100th ANNIVERSARY OF THE BIRTH OF Ya B ZELDOVICH

PACS numbers: 03.65.Ge, 03.65.Pm, 32.30.-r

### Hydrogen atom in a strong magnetic field

V S Popov, B M Karnakov

DOI: 10.3367	/UFNe.0184.201403e.	0273

#### Contents

1.	Introduction	257
2.	Problem formulation and basic equations	258
3.	Asymptotic formulas for the spectrum of even levels of a hydrogen atom in a strong magnetic field	260
4.	Equation (3.7) versus other approximations and numerical calculations	262
	4.1 1s ground state; 4.2 Excited levels	
5.	Odd levels	265
6.	Size and the quadrupole moment of the atom and radiative transition probabilities for $B \gg B_a$	265
7.	Relativistic effects	267
8.	Quantum electrodynamics effects	268
9.	Zeldovich effect in atomic structures	269
10.	Concluding remarks	272
11.	Appendices	273
	A. Critical nuclear charge; B. Extreme magnetic field; C. Coulomb corrections to the scattering length	
	References	278

Abstract. We study the energy spectrum of atomic hydrogen in strong ( $B > B_a \sim 10^9$  G) and ultra-strong ( $B \gtrsim B_{cr} \sim 10^{14}$  G) magnetic fields, in which the hydrogen electron starts to move relativistically and quantum electrodynamics effects become important. Within the adiabatic approximation, highly accurate energy level values are obtained analytically for  $B > 10^{11}$  G, which are then compared with asymptotic and numerical results available in the literature. A characteristic feature noted in electron motion in a strong magnetic field is that for  $B \gtrsim B_{cr}$ , the transverse motion becomes relativistic, while the longitudinal motion (along B) can be described by nonrelativistic theory and is amenable to the adiabatic approximation. Topics discussed include: the qualitative difference in the way odd and even levels change with the magnetic field (for  $B \gg B_a$ ); the removal of degeneracy between odd and even atomic states; spectral scaling relations for different quantum numbers  $(n, n_a, m)$  and different field strengths; the shape, size, and quadrupole moment of the atom for  $B \gg B_a$ ; radiative transitions  $np \rightarrow 1s$  in a strong magnetic field; relativistic QED effects, including the effects of vacuum polarization and of the electron anomalous magnetic moment on the energy level positions; Coulomb potential screening and energy level freezing at  $B \to \infty$ ; and the possibility of the Zeldovich effect in the

Received 6 July 2013

Uspekhi Fizicheskikh Nauk **184** (3) 273–296 (2014) DOI: 10.3367/UFNr.0184.201403e.0273 Translated by E G Strel'chenko; edited by A M Semikhatov hydrogen spectrum in a strong magnetic field. The critical nuclear charge problem is briefly discussed. Simple asymptotic formulas for  $Z_{cr}$ , valid for low-lying levels, are proposed. Some of the available information on extreme magnetic fields produced in the laboratory and occurring in space is given. The Coulomb renormalization of the scattering length is considered in the resonance situation with a shallow level in the spectrum.

#### 1. Introduction

The problem of the spectrum of atomic hydrogen in a strong magnetic field  $B \gg B_a \sim 10^9$  G is of considerable interest for astrophysics<sup>1</sup> [1–4], solid state physics, and atomic physics and has been treated by many investigators starting from the pioneering work of Shiff and Snyder [5], whose adiabatic approximation has been used by all subsequent authors. The literature on this topic contains dozens of studies (see, e.g., Refs [6–26] and the references therein). Because the variables involved in the Schrödinger equation are not separated, various numerical methods have been used to solve the Schrödinger and Dirac equations [13–16], and a number of analytic approximations are available. We consider approximate asymptotic formulas (obtained both by us and by others) for the atomic hydrogen energy levels in a strong magnetic field. Among the topics to be discussed are: the accuracy of these formulas as a function of the field B; the difference in the way odd and even levels change with the magnetic field for  $B \gg B_a$ ; compression and deformation of atoms in a strong magnetic field; how vacuum polarization in an ultrastrong magnetic field and the electron anomalous

V S Popov Alikhanov Institute for Theoretical and Experimental Physics, ul. B. Cheremushkinskaya 25, 117218 Moscow, Russian Federation E-mail: markina@itep.ru

**B** M Karnakov National Research Nuclear University 'MEPhI', Kashirskoe shosse 31, 115409 Moscow, Russian Federation E-mail: karnak@theor.mephi.ru

<sup>&</sup>lt;sup>1</sup> Magnetic white dwarfs (fields up to 350 MG on the surface of a star) and, in particular, neutron stars: pulsars with  $B \gtrsim 10^{12}$  G and magnetars, a special class of neutron stars with magnetic fields reaching record values  $10^{15}$  G.

magnetic moment affect energy level positions; energy level freezing at  $B \ge 10^{16}$  G; and the Zeldovich effect in (or rearrangement of) the hydrogen spectrum. Appendices A, B, and C discuss problems related to the critical nuclear charge, Z > 137, and provide brief information on the maximum magnetic fields obtained in the laboratory and known in astrophysics.

We start by giving formulas for and numerical values of the basic quantities involved in the problem to be considered (below,  $e = 4.80 \times 10^{-10}$  is the electron charge in CGSE,  $m_e = 9.11 \times 10^{-28}$  g is the electron mass, and  $a_B = \hbar^2/m_e e^2 = 0.529 \times 10^{-8}$  cm is the Bohr radius). The atomic unit of magnetic field strength<sup>2</sup> is  $B_a = m_e^2 e^3 c/\hbar^3 = 2.349 \times 10^9$  G;  $\mathcal{H} = B/B_a$  is the dimensionless reduced field;  $B_{cr} = m_e^2 c^3/e\hbar = 4.414 \times 10^{13}$  G is the 'critical' [27, 28] or Schwinger [29] field in quantum electrodynamics (QED) [30] (note that  $B_a/B_{cr} = \alpha^2 = 5.325 \times 10^{-5}$ , with  $\alpha = e^2/\hbar c = 1/137$ ;  $a_H = \sqrt{c\hbar/eB}$  is the magnetic length or Landau radius

$$a_H = \frac{a_{\rm B}}{\sqrt{\mathcal{H}}} = \begin{cases} a_{\rm B} , & B = B_{\rm a} ,\\ \alpha \, a_{\rm B} = l_{\rm C} , & B = B_{\rm cr} , \end{cases}$$
(1.1)

where  $l_{\rm C} = \hbar/m_{\rm e}c = 3.861 \times 10^{-11}$  cm is the Compton electron wavelength; and  $\omega_H = eB/m_{\rm e}c$  is the circular rotation frequency of a classical nonrelativistic electron in a magnetic field *B*. The Landau level separation is given by

$$\hbar\omega_{H} = \frac{e\hbar}{m_{e}c} B = \frac{m_{e}e^{4}}{\hbar^{2}} \mathcal{H}$$

$$= \begin{cases} \frac{m_{e}e^{4}}{\hbar^{2}} = 2 \operatorname{Ry} = 27.21 \text{ eV}, & B = B_{a}, \\ m_{e}c^{2} = 0.511 \text{ MeV}, & B = B_{cr}, \end{cases}$$
(1.2)

and the atomic level energies are

$$E_n = -\lambda_n^2 \operatorname{Ry}, \qquad (1.3)$$

where  $Ry = m_e e^4/2\hbar^2 = 13.61$  eV (Rydberg) and  $\lambda_n$  is the dimensionless momentum of a bound state.

#### 2. Problem formulation and basic equations

The following features should be noted regarding the motion of an electron in a central attracting field U(r) (in particular, in a Coulomb potential  $U_{\rm C} = -e^2/r$ ) in the presence of a uniform magnetic field *B* parallel to the *z* axis. The system Hamiltonian in the nonrelativistic approximation, the Pauli Hamiltonian

$$\hat{H} = \frac{1}{2m_{\rm e}} \left( \hat{\boldsymbol{\sigma}} \left( \hat{\boldsymbol{p}} + \frac{e}{c} \, \boldsymbol{A} \right) \right)^2 + U(r) \,, \tag{2.1}$$

in cylindrical coordinates takes the form (in the atomic units where  $e = \hbar = m_e = 1$ )

$$\hat{H} = -\frac{1}{2} \Delta_{\perp} + \frac{1}{8} \mathcal{H}^2 \rho^2 + \frac{1}{2} \mathcal{H}(\hat{l}_z + \hat{\sigma}_z) - \frac{1}{2} \frac{\partial^2}{\partial z^2} + U(\sqrt{\rho^2 + z^2}), \qquad (2.2)$$

<sup>2</sup> The interaction energy between the magnetic moment  $\mu = e\hbar/m_ec$  and such a field is equal to the Coulomb interaction energy  $e^2/a_B = m_e e^4/\hbar^2 = 27.21$  eV. The magnetic field strength in the hydrogen atom is typically  $B \sim B_{\rm at} \sim \alpha^2 B_{\rm a} \sim 10^5$  G.

 $\rho = \sqrt{x^2 + y^2}$ , and the magnetic field vector potential is  $\mathbf{A} = [\mathbf{Br}]/2$ . Because this Hamiltonian is axially symmetric, its eigenfunctions can be written in the form

$$\Psi_E = \frac{1}{\sqrt{2\pi}} \exp\left(im\phi\right) \varphi_{\sigma_z}(\sigma) \psi(\rho, z) , \qquad (2.3)$$
$$m = 0, \pm 1, \pm 2, \dots, \qquad \sigma_z = \pm 1 ,$$

where  $\varphi_{\sigma_z}(\sigma)$  is the spin function. The functions  $\psi(\rho, z)$  have a definite parity under reflection in the plane perpendicular to the vector **B**, i.e., under the transformation  $z \to -z$ ; however, the variables  $\rho$  and z do not separate.

Whenever the potential U(r) can be considered a perturbation in the background of the magnetic field, the adiabatic approximation [2, 5, 6] can be used to solve the Schrödinger equation. The wave functions  $\psi(\rho, z)$  in Eqn (2.3) are then written (for both even and odd states) in the form

$$\psi(\rho, z) = R_{n_{\rho}m}(\rho) \,\chi_{nn_{\rho}m}(z) \,, \qquad n_{\rho} = 0, 1, 2, \dots, \qquad (2.4)$$

where  $R_{n_{\rho}m}(\rho)$  are known functions for the radial transverse motion of an electron in a purely magnetic field [6], and the spectrum of bound states of the Hamiltonian has the form

$$E_{nn_{\rho}m\sigma_{z}} = N\mathcal{H} - \frac{1}{2}\lambda_{nn_{\rho}m}^{2},$$

$$N = n_{\rho} + \frac{1}{2}\left(|m| + m + \sigma_{z} + 1\right).$$
(2.5)

Here, N = 0, 1, 2, ... labels the Landau levels,  $\sigma_z = \pm 1$  is twice the projection of the electron spin on the magnetic field direction, n = 0, 1, 2, ... is the longitudinal quantum number, and  $\lambda_n^2/2$  determines the corresponding level energies due to the potential U(r). These energies are found from the Schrödinger equation for the longitudinal part of the electron wave function,

$$\left\{-\frac{1}{2}\frac{\partial^2}{\partial z^2} + U_{\text{eff}}(|z|) + \frac{\lambda^2}{2}\right\}\chi_n(z) = 0, \quad -\infty < z < +\infty,$$
(2.6)

in which the effective potential energy [i.e., the potential U(r) averaged over the fast transverse motion of the electron in the magnetic field] is given by [2, 5, 6]

$$U_{\text{eff}}(|z|) = \int_0^\infty U(\sqrt{\rho^2 + z^2}) \left| R_{n_\rho m}(\rho) \right|^2 \rho \,\mathrm{d}\rho \,. \tag{2.7}$$

The specific form of  $U_{\text{eff}}$  depends on the transverse quantum numbers  $n_{\rho}$  and |m|.

Thus, the adiabatic approximation allows separating the variables  $\rho$  and z. We note that the size of the transverse localization region of the electron, determined by  $|R_{n_{\rho}m}(\rho)|^2$ , is of the order of  $\rho \sim a_H \sim a_B/\sqrt{\mathcal{H}}$  (for the lower Landau levels). Therefore, for the potential range  $r_0 \ge a_H$ , the magnetic field strongly confines the transverse motion of the electron, resulting in the electron density distribution  $|\psi(\mathbf{r})|^2$  taking a spoke-like shape (along the z axis) with the center at r = 0. Accordingly, if the central potential U(r) is not too singular at zero, such that  $r|U(r)| \to 0$  as  $r \to 0$ , then  $U(\sqrt{\rho^2 + z^2})$  can be replaced by U(|z|) in the integrand in Eqn (2.7). Therefore, in this case, the one-dimensional effective potential coincides with the original central potential and is no longer dependent on the magnetic field. Hence,

the electron density distribution along the z axis and the spectrum of the longitudinal motion cease to depend on the field strength.

We now compare the spectrum of one-dimensional motion in a symmetric potential U(|z|) defined on the entire axis  $-\infty < z < +\infty$  with the spectrum of the *ns* states in the same central potential U(r),  $r \ge 0$ . As is known, if the wave function of an *ns* state is written as  $\psi_{ns} = \chi_{n_r}(r)/(\sqrt{4\pi}r)$ , then the Schrödinger equation for  $\chi_{n_r}(r)$  takes a form similar to Eqn (2.6). This time, however,  $r \ge 0$ , and the function  $\chi_{n_r}(r)$ satisfies the boundary condition  $\chi_{n_r}(0) = 0$ . For odd states, the  $\chi_n(z)$  in Eqn (2.6) also satisfy the condition  $\chi_n^{(-)}(0) = 0$ , and it then follows that the spectra coincide,

$$E_n^{(-)} = E_{ns}, \quad n = 1, 2, \dots$$
 (2.8)

For even states, this analogy does not hold, because now  $\tilde{\psi}_{ns}(r) = \chi_n^{(+)}(r)/r \propto 1/r \to \infty$  as  $r \to 0$ , and such singular solutions are excluded from consideration (because we then have  $\Delta \tilde{\psi}_{ns}(r) \propto \delta(\mathbf{r})$  as  $r \to 0$ ), with the exception of a three-dimensional zero-radius potential. Such a potential, localized at r = 0, is determined by requiring that the wave function satisfy the boundary condition [31, 32]

$$\frac{\mathrm{d}\ln\chi(r)}{\mathrm{d}r} = -\frac{1}{a_{\mathrm{s}}}, \text{ or } \tilde{\psi}_{n\mathrm{s}}(r) \approx \mathrm{const}\left(\frac{1}{r} - \frac{1}{a_{\mathrm{s}}}\right) \text{ as } r \to 0.$$
(2.9)

Here,  $a_s$  is the scattering length for the short-range potential  $U_s(r)$  of radius  $r_0$  modeled by such a zero-radius potential. The parameter  $\kappa_0 = 1/a_s$  here determines the energy  $E_0 = -\hbar^2 \kappa_0^2/2m$  of a shallow real (for  $a_s > 0$ ) or virtual (for  $a_s < 0$ ) s level, which exists in the system if  $|E_0| \ll \hbar^2/mr_0^2$   $(|a_s| \ge r_0)$ .

It is therefore readily seen that the even solutions  $\chi_n^{(+)}(|z|)$ of Eqn (2.6) on the entire z axis with the one-dimensional potential U(|z|) (solutions for which  $\chi_n^{(+)}(0) = \text{const} \neq 0$  and  $\chi_n^{(+)}(0) = 0$ ) are in a 1:1 correspondence with those solutions  $\bar{\psi}_{ns}(r) = \chi^{(+)}(r)/(\sqrt{2\pi}r)$  of the three-dimensional Schrödinger equation for s states with the same potential U(r) that satisfy boundary condition (2.9) with  $a_s = \infty$ , i.e.,  $\kappa_0 = 0$ . This means that the spherically symmetric potential is a superposition of a potential U(r) [here, the effective potential  $U_{\text{eff}}(r)$ ] and the zero-range potential (at r = 0), for which  $\kappa_0 = 0$ , i.e., at the instant a bound state occurs in this potential.

Importantly, the case  $\kappa_0 = 0$  implies that such a zerorange potential models a 'strong' short-range potential of the radius  $r_0$  at the moment when it gives rise to a bound state (with zero binding energy). We note that the scattering of slow ( $kr_0 \ll 1$ ) particles by such a potential is resonant in character,<sup>3</sup> with the scattering cross section  $\sigma \approx 2\pi\hbar^2/mE$  (*E* is the electron energy) [6].

Such a potential produces a rearrangement in the spectrum of s levels in a long-range potential  $U_L(r)$  of radius  $r_L \ge r_0$ , as can be seen from the classical quantization rule for this case,<sup>4</sup>

$$\frac{1}{\hbar} \int_{0}^{b} \sqrt{2m_{\rm e} \left[ E_{n_r 0} - U(r) \right]} \, \mathrm{d}r = \pi \left( n_r + \frac{3}{4} + \tilde{\gamma} \right), \qquad (2.10)$$

where *b* is the turning point. Corresponding to levels in an isolated potential U(r) is the value  $\tilde{\gamma} = 0$ , and to the shifts due to the zero-radius potential with  $\kappa_0 = 0$ ,  $\tilde{\gamma} = -1/2$  (the levels are strongly shifted downward:  $n_r \to n_r - 1/2$ ).

For the hydrogen atom, however, there is no justification for replacing the Coulomb potential  $U_{\rm C} = -e^2/\sqrt{\rho^2 + z^2}$  in Eqn (2.7) with a one-dimensional Coulomb potential  $U_{\rm eff}(|z|) = -e^2/|z|$ . The reason is that in two and three dimensions, the Coulomb potential at short distances acts as a small correction, which manifests itself such that for  $Z \leq 10$ , the structure of the nucleus has little effect on the level shift. In one dimension, with  $U = -e^2/|z|$  on the entire z axis, the situation is totally different. Two independent solutions of the Schrödinger equation for  $z \to \pm 0$  have the form<sup>5</sup> (to the right and left of z = 0):

$$\psi_{E}^{\pm}(z) \approx C_{E,1}^{\pm} \left[ 1 - \frac{2|z|}{a_{\rm B}} \ln \frac{|z|}{a_{\rm B}} + O\left(\frac{z^{2}}{a_{\rm B}^{2}} \ln \frac{|z|}{a_{\rm B}}\right) \right] + C_{E,2}^{\pm} \left[ |z| + O\left(\frac{z^{2}}{a_{\rm B}}\right) \right], \quad z \to \pm 0, \qquad (2.11)$$

and the conditions (usual for regular potentials) of the continuity of the wave function and its derivative cannot be satisfied at z = 0 because we then have  $|d\psi_E(z)/dz| \rightarrow \infty$  (except for odd states). Therefore, the energy spectrum of even states is strongly dependent on exactly how the potential is cut off at distances  $\sim a_H$ , i.e., on the form of  $U_{\text{eff}}(|z|)$  at  $|z| \ll a_{\text{B}}$ .

For the purpose of further discussion, the following properties of the effective Coulomb potential should be noted. At distances  $|z| \leq L$ , where

$$a_H \ll L \ll \sqrt{a_H a_B} = \mathcal{H}^{1/4} a_H = \mathcal{H}^{-1/4} a_B ,$$
 (2.12)

we have  $|U_{\text{eff}}| \leq e^2/a_H$  for this potential, resulting in

$$|U_{\rm eff}| \ll \frac{\hbar^2}{m_e L^2} \,. \tag{2.13}$$

At such distances,  $U_{\text{eff}}(|z|)$  is therefore a shallow one-dimensional potential well, which allows solving Eqn (2.6) perturbatively. The logarithmic derivative  $\chi'/\chi$  for  $|z| \sim L$  is then obtained as a simple closed expression using the explicit form of the effective potential [see Eqn (3.3) below]. On the other hand, at distances  $|z| \geq L \gg a_H$ , the effective potential already becomes a Coulomb one,  $U_{\text{eff}} \approx -e^2/|z|$ , and the exponentially decaying solution of the Schrödinger equation is the Whittaker function. Matching the logarithmic derivatives in their overlap region, Eqn (2.12), leads to an equation for the energy spectrum of even states.

We note that other studies using the adiabatic approximation to solve Schrödinger equation (2.6) make additional assumptions about the properties of the effective potential, which leads to less accurate results. For example, in Ref. [6], the cut-off Coulomb potential is specifically chosen as  $U_{\rm eff}(|z|) = -e^2/(|z| + a_H)$ . Therefore, a correct use of the adiabatic approximation at distances  $|z| \leq L$  gives simple analytic expressions for the spectrum of even levels, which provide higher accuracy than numerical solutions.

We next discuss the odd levels. The zeroth-order approximation for them is obtained by using the nondistorted

<sup>&</sup>lt;sup>3</sup> As noted in Section 9, a similar distortion in the Coulomb potential U(r) at small distances gives rise to the Zeldovich effect.

<sup>&</sup>lt;sup>4</sup> This rule is a simple generalization of the Bohr–Sommerfeld quantization rule. We note that for a spherical oscillator, Eqn (2.10) yields an exact expression for the spectrum.

<sup>&</sup>lt;sup>5</sup> We note the choice of the argument in the logarithmic term and the fact that the term is energy independent; only the correction terms in asymptotic expression (2.11) depend on energy.

