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# Turbulent flows at very large Reynolds numbers: new lessons learned

G I Barenblatt, A J Chorin, V M Prostokishin

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## Contents



**Abstract.** **The universal (Reynolds-number-independent) von Kármán–Prandtl logarithmic law for the velocity distribution in the basic intermediate region of a turbulent shear flow is generally considered to be one of the fundamental laws of engineering science and is taught universally in fluid mechanics and hydraulics courses. We show here that this law is based on an assumption that cannot be considered to be correct and which does not correspond to experiment. Nor is Landau's derivation of this law quite correct. In this paper, an alternative scaling law explicitly incorporating the influence of the Reynolds number is discussed, as is the corresponding drag law. The study uses the concept of intermediate asymptotics and that of incomplete similarity in the similarity parameter. Yakov Borisovich Zeldovich played an outstanding role in the development of these ideas. This work is a tribute to his glowing memory.**

**G I Barenblatt** Shirshov Institute of Oceanology, Russian Academy of Sciences, Nakhimovskii prosp. 36, 119997 Moscow, Russian Federation; University of California, Berkeley, CA 94720, USA; Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94720, USA
Tel. +7 (499) 380 41 52
E-mail: gibar@math.berkeley.edu, gibarenblatt@gmail.com
**A J Chorin** University of California, Berkeley, CA 94720, USA; Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94720, USA
E-mail: chorin@math.berkeley.edu
**V M Prostokishin** Shirshov Institute of Oceanology, Russian Academy of Sciences, Nakhimovskii prosp. 36, 119997 Moscow, Russian Federation; Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94720, USA; National Research Nuclear University MEPhI, Kashirskoe shosse 31, 115409 Moscow, Russian Federation
E- mail: VMProstokishin@mephi.ru, v.m.prost@gmail.com

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## 1. Introduction: the turbulence problem

Turbulence is the state of a vortex flow of a viscous fluid where the velocity, pressure, and other flow field characteristics vary in time and space sharply and irregularly, such that they can be treated as random. The basic dimensionless parameter governing a turbulent flow is the Reynolds number $\mathrm{Re} = UL/\nu$. Here, $U$ is the characteristic flow velocity, $L$ is the characteristic linear scale, and $\nu$ is the kinematic viscosity of the fluid.

First recognized by Leonardo da Vinci, turbulence has been intensively studied by engineers, mathematicians, and physicists for the past one and a half centuries. And yet, one has to concede that in what concerns the creation of a closed theory of turbulent flows, almost nothing has been obtained from first principles, i.e., the Navier–Stokes equations and the continuity equation, even in the simplest case of an incompressible fluid.

Turbulence at very large Reynolds numbers, $\ln \mathrm{Re} \gg 1$, is traditionally regarded as one of the safer provinces in the realm of turbulence. It was presumed, and is still presumed by many, that two basic results obtained in this field are well established and will enter, basically unchanged, a future complete and self-consistent theory of turbulence. These results are essentially the von Kármán–Prandtl universal logarithmic law for the velocity field in the wall (intermediate) region of a turbulent shear flow and the Kolmogorov–Obukhov 2/3 scaling law for the local structure of 'developed' ($\ln \mathrm{Re} \gg 1$) turbulent flows. These laws, and to a larger extent the former one, are discussed here.

## 2. Developed turbulence. The basic assumption of von Kármán. The universal logarithmic law

The beginning of fundamental research of developed ($\ln \mathrm{Re} \gg 1$) turbulent flows can be dated precisely: it was the plenary talk by von Kármán, "Mechanical similitude and turbulence," given at the Third International Congress for Applied Mechanics on 25 August 1930. Von Kármán considered shear (hydraulic, in his terminology) steady flows, homogeneous in the direction of mean velocity and bounded by a rigid wall. The velocity in such flows depends only on the distance to the wall. These flows include practically important flows in long cylindrical pipes, in boundary layers over plates in a uniform flow far from their leading edge, and in channels.

The main hypothesis adopted by von Kármán was formulated by him in the following form: "On the basis of these experimentally well-established facts, we make the assumption that away from the close vicinity of the wall, the velocity distribution of the mean flow is viscosity independent."

To be precise, we present this assumption in its original language [1]: "Wir gründen auf diese experimentell festgestellten Tatsachen die Annahme, dass, abgesehen von der Wandnähe, die Geschwindichkeitverteilung der mittleren Strömung vor der Zähigkeit unabhängig ist."

