

CONFERENCES AND SYMPOSIA

PACS numbers: 11.10.Hi, 11.25.Uv, 12.60.Jv

100th ANNIVERSARY OF THE BIRTH OF I Ya POMERANCHUK

Limit cycles in renormalization group dynamics

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DOI: 10.3367/UFNe.0184.201402g.0182

Contents

Abstract. **This review discusses the concept of limit cycles in renormalization group flows. Examples from quantum mechanics and field theory are presented.**

1. Introduction

It is usually assumed that the renormalization group (RG) flow connects fixed points, starting at a UV repelling point and terminating at an IR attracting point. However, it turns out that such open RG trajectories do not exhaust all possibilities, and a clear-cut quantum mechanical example of a nontrivial RG limit cycle has been found in [1], confirming the earlier expectations. This example triggered the search for patterns of this phenomenon, which was quite successful. They have been identified both in systems with a finite number of degrees of freedom [2–6] and in the field theory framework [7–9]. The cyclic RG is currently taking its prominent place in the world of RG phenomena; however, the subject certainly deserves much more study.

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Received 17 December 2013
Uspekhi Fizicheskikh Nauk **184** (2) 182 – 193 (2014)
DOI: 10.3367/UFNr.0184.201402g.0182
Translated by K M Bulycheva and A S Gorsky; edited by A M Semikhatov

The appearance of critical points corresponds to second-order phase transitions; hence, there is a natural question concerning the connection between RG cycles and phase transitions. The very phenomenon of the cyclic RG flow has been interpreted in important paper [8] as a kind of generalization of the Berezinsky–Kosterlitz–Thouless (BKT) phase transition in two dimensions. We can start from a regular example of an RG flow connecting UV and IR fixed points and then consider motion in a parameter space, which results in a merging of the fixed points. In [8], it was argued that when the parameter goes into the complex region, the cyclic behavior of the RG flow is manifested and a gap in the spectrum arises. This occurs similarly to the BKT transition case, when a deconfinement of vortices occurs at the critical temperature and conformal symmetry is restored at lower temperatures.

The appearance of RG cycles can also be interpreted as a peculiar anomaly in the classical conformal group [10–13]. This anomaly has its origin in some 'falling to the center' UV phenomena, which can have a quite universal character.

We emphasize one more generic feature of the phenomenon: a cyclic RG usually occurs in a system with at least two couplings. One of them undergoes the RG cyclic flow, while the other determines the period of the cycle.

The collision of UV and IR fixed points can be illustrated in quite a general manner as follows. We assume that there are two couplings (α, g) in the theory and focus on the renormalization of a coupling that has the β-function

$$\beta_g = (\alpha - \alpha_0) - (g - g_0)^2\,, \tag{1}$$

which vanishes on a hypersurface in the parameter space,

$$g = g_0 \pm \sqrt{\alpha - \alpha_0}\,. \tag{2}$$

It was argued in [8] that the collision of two roots at $\alpha = \alpha_0$ can be interpreted as the collision of UV and IR fixed points. Upon the collision, the points move into the complex g plane and an RG cycle emerges. The period of the cycle can be immediately estimated as

$$T \propto \int_{g_{\rm UV}}^{g_{\rm IR}} \frac{dg}{\beta(\alpha; g)} \propto \frac{1}{\sqrt{\alpha - \alpha_0}} \,. \tag{3}$$

The phenomenon is believed to be generic once the β-function has form (1).

Breaking of the conformal symmetry results in the generation of a mass scale, which has a nonperturbative nature. Due to the RG cycles, the scale is not unique, and the whole tower with the Efimov-like scaling is manifested:

$$E_{n+1} = \lambda E_n \,, \tag{4}$$

where λ is fixed by the period of the RG cycle.

In the examples available, we could attempt to trace the physical picture. It turns out that the origin of two couplings is quite general. One coupling does not break the conformal symmetry, which is exact in some subspace of the parameter space. The second coupling plays the role of a UV regularization, which can be imposed in one manner or another. This breaks the conformal symmetry; however, some discrete version of the scale symmetry survives, which is manifested in the cycle structure. The UV regularization has different reincarnations in the examples considered: a manifestation of the finite size of the nuclei, contact interaction in the model of superconductivity, or brane splitting in supersymmetric models.

Historically, the first example of this phenomenon was found a long time ago by Efimov [14, 15] in the context of nuclear physics. He considered a three-body system where two particles are near the threshold and have an attractive potential with the third particle. It was shown that two-particle bound states are absent in the spectrum, but there is a tower of three-particle bound states, with the geometrical scaling corresponding to $\lambda \approx 22.7$. A review of the RG interpretations of the Efimov phenomenon can be found in [16].