Coulomb potential (i.e., by setting  $U_{\text{eff}} = -1/r$ ) in Eqn (2.6), and hence their spectrum for states with various values of the quantum numbers  $n_{\rho}$  and *m* is identical to that of unperturbed Coulomb *n*s levels in the central potential U(r) = -1/r:

$$E_n^{(-)} \approx -\frac{1}{2n^2}, \quad n = 1, 2, \dots$$
 (2.14)

The fact that the effective potential is not purely Coulomb can be taken into account by the perturbation theory, with the result

$$E_{nmn_{\rho}}^{(-)} \equiv -\frac{m_{e}e^{4}}{2\hbar^{2}} \left(\lambda_{nmn_{\rho}}^{(-)}\right)^{2}$$
  
=  $-\frac{m_{e}e^{4}}{2\hbar^{2}n^{2}} + \int \left(U_{eff}(r) + \frac{e^{2}}{r}\right) \psi_{ns}^{2}(r) d^{3}r, \qquad (2.15)$ 

where  $\psi_{ns}(r)$  are the unperturbed wave functions of *n*s states in a Coulomb potential. We note that the integrand in Eqn (2.15) is positive, and hence the levels are slightly upshifted with respect to the unperturbed Coulomb levels.

In concluding this section, we present the radial wave functions of the transverse motion [6]:

$$R_{n_{\rho}m}(\rho) = \frac{1}{a_{H}^{1+|m|}} \left[ \frac{(|m|+n_{\rho})!}{2^{|m|}n_{\rho}!(|m|!)^{2}} \right]^{1/2} \exp\left(-\frac{\rho^{2}}{4a_{H}^{2}}\right) \\ \times \rho^{|m|} F\left(-n_{\rho}, |m|+1, \frac{\rho^{2}}{2a_{H}^{2}}\right)$$
(2.16)

[with F(...) being the degenerate hypergeometric function], normalized by the condition  $\int_0^\infty R_{n_pm}^2(\rho) \rho \, d\rho = 1$ , and the explicit forms of the effective Coulomb potential for  $m = n_\rho = 0$  states,

$$U_{\text{eff}}(|z|) = -\frac{\sqrt{2}}{a_H} \int_0^\infty \exp\left(-\frac{\sqrt{2}|z|x}{a_H} - x^2\right) dx$$
$$= -\sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{|z|}{\sqrt{2}a_H}\right) \exp\left(\frac{z^2}{2a_H^2}\right) \frac{1}{a_H}, \quad (2.17)$$

and its limit expressions

$$U_{\rm eff}(|z|) = \begin{cases} -\left(\sqrt{\frac{\pi}{2}} - \frac{|z|}{a_H} + \dots\right) \frac{1}{a_H}, & |z| \le a_H, \\ -\left(1 - \frac{a_H^2}{z^2} + \dots\right) \frac{1}{|z|}, & |z| \ge a_H. \end{cases}$$
(2.18)

Here, erfc  $(x) = (2/\sqrt{\pi}) \int_{x}^{\infty} \exp(-t^2) dt = 1 - \operatorname{erf}(x)$ . The value of the potential at zero for  $n_{\rho} = 0$  states is

$$U_{\text{eff};n_{\rho}=0,m}(0) = -\sqrt{\frac{\pi}{2}} \frac{(2|m|-1)!!}{2^{|m|}|m|!} \frac{1}{a_{H}}$$
$$= -\sqrt{\frac{\pi}{2}} \frac{1}{a_{H}} \begin{cases} 1, & m=0, \\ 1/2, & m=1, \\ 3/8, & m=2, \\ 63/256, & m=5. \end{cases}$$
(2.19)

As |m| increases, the well becomes shallower,<sup>6</sup> and hence the deepest levels correspond to m = 0.

<sup>6</sup> This is because the repulsive centrifugal energy  $\hbar^2 m^2/2m_e\rho^2$  acts to confine the electron more tightly in the transverse direction as |m| increases, thus decreasing  $U_{\text{eff}}$ . For the same reason, as |m| (and also  $n_\rho$ ) increases, higher magnetic fields are needed for the adiabatic approximation to be valid.

# 3. Asymptotic formulas for the spectrum of even levels of a hydrogen atom in a strong magnetic field

As noted in Section 2, the effective one-dimensional Coulomb potential in the distance range  $|z| \leq L$  has the form of a shallow one-dimensional potential well, and Eqn (2.6) can be solved perturbatively. The wave functions of even levels remain almost unchanged at such distances,  $\chi(z) \approx \text{const.}$  Therefore, neglecting the energy term in Eqn (2.6), setting  $\chi(z) = \text{const}$  in the  $U_{\text{eff}}$  term, and integrating over z, we find the logarithmic derivative of the wave function in the form

$$\frac{\chi'(z)}{\chi(z)} \approx 2 \int_0^z U_{\text{eff}}(z') \, \mathrm{d}z', \quad z > 0 \,, \quad \chi'(0) = 0 \,. \tag{3.1}$$

Substituting expression (2.7) for  $U_{\text{eff}}(z')$  with  $U = -(\rho^2 + z'^2)^{-1/2}$ , changing the order of integrations over  $\rho$  and z', and using the formulas (for  $z \gg \rho \sim a_H$ )

$$\int_{0}^{z} \frac{\mathrm{d}z}{\sqrt{z^{2} + \rho^{2}}} = \ln \frac{z + \sqrt{z^{2} + \rho^{2}}}{\rho} = \ln \frac{2z}{\rho} + O\left(\frac{\rho^{2}}{z^{2}}\right), \quad (3.2)$$
$$\int_{0}^{\infty} x^{s-1} \exp\left(-x\right) \ln x \,\mathrm{d}x = \Gamma'(s) = \Gamma(s) \,\psi(s) \,, \quad (3.2')$$

where  $\psi(s)$  is the logarithmic derivative of the Gamma function, we rewrite Eqn (3.1) as

$$\frac{\chi'(z)}{\chi(z)} \approx -2\ln\frac{z}{a_H} + A_{n_p|m|}, \qquad (3.3)$$

where

$$A_{n_{\rho}|m|} = 2 \int_{0}^{\infty} \rho \ln \frac{\rho}{2a_{H}} R_{n_{\rho}|m|}^{2}(\rho) \,\mathrm{d}\rho \,.$$
(3.4)

We note that  $A_{n_p|m|}$  is independent of both the magnetic field and the sign of *m*.

On the other hand, at distances  $z \gtrsim L$ , where the effective potential is already identical to the Coulomb potential, the solution of Eqn (2.6) that decreases exponentially at infinity is expressed in terms of the Whittaker function,

$$\chi_{\nu}(z) = \text{const } W_{\nu, 1/2}(x) , \quad x = 2\lambda z , \quad \nu = \frac{1}{\lambda} , \quad z > 0 , \quad (3.5)$$

for which the following expansion holds:

$$W_{\nu,1/2}(x) = \frac{1}{\Gamma(1-\nu)} \times \left\{ 1 - c_1 x \ln x - c_2 x + c_3 x^2 \ln x + O(x^2) \right\}, \quad x \to 0,$$
  

$$c_1 = \nu, \quad c_2 = \frac{1}{2} + \nu [\psi(1-\nu) + 2\gamma - 1], \quad c_3 = \frac{1}{2} \nu^2, \dots$$
(3.5')

It hence follows that

$$\frac{\chi'(z)}{\chi(z)} = -\left\{\lambda + 2\left[\ln\left(\lambda|z|\right) + \psi(1-\lambda^{-1}) + 2\gamma + \ln 2\right]\right\} + O\left(z\ln^2\left(\lambda z\right)\right),$$
(3.6)

where  $z \ll 1$ ,  $\lambda = \sqrt{-2E}$ , *E* is the level energy, and  $\gamma = -\psi(1) = 0.5772...$  is the Euler constant.

When logarithmic derivatives (3.3) and (3.6) are matched in region (2.12), where they both apply,  $a \sim L$ , the dependence on the coordinate z disappears, and we arrive at an equation for the spectrum of even levels for the longitudinal motion of the electron:

$$\ln \mathcal{H} = \lambda + 2\ln(2\lambda) + 2\psi\left(1 - \frac{1}{\lambda}\right) + 4\gamma + A_{n_{\rho}|m|}.$$
 (3.7)

A few remarks are in order on the implications this equation has for the properties of the spectrum of even levels  $E = -\lambda^2/2$  in the case of a strong magnetic field (formally, for  $\ln H \ge 1$ ).

(1) For each pair of the  $n_{\rho}$  and m quantum numbers for the transverse electron motion (with any sign of m), there is one level for which  $\lambda \sim \ln \mathcal{H} \ge 1$  and which is deep on the atomic scale (the deeper, the larger |m| is). We emphasize that for  $m \neq 0$ , their binding energy is independent of the sign of m, but they lie below various Landau levels [see Eqn (2.5)].

(2) Besides this deep level, an infinite number of levels are located between neighboring unshifted Coulomb *n*'s levels,  $E_{n'} = -1/(2(n')^2)$ , with the principal quantum numbers *n'* equal to *n* and n + 1 for  $n \ge 1$ .<sup>7</sup> In the limit as  $\ln \mathcal{H} \to \infty$ ,  $n' \to n$ ; however, the transition to the asymptotic regime is extremely slow (see Section 4 below).

(3) For a known value of  $\lambda_{nn_{\rho}|m|}$ , Eqn (3.7) directly determines the magnitude of the corresponding magnetic field.

(4) We next rewrite Eqn (3.7) as  $F(\lambda) = F_{n_o m}(\mathcal{H})$ , where

$$F(\lambda) \equiv \lambda + 2\ln\lambda + 2\psi\left(1 - \frac{1}{\lambda}\right),$$
  

$$F_{n_{\rho}m}(\mathcal{H}) = \ln\mathcal{H} - 4\gamma - 2\ln2 - A_{n_{\rho}m},$$
(3.8)

and note the following consequence of this equation.

As can be seen from Eqns (3.7) and (3.8), the level energy dependence is here determined by the universal function  $F(\lambda)$ , which depends neither on the magnetic field nor on the quantum numbers  $n_{\rho}$ , m, and n [their effect comes through the function  $F_{n_{\rho}m}(\mathcal{H})$  in the right-hand side of the equation]. This means that in the adiabatic approximation, the entire even spectrum  $E_{nn_{\rho}|m|}(\mathcal{H}) = -\lambda^2/2$  lies on this universal curve, independent of the values of the quantum numbers and the magnetic field.

Figure 1 illustrates how scaling is fulfilled for the ground 1s level with the quantum number  $n_{\rho} = 0$  for various values of *m* and the magnetic field [24]. The figure plots the function  $F(\lambda)$  and indicates the values of  $F_{0m}(\mathcal{H})$  for the corresponding levels for a numerical solution of the Schrödinger equation. It can be seen that this scaling is fulfilled to good accuracy for states with different *m* for  $\lambda^2 > 12$ . For  $\lambda^2 < 10$ , a deviation<sup>8</sup> from  $F(\lambda)$  occurs; however, also in this region the positions of the levels are grouped along a smooth curve close to  $F(\lambda)$ .

(5) For even states near the ground Landau band N = 0 $(n_{\rho} = 0, m = -|m| = 0, -1, -2, \dots, \sigma_z = -1),$ 

$$A_{0|m|} = -\ln 2 + \psi (1 + |m|), \quad A_{00} = -(\ln 2 + \gamma), \quad (3.9)$$

<sup>8</sup> This comes as no surprise because, with increasing m, the range of applicability of the adiabatic approximation shifts toward higher magnetic fields due to the increase in the centrifugal energy.



**Figure 1.** Verifying the scaling relations. Solid line, the function  $F(\lambda)$  from (3.8);  $\bigcirc$ , +,  $\Box$ ,  $\bullet$ ,  $\triangle$ ,  $\times$ , numerical results [13–16] for states with quantum numbers  $n_{\rho} = 0$  and m = 0, -1, -2, -3, -4, -5.

 $\psi(n+1) = -\gamma + \sum_{k=1}^{n} k^{-1}, n = 1, 2, ...;$  we let these deepest states be denoted by LLL (lowest Landau levels) [33–35].

(6) The value we obtained for logarithmic derivative (3.3) for even states of Eqn (2.6) at distances  $r_0 \ll z \ll a_B$ , where the potential has a Coulomb form, can be extrapolated to z = +0 with the simultaneous replacement of the effective potential  $U_{\text{eff}}(|z|)$  by the one-dimensional Coulomb potential  $U_{\text{C}}(|z|) = -e^2/|z|$ . This is equivalent to modeling a shortrange distortion of this potential by a point-like interaction (at z = 0).<sup>9</sup> Using the values of logarithmic derivative (3.3) for  $z \to 0$  for even solutions (2.11) of the Schrödinger equation with a Coulomb potential, we then obtain the relation

$$C_1^- = C_1^+, \quad \frac{C_2^+}{C_1^+} = \beta,$$
 (3.10)

where the energy-independent parameter  $\beta$  with the dimension of inverse length is given by

$$\beta = \frac{1}{a_{\rm B}} \left( 2 \ln \frac{a_H}{a_{\rm B}} + 2 + A_{n_{\rho}|m|} \right). \tag{3.11}$$

The dependence of  $\beta$  on the quantum numbers  $n_{\rho}$  and m is related to how these quantum numbers affect the nature of the Coulomb potential distortion at small distances [see  $U_{\text{eff}}(|z|)$  in Eqn (2.7)], whose influence persists in the limit as  $r_0 \rightarrow 0$ .

In the case of a singular Coulomb potential  $U_{\rm C}(|z|)$ , the matching conditions at z = 0, Eqn (3.10), replace the usual continuity conditions for the wave function and its derivative for regular potentials. Mathematically, these conditions specify a self-adjoint extension of a Hermitian Hamiltonian with a one-dimensional Coulomb potential (for more details, see the discussion of problem 8.61 in Ref. [36]<sup>10</sup>). The need to

<sup>&</sup>lt;sup>7</sup> These shifted Coulomb levels are easily understood as resulting from the following property of  $\psi(z)$  in Eqn (3.7): in the interval (-n - 1, -n) of the values of *z* with n = 0, 1, 2, ..., the function  $\psi(z)$  increases monotonically from  $-\infty$  to  $+\infty$ ; the values z = -n are its poles.

<sup>&</sup>lt;sup>9</sup> The fact that solutions with distorted and undistorted Coulomb potentials look totally different at distances  $|z| \leq r_0$  is of no consequence, because this distance range contributes little to the normalization integral.

<sup>&</sup>lt;sup>10</sup> We use this opportunity to draw attention to a misprint in formula (14) [the analog of our Eqn (3.10)] in this problem, which can be corrected by making the replacement  $\beta \rightarrow 2\beta$ . Formula (18) determines the atomic hydrogen spectrum in a strong magnetic field.

introduce additional conditions to ensure the mutual orthogonality and completeness of eigenfunctions of the Hamiltonian in this case was first noted in Ref. [37] in connection with the so-called 'fall onto the center' phenomenon in quantum mechanics [6, 38].

In the limit case where the Coulomb potential is turned off, i.e., for  $a_{\rm B} \rightarrow \infty$ , matching conditions (3.10) at z = 0 take the form

$$\psi(+0) = \psi(-0), \quad \psi'(+0) - \psi'(-0) = 2\beta\psi(0)$$

and are modeled by a point-like interaction in the form of a delta potential  $U(z) = \alpha \delta(z)$  at  $\alpha = \hbar^2 \beta/m$ .

(7) As noted in Section 2, even solutions of the Schrödinger equation with a potential U(|z|) defined on the entire z axis and bounded as  $|z| \rightarrow 0$  correspond to solutions of this equation for s states in a similar central potential U(r), but subject to boundary condition (2.9) that models a zero-range potential. However, in the case of the attractive Coulomb potential  $U_{\rm C} = -e^2/r$  (at small distances), this condition, according to Eqn (3.10), should be written as

$$\frac{\chi'(r)}{\chi(r)} \approx -\frac{2}{a_{\rm B}} \left( \ln \frac{r}{a_{\rm B}} + 1 \right) + \beta \,, \qquad r \to 0 \,. \tag{3.12}$$

The parameter  $\beta$  has a clear physical meaning in the case where the Coulomb potential has the same form for all r > 0. The solution of the Schrödinger equation is then written as

$$\chi(r) = \operatorname{const}\left(F_0(kr)\cot\delta_0(k) + G_0(kr)\right), \qquad (3.13)$$

where  $F_0(z)$  and  $G_0(z)$  are respectively a regular and an irregular Coulomb wave function [6], and  $\delta_0$  is the distortion of the Coulomb scattering phase, due to the short-range potential modeled by boundary condition (3.12). Taking the limit  $r \rightarrow 0$  in Eqn (3.13) and using the effective range approximation in the case of a Coulomb potential distorted at small distances,  $r_0 \ll a_{\rm B}$ , we obtain (for  $k \rightarrow 0$ )

$$\beta = -\frac{2}{a_{\rm B}}(\ln 2 + 2\gamma - 1) - \frac{1}{a_{\rm cs}}.$$
(3.14)

Here,  $a_{cs}$  is the s scattering length of the short-range potential renormalized due to Coulomb interaction at small distances. In the problem under consideration, the value of  $\beta$  is given by Eqn (3.11). Boundary condition (3.12) becomes

$$\frac{\chi'(r)}{\chi(r)} \approx -\frac{2}{a_{\rm B}} \left( \ln \frac{2r}{a_{\rm B}} + 2\gamma \right) - \frac{1}{a_{\rm cs}} , \qquad r \to 0 . \tag{3.15}$$

When the potential is switched off, i.e., in the limit as  $a_{\rm B} \rightarrow \infty$ , Eqn (3.12) becomes Eqn (2.9),  $\beta = -1/a_{\rm s}$ , and, according to Eqn (3.14),  $a_{\rm cs} = a_{\rm s}$ , as should be expected.

The results above for the spectrum of Hamiltonian (2.1) with a Coulomb potential are asymptotically exact for  $B \rightarrow \infty$ . The trouble is, however, that the electron transverse velocity increases to infinity,<sup>11</sup> the Schrödinger equation is no longer applicable, and the Dirac equation should be used instead. As shown in Section 7, the adiabatic approach allows a direct generalization to the Dirac equation. Moreover, formula (3.7) for the spectrum turns out to also be applicable

<sup>11</sup> For  $B = B_{cr}$ , the kinetic energy of the transverse motion is  $T = m_e c^2/2$  [see Eqn (1.2)].

in the relativistic range, up to a magnetic field  $B \sim 10^{16}$  G, at which quantum electrodynamics effects become important

Naturally, the question arises as to the range of applicability and accuracy of Eqn (3.7), whose derivation used the condition  $\ln H \ge 1$ .

## 4. Equation (3.7) versus other approximations and numerical calculations

A number of analytic adiabatic approximations are available in the literature for the hydrogen atom energy levels as a function of the magnetic field. Below, they are listed and compared with Eqn (3.7) and with numerical results.

#### 4.1 1s ground state

(see Section 8 below).

In the book by Landau and Lifshitz [6] (Section 112, problem 3), the value of the ground state binding energy<sup>12</sup> is given as

$$\varepsilon_0 = \frac{m_e e^4}{2\hbar^2} \ln^2 \frac{\hbar^3 B}{m_e^2 e^3 c} = \lambda_0^2 \operatorname{Ry}, \text{ or } \lambda_0 = \ln \mathcal{H}.$$
(4.1)

This is only an order-of-magnitude estimate, however. For  $B = B_{\rm cr}$  (i.e., for  $\mathcal{H} = \alpha^{-2}$ ), Eqn (4.1) yields  $\lambda_0 = 2 \ln (1/\alpha) = 9.840$  and  $\varepsilon_0 \approx 1320$  eV, to be compared with the numerically obtained values  $\lambda_0 = 5.735$  and  $\varepsilon_0 = 448$  eV [13].

Equation (4.1) is a simplification of a formula from Loudon's paper [17] (see also Refs [18, 19]), according to which

 $\ln \mathcal{H} = \lambda + 2\ln(2\lambda). \tag{4.2}$ 

This result was improved in Ref. [20] to give

$$\ln \mathcal{H} = \lambda + 2\ln(2\lambda) + \gamma - \ln 2, \qquad (4.3)$$

where the term  $\gamma - \ln 2 = -0.116$  appears in the right-hand side.

In [21], the equation (see Eqn (6) in Ref. [21] with  $v = 1/\lambda$ )

$$\ln \mathcal{H} = \lambda + 2\psi(1 - \lambda^{-1}) \tag{4.4}$$

is given, which for a deep level,  $\lambda \ge 1$ , becomes

$$\ln \mathcal{H} = \lambda - 2\gamma \tag{4.5}$$

and in fact differs from Eqn (4.1) only by a shift. Finally, from Eqn (3.7) in Section 3, we have [22–24]

$$\ln \mathcal{H} = \lambda + 2 \left[ \ln \lambda + \psi \left( 1 - \frac{1}{\lambda} \right) \right] + 3\gamma + \ln 2.$$
 (4.6)

We now compare these approximations with the exact results (to within 10 digits in Ref. [13]) of numerical work [13– 16]. Figure 2 plots  $\lambda_0(\mathcal{H})$  for values of  $\mathcal{H}$  between ~ 10 and 10<sup>6</sup>. It is seen that Loudon's formula (4.2), although only moderately accurate, does provide a qualitative description of  $\lambda_0(\mathcal{H})$  for values of  $\mathcal{H}$  from ~ 10<sup>3</sup> to 10<sup>6</sup>. On the other hand, expressions (4.1) and (4.5) have no range of applicability.