Relying on this assumption, in 1944 L D Landau gave the derivation of the universal logarithmic law, in the book *Mechanics of continuous media* [2], split later into two books *Fluid mechanics* [3] and *Elasticity theory*. For a flow in a smooth pipe (Fig. 1), this derivation can be carried out as follows. The longitudinal velocity gradient $\partial_y u$ at the distance $y$ from the wall depends on the following governing parameters: the pipe diameter $d$, the distance to the wall $y$, the shear stress $\tau$, and the fluid characteristics, its density $\rho$ and kinematic viscosity $\nu$:

$$\partial_y u = f(y, \tau, d, \rho, \nu)\,. \tag{1}$$

We introduce the 'dynamic velocity' $u_* = (\tau/\rho)^{1/2}$, which plays the role of a velocity scale in what follows. Dimensional analysis [4, 5] shows that the parameter $u_* d/\nu$ is a function of a single dimensionless parameter, the traditional 'global' Reynolds number $\mathrm{Re} = Ud/\nu$. Here, $U$ is the mean flow velocity — the ratio of the fluid volume flux to the pipe cross-sectional area. Further, dimensional analysis leads to the relation

$$\partial_y u = \frac{u_*}{y}\,\Phi\left(\mathrm{Re}, \frac{u_* y}{\nu}\right), \tag{2}$$

or, in a dimensionless form,

$$\partial_\eta \varphi = \frac{1}{\eta}\,\Phi(\mathrm{Re}, \eta)\,. \tag{3}$$

Here, $\varphi = u/u_*$, and $\eta = u_* y/\nu$ is the local Reynolds number. Relations (2) and (3) are indubitable. The independence from viscosity assumed by von Kármán literally implies that because the viscosity enters both Reynolds numbers, the global and local ones, the function $\Phi$ is a universal constant, commonly written as $1/\kappa$; relatedly, $\kappa$ has the name von Kármán constant. Relation (3) takes the form

$$\partial_\eta \varphi = \frac{1}{\kappa\eta}\,. \tag{4}$$

Integrating Eqn (4) leads to

$$\varphi = \frac{1}{\kappa}\ln\eta + B\,, \tag{5}$$

where $B$ is also assumed to be a universal constant. For constants $\kappa$ and $B$, Landau chose the values $\kappa = 0.4$ and $B = 5.1$, and hence, in the dimensional form, law (5) becomes

$$u = u_*\left(2.5\ln\frac{u_* y}{\nu} + 5.1\right). \tag{6}$$

Relation (6) is referred to in the literature as the von Kármán–Prandtl universal logarithmic law. It is traditionally regarded as one of the fundamental laws in engineering science: von Kármán and Prandtl managed to persuade the engineering and scientific communities that this law is fundamental, while power laws used earlier in engineering practice with power exponents dependent on the Reynolds number are nothing more than empirical relations. We show below that just the opposite is true.

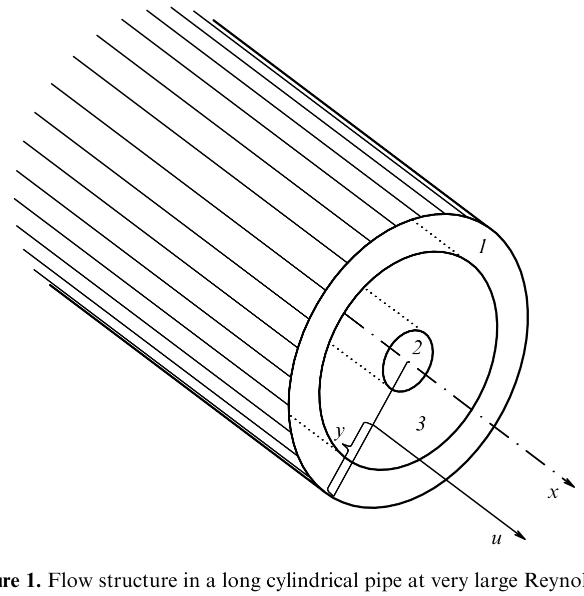


**Figure 1.** Flow structure in a long cylindrical pipe at very large Reynolds numbers.

## 3. Universal logarithmic law disagrees with experimental data

We turn to experimental facts. According to the universal logarithmic law, the constants $\kappa$ and $B$ involved in it should be the same for all experiments carried out in smooth pipes. But the analysis of available experimental data, obtained in reputable laboratories, fails to support this. A logarithmic dependence of the dimensionless velocity on the local Reynolds number has indeed been observed within certain intervals in many series of experiments. And yet, even with the most liberal approach, the values of $\kappa$ and $B$ found experimentally cannot be perceived as universal. The constant $\kappa$ taken to be 0.4 by Landau was found to range between 0.3 and 0.45 in various experiments; the mean value of the constant $B$ taken by Landau as 5.1 ranges in experiments from 3.5 to 6.3. In other words, we can claim that logarithmic law (6) does not hold.

