the wave zone in a pattern like that shown in Fig. 11b, and there they become gravitational waves. During the merger, the horizon vortexes (Fig. 12b) lock onto the same rotational angular velocity as the horizon tendexes and generate gravitational waves in the same manner, with a pattern that looks like the one in Fig. 11a.

In the wave zone, the gravitational waves produced by the rotating near-zone tendexes and those produced by the rotating near-zone vortexes superpose coherently, and the resulting radiation pattern depends on the angle between the

![](_page_7_Figure_3.jpeg)

**Figure 12.** (Color online.) Remnant horizon, shown in the *xy* plane, just after merger, for a superkick simulation tuned for the maximum remnant recoil in the +z direction; from a simulation presented in [20]. The black hole and vortex/tendex structures rotate counterclockwise. (a) Colors show the horizon tendicity; tendex lines are shown emerging from the strongest horizon tendexes. (b) Colors show horizon vorticity; vortex lines are shown emerging from the strongest the horizon vortexes. Figure adapted from [20]. See [28] and [29] for movies of horizon vortexes and tendexes of this simulation.

horizon tendex labeled 'E' and the vortex labeled 'B' in Fig. 12. For the case shown, this angle is 45 degrees, with  $\mathbf{E} \times \mathbf{B}$  in the -z direction (into the page). This is the same as the structure of a gravitational wave propagating in the -z direction (see Fig. 8). As a result, the gravitational waves produced by the vortexes and those produced by the tendexes superpose constructively in the -z direction and destructively in the +z direction, resulting in a maximum gravitationalwave momentum flow in the -z direction and a maximum remnant recoil in the +z direction.

**5.2.4 Generic black-hole collisions.** The geometrodynamic behaviors of more general black-hole collisions are now being explored numerically. For example, Fig. 13 shows the trajectories of two black holes in a fully generic orbit. Vortexes from the larger, rapidly spinning black hole pull the orbit of the smaller black hole into a complicated precession pattern. The spin directions of both black holes also precess as the holes orbit each other. Eventually, a common apparent horizon<sup>2</sup> forms around the two individual apparent horizons, and the two holes merge into one. The Ricci scalar (approximately equal to -2 times the horizon tendicity) is shown on the two individual horizons and on the common horizon at the moment the common horizon forms.

**5.2.5 Tidal disruption of a neutron star by a spinning black hole.** Our final example illustrates the interaction of geometrodynamics and matter. Figure 14 shows a simulation of a neutron star orbiting a black hole, a binary system important for gravitational-wave detectors and possibly for high-energy astrophysical phenomena such as gamma-ray bursts. Here, the black hole has 3 times the mass of the neutron star and a dimensionless spin  $S/M^2 = 0.5$  in a direction inclined approximately 45 degrees to the orbital angular momentum. When the orbit has shrunk sufficiently because of energy lost to gravitational radiation, the black hole's tidal tendexes rip apart the neutron star, and its frame-drag vortexes then pull

![](_page_7_Figure_9.jpeg)

**Figure 13.** (Color online.) The two thin curves are the trajectories of the centers of two black holes in a generic orbit; from a simulation presented in [30]. The mass ratio of the two holes is 6:1, and the dimensionless spins of the larger and smaller holes are  $S/M^2 = 0.91$  and 0.3, respectively (compared to the maximum possible spin  $S/M^2 = 1$ ). The initial black hole positions are shown in black, and the initial spins are shown as arrows. The spins are initially oriented in generic directions, such that the orbital plane precesses. Shown also are the apparent horizons of both holes and the common apparent horizon that encloses them, at the time the scalar Ricci curvature, which is approximately -2 times the horizon tendicity. See [31] for a movie of this simulation.

![](_page_7_Figure_11.jpeg)

**Figure 14.** (Color online.) Four successive snapshots of a collision between a black hole (black) and a neutron star (blue or gray), viewed edge-on in the initial orbital plane; from a simulation in [33]. (a) The initial black-hole spin S is inclined with respect to the initial orbital angular momentum L. (b) The black hole's tendexes begin to rip apart the neutron star. (c) Some of the matter falls on the black hole, but some remains outside the horizon. (d) The black hole's vortexes pull the remaining matter into the black hole's equatorial plane, forming a disk and a tidal tail. Figure adapted from [33]. See [34] for a movie of this simulation.

the stellar debris out of its original orbit and into the black hole's equatorial plane. If the black hole has a small enough spin or a large enough mass, the neutron star is not disrupted, but instead is swallowed whole by the black hole [32].

# 6. Gravitational-wave observations

Geometrodynamics generically produces gravitational waves. We are entering an era in which these waves, generated by sources in the distant universe, will be detected on Earth.

<sup>&</sup>lt;sup>2</sup> An apparent horizon is a surface of zero outgoing null expansion and lies inside or on the event horizon. Apparent horizons are local quantities that are much easier to find in numerical simulations than the event horizons, because the location of the event horizon depends on the full future evolution of the spacetime.

![](_page_8_Figure_2.jpeg)

**Figure 15.** (Color online.) A recent catalog of black-hole binary simulations from [30]. Shown are 174 waveforms, each with two polarizations (shown in two colors or shades of gray) in a sky direction parallel to the initial orbital plane. The unit on the time axis corresponds to 0.1 s for a binary with a total mass of 20 solar masses.

The first generation of interferometric gravitational-wave detectors has operated at sensitivities where it would require luck to see waves. We were not lucky [35, 36]. Secondgeneration detectors, much more advanced and complex in their design, are under construction. The first two of these (the advanced LIGO detectors in the US) will begin operating in 2015 and should reach their design sensitivity by 2019 or perhaps sooner [37]. They will be joined a bit later by the advanced VIRGO detector in Europe, KAGRA in Japan, and an advanced LIGO detector in India [38]. These advanced detectors will cover a volume of the universe 1000 times larger than the initial detectors did, with estimated event rates for mergers of black-hole and neutronstar binaries ranging from a few per year to a few per week [39–41]. Plans are being developed for further improvements, which should increase the event rate by another factor of ten or more, without major changes in detector design.

The most interesting gravitational-wave sources for these detectors, we think, are the dynamically evolving vortexes and tendexes attached to merging black holes and to a black hole tearing apart a neutron star—the geometrodynamic phenomena discussed above.

The numerical relativity community is building a catalog of binary simulations and associated gravitational waveforms to underpin the advanced detectors' searches for these waves. Simulations of binary black holes with several hundred different sets of parameters (mass ratios and initial vectorial spins) have been completed [30, 42–46], and many more are underway or planned. A sample of computed waveforms from our SXS collaboration is shown in Fig. 15.

Once waves are detected, comparisons of observed waveforms with those from simulations will be crucial for understanding the wave sources. By such comparisons, we can deduce the geometrodynamics of the sources and test predictions of general relativity.

# 7. Conclusions

Physicists have barely scratched the surface of geometrodynamics. As numerical simulations continue to improve and are used to explore more complicated and generic situations, we expect to learn more about the geometrodynamics of critical behavior, singularities, dynamical black holes, and other phenomena. We look forward to observations of gravitational waves from strongly gravitating astrophysical sources, which will enable us to test the geometrodynamical predictions of Einstein's equations for the first time.

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