<sup>12 &#</sup>x27;Logarithmically accurate', as noted in Ref. [6].



**Figure 2.** Plot of the momentum  $\lambda(\mathcal{H})$ : curve *1*, numerical computations [13, 15, 16]; dashed curve 2, calculations by Eqn (4.6); curve 3, asymptotic formula (4.7). Corresponding to (4.1), (4.5), and (4.2) are curve 4 and dashed curves *K* and *L*, respectively. Here and in all subsequent figures,  $\lg \mathcal{H} \equiv \log_{10} \mathcal{H}$ .

The highest accuracy is achieved with Eqn (4.6): for  $B \gtrsim 5 \times 10^{11}$  G, curves *l* and *2* are already indistinguishable within the accuracy of Fig. 2, and the larger  $\mathcal{H}$  is, the more accurate this equation becomes. As regards asymptotic formula (4.3), it approaches an exact solution, albeit fairly slowly. We stress that except for Eqns (4.4) and (4.6), the equations presented above relate only to the ground 1s state and do not apply to the excited hydrogen-atom states.

We note the properties of the formal asymptotic expression (4.6),

$$\tilde{\lambda}_0 \approx \ln \frac{\mathcal{H}}{\ln^2 \mathcal{H}}, \quad \tilde{\varepsilon}_0 = \frac{\tilde{\lambda}_0^2}{2} \quad \text{as} \quad \mathcal{H} \to \infty,$$
(4.7)

which is functionally different from Eqn (4.1) and for which

$$\frac{\tilde{\varepsilon}_0}{\varepsilon_0} = 1 - 4 \, \frac{\ln \ln \mathcal{H}}{\ln \mathcal{H}} + O\left(\frac{1}{\ln \mathcal{H}}\right), \quad \varepsilon_0 - \tilde{\varepsilon}_0 \approx 4 \ln \ln \mathcal{H}. \quad (4.8)$$

Formally,  $\tilde{\epsilon}_0/\epsilon_0 \rightarrow 1$  as  $\mathcal{H} \rightarrow \infty$ , but this limit is attained very slowly and occurs in physically unrealistic fields (for example, the difference between  $\epsilon_0$  and  $\tilde{\epsilon}_0$  is of the order of 30% even for  $B = 10^8 B_{\rm cr} \sim 10^{22}$  G).

The values of  $\lambda_0(\mathcal{H})$  for the ground state,  $n_\rho = m = 0$ , were recently obtained in [35] by numerically solving the Schrödinger and Dirac equations. Their comparison with asymptotic formula (4.6) (see Table 1) shows that this formula also remains highly accurate in the region of ultrastrong fields  $B \ge B_{\rm cr}$ , up to  $B \sim m_e^2/e^5 \sim 10^{18}$  G. For  $B = 10^5 B_{\rm cr} \approx 4 \times 10^{18}$  G, the error it introduces is  $\delta \approx 0.2\%$ , and even in fields  $B \le 10^{10} B_{\rm cr}$  G, we have  $\delta \approx 0.5\%$ . We note that in fields  $B \le 10^{15}$  G, the Coulomb potential screening [35] can be neglected, but for  $B > 10^{17}$  G, it becomes important and leads to the 'freezing' of the ground-state energy at  $\lambda_{\infty} = 11.3$ , or  $E_0(\infty) = -1.71$  keV. This determines the range of applicability of Eqn (4.6).

From the perspective of neutron star astrophysics, of particular interest is not  $\lambda(\mathcal{H})$  but its inverse  $\mathcal{H}(\lambda)$ , which

**Table 1.** Dimensionless momentum  $\lambda_0(B)$  for the ground state of the hydrogen atom [35].

$B/B_{\rm cr}$	KP	S	D	$\bar{D}$
$10^{-2}$	3.108	3.032	3.032	_
$10^{-1}$	4.289	4.272	4.272	_
$10^{0}$	5.737	5.735	5.734	5.7
101	7.374	7.374	7.371	7.4
10 <sup>2</sup>	9.141	9.141	9.135	9.1
10 <sup>3</sup>	11.00	11.00	10.99	10.6
104	12.93	12.93	12.91	11.2
105	14.91	14.91	14.88	11.3
106	16.93	16.93	16.89	11.3
10 <sup>8</sup>	21.06	21.05	20.98	11.3

*Note.* The values of  $\lambda_0$  obtained neglecting screening [35]: *KP*, according to Eqn (4.6); *S*, from Schrödinger equation (8.9); *D*, numerical calculation using the Dirac equation. The results in column *D* are obtained from the Dirac equation with the Coulomb potential  $B/B_{\rm cr} = \alpha^2 \mathcal{H}$  screened in accordance with Eqn (8.1).

**Table 2.** Magnetic field calculation from the binding energy of the ground LLL state (1s).

•2				
$\lambda_{\tilde{0}}$	a	b	с	$ E_0 , eV$
3.4956	10	3.50	6.49	47.5
7.5796	100	79.6	15.7	103
11.875	425.5*	388	36.6	161
15.325	1000	962	50.1	208
28.282	1.0(4)	0.9991 (4)	204	385
32.92	1.878 (4)**	1.880(4)	310	448
47.783	1.0(5)	1.003(5)	1.005(3)	650
65.84	5.0(5)***	5.017(5)	3.334(3)	896
74.84	1.0(6)	1.0014(6)	5.69(3)	1017

*Note.* a, numerical results [13–15]; b, calculations with Eqn (4.6); c, Eqn (4.1). The notation used is as follows:  $x(y) \equiv x \times 10^{y}$ .

\*  $B = 10^{12}$  G.

\*\* 
$$B = B_{\rm cr} = 4.414 \times 10^{13} \, {\rm G}.$$

\*\*\*  $B = 2.35 \times 10^{15}$  G.

determines the magnetic field on the surface of a star from the experimental level shift (i.e.,  $\lambda_0^2/2$ ). Referring to the data in Table 2, we see that for fields  $B \gtrsim 10^{13}$  G, the error that Eqn (4.6) introduces into  $\mathcal{H}$  does not exceed 0.3%, and even for  $B \sim 10^{12}$  G, a typical value for a neutron star, the error is of the order of 10%, an apparently acceptable value for many astrophysical applications. On the other hand, Eqns (4.1) and (4.5) lead to large errors (of 2–3 orders of magnitude, growing larger as the field increases), and cannot therefore compete with Eqn (4.6).

The preceding equations assumed that  $n_{\rho} = m = 0$  (the nodeless ground state). Among the LLL states also are states with  $n_{\rho} = 0$  and m = -|m|. The binding energies of these are given by Eqn (3.7) with the constant  $A_{0|m|}$  defined in Eqn (3.9). Table 3 compares the results of numerical calculations [13–16] with the solution of Eqn (3.7) for states with quantum numbers  $n_{\rho} = 0$  and m = 0, -1, -2, -4, and -5. As |m| increases, the binding energy of these states decreases, and for a quite understandable reason: because the centrifugal energy  $\hbar^2 m^2 / 2m_e \rho^2$  increases, the electron moves away from the nucleus in the direction perpendicular to **B**, leading to a decrease in the effective potential  $U_{\text{eff}}(z)$  (for a given  $\mathcal{H}$ ).

	m = 0		<i>m</i> =	-1	m = -2	
Ĥ	а	b	а	b	а	b
10 <sup>2</sup>	7.580	8.082	5.270	6.132	4.376	5.374
$10^{3}$	15.32	15.49	11.28	11.70	9.610	10.15
$10^{4}$	28.28	28.29	21.83	21.97	19.05	19.26
$10^{5}$	47.78	47.76	38.41	38.42	34.22	34.26
$\infty$	—	125.7	—	108	—	99.7
	m = -3 m			= -4	<i>m</i> =	-5
н	а	b	а	b	а	b
10 <sup>2</sup>	3.860	4.938	3.510	4.643	3.251	4.425
$10^{3}$	8.617	9.230	7.929	8.598	7.412	8.125
$10^{4}$	17.35	17.61	16.16	16.45	15.25	15.57
1						
105	31.60	31.68	29.74	29.84	28.32	28.42

**Table 3.** Binding energies for the lower LLL states,  $n_{\rho} = 0$  and m = -|m|.

*Note.* Presented are the values of  $\lambda_m^2$ : a, numerical calculations [13, 16]; b, from Eqn (4.6), the limit at  $\mathcal{H} = \infty$  is obtained from Eqn (8.8).

#### 4.2 Excited levels

As noted above, Eqn (3.7) describes the entire discrete spectrum of even states. Importantly, the excited levels, both even and odd, lie in the energy range of the unperturbed hydrogen atom  $\lambda = \lambda_n(\mathcal{H}) < 1$ , with  $\lambda_n \to 1/n$  as  $\mathcal{H} \to \infty$ [here, n = 1, 2, ... are the poles of the function  $\psi(1 - 1/\lambda) =$  $\psi(1-n)$ ]. A remark is in order here. In the absence of a magnetic field, Coulomb states are characterized by the quantum numbers n, l, m. In a magnetic field, l is no longer a 'good' quantum number, except for its parity  $(-1)^{l}$ . Still, for the purpose of classifying states, it makes sense [13] even in this case to use (nl) instead of the notation  $n^{\pm}$  suitable for the adiabatic approximation. Thus, levels are labeled by l in the order of their energy positions as the magnetic field is varied. Without delving into the details (for which the reader is referred to Refs [22-25]), we note the correspondence between different notations for the lower levels:  $1s \rightarrow 0^+$ (LLL),  $2s \rightarrow 1^+$ , 3d (not  $3s(!)) \rightarrow 2^+$ ,  $2p \rightarrow 1^-$ , etc. (see, in particular, Fig. 8 in Ref. [24]).

A comparison with numerical calculations for 2s and 3d states shows (see Fig. 3 and the details in Table 4) that the binding energies of these levels  $\lambda_n^2/2$  for  $\mathcal{H} \gtrsim 100$  are determined by Eqn (3.7) to a few percent or even higher accuracy. From Fig. 3, we also see a marked difference in the behavior of  $\lambda_{nl}(\mathcal{H})$  and  $\varepsilon_{nl}(\mathcal{H})$  for even and odd levels (see also Section 5).

Table 4. $\lambda_i^2$	for excite	d levels.
------------------------	------------	-----------



**Figure 3.** Dimensionless momentum  $\lambda_{nl}(\mathcal{H})$  for excited s, p, and d levels: solid curves, numerical calculations [13, 16]; dashed-dotted curves, from Eqns (3.7) and (5.6) for even and odd levels. Dashed curves indicate the limit  $(\mathcal{H} \to \infty)$  values of  $\lambda_{nl}$ .

A number of asymptotic formulas for the energies of even excited levels of the hydrogen atom are available in the literature. According to Eqn (3.24) in Ref. [19],

$$\ln \mathcal{H} = 2 \left[ \ln \left( 2\lambda \right) + \psi (1 - \lambda^{-1}) \right], \quad 0 < \lambda < 1; \quad (4.9)$$

for  $\mathcal{H} \leq 10^3$ , this formula has only a qualitative value. The recently obtained asymptotic formula [25, 34]

$$\lambda_n = \frac{1}{n} - \frac{2/n^2}{\ln \mathcal{H} - \ln 2 - \gamma + 1/n + 2\left[\ln n - \sum_{k=1}^{n-1} (1/k)\right]},$$
  
$$n = 2, 3, \dots \qquad (4.10)$$

is valid for  $\ln H \ge 1$ . For the first excited states, the sum over *k* in Eqn (4.10) should be omitted, giving

$$\lambda_1 = 1 - \frac{2}{\ln \mathcal{H} - 0.2704} + O\left(\frac{1}{(\ln \mathcal{H})^2}\right), \quad n = 1.$$
 (4.11)

While not yet sufficiently accurate in the range  $\mathcal{H} = 10^2 - 10^4$  typical of neutron stars, these asymptotic formulas are valid for ultrastrong fields  $\mathcal{H} > 10^5$ , or  $B \gtrsim 10^{14}$  G.

	2s		3d		2p		3р	
Ĥ	а	b	а	b	а	b	а	b
0	0.2500	_	0.1111	_	0.2500	_	0.111	_
10	0.4179	0.4429	0.1543	0.1594	0.7653	0.6708	0.2197	0.2122
10 <sup>2</sup>	0.5124	0.5176	0.1738	0.1747	0.9272	0.9162	0.2419	0.2400
10 <sup>3</sup>	0.5917	0.5921	0.18870	0.18876	0.9850	0.9843	0.2482	0.2481
10 <sup>4</sup>	0.6554	0.6552	0.19986	0.19983	0.9976	0.9976	0.2497	0.2497
10 <sup>5</sup>	0.7054	0.7053	0.20814	0.20812	0.9996	0.9997	0.2499	0.2500
10 <sup>6</sup>	0.7444	0.7443	_	0.21433	0.9999	1.0000	0.2500	0.2500
Note a numar	ical calculations	[12 14 16] b o	loulations using	Eans (16) and	(5.6) for over an	d add states res	nostivaly Ac rag	ards the mutual

*Note.* a, numerical calculations [13, 14, 16]; b, calculations using Eqns (4.6) and (5.6) for even and odd states, respectively. As regards the mutual position of the 3s and 3d levels, see Fig. 8 in Ref. [24].

We close this section by giving the characteristic values of the Landau radius  $a_H = \sqrt{c\hbar/eB}$ :  $4 \times 10^{-11}$  cm and  $1 \times 10^{-12}$  cm for respective fields  $B_{cr} \approx 4 \times 10^{13}$  G and  $B = 6 \times 10^{16}$  G. The large values of  $a_H$  compared to the proton radius  $r_p \approx 10^{-13}$  cm justify approximating the nucleus by a point-like charge. However, as we show in Section 8, in fields  $B \gtrsim \alpha^{-1}B_{cr} \sim 5 \times 10^{15}$  G, the Coulomb potential of a point-like charge is distorted (screened) at small distances due to the vacuum polarization by the magnetic field, a phenomenon that considerably changes the dependence of the spectrum of even levels on the magnetic field *B*, as opposed to the case of a point-like charge.

#### 5. Odd levels

As noted at the end of Section 2, the spectrum of odd levels (2p, 3p, 4f, etc., according to the classification in Ref. 13]) is described by Eqn (2.15), in which  $\psi_{ns}(r)$  are the unperturbed Coulomb potential wave functions [6]

$$\psi_{ns}(r) = \frac{1}{\sqrt{\pi n^3}} \exp\left(-\frac{r}{n}\right) F\left(1-n, 2; \frac{2r}{n}\right)$$
$$= \frac{1}{\sqrt{\pi n^3}} \left(1-r + \frac{2n^2+1}{6n^2}r^2 + \dots\right), \tag{5.1}$$

where F(...) is the degenerate hypergeometric function. We note a slight upward shift with respect to the unperturbed Coulomb levels  $E_n^{(0)} = -(1/n^2)$  Ry. Figure 3 illustrates the qualitative difference between the spectra of even and odd levels in a strong magnetic field  $\mathcal{H} \ge 1$ .

Equation (2.15) implies the following asymptotic expansion for the binding energy [24]:

$$\left(\lambda_{nmn_{\rho}}^{-}\right)^{2} = \frac{1}{n^{2}} - \frac{4}{n^{3}} \left\{ a_{nm_{\rho}} \frac{\ln \mathcal{H}}{\mathcal{H}} + \frac{a_{nmn_{\rho}}^{(1)}}{\mathcal{H}} + \frac{a_{nm_{\rho}}^{(3/2)}}{\mathcal{H}^{3/2}} + \dots \right\}, \quad (5.2)$$
$$\mathcal{H} \to \infty.$$

The leading logarithmic term in the expression for the level shift is determined by the integration region  $a_H \leq r \leq a_B$  in Eqn (5.1). A logarithmically accurate expression for the shift of an odd level can be obtained by setting

$$U_{\rm eff}(r) \approx -\frac{e^2}{r} + \frac{e^2 \rho_{mn_{\rho}}^2}{2r^3}, \quad \psi_{ns}^2(r) \approx \psi_{ns}^2(0) = \frac{1}{\pi n^3 a_{\rm B}^3}$$
(5.3)

to give

$$\delta E_{nmn_{\rho}}^{-} \approx \int \left( U_{\text{eff}} + \frac{e^2}{r} \right) \psi_{ns}^2(r) \, \mathrm{d}^3 r \approx \frac{2e^2 \,\overline{\rho_{nm_{\rho}}^2}}{a_{\text{B}}^3 n^3} \int_{a_H}^{a_{\text{B}}} \frac{\mathrm{d}r}{r}$$
$$= \frac{e^2 \,\overline{\rho_{nm_{\rho}}^2}}{a_{\text{B}}^3 n^3} \left( \ln \mathcal{H} + O(1) \right). \tag{5.4}$$

Because

$$\overline{\rho_{mn_{\rho}}^{2}} = \iint \rho^{2} \left| R_{n_{\rho}m}(\mathbf{p}) \right|^{2} \mathrm{d}^{2}\rho = 2 \big( 2n_{\rho} + |m| + 1 \big) a_{H}^{2} \,,$$

the (*n*-independent) coefficient  $a_{mn_o}$  is found to be

$$a_{mn_{\rho}} = 2n_{\rho} + |m| + 1.$$
(5.5)

Determining the subsequent expansion coefficients involves more tedious calculations (for which the reader is

**Table 5.** Rydberg corrections  $\Delta_{nl}^{\pm}$  for even and odd levels.

	$arDelta_{nl}^{+}(\mathcal{H})$				$\varDelta_{nl}^{-}(\mathcal{H})$				
Ĥ	1s	2s	3s	3d	2p	3p	4f		
1	0.7756	0.7652	0.7407	0.7476	0.3867	0.3541	0.3435		
10	0.5349	0.5469	0.5452	0.5456	0.1431	0.1335	0.1312		
$10^{2}$	0.3632	0.3970	0.3994	0.3989	0.0385	0.0368	0.036		
$10^{3}$	0.255	0.300	0.303	0.302	0.0076	0.0074	—		
$10^{4}$	0.188	0.235	0.237	0.237	0.0012	0.0012	—		
$10^{5}$	0.145	0.191	0.192	0.192	0.0002	0.0002	0		
$10^{6}$	0.116	0.159	0.160	0.160	0	0	0		
<i>Note</i> . the to	<i>Note.</i> The values $\Delta_{nl}^{\pm}$ [13, 14, 16] refer to the ground Landau level 1s and the tower of its adjacent levels: 2s, 3s, etc.								

referred to Ref. [25]). In the simplest case  $m = n_{\rho} = 0$ , we have

$$\left(\lambda_{n00}^{-}\right)^{2} = \frac{1}{n^{2}} - \frac{4}{n^{3}} \left[ \frac{\ln \mathcal{H}}{\mathcal{H}} - \frac{k_{n}}{\mathcal{H}} + \frac{4\sqrt{2\pi}}{\mathcal{H}^{3/2}} + \dots \right], \qquad (5.6)$$

where n = 1, 2, 3, ... is the quantum number for the longitudinal electron motion, and the values of  $k_n$  are

$$k_1 = 2 + \ln 2 + \gamma = 3.270, \quad k_2 = 3.384,$$

$$k_3 = 3.407, \quad k_4 = 3.417, \dots.$$
(5.7)

The point to note here is the marked energy difference between even and odd levels,  $E_n^{\pm} = -(\lambda_n^{\pm})^2/2$ , in an ultrastrong magnetic field (see Table 4), which is easily seen by changing from the energies  $E_{nl}^{\pm}$  to the Rydberg corrections  $\Delta_{nl}^{\pm}$ used in atomic physics (see Table 5 and its more detailed discussion in Section 9).

For the even levels 2s, 3s, and 3d for  $\mathcal{H} \geq 100$ , the numerically obtained values of  $\Delta_{nl}^+(\mathcal{H})$  [13, 14, 16] and those calculated from Eqn (3.7) are virtually identical and slowly decrease with  $\mathcal{H}: \Delta_{nl}^+ \propto 1/\ln \mathcal{H}$ , such that even at  $\mathcal{H} = 10^5$  (i.e., in fields  $B \geq 10^{14}$  G), they are still far enough from their  $\mathcal{H} \to \infty$  limit, and therefore the spectrum of even levels is strongly perturbed compared to the Coulomb spectrum  $E_n^{(0)} = -1/2n^2$ . At the same time, for the odd levels 2p, 3p, 4f, etc., the Rydberg corrections  $\Delta_{nl}^-(\mathcal{H})$  are very small numerically for  $\mathcal{H} \geq 100$ ,  $\Delta_{nl}^- \propto (\ln \mathcal{H})/\mathcal{H} \ll \Delta_n^+$  and  $E_n^- \approx E_{n-1}^{(0)}$ , which suggests a revision of the claim in [6, 19] about the double degeneration of the even and odd levels of the hydrogen atom in a strong magnetic field.