## 4. Cause of the mismatch: analysis and alternative proposals

A question arises as to what the defect of the derivation presented above is. In reality, there is a delicate mathematical issue that has escaped attention. Indeed, the function $\Phi$ [see Eqn (3)] depends on two dimensionless arguments: the global and local Reynolds numbers. But these are essentially different: in each given experiment, the global Reynolds number $\mathrm{Re} = Ud/\nu$ is constant and fixed, whereas the local Reynolds number $\eta = u_* y/\nu$ varies in wide limits from several units to many thousand, because it corresponds to the local value of the averaged velocity, which is measured by an experimentalist over the whole cross section. In the derivation above, the influence of the global Reynolds number was neglected, and it was implicitly assumed that there is a finite limit for the function $\Phi$ as $\eta \to +\infty$, and hence at large values of the local Reynolds number $\eta$, the function $\Phi$ can be replaced with a constant. In the existing terminology, this is referred to as the complete similarity in the local Reynolds number.

In our work, we abandoned the hypothesis that the velocity gradient is independent of the global Reynolds number. The hypothesis of incomplete similarity over the local Reynolds number was assumed, by virtue of which the function $\Phi$ for large $\eta$ is written as

$$\Phi = A(\mathrm{Re})\, \eta^{\alpha(\mathrm{Re})} \,. \tag{7}$$

Here, $A$ and $\alpha$ are some functions of Re to be determined. In a particular case of complete similarity, which led to the disagreement with experiment, $A = \mathrm{const}$ and $\alpha = 0$. There are obviously two behavior types for the function $\Phi$ for large $\eta$ linked to the existence or lack of a finite limit as $\eta \to +\infty$. The particular case of the second type is the incomplete similarity, whose existence is related to the invariance with respect to an additional renormalization group (see Refs [7, 8]).

Next, a hypothesis called the ''principle of vanishing viscosity'' has been assumed: *as* $\nu \to 0$*, the velocity gradient tends to a finite limit*. The hypotheses of incomplete similarity in the local Reynolds number $\eta = u_* y/\nu$ and of vanishing viscosity allow determining the function $\Phi$ up to three constants, whose universality has been tested in a large suite of experiments (about one thousand) carried out in various laboratories using different setups, and has been confirmed.

## 5. Alternative nonuniversal law

In agreement with the assumptions made, the function $\Phi$ at very large Re can be written as

$$\Phi = (A_0 + A_1 \alpha)\, \exp(\alpha \ln \eta) \,, \quad \eta = \frac{u_* y}{\nu} \,, \tag{8}$$

and $\alpha = \alpha(\mathrm{Re})$ tends to zero as $\mathrm{Re} \to \infty$. Here, $A_0$ and $A_1$ are universal constants. Three cases are possible. In the first, $\alpha(\mathrm{Re})$ tends to zero faster than $1/\ln \mathrm{Re}$, implying the complete similarity in both parameters Re and $\eta$ as they tend to infinity. As shown above, this contradicts the experiment. If $\alpha(\mathrm{Re})$ tends to zero more slowly than $1/\ln \mathrm{Re}$, then $\Phi$ tends to infinity as $\nu \to 0$, which contradicts the principle of vanishing viscosity. We have to accept that a special case is realized as $\mathrm{Re} \to \infty$:

$$\alpha = \frac{A_2}{\ln \eta} \,, \tag{9}$$

where $A_2$ is also a universal constant. Under all the assumptions made, the asymptotic relation for the dimensionless velocity gradient takes the form

$$\partial_\eta \varphi = \frac{1}{\eta} \left( A_0 + \frac{A_1 A_2}{\ln \mathrm{Re}} \right) \eta^{A_2/\ln \mathrm{Re}} \,. \tag{10}$$

Integrating Eqn (10) and adopting an additional assumption that

$$\varphi(0) = 0 \,, \tag{11}$$

which is substantiated experimentally, we arrive at a power (nonlogarithmic) law incorporating the dependence on the global Reynolds number, i.e., lacking universality:

$$\varphi = \left( \frac{A_0}{A_2} \ln \mathrm{Re} + A_1 \right) \eta^{A_2/\ln \mathrm{Re}} \,, \tag{12}$$

or, in dimensional form,

$$u = u_* \left( \frac{A_0}{A_2} \ln \mathrm{Re} + A_1 \right) \left( \frac{u_* y}{\nu} \right)^{A_2/\ln \mathrm{Re}} \,. \tag{13}$$

Processing the experimental data (see Ref. [9] and the references therein to other papers by the authors) lent support to the universality of the constants $A_0$, $A_1$, and $A_2$ and led to the following values:

$$A_0 = \frac{\sqrt{3}}{2} \,, \quad A_1 = \frac{5}{2} \,, \quad A_2 = \frac{3}{2} \,. \tag{14}$$

Hence, the nonuniversal power law proposed by us for the velocity profile in a pipe cross section (outside the close vicinity of the pipe wall and axis) becomes

$$u = u_* \left( \frac{1}{\sqrt{3}} \ln \mathrm{Re} + \frac{5}{2} \right) \left( \frac{u_* y}{\nu} \right)^{3/(2\ln \mathrm{Re})} \,, \tag{15}$$

or, in dimensionless form,

$$\varphi = \left( \frac{\sqrt{3} + 5\alpha}{2\alpha} \right) \eta^{\alpha} \,, \quad \alpha = \frac{3}{2 \ln \mathrm{Re}} \,. \tag{16}$$

We stress that the scaling (power) law (15) cannot be derived solely from dimensional analysis: its existence relies on the invariance with respect to an additional renormalization group.

## 6. Comparison of the universal logarithmic law and the alternative power law

We summarize the intermediate results.

*The universal logarithmic law* is expressed in the form

$$u = u_* (2.5 \ln \eta + 5.1) \,, \quad \varphi = \frac{u}{u_*} \,, \quad \eta = \frac{u_* y}{\nu} \tag{17}$$

(Landau and Lifshits [2, 3], Monin and Yaglom [6], and many others).

*The nonuniversal power law* is expressed as

$$\varphi = \left( \frac{\sqrt{3} + 5\alpha}{2\alpha} \right) \eta^{\alpha} \,, \quad \alpha = \frac{3}{2 \ln \mathrm{Re}} \,. \tag{18}$$