Asymptotic formula (3.7) provides a higher accuracy for excited even states than for the ground state. The physical explanation is simple: the localization region of the wave function in the longitudinal direction increases as  $n^2$ , thus decreasing the Coulomb interaction and extending the adiabatic approximation to lower magnetic fields.

#### 6. Size and the quadrupole moment of the atom and radiative transition probabilities for $B \gg B_a$

As noted in Sections 2 and 3, the wave function  $\chi(z)$  that enters Eqn (2.4) and describes the longitudinal motion of an electron in its main localization region  $|z| \gtrsim a_H$  is given by Whittaker function (3.5). Because distances  $|z| \leq a_H$  contribute little to the normalization integral, the spatial characteristics of the electron cloud over the entire *z* axis can be calculated from Eqn (3.5). The normalization integral for

		$a_\parallel/a_\perp$			
Ĥ	$a_{\perp}/a_B$	1s	2s		
10 <sup>2</sup>	0.100	3.52	32.5		
10 <sup>3</sup>	0.032	7.12	89.6		
104	0.010	15.7	257		
10 <sup>5</sup>	3.16(-3)	36.7	759		

 Table 6. Characteristic dimensions of the hydrogen atom in a strong magnetic field.

 $\psi_{\nu}(z)$  and the average radius of this state can be calculated using the results obtained previously [39, 40] in the theory of a pp atom. Eventually, for a hydrogen atom in a strong magnetic field, we find that

$$\frac{a_{\parallel}}{a_{\rm B}} = \frac{\lambda}{\sqrt{2}(\lambda - 1/2)(\lambda - 1/3)} \times \left\{ \frac{{}_{3}F_{2}(-\lambda^{-1}, -\lambda^{-1}, 3; 4 - \lambda^{-1}, 4 - \lambda^{-1}; 1)}{{}_{3}F_{2}(-\lambda^{-1}, -\lambda^{-1}, 1; 2 - \lambda^{-1}, 2 - \lambda^{-1}; 1)} \right\}^{1/2}, \quad (6.1)$$

where  ${}_{3}F_{2}$  is the generalized hypergeometric series [41]. For a deep  $(v \to 0)$  level,  $\psi_{0}(z) \approx \sqrt{\lambda} \exp(-\lambda |z|)$  and

$$\frac{a_{\parallel}}{a_{\rm B}} = (2\lambda)^{-1/2} \approx \frac{1}{\sqrt{2}} \left( \ln \frac{\mathcal{H}}{\ln^2 \mathcal{H}} \right)^{-1},\tag{6.2}$$

where we have used asymptotic formulas (4.7). As  $\mathcal{H}$  increases, the longitudinal size of the atom  $a_{\parallel}$  decreases much more slowly than  $a_{\perp}$ :

$$\frac{a_{\parallel}}{a_{\perp}} = \frac{\sqrt{\mathcal{H}}}{2\lambda} \sim \frac{\sqrt{\mathcal{H}}}{\ln \mathcal{H}} , \qquad \mathcal{H} \gg 1$$
(6.3)

[such that the atom becomes spoke or needle shaped (see Table 6)].

There is a clear physical explanation for the abovementioned longitudinal contraction  $a_{\parallel} \propto 1/\ln \mathcal{H}$ . As noted in Section 2, the 1s state under consideration is a state with a small binding energy,  $\lambda^2 \sim (\ln \mathcal{H})^2$  (shallow level), in a potential well of the radius  $r_0 \sim a_H = 1/\sqrt{\mathcal{H}} \ll 1/\lambda$ , which describes a distortion of the Coulomb potential. In this case, the exponential decay length of the wave function  $\psi \propto \exp(-\lambda |x|)$  outside the well,  $|x| \sim 1/\lambda \gg r_0$ , is much larger than the well width. Therefore, we here have a manifestation of the 'anticlassical' quantum mechanical property of the bound states of a weakly bound particle in a short-range potential: even though the particle is bound in the well, the probability of finding it within the well is small,  $w \sim \lambda r_0 \ll 1$  (i.e., most of the time it is outside the well). Systems with this property are the deuteron<sup>13</sup> and the negative hydrogen ion H<sup>-</sup>. In the latter case, the outer electron has the binding energy  $\varepsilon = 0.75$  eV (the electron moves in the electrostatic field of a proton completely screened at distances  $r \gtrsim a_{\rm B}$  by the inner atomic electron).

The deformation of the electron cloud  $|\psi(\mathbf{p}, z)|^2$  by a magnetic field results in the atom acquiring a quadrupole moment. In a strongly magnetized plasma, in addition to the

usual van der Waals interaction, a quadrupole–quadrupole interaction arises between the atoms. The quadrupole moment Q of the ground state of the hydrogen atom was found numerically [26] by the variational method. Below, we give a simple asymptotic formula for Q, valid for  $\mathcal{H} \gtrsim 100$ .

Due to the axial symmetry, only the diagonal components of the quadrupole moment tensor are nonzero:

$$Q_{xx} = Q_{yy} = \frac{1}{2} Q, \quad Q_{zz} = -Q, \quad Q = 2\langle z^2 \rangle - \langle \rho^2 \rangle \quad (6.4)$$

(the z axis is taken to be along the magnetic field;  $Q_{zz} < 0$  because the electron charge is negative). Hence,

$$Q(\mathcal{H}) = 2\left\{a_{\parallel}^{2}(\mathcal{H}) - \mathcal{H}^{-1}\right\}, \quad m = n_{\rho} = 0, \qquad (6.5)$$

or asymptotically,

$$Q = \Lambda^{-2} + \left[2(\gamma + \ln 2) + \frac{5}{3}\right]\Lambda^{-3} + \dots,$$

$$\Lambda = \ln \frac{\mathcal{H}}{\ln^2 \mathcal{H}} \ge 1$$
(6.6)

(we set  $a_{\rm B} = 1$  here). As can be seen from Fig. 4, analytic formulas (6.1)–(6.4) yield virtually the same results as given by numerical calculations [26] if  $\mathcal{H} \gtrsim 10$ . As regards expansion (6.6), we note that for  $\mathcal{H} < 1000$ , it gives only a qualitative behavior of  $Q(\mathcal{H})$ , but its accuracy increases with increasing  $\mathcal{H}$ . The decrease in the quadrupole moment with increasing  $\mathcal{H} \gg 1$  is explained by the fact that an ultrastrong magnetic field squeezes the atom both transversely and longitudinally.

A brief discussion is appropriate of radiative transitions  $np \rightarrow 1s$  near the ground Landau band, where  $\mathcal{H} \ge 1$ ,  $a_H \ll a_{\parallel} \ll a_B$ . In this limit, the shift of the odd np level,

$$E_n^- = -\frac{1}{2n^2}$$
,  $\psi_n^-(z) = \sqrt{\frac{2}{n^3}} \exp\left(-\frac{z}{n}\right) F\left(1-n,2;\frac{2z}{n}\right)$ ,

can be neglected, the wave function of the deep ground state can be approximated by  $\chi_{\lambda}(z) \approx \sqrt{\lambda} \exp(-\lambda |z|)$  (which



Figure 4. Quadrupole moment Q for the ground state of the hydrogen atom. The solid line is obtained from Eqns (6.4) and (6.1);  $\circ$  and +: numerical results [22].

<sup>&</sup>lt;sup>13</sup> The deuteron, the nucleus of the heavy hydrogen atom, is a protonneutron system weakly bound by the nuclear interaction. Its binding energy is  $\varepsilon_d = 2.23$  MeV, the interaction potential is typically  $U_0 \approx 50$  MeV, and the radius is  $R \sim 2 \times 10^{-13}$  cm.

corresponds to a one-dimensional  $\delta$  potential), and the transition probability is estimated to be

$$w(n\mathbf{p}\to \mathbf{1s}) = \frac{16\alpha^3}{9n^3\lambda} \left[ 1 - \frac{2}{3\lambda} + O(\lambda^{-2}) \right] \omega_0 , \qquad (6.7)$$

where  $\omega_0 = m_e e^4 / \hbar^3 = 4.13 \times 10^{16} \text{ s}^{-1}$  and  $\lambda = \lambda(\mathcal{H})$  is determined from Eqn (3.7). For  $B \sim 10^{13}$  G, we have  $|E_{1s}| \approx 300 \text{ eV}, \lambda \approx 5, \text{ and } w(2p \rightarrow 1s) \sim 10^{12} \text{ s}^{-1}, \text{ which is a}$ few orders of magnitude larger than the probability of a similar transition in a free  $(\mathcal{H} = 0)$  hydrogen atom,  $w_0 = 6.27 \times 10^8 \text{ s}^{-1}$ .

In the dipole approximation, the angular distribution of escaping photons has the form

$$w(\theta) \propto [\mathbf{n} \times \mathbf{d}_{fi}]^2 \sim \sin^2 \theta$$
, (6.8)

where  $\theta$  is the angle between the magnetic field and the phonon momentum k (notably, photons are not emitted along the magnetic field **B**). It can be shown (as noted by M I Vysotsky) that these photons are 100 percent linearly polarized in the (**k**, **B**) plane.

Detailed computational research on radiative transition probabilities can be found in [16] and the studies cited therein.

#### 7. Relativistic effects<sup>14</sup>

As noted in Section 3, in fields  $B \gtrsim 10^{13}$  G, the transverse motion of an electron becomes relativistic, requiring the use of the Dirac equation.

We first note the following property of bound states in this case. There is a wide range of high magnetic field strengths,  $B \gtrsim B_{\rm cr} \sim 10^{14}$  G, in which the adiabatic approximation is valid as before, and the variables of the transverse and longitudinal motions separate. The transverse motion of an electron here is relativistic (or even ultrarelativistic), whereas its longitudinal motion remains classical and can be treated in the framework presented in Sections 2 and 3. An example to illustrate this point is the bound LLL states of an electron that arise from the transverse ground Landau state of the Coulomb potential.

The Dirac Hamiltonian for an electron in the problem under discussion has the form [30]

$$\hat{H}_{\rm D} = \alpha \left( \hat{\mathbf{p}} + \frac{e}{2} \left[ \mathbf{B} \times \mathbf{r} \right] \right) + \beta m_{\rm e} + U(r) \,, \tag{7.1}$$

where  $U(r) = -e\varphi(r)$  is a central electrostatic potential. From the Dirac equation  $\hat{H}_{\rm D}\Psi_{\varepsilon} = \varepsilon \Psi_{\varepsilon}$  for the bispinor

$$\Psi_{\varepsilon} = \begin{pmatrix} \varphi_{\rm D}(\mathbf{r}) \\ \chi_{\rm D}(\mathbf{r}) \end{pmatrix}$$

we find the following equations for two-component spinors:

$$\boldsymbol{\sigma}\left(\hat{\mathbf{p}} + \frac{e}{2}\left[\mathbf{B} \times \boldsymbol{\rho}\right]\right) \chi_{\mathrm{D}} = \left(\varepsilon - m_{\mathrm{e}} - U(r)\right) \varphi_{\mathrm{D}}, \qquad (7.2')$$

$$\boldsymbol{\sigma}\left(\hat{\mathbf{p}} + \frac{e}{2}\left[\mathbf{B} \times \mathbf{\rho}\right]\right) \varphi_{\mathrm{D}} = \left(\varepsilon + m_{\mathrm{e}} - U(r)\right) \chi_{\mathrm{D}}, \qquad (7.2'')$$

<sup>14</sup> In Sections 7 and 8, we use relativistic units for which  $\hbar = 1, c = 1$ , and  $e^2 = \alpha = 1/137.$ 

where  $\varepsilon$  is the energy eigenvalue of the Dirac equation ( $\varepsilon$ includes the rest-mass energy  $m_{\rm e}$ ).

The characteristic values of the magnetic field are

$$B_{\rm a} = m_{\rm e}^2 e^3$$
,  $B_{\rm cr} = \frac{m_{\rm e}^2}{e}$ ,  $B_{\infty} = 3\pi\alpha^{-1}B_{\rm cr} = 5.70 \times 10^{16} \,{\rm G}$ .  
(7.3)

With the potential U(r) neglected, Eqn (7.2) yields

$$\frac{1}{2m_{\rm e}} \left( \hat{\boldsymbol{\sigma}} \left( \hat{\mathbf{p}} + \frac{e}{2} \left[ \mathbf{B} \times \boldsymbol{\rho} \right] \right) \right)^2 \varphi_{\rm D} = \frac{1}{2m_{\rm e}} [\varepsilon^2 - m_{\rm e}^2] \varphi_{\rm D} \quad (7.4)$$

and a similar equation for the spinor  $\chi_{\rm D}$  [equations of a similar form but for different spinors that are related by Eqns (7.2)]. Equation (7.4) is identical to the Pauli equation for a nonrelativistic electron, with the nonrelativistic energy E replaced by  $(\varepsilon^2 - m_e^2)/2m_e$ . This immediately yields the wellknown spectrum of a relativistic electron in a uniform magnetic field.

$$\varepsilon_{n_{\rho}m\sigma_{z}p_{z}} = \sqrt{m_{e}^{2} + 2m_{e}\left(\hbar\omega_{H}N + \frac{1}{2m_{e}}p_{z}^{2}\right)}$$
$$= \sqrt{m_{e}^{2}\left(1 + \frac{2NB}{B_{cr}}\right) + p_{z}^{2}}$$
(7.5)

[see Eqns (2.5) and (2.6) with  $U \equiv 0$  and with the kinetic energy  $p_z^2/2m_e$  of free longitudinal motion], where the values of N are given by Eqn (2.5) and the transverse part of  $\varphi_{\rm D}$  is identical to the nonrelativistic spinor in Eqns (2.3) and (2.4).

According to Eqn (7.5), the lower bound of the spectrum  $(N = 0, p_z = 0)$  coincides with the electron rest-mass energy (here,  $n_{\rho} = 0$ , m = 0, -1, -2, ..., and  $\sigma_z = -1$  for LLL states). Using this fact and the first-order perturbation theory in the potential  $U(r) = -e^2/r$ , we obtain a formula for the spectrum of LLL states, which is identical to nonrelativistic expression (3.7). Namely, as seen from Eqn (7.2'), the spinor  $\chi_D$  is in this case small compared with  $\varphi_{\rm D}$  and, neglecting the potential U(r) in Eqn (7.2"), we obtain

$$\chi_{\rm D} \approx \frac{1}{2m_{\rm e}} \, \boldsymbol{\sigma} \left( \hat{\boldsymbol{p}} + \frac{e}{2} \left[ \mathbf{B} \times \boldsymbol{\rho} \right] \right) \varphi_{\rm D} \,, \tag{7.6}$$

which, when substituted in Eqn (7.2'), yields the Pauli equation with the energy  $E = \varepsilon - m_{\rm e}$  and the already known Eqn (3.7) for the spectrum of nonrelativistic LLL states,

$$\varepsilon_{n_{\rho}=0, m, \sigma_{z}=-1} = m_{e} - \frac{1}{2} m_{e} e^{4} \lambda^{2} = m_{e} \left(1 - \frac{1}{2} \alpha^{2} \lambda^{2}\right).$$
 (7.7)

It hence follows that the relativistic corrections to Eqn (3.7)are extremely small until the field B greatly exceeds  $B_{cr}$  (see Table 1). This is explained by the relation

$$\frac{\left|U_{\rm eff}(0)\right|}{m_{\rm e}c^2} = \alpha \sqrt{\frac{\pi}{2} \frac{B}{B_{\rm cr}}},\tag{7.8}$$

whence we see that for  $B < 5 \times 10^{17}$ , the depth of the potential well in  $U_{\rm eff}$  is smaller than  $m_{\rm e}c^2$ , and the transverse size of the hydrogen atom  $a_H$  decreases to the Compton length  $\lambda_{\rm C}$  for  $\mathcal{H} = \alpha^{-2}$  (or  $B = B_{\rm cr}$ ) and to the proton radius  $r_{\rm p} \approx 10^{-13}$  cm for  $\mathcal{H} \sim (a_{\rm B}/r_{\rm p})^2 \sim 10^{10}$ , i.e.,  $B \sim 10^{19}$  G. Thus, the fields  $B \leq 10^{18}$  G still allow neglecting the finite

size of the nucleus.

#### 8. Quantum electrodynamics effects

We now address the question of how ultrastrong magnetic fields  $B \gtrsim B_{\infty} = 5.7 \times 10^{16}$  G affect the positions of atomic levels.

An interesting recent discovery is that such fields can influence atomic spectra: due to the vacuum polarization by an ultrastrong magnetic field, the Coulomb potential of a point-like charge is distorted (screened), and hence the deepening of a level with increasing the field is slowed down, and as  $B \to \infty$ , the levels are 'frozen' at the finite limit value  $E_{\infty} = -\lambda_{\infty}^2/2$ .

This surprising effect was first observed by Shabad and Usov [42, 43] and subsequently investigated by Vysotsky, Machet, and Godunov [33–35, 44]. While qualitatively similar, the results of the two groups differ in some details and numerical estimates. For example, for the ground level of the hydrogen atom, Refs [42, 43] give  $E_{\infty} = -4.0$  keV, whereas Ref. [34] gives  $E_{\infty} = -1.7$  keV. The reasons for this discrepancy are explained in Section 4.2 in Ref. [34]. Our discussion below follows Refs [34, 35].

With screening taken into account, the effective Coulomb potential (2.7) for  $n_{\rho} = 0$  states takes the form [see Ref. [34], Eqn (52)]

$$\begin{split} \tilde{U}_{\rm eff}(z) &= U_{\rm eff} + \delta U_{\rm eff} \,, \end{split} \tag{8.1} \\ \delta U_{\rm eff}(z) &= e^2 \int_0^\infty \frac{1}{\sqrt{\rho^2 + z^2}} \, R_{0m}^2(\rho) \\ &\times \left[ \exp\left(-\mu |z|\right) - \exp\left(-\sqrt{1+b} \, \mu |z|\right) \right] \rho \, \mathrm{d}\rho \,, \end{split}$$

where  $\rho = \sqrt{x^2 + y^2}$ ,  $U_{\text{eff}}(z)$  is, as before, given by Eqn (2.7),

$$\mu = \sqrt{6} m_{\rm e}, \quad b = \frac{\alpha^3}{3\pi} \mathcal{H} = \frac{\mathcal{H}}{\mathcal{H}_{\infty}}, \quad \mathcal{H}_{\infty} = 2.423 \times 10^7 \quad (8.2)$$

(or  $b = B/B_{\infty}$ , where  $B_{\infty} = 3\pi\alpha^{-1}B_{cr} \sim 10^{17}$  G). According to Eqn (8.1),  $\delta U_{\rm eff}(z) > 0$ , and hence the inclusion of screening shifts the levels upward.

We note the following properties of the potential  $\delta U_{\text{eff}}(z)$ . Due to the difference in exponential terms in Eqn (8.1), the screening of the potential is most important at  $|z| \sim 1/\mu$  (at larger distances, the screening is exponentially small). On the other hand, the integral over  $\rho$  is dominated by  $\rho \sim a_H$ . Therefore, in the case  $1/\mu \ge a_H$ , i.e., for  $\mathcal{H} \ge \alpha^{-2} \sim 10^4$ ,  $\rho^2$ can be omitted under the square root in Eqn (8.1), giving<sup>15</sup>

$$\delta U_{\rm eff}(z) \approx \frac{e^2}{|z|} \left[ \exp\left(-\mu|z|\right) - \exp\left(-\sqrt{1+b}\,\mu|z|\right) \right]. \quad (8.3)$$

As a result of screening, the logarithmic derivative in Eqn (3.3) changes by an amount that, according to Eqn (3.1), is given by (in atomic units)

$$\delta\left(\frac{\chi'(z)}{\chi(z)}\right) \approx 2 \int_0^z \delta U_{\text{eff}}(z) \, \mathrm{d}z$$
  
$$\underset{(z \gtrsim L)}{=} 2 \int_0^\infty \frac{e^2}{z} \left[ \exp\left(-\mu z\right) - \exp\left(-\sqrt{1+b}\,\mu z\right) \right] \, \mathrm{d}z \quad (8.4)$$

<sup>15</sup> In fields  $\mathcal{H} \lesssim 10^4$ , this expression does not hold. However, the screening of the potential is important only in fields  $\mathcal{H} \gtrsim 10^7$ .

and which is independent of z in the region where the logarithmic derivatives are matched. Hence,

$$\delta\left(\frac{\chi'(z)}{\chi(z)}\right) = -2\ln\mu + 2\ln\left(\mu\sqrt{1+b}\right)$$
$$= \ln\left(1 + \frac{\alpha^3}{3\pi}\mathcal{H}\right), \quad z \sim L.$$
(8.5)

Therefore, mathematically, the screening effect amounts to adding this term to logarithmic derivative (3.3), i.e., to replacing

$$\mathcal{H} \to \frac{\mathcal{H}}{1 + (\alpha^3/3\pi)\mathcal{H}}$$
 (8.6)

in the logarithmic derivative. As a result, the equation for the spectrum of even levels takes the form [34, 35] [cf. Eqn (3.7)]

$$\ln\left(\frac{\mathcal{H}}{1+\mathcal{H}/\mathcal{H}_{\infty}}\right) = \lambda + 2\left[\ln\left(2\lambda\right) + \psi\left(1-\frac{1}{\lambda}\right)\right] + 4\gamma + A_{0m},$$
(8.7)

where  $\alpha = 1/137$ ,  $n_{\rho} = 0$ , and  $A_{0m} = -\ln 2 + \psi(1 + |m|)$ .