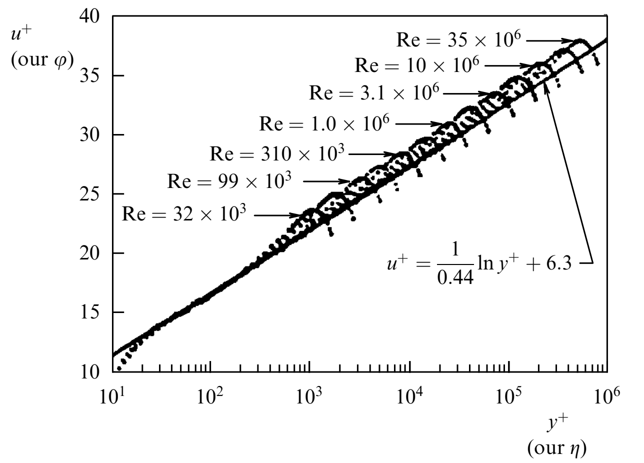


**Figure 2.** The universal logarithmic law is not confirmed by Zagarola's experiments.

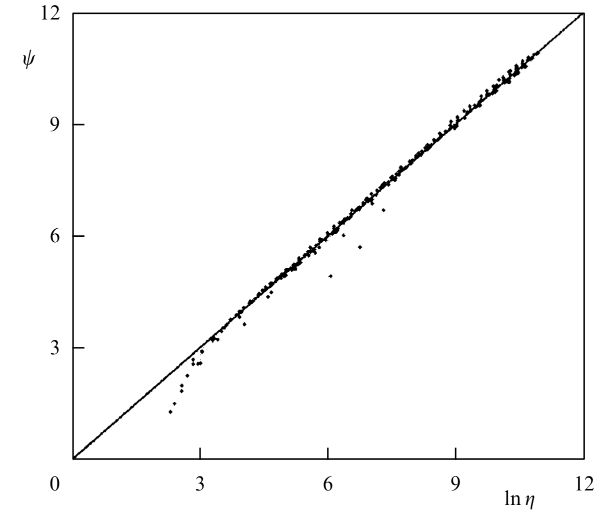


**Figure 3.** Nikuradse's data confirm the new power law.

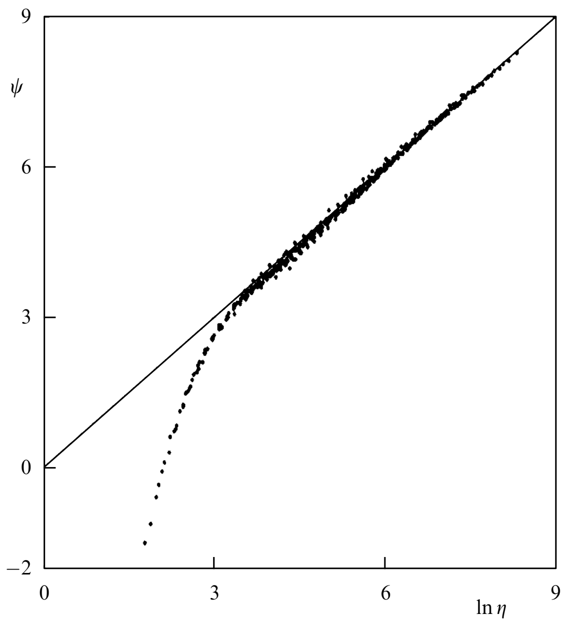


**Figure 4.** Zagarola's data confirm the new power law.

We have estimated constants (14) of power laws (15) and (18) by processing experiments by Nikuradse [10] carried out in Prandtl's laboratory at the University of Göttingen (Germany); the results of these experiments have been presented by the author in numerical form. We emphasize that the values of constant (14) are understood as fixed and cannot be subject to tuning in various experiments. Power law (18) can obviously be written in the form

$$\psi = \ln \eta \,, \quad \text{where } \psi = \frac{1}{\alpha} \ln \frac{2\alpha\varphi}{\sqrt{3} + 5\alpha} \,, \quad \alpha = \frac{3}{2 \ln \mathrm{Re}} \,. \tag{19}$$

In 1996, more than 60 years after Nikuradse, at Princeton University, Zagarola [11], using a fundamentally new setup called Superpipe, carried out a large series of experiments on turbulent flow in pipes, reaching a quality comparable to Nikuradse's experiments of 1932. The idea for this setup was suggested by the outstanding experimentalist G Brown. In contrast to Nikuradse, who worked with a water flow, Zagarola used a gas flow under high pressure. Figure 2 reproduces the results of Zagarola's experiments on the Superpipe [11]. It is seen that they do not confirm the universal logarithmic law: there is a clear splitting of the experimental data over the Reynolds number, while the experimental points do not fit the straight line $\varphi = (1/0.44) \ln \eta + 6.3$, expressing, in the author's opinion, the universal logarithmic law; the constants describing this line are noticeably different from those generally accepted in the literature (see above).

At the same time, fitting the experimental data by power law (19) in both cases (in the experiments by both Nikuradse and Zagarola) reveals a convincing agreement: beginning from $\ln \eta \sim 3$, the experimental points (except four out of more than 250 in Nikuradse's experiments) fit the unique line $\psi = \ln \eta$ of the power law, the bisector (Fig. 3) of the first quadrant. We note that we have elucidated the role of roughness at large Reynolds numbers in the Superpipe setup (see Ref. [9]).

We stress one more principal difference between the universal and power laws: in the coordinates ($\ln \eta$, $\varphi$), the experimental points according to the universal logarithmic law should be fitted by the unique line $\varphi = (1/0.4) \ln \eta + 5.1$. This does not find support in experimental data (see Fig. 2 [11]). At the same time, according to the power law, the experimental points in the coordinates ($\ln \eta$, $\psi$) should lie on a straight line, which is corroborated by the experiments of Nikuradse and Zagarola (Fig. 3 and 4). The experimental points should occupy some domain in the coordinates ($\ln \eta$, $\varphi$), which is confirmed by the data in Fig. 2.

## 7. Envelope of a family of power laws and intermediate nonuniversal logarithmic laws

The family of power-law curves (15) has an envelope, as indicated by the analysis (see, e.g., [12] and the references therein to the previous work by the authors). In the range of Reynolds numbers explored by Nikuradse, this curve is close to the straight line $\varphi = 2.5 \ln \eta + 5.1$ in the coordinates ($\ln \eta$, $\varphi$). Apparently, the measurements performed (or the data presented) in Nikuradse's work are only those that were close to the envelope, which led the author and those who followed him to assert the validity of the universal logarithmic law. The

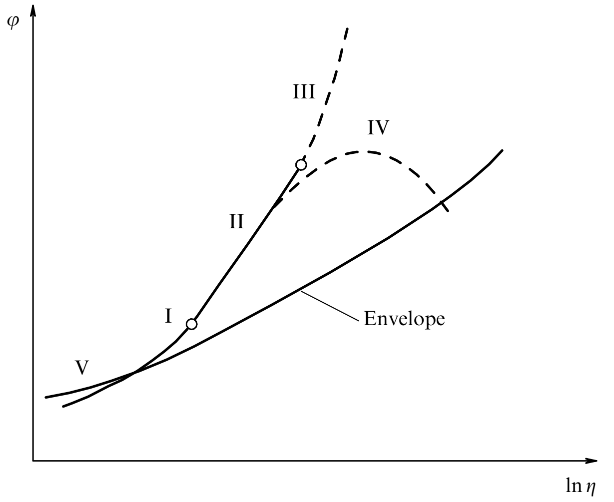


**Figure 5.** Envelope of power laws and nonuniversal logarithmic laws.

actual picture is as follows (Fig. 5, fragment II): In the plane $(\ln\eta, \varphi)$, the curves of the power-law family have intermediate asymptotic intervals described by the formula