It follows from Eqn (8.7) that in fields  $\mathcal{H} \ll \mathcal{H}_{\infty}$ , the influence of vacuum polarization is negligibly small. With increasing the field, for  $\mathcal{H} \gtrsim \mathcal{H}_{\infty}$ , the polarization effect slows the deepening of the levels and 'freezes' them at fields  $\mathcal{H} \gg \mathcal{H}_{\infty}$  [35] (Fig. 5 for the ground level clearly illustrates the freezing effect for  $B \to \infty$ ).

The limit values  $\lambda = \lambda_{\infty}$  for  $\mathcal{H} \to \infty$  follow from Eqn (8.7) as

$$\lambda_{\infty} + 2\left[\ln\left(2\lambda_{\infty}\right) + \psi\left(1 - \frac{1}{\lambda_{\infty}}\right)\right] = \ln\mathcal{H}_{\infty} - 4\gamma - A_{0m}.$$
 (8.8)

Table 7 lists the  $\lambda_{\infty}$  values and the limit binding energy values  $\varepsilon_{\infty}$  for a number of deep (on the atomic scale) LLL levels and their adjacent excited levels from among those in the 'tower' [33–35] of states with n = 1, 2, 3, ... and with the magnetic quantum number  $m \leq 0$ .



**Figure 5.**  $\lambda_0(\mathcal{H})$  for the ground level in the region of ultrastrong magnetic fields without (1) and with (2) taking the vacuum polarization into account; the limit value is  $\lambda_{\infty} = 11.2$  ( $E_{\infty} = -1.71$  keV). Values marked on the horizontal axis are  $h_{\rm cr} = \log (B_{\rm cr}/B_{\rm a})$  and  $h_{\infty} = \log \mathcal{H}_{\infty}$ .

	LLL state $n_{\rho} = 0, m$	es, ≼ 0	Excited levels, $n_{\rho} = m = 0, n = 1, 2, \dots$					
т	$\lambda_\infty$	$\lambda_{\infty}$ $\varepsilon_{\infty}$ , keV		$\lambda_\infty$	$\varepsilon_{\infty}^{(n)},\mathrm{eV}$			
0	11.21	1.71	1	0.877	10.7			
-1	10.39	1.47	2	0.470	3.00			
-2	9.986	1.36	3	0.320	1.39			
-3	9.718	1.28	4	0.242	0.798			
-4	9.518	1.23	5	0.195	0.517			
-5	9.359	1.19	10	0.0985	0.132			

 Table 7. 'Freezing' of the atomic level with the vacuum polarization taken into account.

*Note.* Presented are the limit  $(B \to \infty)$  values of  $\lambda_{\infty}$  and the binding energies  $\varepsilon_{\infty}$  for hydrogen atom levels calculated including the screening of the potential; calculations for LLL states are made using Eqn (8.8) (neglecting relativistic corrections).

In [35], the Dirac equation averaged over the transverse motion of the electron was used to numerically calculate the spectrum. A trick known from the theory of the critical nuclear charge [45–50] was also used in [35] (with respect to  $Z_{\rm cr}$ , see Appendix A), which allowed reducing the Dirac equation for two-component spinors to a second-order Schrödinger-type equation:

$$\frac{d^2\chi}{dz^2} + 2m_e [E - V(z)]\chi = 0, \quad E = \frac{\varepsilon^2 - m_e^2}{2m_e} = -\frac{1}{2} m_e e^4 \lambda^2$$
(8.9)

In this equation, the potential V(z) is expressed in terms of  $\overline{U}$  (or  $\widetilde{U}$ ) in an explicit, although somewhat cumbersome form as [45–47]

$$V = \frac{\varepsilon}{m_{\rm e}} U - \frac{1}{2m_{\rm e}} U^2 + \frac{U_{zz}}{4m_{\rm e}(\varepsilon + m_{\rm e} - U)} + \frac{3}{8} \frac{U_z^2}{m_{\rm e}(\varepsilon + m_{\rm e} - U)^2},$$
(8.10)

where  $U_z = dU/dz$ , etc.

We compare the accuracy of asymptotic formulas (3.7) and (8.7) against the numerical calculations in [35] using, in particular, the solution of the Dirac equation including the screening of the Coulomb potential in a strong magnetic field (see Table 1). As is seen, already in fields  $B/B_{\rm cr} \approx 10^{-2}$  (for which  $a_H \approx 0.1 a_B$ ), the error in Eqn (3.7) is of the order of one percent and rapidly decreases as B increases to the values  $B \approx 10^2 B_{\rm cr} \approx 4 \times 10^{15}$  G, which exceed the highest magnetic field values known in astrophysics (see Appendix B). In this range of fields, the effects of the screening of the Coulomb potential on the positions of the atomic levels can be neglected. Bur the influence becomes stronger as the magnetic field increases further. In this case, instead of Eqn (3.7), we should use Eqn (8.7), which for  $B \gtrsim 10^5 B_{\rm cr}$ causes the atomic spectrum to freeze in accordance with the Dirac equation  $\overline{D}$  (see Table 1). We note that  $\lambda_0 = 11.2$ according to Eqn (8.7) and that the Dirac equation  $\overline{D}$  yields  $\lambda_0 = 11.3.$ 

Until this point, we have neglected the anomalous magnetic moment of the electron  $\Delta \mu$ , which arises due to radiative QED corrections [30]; however, for  $B \gtrsim B_{cr}$ , its contribution becomes significant. In such strong fields, the magnetic moment is itself field dependent; in the first order

in α [51, 52],

$$\Delta \mu(B) = \frac{\alpha}{2\pi} \,\mu_{\rm B} \begin{cases} 1 + \frac{8}{3} \,L \ln L \,, & L \ll 1 \,, \\ -L^{-1} \ln^2 L \,, & L \gg 1 \,, \end{cases}$$
(8.11)

where  $L = B/B_{cr} = \alpha^2 \mathcal{H}$  and  $\mu_{\rm B} = |e|\hbar/2m_{\rm e}c$  is the Bohr magneton. The value  $\Delta \mu/\mu_{\rm B} = \alpha/2\pi$  for  $B \to 0$  was obtained by Schwinger [30]; as *B* increases, the anomalous magnetic moment of the electron changes sign [51, 52].

The energy of the ground Landau level for  $B \gg B_{cr}$  is

$$E_{0} = m_{e} + \frac{1}{2} \omega_{H} + (\mu_{B} + \Delta \mu) B - \frac{1}{2} m_{e} e^{4} \lambda^{2}$$
$$= m_{e} + \left(\frac{1}{2\pi\alpha} \ln^{2} L - \lambda^{2}\right) \frac{m_{e} e^{4}}{2}, \quad L \ge 1.$$
(8.12)

For a 'Dirac' electron, i.e., for  $\Delta \mu \equiv 0$ , for the LLL states  $(N = 0, \sigma_z = -1)$ , the zero-point energy of oscillations in the (x, y) plane completely cancels the spin magnetic energy,

$$\frac{\omega_H}{2} = \mu_{\rm B} B = \frac{1}{2} \frac{m_{\rm e} L}{2} .$$
 (8.13)

But the anomalous magnetic moment leads to an energy shift  $\Delta E_0 = -\Delta \mu B$ , universal for all atomic levels. For  $L \ge 1$ , the contribution of the anomalous moment  $\Delta \mu$  to Eqn (8.12) increases in proportion to  $\ln^2 L$  and is comparable to the level binding energy for  $\ln L = \sqrt{2\pi\alpha} \lambda$  or  $L = L_*$ ,<sup>16</sup>

$$L_* = \exp \frac{2\ln(1/\alpha)}{(2\pi\alpha)^{-1/2} - 1} \approx 15.$$
(8.14)

For  $L > L_*$ , i.e.,  $B \ge 10^{15}$  G, the level shift exceeds the binding energy  $\lambda^2$ . This shift, however, is determined by QED rather than by atomic physics; it is the same for all levels and does not enter energy differences. Therefore, the inclusion of  $\Delta \mu$  leaves the potentially measurable energies of the radiative transitions  $np \rightarrow 1s$  in the hydrogen atom virtually unchanged.

#### 9. Zeldovich effect in atomic structures

In his paper "Energy levels in a distorted Coulomb field" [53] (see also Ref. [54]), Zeldovich considered the energy spectrum of valence electrons in an impurity semiconductor with a dielectric constant  $\varepsilon \gg 1$  and predicted a curious physical effect to be observed when the interaction potential can be divided into parts with strongly different ranges,  $V(r) = V_{\rm sh} + U_{\rm L}$ . Namely, at those values of the coupling constant  $(g = g_0)$  for which the short-range ('strong') potential  $V_{\rm sh}(r) = -g(\hbar^2/2mr_0^2) v(r/r_0)$  produces a bound s level (or gives rise to a scattering resonance for low-energy particles, i.e., to a real or virtual level with the angular momentum l = 0 and near-zero energy), a small change in g causes a dramatic change in the spectrum of the atom: the Coulomb levels  $E_{ns}$  with n = 1, 2, ... go down to take the positions of  $E_{(n-1),s}$ , and the ground level  $E_{1s}$  falls deeply (on the atomic scale). The width of the rearrangement region (measured in terms of the constant g of the strong potential)

<sup>16</sup> We have set  $\lambda \approx \ln \mathcal{H} = \ln L - 2 \ln \alpha$  for estimation purposes.

Rediscovered later in the context of the nonrelativistic Coulomb problem with a charge Z > 137 [55] and in the theory of the lightest hadron atoms (atomic spectrum rearrangement [39, 40, 56, 57]), this effect came to be known as the Zeldovich effect [58–60]. The cited papers used specific models of the strong potential  $V_{\rm sh}(r)$ : a rectangular well [53], a parabolic potential corresponding to the uniform proton charge density inside the nucleus [55], and separable finiterank potentials [56].

The following model-free equation to describe the Zeldovich effect for s states was given in [57]:

$$F(\lambda) \equiv \lambda + 2\left[\ln \lambda + \psi(1 - \lambda^{-1})\right] = \frac{a_{\rm B}}{a_{\rm cs}} \,. \tag{9.1}$$

Here, l = 0,  $a_{\rm B} = \hbar^2/m_{\rm e}e^2$ , and  $a_{\rm cs}$  is the Coulomb interaction renormalized s-scattering length for a short-range potential. The parameter  $a_{\rm B}/a_{\rm cs}$  determines not only the spectrum of states in a distorted Coulomb potential but also the change in the Coulomb phase of the s-scattering in the radial function

$$\chi_0(r) \approx \sin\left[kr + \frac{1}{ka_{\rm B}}\ln\left(2kr\right) + \delta_0^{\rm Coul}(k) + \delta_0(k)\right], \quad (9.2)$$
$$\delta_0^{\rm Coul}(k) = \arg\Gamma\left(1 - \frac{i}{ka_{\rm B}}\right), \quad r \to \infty,$$

in the case of slow  $(kr_0 \ll 1)$  particles,

$$\delta_0(k) \approx -\operatorname{arccot} \frac{a_{\rm B}}{2\pi a_{\rm cs}}, \quad k \to 0.$$
 (9.3)

Equation (3.7) is identical to Eqn (9.1) for the scattering length

$$a_{\rm cs}(\mathcal{H}) = \left(\ln \mathcal{H} - 4\gamma - 2\ln 2 - A_{n_\rho m}\right)^{-1} a_{\rm B}, \qquad (9.4)$$

or

$$\frac{1}{a_{\rm cs}(\mathcal{H})} = \left(\ln \mathcal{H} - 4\gamma - 2\ln 2 - A_{n_{\rho}m}\right) a_{\rm B}^{-1}.$$
(9.5)

We note that the analogy between the level arrangements in these two problems (i.e., the hydrogen atom for  $B \ge B_a$ ,  $a_H \ll a_B$  and a short-range distortion of the Coulomb potential for  $r_0 \ll a_B$ ) may have first been noticed in Refs [22–24]. For states with  $n_{\rho} = 0$ ,

$$a_{\rm cs}(\mathcal{H}) = \left[\ln \mathcal{H} - 4\gamma - \ln 2 - \psi (1 + |m|)\right]^{-1} a_{\rm B}, \qquad (9.6)$$

where the short-range cutoff radius  $r_0$  for the Coulomb potential is  $a_H = a_B/\sqrt{\mathcal{H}}$ .

It is shown in [57] that the shifts of the Coulomb *n*s states in the Zeldovich effect for the hydrogen atom are suitably described by the effective principal quantum number  $n^*$  and its associated Rydberg corrections  $\Delta_n$  known in atomic physics [6] ( $\hbar = m_e = e = 1$ ):

$$E_{ns} = -\frac{1}{2} \lambda_n^2 \equiv -\frac{1}{2(n^*)^2} = -\frac{1}{2(n+\Delta_n)^2},$$
  
or  $n^* = \frac{1}{\lambda_n} = n + \Delta_n.$  (9.7)

For excited states, irrespective of the Coulomb potential distortion at small distances,  $\Delta_n \approx \text{const}$  (depends weakly

on *n*). The validity conditions for Eqn (9.1) are  $r_0 \ll a_{\rm B}$  and  $\lambda r_0 \ll 1$  for any value of the scattering length  $a_{\rm cs}$ . In the problem considered by Zeldovich [53], these conditions are ensured by the fact that the Bohr radius  $a_{\rm B} = \varepsilon (m_{\rm e}/m_{\rm eff}) \gg r_0$ , where  $m_{\rm e}$  is the electron mass,  $m_{\rm eff} \ll m_{\rm e}$  is the electron effective mass in a lattice, and the ion radius  $r_0$  is of the order of  $a_{\rm B}$ .

We next discuss some level shifting aspects that follow from Eqn (9.1) in the case where the scattering length is given by Eqn (9.6) with m = 0; we consider the difference between the Coulomb potential and an arbitrary strong short-range potential in terms of how their distortions work. The following table shows the relation between numerical values of the magnetic field and the scattering length:

where  $a_{cs}(\mathcal{H}^*) = \infty$  and  $\ln \mathcal{H}^* = 3\gamma + \ln 2 = 2.4248$ . Equation (9.6) for the scattering length can be rewritten as

$$a_{\rm cs}(\mathcal{H}) = \frac{1}{\ln\left(\mathcal{H}/\mathcal{H}^*\right)} a_{\rm B}, \qquad (9.9)$$

showing that the scattering length depends very sensitively on  $\mathcal{H}$  near  $\mathcal{H}^*$ :  $a_{cs}(\mathcal{H}) \approx [\mathcal{H}^*/(\mathcal{H} - \mathcal{H}^*)]a_B$  and that it slowly decreases as  $\mathcal{H} \to \infty$ :  $a_{cs}(\mathcal{H}) \propto 1/\ln \mathcal{H}$ .

We first consider the most impressive manifestation of the Zeldovich effect, with the scattering length  $a_{cs} = \infty$ . The table that follows presents the values of the Rydberg corrections for this (resonance) case:

Level
 1s
 2s
 3s
 4s
 5s
 
$$\infty$$
 s

  $\Delta_n$ 
 0.4696
 0.4964
 0.4987
 0.4993
 0.4996
 0.5

 (9.10)

It is seen that in the region of the unperturbed Coulomb spectrum,  $E_n^{(0)} = -1/2n^2$ , the energy levels shift most significantly (in terms of  $n^*$ ): they are located at the middle between neighboring levels with quantum numbers n and n + 1, and for them  $\Delta_n \approx 0.5$ . It is interesting that in this case, for the shift of the ground ('deep') 1s level, we also have  $\Delta_{1s} \approx 0.5$ , and  $\lambda_{1s}^2 = 4.535$ , i.e., this level also lies close to the unperturbed Coulomb spectrum as, according to Zeldovich, it should. Finally, it is in this case that, according to Eqn (9.3), the Coulomb s-scattering phase is distorted most,  $\delta_0(0) = \pm \pi/2$ .

As  $a_{cs}$  decreases ( $\mathcal{H}$  increases), the levels lower. For  $\ln \mathcal{H} \ge 1$ , the ground state falls deeply (for it,  $\lambda_{1s} \approx \ln \mathcal{H}$ ), and the atomic levels  $\lambda_n(\mathcal{H})$  with  $n \ge 2$  approach the lower adjacent Coulomb levels (n-1) with equal values of  $\Delta_n \to 0$ .

This is clearly seen for small scattering lengths,  $a_{cs} \ll a_B$ , and hence for large values of the right-hand side of Eqn (9.1). As already noted, the function  $\psi(1 - 1/\lambda)$  in the region of the Coulomb spectrum,  $\lambda \leq 1$ , has poles at the points  $\lambda_n = n$  with n = 1, 2, ... (the unperturbed spectrum for  $a_{cs} = 0$ ). From Eqn (9.1), using

$$\psi\left(1-\frac{1}{z}\right) = \psi\left(\frac{1}{z}\right) + \pi \cot \frac{\pi}{z}$$
 (9.11)

with  $z = 1/\lambda_n = n^* = n + \Delta_n$ , we obtain

$$\Delta_n \approx 2a_{\rm cs} = {\rm const}\,, \qquad |a_{\rm cs}| \ll 1\,. \tag{9.12}$$

From Eqns (9.7) and (9.12), the shifted Coulomb *n*s levels are given by

$$E_{ns} = -\frac{1}{2(n+\Delta)^2} = -\frac{1}{2n^2} + \frac{\Delta}{n^3} + \dots, \qquad (9.13)$$

and noting that  $\psi_{ns}(0) = 1/\sqrt{\pi n^3}$  for unperturbed Coulomb *ns* states [6], we obtain the level shifts

$$E_{ns} - E_n^{(0)} \approx \frac{\Delta}{n^3} = 2\pi \psi_{ns}^2(0) a_{cs} \,.$$
 (9.14)

It was noted by Zeldovich [53] that although the perturbation theory in a strong, short-range distorting potential does not work here for  $r \sim r_0 \ll a_B$ , the shift of a level is nevertheless proportional to the probability of finding the electron in this region in the unperturbed state  $\psi_{ns}^2(0)$ . We recall in this connection that in the first-order perturbation theory, the shift of an *ns* level due to a weak short-range potential  $V_{sh}(r)$  is given by [6]

$$\Delta E_{ns} = \int V_{sh}(r)\psi_{ns}^{2}(r) \,\mathrm{d}^{3}r \approx \psi_{ns}^{2}(0) \int V_{sh}(r) \,\mathrm{d}^{3}r$$
$$= \frac{2\pi\hbar^{2}}{m}\psi_{ns}^{2}(0)\tilde{a}_{\mathrm{B}}, \qquad (9.15)$$

where

$$\tilde{a}_{\rm B} = -f_{\rm B}(E=0) = \frac{m}{2\pi\hbar^2} \int V_{\rm sh}(r) \,\mathrm{d}^3 r \,. \tag{9.16}$$

Here,  $\tilde{a}_B$  and  $f_B(E=0)$  are the scattering length and the scattering amplitude with the energy E=0 in the Born approximation [6].

Zeldovich's work provides justification for extending perturbative formula (9.15) to the nonresonant case (9.14) of a short-range distortion of the Coulomb potential, a distortion which amounts to replacing the Born s-scattering length  $\tilde{a}_{\rm B}$  with the exact (nonperturbative) value  $a_{\rm cs}$  (see also Refs [61–63]).

Formula (9.14) (known as the *perturbation theory with respect to the scattering length*) applies when the short-range distortion of the potential allows inelastic processes, such as annihilation into pions in a p $\bar{p}$ -hadron atom or the annihilation of a positronium  $e^-e^+$  into two or three  $\gamma$ -quanta. In these cases, the scattering length  $a_{cs}$  and the level shift  $\Delta E$  are complex quantities, and the corresponding states are quasistationary, with the lifetime  $\tau = \hbar/\Gamma$ ,  $\Gamma = -2 \text{ Im} (\Delta E)$ . We note that Eqn (9.14) relates the lifetime of a state to the behavior of the corresponding inelastic (annihilation) cross section in the limit  $kr_0 \ll 1$  (see Ref. [36] for the application of this equation).

Table 5 lists Rydberg corrections  $\Delta_{nl} \equiv \Delta_n^{\pm}$  to the hydrogen atom spectrum in an ultrastrong magnetic field that were obtained from the numerical calculations of level binding energies [13–16]. We note that in Ref. [13], states of an atom in a magnetic field were classified in terms of the states *nl* (2s, 3d, 2p, etc.) of the unperturbed hydrogen atom, instead of labeling them as  $n^{\pm}$  with n = 1, 2, ... for even and odd states in the presence of a magnetic field (1<sup>+</sup>  $\equiv$  1s is a deep level) (see the remark on this point in Section 4). Therefore, the energy



Figure 6. Zeldovich effect. The perturbation of the atomic spectrum by a nearby resonant ('quasinuclear' [46–48]) level. Figure 1 from Ref. [57] is used.

of a level is given not by Eqn (9.7) but by <sup>17</sup>

$$n^* = \frac{1}{\lambda_{nl}} = n - 1 + \Delta_n ,$$
 (9.17)

i.e., the value of *n* is shifted by 1.