$$\varphi = \exp\left(\frac{3}{2}\right)\left[\left(\frac{\sqrt{3}}{2}+\frac{15}{4\ln \mathrm{Re}}\right)\ln\eta - \frac{1}{2\sqrt{3}}\ln \mathrm{Re} - \frac{5}{4}\right], \tag{20}$$

i.e., by relations in the form of logarithmic law (12),

$$\varphi = \frac{1}{\kappa_{\mathrm{eff}}(\mathrm{Re})}\ln \mathrm{Re} + B_{\mathrm{eff}}(\mathrm{Re}), \tag{21}$$

but with the constants

$$\kappa_{\mathrm{eff}}(\mathrm{Re}) = \frac{\exp(-3/2)}{\left(\sqrt{3}/2 + 15/(4\ln \mathrm{Re})\right)},$$

$$B_{\mathrm{eff}}(\mathrm{Re}) = -\exp\left(\frac{3}{2}\right)\left(\frac{\ln \mathrm{Re}}{2\sqrt{3}}+\frac{5}{4}\right), \tag{22}$$

depending on the global Reynolds number.

As $\mathrm{Re}\to\infty$, the value of $\kappa(\mathrm{Re})$ tends very slowly to the limit $\kappa_\infty = 2/(\sqrt{3}\exp(3/2)) \approx 0.2776$, and the additive constant $B(\mathrm{Re})$ tends to $-\infty$. Thus, if the measurement were carried out at points distant from the envelope, the plots in coordinates $(\ln\eta, \varphi)$ would contain linear intervals with the slope being substantially different from the slope of the envelope, with their positions dependent on the Reynolds number. This, in fact, was revealed by Zagarola in experiments on the Superpipe (see Fig. 2). But for reasons not fully clear, Zagarola interpreted his results as supporting the von Kármán–Prandtl logarithmic law, albeit with different values of the constants, $\kappa = 0.44$ and $B = 6.3$; by contrast, we here see an unambiguous confirmation of the nonuniversality.

## 8. Analysis of turbulent flows in a boundary layer

The analysis of the distribution of the averaged longitudinal velocity in the intermediate domain in smooth pipes should, according to the underlying idea, be valid for arbitrary shear flows bounded by a smooth rigid wall, for very large Reynolds numbers. More precisely, our prediction is formulated as follows: in the intermediate region of an arbitrary shear flow, adjacent to its viscous sublayer, the nonuniversal (Reynolds-number-dependent) power law (18) should be observed. The most pertinent and important particular case to verify this conjecture is the flow in a turbulent boundary layer over a planar plate in a zero-pressure-gradient flow with a velocity $U$. We emphasize that the constants in (14) remain fixed. Here, however, the question arises as to what Re for the boundary-layer flow is.[1]

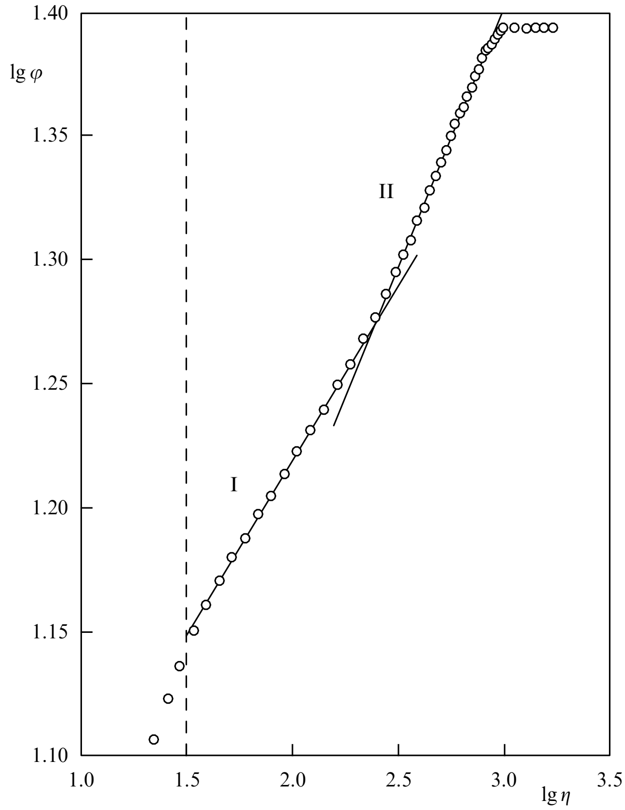


**Figure 6.** Flow structure in a turbulent boundary layer: two power laws, one sharply joining the other.

Traditionally, in the experimental research on flows in a turbulent boundary layer, the "momentum thickness"