It follows from Table 5 and Eqn (9.8) that even in fields  $\mathcal{H} \approx 10$ , Eqn (9.1) with the scattering length given by Eqn (9.5) is applicable to describing Zeldovich effect manifestations in the problem under discussion (even though the Coulomb potential cutoff radius  $r_0 \approx 0.3 a_{\rm B}$  is not quite small in this case).

On the other hand, it may seem odd that as the magnetic field increases to  $\mathcal{H} \sim 10^6$ , with the corresponding Coulomb potential cutoff radius  $r_0 \sim a_{\rm B}/\sqrt{\mathcal{H}} \sim 10^{-3}a_{\rm B}$ , the scattering length  $a_{\rm cs}$  and the (weakly *n*-dependent) Rydberg corrections  $\Delta_n$  for excited even states decrease very slowly, to  $\Delta_n \approx 0.16$ . At the same time, the Rydberg correction for the ground 1s level is already markedly different from them (by a factor of 1.5; cf. the resonance-case  $\mathcal{H}^*$  considered above). In this case, the perturbative scattering length formula (9.14) gives only a qualitative description of level shifts. For  $\mathcal{H} = 10^6$ ,  $\Delta_n = 0.160$  according to Table 5 and  $\Delta_n = 0.176$  according to Eqns (9.8) and (9.12) (clearly, as the field is further increased, these values come closer together and become zero).

It is easy to explain why  $a_{cs}$  and  $\Delta_n$  behave in this way. As  $\mathcal{H}$  increases, the distortion of the Coulomb potential at small distances leads not only to an increase in the depth but also to a decrease in the width of the well. As a result, the resonant 1s level also lowers; however, its depth, which is proportional to  $\ln^2(1/a_H)$ , is much less than the well depth  $\propto 1/a_H$ , and hence, as previously, the level is a weakly bound state in a short-range potential,<sup>18</sup> and for it

$$|E_{1s}| \ll \frac{\hbar^2}{m_{\rm e} r_0^2},$$
 (9.18)

and the scattering length  $a_{cs}$  is much greater than the radius  $r_0$  of this potential.

<sup>&</sup>lt;sup>17</sup> In Fig. 6, marked on the plot of the graphical solution of Eqn (9.1) are the positions of the levels shifted due to the distorted Coulomb potential (ground 1s and a series of the Coulomb levels 2s, 3s, 4s, ...). It is seen that in the presence of the Zeldovich effect, the spectrum of the atom is periodic in  $n^*$  to good accuracy.

<sup>&</sup>lt;sup>18</sup> For this reason, the natural estimate  $a_{cs} \sim r_0$  for the resonance case of a strongly distorted Coulomb potential in this problem in fields  $\mathcal{H} \sim 10^6$  does not yet apply.

Importantly, this level corresponds to a weakly coupled state, not only in the distorted potential  $U(r) = V_{\rm sh}(r) + U_{\rm C}(r)$  but also in an isolated short-range potential  $V_{\rm sh}(r)$  with  $U_{\rm C}$  'switched off'. With  $a_{\rm s}$  denoting the s-scattering length in the potential  $V_{\rm sh}$ ,  $a_{\rm cs}$  and  $a_{\rm s}$  in the resonance case are given by

$$\frac{1}{a_{\rm cs}(\mathcal{H})} - \frac{1}{a_{\rm s}} = -\frac{2}{a_{\rm B}} \left[ \ln \frac{r_0(\mathcal{H})}{a_{\rm B}} + O(1) \right]. \tag{9.19}$$

This is Schwinger's formula [64] for the renormalization of the s-scattering length  $a_s$  in a short-range potential  $V_{sh}(r)$  by a Coulomb attraction potential. We note that its logarithmically accurate derivation uses the perturbation theory to take the Coulomb potential into account within the range of the short-range potential  $V_{sh}(r)$  [see Appendix C on the scattering lengths  $a_s$  and  $a_{cs}$  at resonance  $(E_l \rightarrow 0, g \approx g_l)$ ].

The reason why Eqn (9.19) is especially useful in the present context is the knowledge that  $r_0(\mathcal{H}) \propto 1/\sqrt{\mathcal{H}}$ . This allows applying Eqn (9.19) to the field values  $\mathcal{H}$  and  $\mathcal{H}^*$  to exclude incalculable constants entering that formula and using Eqn (9.9) to obtain the scattering length  $a_{cs}$ . In particular, for  $\ln \mathcal{H} \ge \ln \mathcal{H}^*$ , Eqn (9.19) yields  $a_{cs} \approx a_{B}/\ln \mathcal{H}$ , which is the perturbative result for a small scattering length, Eqn (9.14).

Equation (9.1) can be extended to states with an arbitrary angular moment  $l \neq 0$  [65, 66]. In this case, the effect of the short-range potential  $V_{\rm sh}(r)$  on the states in the long-range potential  $U_{\rm L}$  cannot be included by imposing a boundary condition on the wave function at  $r \rightarrow 0$ , as was the case for s states according to Eqns (2.9) and (3.1) (when the long-range potential had a Coulomb singularity at zero). The reason is that as  $r \rightarrow 0$ , the general solution of the radial Schrödinger equation has the form

$$R_l(r) = c_1 r^l + c_2 r^{-l-1}$$

E

and for  $l \ge 1$ , the singular solution should be dropped because it is not square integrable. The method of zerorange potentials [31, 32] was extended to the case where  $l \ne 0$  in [67, 68].

We consider Fig. 7, which illustrates the difference between the Zeldovich effect manifestations in the case of s states (l = 0) and for an orbital moment  $l \ge 1$ . In this figure, the horizontal axis corresponds to the coupling constant g in the short-range potential  $V_{\rm sh}(r) = -(g/2r_0^2)v(r/r_0)$  (this constant determines the position of a resonance level when it

Ε

l = 0

b

 $l \ge 1$ 



**Figure 7.** Rearrangement of the atomic spectrum (a) for s levels and (b) for  $l \neq 0$ .

is in the atomic region), and the vertical axis shows the positions of the unperturbed Coulomb levels  $E_{nl}^{(0)} = -1/2n^2$  with their corresponding angular momentum values l (for the lowest n = l + 1). In the case of a resonance, all excited *ns* levels<sup>19</sup> are shifted downward with equal values of the Rydberg correction  $\Delta$  (see Table 5).

In the case  $l \neq 0$ , the Zeldovich effect shows up in a totally different way [65, 66]. The spectrum then consists of weakly shifted *nl* levels and a resonant level in  $V_{\rm sh}(r)$ . Such a spectrum undergoes a rearrangement only when an increase in the coupling constant g causes the resonant level to approach one of the atomic levels.<sup>20</sup> (We note the *intersection-of-terms* nature of the level interaction in a certain narrow range of g values.)

The difference in the manifestation of the Zeldovich effect between the l = 0 and  $l \neq 0$  cases is explained by the presence of a low penetrability barrier  $\hbar^2 l(l+1)/2mr^2$  at  $l \neq 0$ , which separates two regions with attractive potentials: one with a short-range potential  $V_{\rm sh}(r)$  with a shallow resonance level and the other with a Coulomb potential with nl levels. As long as the resonance level and the nl levels are not close in energy, the low penetrability of the barrier prevents them from affecting each other noticeably.

#### **10.** Concluding remarks

We have discussed various approaches to studying the energy spectrum of the hydrogen atom in a strong magnetic field  $B \ge B_a$ . It is shown that using the explicit form of the effective potential within the adiabatic approximation to solve the Schrödinger equation (or the Dirac equation in the case  $B \ge B_{cr} \sim 10^{14}$  G) yields simple analytic expressions for the spectrum. A detailed discussion is given of atomic levels adjacent to the ground Landau level with  $n_{\rho} = 0$  and magnetic quantum numbers  $m = 0, -1, -2, \ldots$ . A comparison with available numerical simulations shows that for  $\mathcal{H} = B/B_a \ge 100$ , Eqn (3.7) has a few percent or better accuracy.

A few concluding remarks are in order.

(1) There is a marked difference between the energies of even and odd excited levels,  $E_n^{\pm} = -(\lambda_n^{\pm})^2/2$ , as is already qualitatively seen from the asymptotic expressions (neglecting vacuum polarization)

$$\lambda_n^+ \approx \frac{1}{n} \left( 1 - \frac{2}{n \ln \mathcal{H}} + \dots \right), \qquad (10.1)$$
$$\lambda_n^- \approx \frac{1}{n} \left( 1 - \frac{2}{n} \frac{\ln \mathcal{H}}{\mathcal{H}} + \dots \right), \qquad \mathcal{H} \ge 1,$$

n = 1, 2, ... We see that the excited even levels are much higher than the odd ones, and the Rydberg corrections to them are given by

$$\Delta^+ \approx \frac{2}{\ln \mathcal{H}}, \quad \Delta^- \approx \frac{2\ln \mathcal{H}}{\mathcal{H}}.$$
 (10.2)

(2) For  $\mathcal{H} \sim 10^3 - 10^4$ , the odd levels are already close to the unperturbed Coulomb values  $E_n = -1/2n^2$ , whereas the even ones are still far from this limit. There is, therefore, a need for further elaboration of the statement (see Refs [6, 19]) about the approximate twofold degeneration of even and odd excited levels.

<sup>&</sup>lt;sup>19</sup> The level s shifts more significantly.

<sup>&</sup>lt;sup>20</sup> The trajectory of the level is shown dashed.

(3) In Section 3, we noted the occurrence of a peculiar kind of scaling in the energy spectrum, according to which for any level  $E = -\lambda^2/2$ , the dependence  $F(\lambda)$  [see Eqn (3.8)] is, in the adiabatic approximation, a universal function (i.e., its value is the same irrespective the magnetic field and the level quantum number).

As can be seen from Fig. 1, this scaling holds well for  $\lambda^2 > 12$ . Interestingly, as  $\lambda^2$  decreases and the adiabatic approximation ceases to apply, the levels still continue to group together along a curve close to  $F(\lambda)$ .

(4) Spectrum equation (3.7) was obtained by solving the Schrödinger equation. However, in fields  $B \ge 10^{13}$  G, the transverse motion of the electron becomes relativistic. The solution of the Dirac equation with a point-charge nucleus showed that Eqn (3.7) for the electron binding energy remains valid for ultrastrong fields up to  $B > 10^{17}$  G (see Table 1).

(5) It was found in [33–35, 42, 43] that, interestingly, a strong magnetic field  $B > 10^{16}$  G influences the hydrogenatom spectrum via the vacuum polarization and the screening of the proton's Coulomb field.

With the vacuum polarization taken into account in the limit as  $B \to \infty$ , the lowest even LLLs with different *m* do not decrease to zero but rather approach finite limit values (level 'freezing' effect), whose magnitudes  $E_{\infty}^+ = -\lambda_{\infty}^2/2$  depend on the magnetic quantum number *m* and are large compared to the levels of the discrete spectrum of the unperturbed hydrogen atom. Their binding energy remains nonrelativistic, however, and (neglecting relativistic corrections) is determined by Eqn (8.7), which includes the screening of the effective Coulomb potential. For example,  $\varepsilon_{\infty} = \lambda_{\infty}^2 Ry = 1.71$  keV according to Ref. [34].

Similarly, in the case of excited cases, the asymptotic form of the energies  $E_n^+$  for  $B \to \infty$  differs from  $E_n^0 = -1/2n^2$  (see Table 7).

(6) Including the QED vacuum polarization effect usually yields small corrections (the contribution to the Lamb shift  $\Delta_{LS} = 1058$  MHz for the atomic hydrogen levels  $2s_{1/2}$  and  $2p_{1/2}$  [30]). In an ultrastrong magnetic field  $\mathcal{H} \gtrsim \mathcal{H}_{\infty}$ , vacuum polarization produces a qualitative change in the spectrum of even levels.

(7) Up to this point in our discussion, the hydrogen atom has been assumed to be at rest and the magnetic field to be static. Situations exist, however, where an atom should be considered as moving, a noted example (from astrophysics) being the motion of neutron stars (with magnetic fields  $B \sim 10^{11} - 10^{13}$  G and higher) through a cloud of interstellar gas. Substantial literature exists on how this motion (in the presence of a magnetic field) influences atomic spectra in terms of level energies, radiative transition probabilities, photoionization cross sections, etc. These problems are beyond the scope of this paper, however, and we limit ourselves to referring the reader to Refs [69–72] and the references therein.

(8) L D Landau, I Ya Pomeranchuk, and K A Ter-Martirosyan, in their (unpublished) 1957 paper "Interaction and annihilation of antinucleons with nucleons" considered the question of the spectrum of the proton–antiproton atom. Including the distortion of the Coulomb field due to the strong  $\bar{p}-p$  interaction at small distances and matching the inner and outer wave functions in the region where the strong interaction can already be neglected, these authors obtained results similar to those in Refs [39, 40, 57]. One of the present authors (V P) gratefully remembers the discussions on this subject with Ter-Martirosyan (who told V P about the 1957 paper and showed him its manuscript) and with E M Lifshitz, who made a number of useful comments. It should be noted, however, that when writing papers [39, 40, 57], we were unaware of, and hence did not refer to, the work of Landau, Pomeranchuk, and Ter-Martirosyan.

#### Acknowledgements

Our sincere thanks are due to M I Vysotsky and S I Godunov for drawing our attention to papers [33–35] prior to their publication and for the useful comments; and to B O Kerbikov, A E Kudryavtsev, V D Mur, S V Popruzhenko, and Yu A Simonov for discussing our results. V P also thanks L S Minkin and D V Popov for support and assistance in preparing this paper.

#### **11.** Appendices

#### A. Critical nuclear charge

The Dirac equation with an attractive point-like potential,

$$V(r) = -\frac{Z\alpha}{r}, \qquad 0 < r < \infty, \qquad (A.1)$$

is solved analytically [30] to give the well-known Sommerfeld formula

$$\varepsilon_{nj} = m_{\rm e} \left\{ 1 + \frac{\zeta^2}{\left[ n - (j+1/2) + \sqrt{(j+1/2)^2 - \zeta^2} \right]^2} \right\}^{-1/2},$$
  
$$0 < \zeta < j + \frac{1}{2},$$
  
(A.2)

where  $0 \le \varepsilon_{nj} < m_e$ ,  $n = n_r + l + 1 = 1, 2, 3, ...$  is the principal quantum number and  $j = l \pm 1/2 = 1/2, 3/2, ..., n - 1/2$  is the total angular momentum of the level.

Notably, the energy  $\varepsilon_0$  (including the rest mass  $m_e$ ) of the  $1s_{1/2}$  ground state is<sup>21</sup>

$$\varepsilon_0(Z) = m_{\rm e} \sqrt{1 - \zeta^2}, \quad \zeta = Z \alpha = \frac{Z}{137},$$
 (A.3)

and for j = 1/2 as  $\zeta \to 1$ ,

$$\varepsilon_{n,1/2} \approx \sqrt{1 - \frac{1}{N^2}} + \frac{1}{N^3} \sqrt{1 - \zeta^2} + O(1 - \zeta^2),$$

where

$$N = \sqrt{n^2 - 2n + n}$$

For Z > 137, the energies  $\varepsilon_0$  become imaginary, indicating that the Dirac Hamiltonian for a point-like charge, Eqn (A.1), is no longer a self-adjoint operator and is therefore physically unacceptable; the reason for this is the 'falling onto the center' phenomenon, well known in quantum physics [6, 36–38], which arises in the Coulomb problem for the Dirac equation with Z > 137. As shown by Pomeranchuk and Smorodinskii [73, 74], including the finite size of the nucleus remedies this difficulty.

The term 'critical nuclear charge  $Z_{cr}$ ' is used in QED to refer [45–50, 75–78] to the value of Z at which a discrete level

<sup>&</sup>lt;sup>21</sup> As  $\zeta \to 1$ , the function  $\varepsilon_0(\zeta)$  has a singularity and the curve undergoes a discontinuity before reaching the lower continuum boundary. Similar singularities arise for all excited states.

of the atomic spectrum drops to the lower continuum ('Dirac sea') boundary

$$\varepsilon_0(Z_{\rm cr}) = -m_{\rm e}c^2 \,. \tag{A.4}$$

The factors that determine the value of  $Z_{cr}$  are the nuclear radius  $r_N$ , the level quantum numbers (njl), the way the electrical charge is distributed over the volume of the nucleus, and the degree of ionization of the atom's outer electron shells (see Refs [48, 50, 77, 78]).

Simple asymptotic estimates for  $Z_{cr}$  obviate the need for tedious numerical work. If the nuclear radius is taken to be  $r_N > 0$ , the square root singularity in Eqn (A.3) disappears, giving [48, 79]

$$\varepsilon_0(\zeta) = m_{\rm e} \sqrt{1 - \zeta^2} \, \coth\left(\Lambda \sqrt{1 - \zeta^2}\right),\tag{A.5}$$

where  $\Lambda = \ln (1/m_e r_N) \ge 1$ . Unlike Eqn (A.3), Eqn (A.5) can be analytically continued to the region  $\zeta > 1$ ,

$$\varepsilon_0(\zeta) = m_e g \cot(\Lambda g), \quad g = \sqrt{\zeta^2 - 1}, \quad \zeta > 1, \quad (A.6)$$

such that, for example, for Z = 137 we have  $\varepsilon_0(\zeta = 1) = m_e/\Lambda > 0$ .

From Eqn (A.6), the lower level  $\varepsilon_0(\zeta)$  is in the middle between the upper and lower continuum boundaries for  $Ag_0 = \pi/2$ , or

$$\zeta = \zeta_0 = \sqrt{1 + \frac{\pi^2}{4\Lambda^2}} = 1 + \frac{\pi^2}{8\Lambda^2} + \dots, \quad \Lambda \ge 1, \quad (A.7)$$

[for  $r_N \rightarrow 0$ , this gives  $\zeta_0 = 1$  in accordance with Eqn (A.3)]. Noting that function (A.6) has a pole at  $g = g_1 = \pi/\Lambda$ , we obtain the estimate<sup>22</sup>

$$\zeta_{\rm cr} = Z_{\rm cr} \alpha = \sqrt{1 + g_1^2} \approx 1 + \frac{\pi^2}{2\Lambda^2} + \dots$$
 (A.8)

This formula can be improved in accuracy by calculating the next ( $\propto \Lambda^{-3}$ ) term in the asymptotic expansion of  $\zeta_{cr}$ in powers of  $1/\Lambda$ . This is achieved by substituting Eqn (A.6) in boundary condition (A.4), with the result  $\tan (\Lambda g)/\Lambda g = -1/\Lambda$ , whence  $g = \pi/(\Lambda + 1) + O(\Lambda^{-3})$ ,

$$\zeta_{\rm cr}(1s_{1/2}) = 1 + \frac{\pi^2}{2\Lambda(\Lambda+2)} + O(\Lambda^{-4}), \quad n = 1,$$
 (A.9)

and for the excited ns levels [79]

$$\zeta_{\rm cr}(ns_{1/2}) = 1 + \frac{n^2 \pi^2}{2\Lambda(\Lambda + 2n)} + \dots$$
 (A.10)

Similarly, for the  $2p_{1/2}$  level,

$$\varepsilon_{2p} = m_{\rm e} \sqrt{\frac{1+\sqrt{1-\zeta^2}}{2}} \xrightarrow[(\zeta>1)]{} \frac{m_{\rm e}}{\sqrt{2}} \left(1+\frac{1}{2} \operatorname{g} \operatorname{cot} \left(\Lambda g\right)+\ldots\right),$$
(A.11)

<sup>22</sup> While correctly describing the qualitative dependence of  $Z_{\rm cr}$  on the Coulomb cutoff radius, this formula is not accurate. For example, for  $r_{\rm N} = 10$  Fm ( $\Lambda = 3.653$ ), Eqn (A.8) gives  $Z_{\rm cr} \approx 190$ , whereas the improved-accuracy asymptotic formula (A.9) gives  $\zeta_{\rm cr} = 1.24$  and  $Z_{\rm cr} = 170$ , which is already close to the exact numerical values (for the ground level).

Table 8. Critical charge of a superheavy nucleus for lower levels.

Level	<i>r</i> <sub>N</sub> , Fm	Λ	$Z_{ m cr}^{(0)}$	$Z_{\rm cr}$	$Z_{ m cr}^{( m as)}$	$\zeta_{\rm cr}/(j+1/2)$
$1s_{1/2}$	9.14	3.743	168.8	172	169	1.23
$2p_{1/2}$	9.38	3.717	181.3	184	181	1.32
$2s_{1/2}$	10.1	3.643	232	239	234	1.69
$3p_{1/2}$	10.5	3.604	254	263	—	1.85

*Note.* Nuclear radius  $r_{\rm N} = 1.2A^{1/3}$  Fm, A = 2.6Z,  $A = \ln (1/m_{\rm e}r_{\rm N})$ ,  $1/m_{\rm e} = 386$  Fm;  $Z_{\rm cr}^{(0)}$  is the critical charge of a bare nucleus;  $Z_{\rm cr}$  is the same including the screening of the Coulomb potential by the outer electron shells;  $Z_{\rm cr}^{(as)}$  is obtained from asymptotic formulas (A.8) and (A.11). The values  $Z_{\rm cr}^{(0)}$  and  $Z_{\rm cr}$  are taken from numerical computations [46, 75, 80].

whence  $g \cot(\Lambda g) = -2(1 + \sqrt{2})$ , or

$$\zeta_{\rm cr}(2p_{1/2}) = 1 + \frac{\pi^2}{2\Lambda(\Lambda + \sqrt{2} - 1)} + O(\Lambda^{-4}). \tag{A.12}$$

Here, it is assumed that  $\Lambda \ge n$ , and therefore the asymptotic expressions apply only to the lower levels; we note that these expressions are reasonably accurate (Table 8), even though the expansion parameter  $1/\Lambda \approx 0.25$  is not very small. Subsequent terms in these expansions depend not only on the radius  $r_{\rm N}$  but also on the specific way the Coulomb potential is cut off inside the nucleus.

We note that the values of  $Z_{\rm cr}$  depend weakly on the cutoff details. For example, if all the shells of superheavy atoms (except for  $1s_{1/2}$ ) are filled, then the Thomas–Fermi screening calculation gives the value  $Z_{\rm cr} \approx 171.5$  [80], instead of  $Z_{\rm cr} = 170$  (see also Ref. [77]).

From Eqn (A.2), for the excited state with j = n - 1/2 (the maximum possible value of *j* for a given *n*), we have

$$\varepsilon_{n,n-1/2} = m_{\rm e} \sqrt{1 - \frac{\zeta^2}{n^2}}, \quad 0 < \zeta < n.$$
 (A.13)

With the Coulomb potential cut off for  $r < r_N$ , the level energies smoothly continue to the region  $\varepsilon < 0$ , reaching the lower continuum boundary at  $\zeta = \zeta_{cr}(njl)$ , and the (finite) slope of entering the lower continuum is

$$\frac{\mathrm{d}\varepsilon_0}{\mathrm{d}\zeta} = m_\mathrm{e}\zeta_\mathrm{cr} \left[ \frac{1}{g} \cot\left(\Lambda g\right) - \frac{\Lambda}{\sin^2\left(\Lambda g\right)} \right] \approx -\frac{m_\mathrm{e}}{\pi^2} \Lambda(\Lambda + 1)(\Lambda + 2) \tag{A.14}$$

[here,  $g = \sqrt{\zeta_{cr}^2 - 1} = \pi/\sqrt{\Lambda(\Lambda + 2)} \ll 1$ ; see Eqns (A.6) and (A.9)].

A similar problem of the critical distance  $R_{\rm cr}$  between colliding nuclei with  $Z_1 + Z_2 > Z_{\rm cr} \approx 170$  was considered in Refs [81–87]. We refer the reader to these references for the details and present a few characteristic results here.<sup>23</sup> For two uranium nuclei,  $R_{\rm cr} = 35$  Fm, we have  $R_{\rm cr} \approx 43$  Fm for the system (Cm + U),  $R_{\rm cr} \approx 51$  Fm for (Cm+Cm), and  $R_{\rm cr} \approx 70$  Fm for (Fm+Fm). The values of  $R_{\rm cr}$  exceed the sum of the two nuclear radii, which, in principle, offers the hope of performing experiments on spontaneous positron creation in slow (adiabatic) collisions of heavy nuclei [48, 50,

<sup>23</sup> Because  $Z_{1,2} < 137$ , the 'falling onto the center' phenomenon does not occur in the Coulomb field of either of the two nuclei, allowing them to be considered point-like (relativistic two-center problem); to correct for the finite nuclear size, perturbation theory is used.

77]. We also note the availability [81] of an approximate analytic formula for  $R_{\rm cr}(Z)$ , which was obtained by matching the asymptotic behaviors of the wave function  $\psi(\mathbf{r})$  at small and large distances from the nuclei and which shows sufficient accuracy for  $Z_{1,2} < 100$ . For example, for  $Z_1 = Z_2 = 92$ , it gives  $R_{\rm cr}^{\rm cs} = 35.5$ Fm, whereas exact numerical calculations with the Dirac equation [83, 86] yield  $R_{\rm cr} = 34.3$  Fm (correspondingly,  $R_{\rm cr}^{\rm cs} = 42.8$  and 42.6 Fm for  $Z_1 = Z_2 = 94$ , and we also have close values for the U + Cf system,  $Z_1 + Z_2 = 188$ ).

A detailed discussion of the problem of the critical nuclear charge and of other problems that arise in strong-field QED is given in [48–50, 75–77]. A physically transparent description of supercharged electrons in the lower continuum ( $Z > Z_{\rm cr}$ ,  $\varepsilon < -m$ ) is given in Refs [48, 50, 77, 87]. The value  $Z_{\rm cr}^{\mu} \approx 2300$  for a muon ion [88] should be mentioned here.

We note that for  $Z > Z_{cr}$ , spontaneous creation of positrons is only possible when the level with the energy  $E_0 < -m_{\rm e}c^2$  that has descended to the lower continuum is not filled with electrons. This level can, in theory, be ionized via the collision of two heavy ions with the total charge  $Z_1 + Z_2 > Z_{cr} \approx 170$  because their collision velocity is finite (a well-known process in atomic physics). In the field of a 'bare'<sup>24</sup> nucleus with  $Z > Z_{\rm cr}(1s_{1/2}) \approx 170$ , the spontaneous creation of e<sup>+</sup>e<sup>-</sup> pairs from the vacuum is possible, after which the electrons go to and fill the vacant  $1s_{1/2}$  level  $(s_z = \pm 1/2)$ , whereas the two positrons penetrate through the Coulomb barrier and escape to infinity, where they can be detected. The charge of the nucleus decreases in this process by two,  $Z \rightarrow Z - 2$  (for an outer observer). As Z increases, new positrons appear only for  $Z > Z_{cr}(njl)$ , when the next level (njl) of the discrete spectrum goes down to the lower continuum (see Table 8).

The effect of an ultrastrong magnetic field on the critical nuclear charge  $Z_{\rm cr}(B)$  is considered in [34, 44, 78, 89]. It is found that with increasing the magnetic field, the value of  $Z_{\rm cr}(B)$  decreases:  $Z_{\rm cr} = 92$  (uranium) for  $B \approx 5.5 \times 10^{15}$  G [89],  $Z_{\rm cr} = 80$  for  $B \approx 10^{16}$  G [78],  $Z_{\rm cr} = 41$  for  $B = 10^{18}$  G [34, 35], etc. The basis for this conclusion was the following property of the energy spectrum of a Dirac particle in a uniform magnetic field. As noted in Section 8, in the case of the Dirac equation, the lower boundary of the upper continuum remains equal to  $m_ec^2$  (as is the case for a free particle), because for a 'Dirac' electron (and hence for a positron), the zero-point energy of oscillations in the (x, y) plane is compensated by the spin magnetic energy.

In Refs [78, 89], this effect was considered without including the Coulomb potential screening due to vacuum polarization. But this screening has the consequence that an energy level is 'frozen' and thus becomes unable to reach the lower continuum boundary as long as the limit energy is  $E_{\infty}^{Z}(B \rightarrow \infty) > -m_{\rm e}$ ; this condition is satisfied [35] for Z < 52. For such nuclei, the ground level remains subcritical in an arbitrarily strong magnetic field, and the spontaneous creation of positrons in the field of a charge Z is impossible. If Z > 52, the 'critical' state of an electron at the edge of the lower continuum can be achieved with increasing B, but this requires stronger magnetic fields than when screening is neglected.

#### **B.** Extreme magnetic field

In this appendix, we briefly discuss the maximal magnetic fields that can be achieved in the laboratory or encountered in space.

(1) The atomic unit of the magnetic field,  $B_a$ , is defined from the condition  $\mu_B B_a = m_e e^4 / 2\hbar^2 = 1$  Ry, whence  $B_a = m_e^2 c e^3 / \hbar^3 = 2.349 \times 10^9$  G<sup>25</sup>. A permanent magnetic field does not lead to the creation of  $e^+e^-$  pairs from the vacuum [29]; therefore (unlike with the electric field), the production of arbitrarily strong magnetic fields  $B \gtrsim B_{cr}$  is not inconsistent with QED and is possible in principle.

(2) Permanent magnetic fields produced in the laboratory do not exceed a few kilogauss.

In 1951, Sakharov proposed [90–92] the method of magnetic cumulation, in which the magnetic field enclosed in a well-conducting capsule (a cylindrical tube of a high-conductivity metal like copper) is compressed by a shock wave due to the explosion of an explosive surrounding the capsule. The magnetic flux conservation condition  $\Phi = \pi R^2 B$  implies that  $B/B_0 \approx (R_0/R)^2 \ge 1$ , where R(t) is the capsule radius and  $B_0$  is the initial magnetic field (which is created by conventional means). Experimentally, the magnetic field strength  $B \approx 25$  MG ( $\mathcal{H} = B/B_a \approx 0.01$ ) was reached [90], which is a record high value in terrestrial conditions, and further progress is possible [93].

Equations that describe the dynamics of magnetic cumulation are considered in Ref. [94]. Using energy conservation and Maxwell's equations for a quasistationary field in a well-conducting medium, we arrive at the equation

$$\ddot{\xi} = \frac{1}{K\xi^3} \left( 1 + \frac{\mu}{\sqrt{-\xi\dot{\xi}}} \right)^2, \quad 0 < \tau < \tau_{\rm m} \,, \tag{B.1}$$

with the boundary conditions

$$\begin{aligned} \xi(0) &= -\dot{\xi}(0) = 1 \,, \\ \xi(\tau) &= 1 - \tau + \frac{(1+\mu)^2}{2K} \tau^2 + \dots, \quad \tau \to 0 \,, \end{aligned} \tag{B.2}$$

where  $\xi = R/R_0$ ,  $R_0 = R(t = 0)$ ,  $\dot{\xi} = d\xi/d\tau$ ; *K* is the ratio of the capsule kinetic energy to the magnetic energy contained in the capsule at the start of the explosion t = 0;  $\mu = c/\sqrt{2\pi\sigma R_0 v_0}$  is the (dimensionless) coefficient of the capsule Ohmic loss;  $\sigma$  is the capsule conductivity, with the dot denoting the derivative with respect to the dimensionless time  $\tau = v_0 t/R_0$ ; and the values  $\xi_m$  and  $\tau_m$  refer to the moment of maximum compression.<sup>26</sup> In the ideal case, i.e., in the absence of Ohmic loss, for  $\sigma = \infty$  (superconductor) and  $\mu = 0$ , Eqn (B.1) has the energy integral

$$\dot{\xi}^2 = 1 + K^{-1} \left( 1 - \frac{1}{\xi^2} \right) \tag{B.3}$$

<sup>25</sup> For comparison, for the electric field  $\mathcal{E}_{a} = m_{e}^{2} e^{5} / \hbar^{4} = 5.142 \times 10^{9} \text{ V cm}^{-1}$ , which is equal to the electric field strength on the K orbit (1s) of the hydrogen atom,  $\mathcal{E}_{a} = e/a_{B}^{2} = \alpha B_{a}$ .

<sup>26</sup> Equation (B.1) refers to cylindrical geometry and applies only at the compression stage, when  $\dot{\xi} < 0$ . For  $\tau > \tau_m$ , the capsule flies to pieces due to the magnetic field pressure,  $\dot{\xi} > 0$ , and this equation no longer holds. In this case, calculating the magnetic field requires solving the equation of thermal conductivity (or the equation of magnetic field diffusion), which does not reduce to a simple form like Eqn (B.1). This, however, is not necessary if we are interested in the maximum achievable magnetic field  $B_m$  rather than (expressly) the motion of the capsule at the fly-away stage.

<sup>&</sup>lt;sup>24</sup> That is, nuclei with a stripped (fully ionized) K shell.

V S Popov, B M Karnakov

and is solved analytically to give

$$\xi^{(0)}(t) = \sqrt{\xi_{\rm m}^2 + (1 + K^{-1})(\tau - \tau_{\rm m})^2} \; .$$

In this approximation, at the moment when the capsule is compressed most strongly and when the magnetic field is at its maximum,  $\tau = \tau_m^{(0)}$ ,

$$\begin{aligned} \xi_{\rm m}^{(0)} &= (K+1)^{-1/2} \approx K^{-1/2} \,, \\ \tau_{\rm m}^{(0)} &= \frac{K}{K+1} \approx 1 - \frac{1}{K} \,, \\ B_{\rm m}^{(0)} &= (K+1)B_0 \,, \end{aligned} \tag{B.4}$$

where  $B_m^{(0)}$  is the maximum field (in an actual experiment,  $K \ge 1$ ). With the Ohmic losses included, the last of these equations becomes

$$\frac{B_{\rm m}}{B_0} = (K+1) \left[ 1 + 2\mu \int_0^{\tau_{\rm m}} \left( 1 + \frac{2}{K\xi^2} \right) \sqrt{-\frac{\dot{\xi}}{\xi^3}} \, \mathrm{d}\tau \right]^{-1}.$$
 (B.5)

Numerical calculations using these equations were performed in Ref. [95]. We refer the reader to that paper for the details and only present some estimates here. For the initial capsule radius  $R_0 = 3$  cm, the initial compression rate  $v_0 = 10$  km s<sup>-1</sup>, and K = 1000, we respectively have  $\mu = 0.01$  and  $\mu = 0.037$ for  $\sigma = 6 \times 10^5 \ \Omega^{-1} \ \text{cm}^{-1}$  (com temperature conductivity of copper) and  $\sigma = 4 \times 10^4 \ \Omega^{-1} \ \text{cm}^{-1}$  at  $T = 1500 \ \text{°C}$  (the metal is already in a liquid state). Taking the initial field to be  $B_0 \sim 10^5$  G (which is achievable by conventional means) for  $K = 100, 10^4, 10^8$ , we respectively find

$$B_{\rm max} \sim 10^7 {\rm G}, \ 10^9 {\rm G} \sim B_{\rm a}, \ 10^{13} {\rm G} \sim B_{\rm cr}.$$
 (B.6)

The last value of *K* corresponds to the magnetic field being compressed by a relatively low-power underground nuclear explosion (see the note on p. 85 in Ref. [92]) rather than by a conventional explosion.

Equation (B.5) includes Ohmic losses (due to Foucault currents) that arise as the capsule is being compressed; as a result, for example, for  $K \sim 1000$  and  $\mu \approx 0.05$ , the magnetic flux  $\phi$  decreases in the course of compression by a factor of 1.5–2.0 and the maximum achievable field  $B_{\text{max}}$ , by a factor of 2–3 compared to the ideal case where  $\Phi(t) = \text{const}$ ,  $\mu = 0$ ; these estimates are consistent with the brief remarks made in Refs [90, 91].

Hence, the magnetic cumulation method, while in theory suitable for generating fields  $B > B_a$ , requires an extremely high initial energy for the compressing shock wave; another experimental difficulty is that this method is pulsed in nature: it takes a short time  $t_a = a_B/v_a = \hbar^3/m_e e^4 = 2 \times 10^{-17}$  s to create a field of the order of  $B_{max}$ . However, the magnetic field lifetime in this experiment is orders of magnitude larger than the characteristic atomic time  $t_a = a_B/v_a = \hbar^3/m_e e^4 = 2 \times 10^{-17}$  s. Therefore, from an atomic physics perspective (in particular, for the Lorentz ionization of atoms and ions [95]), this field can be regarded as permanent (in the adiabatic approximation).

(3) The maximum magnetic fields currently achievable in the laboratory are apparently those resulting from collisions of heavy relativistic particles using the RHIC and LHC [96]:  $B_{\rm st} \sim m_{\pi}^2 c^3/e\hbar \sim 10^{18}$  G. It goes without saying that the lifetime of such fields is extremely short,  $\tau \leq \hbar/m_{\rm e}c^2 \sim 10^{-21}$  s. High-power lasers are another possible source of strong magnetic fields, and there is considerable recent research on this topic. Values  $B \sim 50$  MG achieved on the surface of plasma blobs in the vacuum are reported in [97] (the field lifetime being  $\tau \sim 10^{-10}$  s), and according to Ref. [98], quasistatic fields  $B \leq 700$  MG with the lifetime  $\tau \sim 10^{-10} - 10^{-9}$  s (which is comparable to the laser pulse duration) can be produced in a laser plasma. The authors thank S V Popruzhenko for drawing their attention to [97, 98].

(4) Superstrong magnetic fields are encountered in astrophysics. For illustration, the surface magnetic field of a magnetic white dwarf can be as high as  $\sim 1000 \text{ MG} (\mathcal{H} \sim 1)$ .

Neutron stars observed as pulsars have much higher magnetic fields  $B \sim 10^{12} - 10^{13}$  G [2, 3] and more, and surface fields in so-called magnetars<sup>27</sup> seem to be even higher, up to  $10^{14} - 10^{15}$  G. Such fields cannot be achieved in terrestrial experiments, which makes the neutron star a unique natural laboratory for testing theoretical ideas about the properties of matter under extreme conditions.

Some cosmological models—e.g., those of gamma-ray bursts—predict the existence of fields  $B \sim 10^{16} - 10^{17}$  G. For these values of *B*, our asymptotic formulas for the spectrum of hydrogen atoms are highly accurate.

Solid state physics — in particular, the theory of excitons and shallow levels in semiconductors (see, e.g., Refs [19, 20])—is yet another application area for the formulas presented here. Here, the effective mass of an electron in a lattice is  $m_{\text{eff}} \ll m_{\text{e}}$ , and the dielectric constant  $\varepsilon \gg 1$ , and hence the characteristic magnetic fields are much lower than  $B_{\text{a}}$  and are quite feasible experimentally. It is for such systems that the Zeldovich effect was predicted [53].

(5) The critical (or characteristic) field in QED is [27–29]

$$F_{\rm cr} = \frac{m_{\rm e}^2 c^3}{c\hbar} = \begin{cases} 1.32 \times 10^{16} \,\,{\rm V}\,{\rm cm}^{-1} \equiv \mathcal{E}_{\rm cr} \\ 4.41 \times 10^{13} \,\,{\rm G} \equiv B_{\rm cr} \end{cases}$$

for the electric and the magnetic field. We note that  $eF_{\rm cr}I_{\rm C} = m_{\rm e}c^2$  and  $\mathcal{E}_{\rm a} = \alpha B_{\rm a}$ . It is useful to remember that  $\mathcal{E} [\rm V \, cm^{-1}] = 300B [\rm G]$ .

The QED characteristic field  $\mathcal{E}_{cr}$  seems to have been introduced by Sauter [27], who, in connection with the 'Klein paradox' [30], found a solution of the Dirac equation in the presence of a constant and uniform electric field and showed<sup>28</sup> that the probability of the production of  $e^+e^-$  pairs per unit volume per unit time is  $w \propto \exp(-\pi \mathcal{E}_{cr}/\mathcal{E})$ .

(6) Finally, we estimate how much energy W is contained in a magnetic field. Noting that  $B_a^2 a_B^3 = m_e c^2$  and changing to the dimensionless variables  $\mathcal{H} = B/B_a$  and  $l = L/a_B (L^3$  is the volume occupied by the magnetic field), we find

$$W \sim B^2 L^3 = \mathcal{H}^2 l^3 m_{\rm e} c^2 , \qquad \mathcal{H} \gg 1 , \qquad (B.7)$$

where  $m_c c^2 = 511 \text{ keV} \equiv 9.11 \times 10^{-28} \text{ g}$ . Hence, for  $B = B_a$ and  $L = a_B$ , we have  $W \sim m_c c^2 \approx 10^4 \text{ Ry}$  (we recall that the characteristic atomic level energies are less than 1 Ry), and for  $B = B_{cr} = \alpha^{-2}B_a$ , the magnetic energy is  $W \approx 137 m_e c^2$  for  $L = 1/m_e = \alpha a_B$  and  $W \approx 3 \times 10^8 m_e c^2$  for  $L = a_B$ . Finally, for a magnetized neutron star with  $B \sim B_{cr}$ ,  $L \sim 10 \text{ km}$  $(l \sim 10^{14})$ , we obtain the estimate  $W \approx 10^{50} m_e c^2$  (which

<sup>&</sup>lt;sup>27</sup> A special class of neutron stars. Among the astrophysical objects currently known, the magnetar SGR (Soft Gamma Repeater) 1806-20 seems to have the strongest magnetic field ( $2 \times 10^{15}$  G on the surface). <sup>28</sup> To exponential accuracy, in the semiclassical approximation.

corresponds to a meteorite with a mass  $M \sim 10^{23}$  g; for comparison, the mass of Earth is  $M \approx 6 \times 10^{27}$  g), and even for a more moderate  $B \sim B_a$  but a radius of the same order (10 km), we have  $W \approx 10^{42} m_e c^2 \sim 10^{15}$  g.

(7) In conclusion, we note that in Ref. [99], the problem of two interacting relativistic particles  $(q\bar{q})$  in ultrastrong magnetic fields was solved.