$$\theta = \frac{1}{U^2}\int_0^\infty u(U-u)\,\mathrm{d}y \tag{23}$$

is taken as the length scale and the corresponding Reynolds number $\mathrm{Re}_\theta = U\theta/\nu$ is used. But for a comparison with the nonuniversal law, we cannot resort to this definition, because it contains an element of arbitrariness. The following procedure was therefore selected [13]: the experimentally obtained velocity distribution was presented in the bilogarithmic axes $(\lg\eta, \lg\varphi)$. In all cases, it was found that this distribution has the form of a broken line outside the viscous sublayer (Fig. 6). Hence, the flow in the intermediate layer adjacent to the viscous sublayer (I in Fig. 6) is described by a self-similarity law $\varphi = A\eta^\alpha$, where the coefficients $A$ and $\alpha$ can be computed independently by statistically processing the experimental data; the scatter in this case proves to be

[1] As Zeldovich used to say: "If you ask a barman to serve water without syrup, he does not ask without which syrup to serve. If you ask to serve with syrup, then the question 'with which one?' is natural."

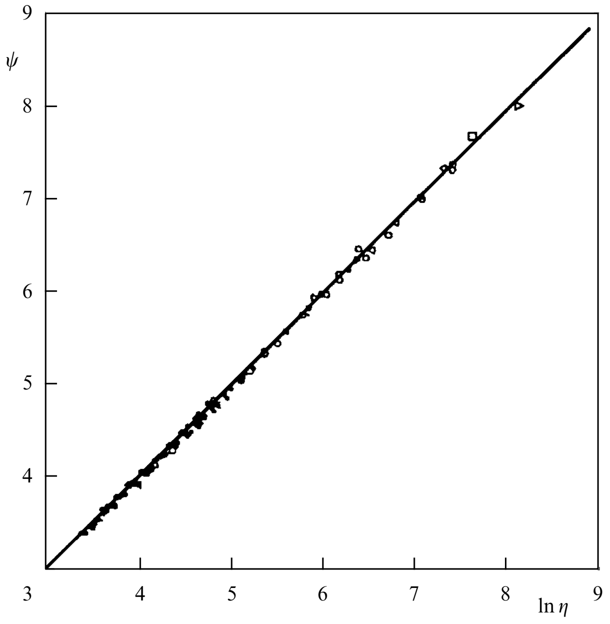


**Figure 7.** Flow in the lower intermediate region of the boundary layer satisfies the new power law.

sufficiently small. We choose the expression for the Reynolds number in the form $\mathrm{Re} = U\Lambda/\nu$, where $\Lambda$ is some linear scale. We then compute two values of $\ln \mathrm{Re}$, $\ln \mathrm{Re}_1$ and $\ln \mathrm{Re}_2$, by solving two equations that correspond to law (18):

$$\frac{1}{\sqrt{3}} \ln \mathrm{Re}_1 + \frac{5}{2} = A\,, \quad \frac{3}{2\ln \mathrm{Re}_2} = \alpha\,. \tag{24}$$

If the proposed power law (18), (19) is valid for shear flows, the values of $\ln \mathrm{Re}_1$ and $\ln \mathrm{Re}_2$ should be close.

Indeed, in all cases, the values of $\ln \mathrm{Re}_1$ and $\ln \mathrm{Re}_2$ turned out to be close [9, 13], and it was therefore natural to choose the definition of the effective Reynolds number as

$$\ln \mathrm{Re} = \frac{\ln \mathrm{Re}_1 + \ln \mathrm{Re}_2}{2}\,, \quad \mathrm{Re} = \sqrt{\mathrm{Re}_1 \mathrm{Re}_2}\,. \tag{25}$$

It turned out (Fig. 7) that in the plane ($\ln\eta$, $\psi$) (see Ref. [13]), in agreement with power law (19), the experimental points lie with satisfactory accuracy on the bisector of the first quadrant.

The results obtained are sufficient to argue that the von Kármán–Prandtl universal logarithmic law cannot be recognized as correct and cannot serve as the basis for teaching and engineering practice.

For computations in smooth pipes, we recommend power law (18),

$$\varphi = \frac{u}{u_*} = \left(\frac{\sqrt{3} + 5\alpha}{2}\right)\eta^{\alpha}\,, \quad \eta = \frac{u_* y}{\nu}\,,$$

$$\alpha = \frac{3}{2\ln \mathrm{Re}}\,, \quad \mathrm{Re} = \frac{\bar{u}d}{\nu}\,, \tag{26}$$

and the related skin-friction law (see its derivation in Ref. [9])

$$\lambda = 8\frac{u_*^2}{\bar{u}^2} = \frac{8}{\Psi^{2/(1+\alpha)}}\,, \quad \Psi = \frac{\exp(3/2)(\sqrt{3} + 5\alpha)}{2^{\alpha}\alpha(1+\alpha)(2+\alpha)}\,. \tag{27}$$

## 9. Local structure of well-developed turbulent flows. The Kolmogorov–Obukhov power laws

The Kolmogorov–Obukhov 'two-thirds' law [14–17],

$$D_{LL} = C\varepsilon^{2/3} r^{2/3}\,, \tag{28}$$

and its spectral analog, the 'five-thirds' law, are considered to be the fundamental laws for the local structure of turbulent flows at very large Reynolds numbers Re. Here,

$$D_{LL}(r) = \overline{\left[u_L(\mathbf{x} + \mathbf{r}) - u_L(\mathbf{x})\right]^2}$$

is the longitudinal structure function, in terms of which we can express all components of the tensor formed by second moments of the velocity increments $\mathbf{u}(\mathbf{x})$ between points $\mathbf{x} + \mathbf{r}$ and $\mathbf{x}$ in an incompressible fluid, $u_L$ is the velocity component in the direction of the vector $\mathbf{r}$, and $\varepsilon$ is the mean energy dissipation rate per unit mass. The analogy between the laws of local structure and shear flows at very large Reynolds numbers has been noted previously [18].

A question naturally arises on the universality of the two-thirds law, i.e., on the independence of the constant $C$ and the power-law exponent from the Reynolds number. Arguments similar to those used above for shear flows hint at the form of the dependence on Re:

$$D_{LL} = A(\mathrm{Re})\,(\varepsilon r)^{2/3}\left(\frac{r}{\lambda}\right)^{\alpha}\,, \tag{29}$$

where $\lambda = \nu^{3/4}/\varepsilon^{1/4}$ is the Kolmogorov scale,

$$A(\mathrm{Re}) = A_0 + \frac{A_1}{\ln \mathrm{Re}}\,, \quad \alpha = \frac{A_2}{\ln \mathrm{Re}} \tag{30}$$

(see also Ref. [19]). In contrast to shear flows, the amount of reliable data for the local structure is very small. Experiments, however, have long pointed to the nonuniversality of the constant $C$ (the dependence on the Reynolds number!) (see Ref. [17]).

***A telling example.*** Figure 8 plots the dependence of $C$ on the Reynolds number as obtained from experimental data [20] in the wind tunnel of TsAGI (Zhukovsky Central Aerohydrodynamic Institute). In contrast to shear flows, the correction $\alpha$ may go unnoticed in this case, compared to $2/3$, because $\alpha \sim 1/\ln \mathrm{Re}$, $\ln \mathrm{Re} \gg 1$.

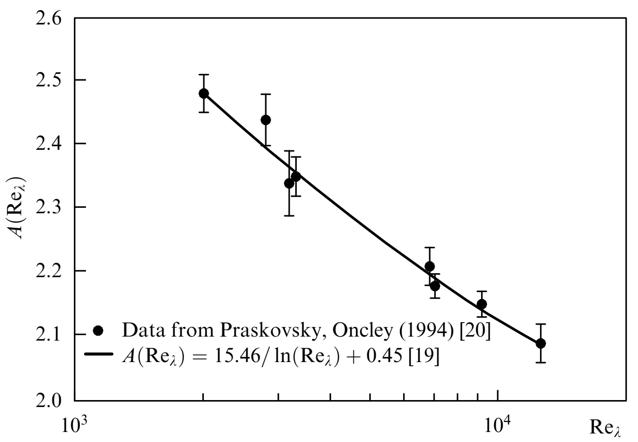


**Figure 8.** Experimental data demonstrate the dependence of the constant in the Kolmogorov–Obukhov law on the Reynolds number.

***A historical note.*** A M Obukhov, in a comment on his paper [16], reprinted in his collection of papers [21], mentions the talk by A N Kolmogorov given already the end of 1939, soon after the beginning of their work on turbulence. In this talk, according to Obukhov, a statement was made which is in fact equivalent to the hypothesis of incomplete similarity (29). However, as Obukhov writes, Kolmogorov did not succeed in 1939 in determining the power-law exponent — the constant $\alpha$: this was done by Kolmogorov "based on similarity arguments" after Obukhov had independently (as was emphasized by Kolmogorov) considered the balance in the spectral energy distribution. Apparently, the similarity arguments required neglecting the impact of viscosity.

## 10. Conclusions

Like Lord Rayleigh, Yakov Borisovich Zel'dovich was unique in classical physics of the 20th century. Turbulence was the field of his imperishably active interest. The authors deeply regret that they were not fated to present this work to Zeldovich, work which is in many respects consonant with his ideas.

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