#### C. Coulomb corrections to the scattering length

The Coulomb interaction  $U_{\rm C}(r) = -\zeta e^2/r$  renormalizes the parameters  $a_l$  and  $r_l$  of low-energy scattering on the 'strong' (short-range) potential

$$V_{\rm sh}(r) = -\frac{g}{2r_0^2} v\left(\frac{r}{r_0}\right), \quad \hbar = m = 1,$$
 (C.1)

in the range  $r_0 \ll a_{\rm B} = \hbar^2/me^2$ . The Coulomb renormalization problem,  $a_{\rm s}^{(l)} \rightarrow a_{\rm cs}^{(l)}$ , requires the (necessarily numerical) solution of the Schrödinger equation with the potential  $U(r) = V_{\rm sh}(r) + U_{\rm C}$  and orbital momentum *l*. However, in our particular case, the potential  $V_{\rm sh}(r)$  produces a shallow level (which can be real, virtual, or quasistationary with  $l \ge 1$ and which perturbs the Coulomb spectrum)<sup>29</sup> and the lowenergy scattering has a resonant nature [6], and therefore the general renormalization formulas [29, 100, 101] considerably simplify. For the s states, we have [102]

$$\frac{1}{a_{\rm cs}} - \frac{1}{a_{\rm s}} = -\frac{2\zeta}{a_{\rm B}} \left[ \ln \frac{r_{\rm s}}{a_{\rm B}} + c_0 + O\left(\frac{r_{\rm s}}{a_{\rm B}}, \frac{r_{\rm s}}{a_{\rm s}}\right) \right],$$

$$c_0 = 2\gamma + \ln \frac{2r_{\rm C}}{r_{\rm s}}, \quad \gamma = 0.5772\dots.$$
(C.2)

(it is assumed that the effective radius  $r_s \ll a_B$ ,  $r_s \ll a_s$ ). Corrections to Eqn (C.2) that are linear in  $r_s$  can be found in Ref. [103]. In [102], Eqn (C.2) is extended to angular momenta  $l \neq 0$ :

$$\frac{1}{a_{\rm cs}^{(l)}} - \frac{1}{a_{\rm s}^{(l)}} = -\frac{2\zeta}{a_{\rm B}} \left[ \frac{(2l)!}{2^l l!} \right]^2 \int_0^\infty \chi_l^2(r) \, \frac{\mathrm{d}r}{r} + \dots \, (C.3)$$

In the above formulas,<sup>30</sup>  $\zeta = -\text{sgn}(Z_1Z_2)$ , and  $r_s$  and  $r_c$  are respectively the effective [6] and Coulomb [40, 103] radii of the system at the moment a bound s state  $(l = 0, g = g_0)$  is created:

$$r_{0} = 2 \int_{0}^{\infty} \left[ 1 - \chi_{0}^{2}(r) \right] dr , \quad \chi_{0}(0) = 0 , \quad \chi_{0}(\infty) = 1 ,$$
  

$$r_{C} = \exp\left\{ \ln R + \int_{0}^{\infty} \left[ \Theta(r - R) - \chi_{0}^{2}(r) \right] \frac{dr}{r} \right\}$$
(C.4)

(it is readily seen that  $r_{\rm C}$  is independent of the choice of R > 0). A point to note in connection with Eqns (C.3) and (C.4) is that the scattering length has the dimension of length [L] only in the case of an s wave and that in general the parameters involved have the following dimensions:

$$[a_l] = L^{2l+1}, \quad [r_l] = L^{1-2l}, \quad [\chi_l] = L^{-l}.$$
 (C.5)

At the moment when a bound *l* level  $(E_l = 0, g = g_l)$  arises, the functions  $\chi_l(r)$ , l = 0, 1, 2, ..., in Eqns (C.3) and (C.4) satisfy the Schrödinger equation

$$\frac{d^2 \chi_l}{dx^2} + \left[ g_l v(x) - \frac{l(l+1)}{x^2} \right] \chi_l = 0, \qquad x = \frac{r}{r_0}, \qquad (C.6)$$

with the boundary conditions

$$\chi_l(r) \propto r^{l+1} \quad \text{as} \quad r \to 0 ,$$
  
 $\lim \left( r^l \chi_l(r) \right) = 1 \quad \text{as} \quad r \to \infty .$ 
(C.7)

For the coupling constant g significantly different from  $g_l$ , renormalization has a more complicated form.

In the case of s states, according to Eqn (C.2), the main parameters to determine the renormalization of  $a_s$  are  $r_C$  and  $r_s$  and their ratio  $\rho = r_C/r_s$  in the formula for  $c_0$ . Below are the numerical values of these parameters for a number of widely used model potentials.

For the Yukawa potential,

$$v(x) = rac{\exp(-x)}{x}$$
,  $g_0 = 1.680$ ,  $r_s = 2.12 r_0$ ,  
 $ho = 0.364$ ,  $c_0 = 0.837$ .

For the Hulthen potential,

$$u(x) = (\exp x - 1)^{-1}, \quad g_0 = 1, \quad r_s = 3r_0,$$
  
 $u = 0.374, \quad c_0 = 0.865.$ 

For the Gauss potential,

ŀ

$$v(x) = \exp(-x^2), \quad g_0 = 2.684,$$
  
 $r_s = 1.44r_0, \quad \rho = 0.418, \quad c_0 = 0.976.$ 

Further examples can be found in Refs [103, 104].

For l = 0, the Coulomb renormalization of the scattering length contains a 'large' logarithm ln  $(r_s/a_B)$ , which is absent for  $l \neq 0$ , and hence this renormalization is especially large for the s states. One example is the nucleon–nucleon pn and pp scattering in the singlet <sup>1</sup>S<sub>0</sub> state, where [105]

$$a_{\rm s}({\rm pn}) = -23.75$$
,  $a_{\rm cs} = -7.84$ ,  
 $r_{\rm s} = 2.75$ ,  $r_{\rm cs} = 2.77$ ,  $a_{\rm B} = 57.8$ ,  
(C.8)

with all values given in Fm units. The large difference in the scattering length between pn and pp scattering does not imply a strong violation of the isotopic invariance of the nuclear interaction of nucleons and is naturally accounted for by including the Coulomb interaction in the pp system. The percent difference between the effective interaction radii of the systems under consideration is due to the small violation of the isotopic invariance by the Coulomb interaction of the protons.

In conclusion, we note that for  $l \neq 0$ , a term singular as  $r \rightarrow 0$  and proportional to  $\ln (r_0/a_B)$  enters higher-order terms in the expansion of the difference  $1/a_{cs}^{(l)} - 1/a_{s}^{(l)}$ . For example, in the right-hand side of Eqn (C.3), the singular term has the form [106]

$$\frac{1}{(l!)^2} \zeta^{2l+1} \ln (\zeta r_0) \sim \frac{1}{(l!)^2} \left(\frac{r_0}{a_{\rm B}}\right)^{2l+1} \ln \frac{r_0}{r_{\rm B}} \,. \tag{C.9}$$

<sup>&</sup>lt;sup>29</sup> Meaning that the Zeldovich effect arises.

<sup>&</sup>lt;sup>30</sup> For a hadron p $\bar{p}$  atom,  $\zeta = 1$ , for a pp system,  $\zeta = -1$ . For s states, the index l = 0 on the scattering length is dropped throughout.

The renormalization of the effective radius  $r_s \rightarrow r_{cs}$ , even in the case of an s wave, contains the logarithm  $\ln (r_0/a_B)$  only in the correction term [see Eqn (C.8)].

#### References

- 1. Garstang R H Rep. Prog. Phys. 40 105 (1977)
- 2. Lai D Rev. Mod. Phys. 73 629 (2001)
- 3. Harding A K, Lai D Rep. Prog. Phys. 69 2631 (2006)
- 4. Lattimer J M, Prakash M Phys. Rep. 442 109 (2007)
- 5. Shiff L I, Snyder H Phys. Rev. 55 59 (1939)
- Landau L D, Lifshitz E M Quantum Mechanics: Non-Relativistic Theory (Oxford: Pergamon Press, 1977); Translated from Russian: Kvantovaya Mekhanika: Nerelyativistskaya Teoriya (Moscow: Fizmatlit, 2002)
- 7. Bureeva L A, Lisitsa V S Vozmushchennyi Atom (Perturbed Atom) (Moscow: IzdAT, 1997)
- Liberman M A, Johansson B Phys. Usp. 38 117 (1995); Usp. Fiz. Nauk 165 121 (1995)
- 9. Lindgren K A U, Virtamo J T J. Phys. B At. Mol. Phys. 12 3465 (1979)
- 10. Virtamo J T, Lindgren K A U Phys. Lett. A 71 329 (1979)
- 11. Goldman S P, Chen Z Phys. Rev. Lett. 67 1403 (1991)
- 12. Chen Z, Goldman S P Phys. Rev. A 48 1107 (1993)
- 13. Wang J-H, Hsue C-S Phys. Rev. A 52 4508 (1995)
- 14. Lai D, Salpeter E E *Phys. Rev. A* **53** 152 (1996)
- 15. Kravchenko Yu P, Liberman M A, Johansson B *Phys. Rev. A* **54** 287 (1996)
- Potekhin A Yu, Dissertation for Doctor of Physics and Mathematics (St. Petersburg: Ioffe Physical Technical Institute of the Russian Academy of Sciences, 2009)
- 17. Loudon R Am. J. Phys. 27 649 (1959)
- 18. Andrews M Am. J. Phys. 34 1194 (1966)
- 19. Elliott R J, Loudon R J. Phys. Chem. Solids 15 196 (1960)
- 20. Hasegawa H, Howard R E J. Phys. Chem. Solids 21 179 (1961)
- 21. Khriplovich I B, Ruban G Yu Laser Phys. 14 426 (2004)
- 22. Karnakov B M JETP Lett. 77 68 (2003); Pis'ma Zh. Eksp. Teor. Fiz. 77 73 (2003)
- Popov V S JETP Lett. 77 74 (2003); Pis'ma Zh. Eksp. Teor. Fiz. 77 79 (2003)
- 24. Karnakov B M, Popov V S JETP 97 890 (2003); Zh. Eksp. Teor. Fiz. 124 996 (2003)
- 25. Popov V S, Karnakov B M JETP 114 1 (2012); Zh. Eksp. Teor. Fiz. 141 5 (2012)
- 26. Potekhin A Y, Turbiner A V Phys. Rev. A 63 065402 (2001)
- 27. Sauter F Z. Phys. 69 742 (1931)
- 28. Heisenberg W, Euler H Z. Phys. 98 714 (1936)
- 29. Schwinger J Phys. Rev. 82 664 (1951)
- Akhiezer A I, Berestetskii V B Quantum Electrodynamics (New York: Interscience Publ., 1965); Translated from Russian: Kvantovava Elektrodinamika 4th ed. (Moscow: Nauka, 1981)
- Baz' A I, Zel'dovich Ya B, Perelomov A M Scattering, Reactions and Decay in Nonrelativistic Quantum Mechanics (Jerusalem: Israel Program for Scientific Translations, 1969); Translated from Russian: Rasseyanie, Reaktsii i Raspady v Nerelyativistskoi Kvantovoi Mekhanike 2nd ed. (Moscow: Nauka, 1971)
- Demkov Yu N, Ostrovskii V N Zero-range Potentials and Their Applications in Atomic Physics (New York: Plenum Press, 1988); Translated from Russian: Metod Potentsialov Nulevogo Radiusa v Atomnoi Fizike (Leningrad: Izd. LGU, 1975)
- Vysotsky M I JETP Lett. 92 15 (2010); Pis'ma Zh. Eksp. Teor. Fiz. 92 22 (2010)
- 34. Machet B, Vysotsky M I Phys. Rev. D 83 025022 (2011)
- 35. Godunov S I, Machet B, Vysotsky M I Phys. Rev. D 85 044058 (2012)
- Galitski V, Karnakov B, Kogan V, Galitski V (Jr.) Exploring Quantum Mechanics: A Collection of 700+ Solved Problems for Students, Lectures, and Researches (Oxford: Oxford Univ. Press, 2013); Galitskii V M, Karnakov B M, Kogan V I Zadachi po Kvantovoi Mekhanike 2nd ed. (Moscow: Nauka, 1992); Zadachi po Kvantovoi Mekhanike 3rd ed. (Moscow: URSS, 2001)
- 37. Case K M Phys. Rev. 80 797 (1950)

- Perelomov A M, Popov V S Theor. Math. Phys. 4 664 (1970); Teor. Mat. Fiz. 4 (1) 48 (1970)
- Popov V S, Kudryavtsev A E, Mur V D Sov. Phys. JETP 50 865 (1979); Zh. Eksp. Teor. Fiz. 77 1727 (1979)
- Popov V S et al. Sov. Phys. JETP 53 650 (1981); Zh. Eksp. Teor. Fiz. 80 1271 (1981)
- Bateman H Higher Transcendental Functions (Director A Erdélyi) Vol. 1 (New York: McGraw-Hill, 1953); Translated into Russian: Bateman H, Erdélyi A Vysshie Transtsendentnye Funktsii Vol. 1 (Moscow: Nauka, 1973)
- 42. Shabad A E, Usov V V Phys. Rev. Lett. 98 180403 (2007)
- 43. Shabad A E, Usov V V Phys. Rev. D 77 025001 (2008)
- 44. Vysotsky M I, Godunov S I Phys. Usp. 57 194 (2014); Usp. Fiz. Nauk 184 206 (2014)
- 45. Popov V S JETP Lett. 11 162 (1970); Pis'ma Zh. Eksp. Teor. Fiz. 11 254 (1970)
- 46. Popov V S Sov. J. Nucl. Phys. 12 235 (1971); Yad. Fiz. 12 429 (1970)
- 47. Popov V S Sov. Phys. JETP **32** 526 (1971); Zh. Eksp. Teor. Fiz. **59** 965 (1970)
- 48. Zeldovich Ya B, Popov V S Sov. Phys. Usp. 14 673 (1972); Usp. Fiz. Nauk 105 403 (1971)
- 49. Gershtein S S, Popov V S Lett. Nuovo Cimento 6 593 (1973)
- 50. Popov V S Phys. Atom. Nucl. 64 367 (2001); Yad. Fiz. 64 421 (2001)
- 51. Ternov I M et al. Sov. Phys. JETP 28 1206 (1969); Zh. Eksp. Teor. Fiz. 55 2273 (1968)
- 52. Newton R G Phys. Rev. D 3 626 (1971)
- 53. Zel'dovich Ya B et al. Sov. Phys. Solid State 1 1497 (1960); Fiz. Tverd. Tela 1 1637 (1959)
- Zeldovich Ya B Selected Works of Yakov Borisovich Zeldovich Vol. 2 Particles, Nuclei, and the Universe (Princeton, N.J.: Princeton Univ. Press, 1993); Translated from Russian: Chastitsy, Yadra, Vselennaya: Izbrannye Trudy (Moscow: Nauka, 1985)
- Popov V S Sov. Phys. JETP 33 665 (1971); Zh. Eksp. Teor. Fiz. 60 1228 (1971)
- Kudryavtsev A E, Markushin V E, Shapiro I S Sov. Phys. JETP 47 225 (1978); Zh. Eksp. Teor. Fiz. 74 432 (1978)
- Kudryavtsev A E, Popov V S JETP Lett. 29 280 (1979); Pis'ma Zh. Eksp. Teor. Fiz. 29 311 (1979)
- Karnakov B M, Kudryavtsev A E, Mur V D, Popov V S Nuovo Cimento B 98 63 (1987)
- Karnakov B M, Kudryavtsev A E, Mur V D, Popov V S Sov. Phys. JETP 67 1333 (1988); Zh. Eksp. Teor. Fiz. 94 65 (1988)
- 60. Badalyan A M et al. Phys. Rep. 82 31 (1982)
- 61. Deser S et al. *Phys. Rev.* **96** 744 (1954)
- 62. Karnakov B M, Mur V D Sov. Phys. JETP 60 657 (1984); Zh. Eksp. Teor. Fiz. 87 1150 (1984)
- 63. Karnakov B M, Krainov V P *WKB Approximation in Atomic Physics* (Heidelberg: Springer, 2013)
- 64. Schwinger J Phys. Rev. 78 135 (1950)
- Karnakov B M, Mur V D, Popov V S Sov. Phys. Dokl. 29 938 (1984); Dokl. Akad. Nauk SSSR 279 345 (1984)
- 66. Karnakov V M, Mur V D, Popov V S Yad. Fiz. 41 1258 (1985)
- Andreev S P, Karnakov B M, Mur V D JETP Lett. 37 187 (1983); Pis'ma Zh. Eksp. Teor. Fiz. 37 155 (1983)
- Andreev S P, Karnakov B M, Mur V D, Polunin V A Sov. Phys. JETP 59 506 (1984); Zh. Eksp. Teor. Fiz. 86 866 (1984)
- Gor'kov L P, Dzyaloshinskii I E Sov. Phys. JETP 26 449 (1968); Zh. Eksp. Teor. Fiz. 53 717 (1967)
- 70. Potekhin A Y J. Phys. B At. Mol. Opt. Phys. 27 1073 (1994)
- 71. Potekhin A Y J. Phys. B At. Mol. Opt. Phys. 31 49 (1998)
- 72. Potekhin A Y, Pavlov G G Astrophys. J. 483 414 (1997)
- 73. Pomeranchuk I, Smorodinsky Ya J. Phys. USSR 9 97 (1945)
- Pomeranchuk I, Smorodinsky Ya, in Pomeranchuk I Ya Sobranie Nauchnykh Trudov (Collected Works) Vol. 2 (Moscow: Nauka, 1972) p. 21
- 75. Pieper W, Greiner W Z. Phys. 218 327 (1969)
- 76. Brodsky S J Comm. Atom. Mol. Phys. 4 109 (1973)
- 77. Greiner W, Müller B, Rafelski J Quantum Electrodynamics of Strong Fields (Berlin: Springer-Verlag, 1985)
- 78. Schluter P et al. J. Phys. B At. Mol. Phys. 18 1685 (1985)
- 79. Popov V S AIP Conf. Proc. 1205 132 (2010)
- Marinov M S, Popov V S JETP Lett. 17 368 (1973) Pis'ma Zh. Eksp. Teor. Fiz. 17 511 (1973)

- Popov V S JETP Lett. 16 251 (1972); Pis'ma Zh. Eksp. Teor. Fiz. 16 355 (1972)
- 82. Müller B, Rafelski J, Greiner W Phys. Lett. B 47 5 (1973)
- 83. Rafelski J, Müller B Phys. Lett. B 65 205 (1976)
- Marinov M S, Popov V S, Stolin V L JETP Lett. 19 49 (1974); Pis'ma Zh. Eksp. Teor. Fiz. 19 76 (1974)
- 85. Lisin V I, Marinov M S, Popov V S Phys. Lett. B 69 141 (1977)
- 86. Lisin V I, Marinov M S, Popov V S Phys. Lett. B 91 20 (1980)
- 87. Okun' L B Comments Nucl. Part. Phys. 6 25 (1974)
- Popov V S, Eletskii V L, Mur V D Sov. Phys. JETP 44 451 (1976); Zh. Eksp. Teor. Fiz. 71 856 (1976)
- Oraevskii V N, Rex A I, Semikoz V B Sov. Phys. JETP 45 428 (1977); Zh. Eksp. Teor. Fiz. 72 820 (1977)
- Sakharov A D Sov. Phys. Dokl. 10 1045 (1966); Dokl. Akad. Nauk SSSR 165 65 (1965)
- 91. Sakharov A D Sov. Phys. Usp. 9 294 (1966); Usp. Fiz. Nauk 88 725 (1966)
- 92. Sakharov A D *Nauchnye Trudy* (Collected Works) (Moscow: OTF FIAN. Tsentrkom, 1995)
- 93. Pulsar Power Conf. 2 746 (1999)
- Pavlovskii A I, in Sakharov A D Nauchnye Trudy (Collected Workds) (Moscow: OTF FIAN. Tsentrkom, 1995) p. 85
- 95. Popov V S, Karnakov B M, Mur V D JETP 88 902 (1999); Zh. Eksp. Teor. Fiz. 115 1642 (1999)
- Itakura K, Iso S, Takahashi T (Eds) Proc. of Intern. Conf. on Physics in Intense Fields, PIF 2010, 24 – 26 November 2010, KEK, Thukuba, Japan (KEK Proc. 2010-13) (Tsukuba, Japan: KEK, 2011)
- 97. Sarri G et al. Phys. Rev. Lett. 109 205002 (2012)
- 98. Wagner U et al. Phys. Rev. E 70 026401 (2004)
- Kerbikov B O, Andreichikov M A JETP Lett. 96 361 (2012); Pis'ma Zh. Eksp. Teor. Fiz. 96 397 (2012)
- Landau L, Smorodinskii Ya J. Phys. USSR 8 154 (1944);
   Landau L D, Smorodinskii Ya A Zh. Eksp. Teor. Fiz. 14 269 (1944)
- Goldberger M L, Watson K M Collision Theory (New York: Wiley, 1964); Translated into Russian: Teoriya Stolknovenii (Moscow: Mir, 1967)
- 102. Kudryavtsev A E, Mur V D, Popov V S Sov. J. Nucl. Phys. 37 844 (1983); Yad. Fiz. 37 1417 (1983)
- Mur V D, Popov V S, Sergeev A V Phys. Atom. Nucl. 59 62 (1996); Yad. Fiz. 59 68 (1996)
- Mur V D, Dissertation for Doctor of Physics and Mathematics (Moscow: Moscow Engineering Physics Institute, 1991)
- 105. Dumbrajs O et al. Nucl. Phys. B 216 277 (1983)
- 106. Mur V D, Popov V S Sov. Phys. Dokl. 20 782 (1985); Dokl. Akad. Nauk SSSR 284 586 (1985)