When considering a system with a finite number of degrees of freedom, the meaning of the RG flows has to be clarified. For this, some UV cutoff must be introduced. In the first example in [1], the step of the RG corresponds to integrating out the highest energy level, taking its correlation with the rest of the spectrum into account. This approach has much in common with the renormalization procedure in matrix models considered in [17]. The same UV cutoff for the formulation of the RG procedure has been used for the Russian Doll (RD) model describing the restricted BCS model of superconductivity [18]. In this case, the coupling providing the Cooper pairing undergoes an RG cycle, while the CP-violating parameter defines the period.

In the second class of examples, the UV cutoff is introduced not at a high energy scale but at small distances. RG cycles have been found in nonrelativistic Calogero-like models with a $1/r^2$ potential, which has a naive conformal symmetry [3–6]. The RG flow is formulated in terms of the short-distance regularization of the model. It is assumed that the wave function with $E = 0$ at large r does not depend on the UV cutoff at small r. This condition yields an equation for the cutoff parameter in the regularization potential. This equation has multiple solutions, which can be interpreted as the manifestation of a tower of shallow bound states with the Efimov scaling in a regularized Calogero model with attraction. The scaling factor in the tower is determined by the Calogero coupling constant, which reflects the remnant of the conformal group upon regularization.

The list of field theory examples in different dimensions with cyclic RG flows is short but quite representative. In two dimensions, an explicit example with an RG cycle has been found in some range of parameters in the sine-Gordon model. The cycle manifests itself in the pole structure of the S-matrix. The Efimov-like tower of states corresponds to specific poles with a Regge-like behavior of resonance masses [7],

$$m_n = m_s \exp \frac{n\pi}{h} \,, \tag{5}$$

where h is a certain parameter of the model. Moreover, it was argued that the S-matrix behaves universally under cyclic RG flows. The tower of Efimov states scales in the same manner as in the quantum mechanical case.

The origination of the cyclic RG behavior in the sine-Gordon model is not surprising. Indeed, it was argued in [8] that the famous BKT phase transition in an XY system belongs to this universality class. On the other hand, we can map the XY system at a temperature T into the sine-Gordon theory with the parameters

$$L_{\rm SG} = T(\partial\phi)^2 - 4z\cos\phi \tag{6}$$

and consider the renormalization of the coupling. The β-functions are

$$\beta_u = -2v^2 \,, \qquad \beta_v = -2uv \,, \tag{7}$$

where

$$u = 1 - \frac{1}{8\pi T} \,, \qquad v = \frac{2z}{T\Lambda^2} \,, \tag{8}$$

and Λ is the UV cutoff introduced to regularize the vortex core. The form of the β-functions implies the existence of a limit cycle, with the correlation length

$$\xi_{\rm BKT}\Lambda \propto \exp \frac{c}{\sqrt{|T - T_{\rm c}|}} \tag{9}$$

above the phase transition. This RG behavior is mapped onto the RG cycle in the sine-Gordon model.

An example of the Efimov tower in $(2+1)$ dimensions has been found in [19] in a holographic representation. The model is based on the D3 – D5 brane configuration and corresponds to the large-N 3d gauge theory with fundamentals having $\mathcal{N} = 4$ supersymmetry. In addition, the magnetic field and the finite density of conserved charge are present. At strong coupling, the gauge theory is described in terms of the probe $N_{\rm f}$ flavor branes in the nontrivial $AdS_5 \times S^5$ geometry when the U(1) bulk gauge field is added, providing the magnetic field in the boundary theory.

A tower of Efimov states is generated at some value of the 'filling fraction' ν in an external magnetic field. The phase transition corresponds to the change in the minimal embedding of the probe D5 branes in the bulk geometry with a BKT critical behavior of the order parameter. In that case, the order parameter is identified with the condensate σ, which behaves as

$$\sigma \propto \exp\left(-\frac{1}{\nu}\right). \tag{10}$$

Above the phase transition, the embedding changes and the brane becomes extended in one more coordinate. The scale associated with this extension into a new dimension is nothing but the nonperturbative scale amounting to a mass gap. The phenomenon of the cyclic RG flow in this case has the Breitenlohner–Freedman instability as the gravitational counterpart.

In four dimensions, the most famous example of the Efimov tower is the so-called Miransky scaling for the condensate in a magnetic field. It was argued in [20, 21] that the chiral condensate is generated in an external magnetic field in the Abelian theory with the behavior

$$\langle\bar{\Psi}\Psi\rangle \propto \Lambda^3 \exp\left(-\frac{c}{\sqrt{\alpha-\alpha_{\rm cr}}}\right), \tag{11}$$

where α is the fine structure constant and c is some parameter of the model.

A more recent example [22] of the Efimov tower in four dimensions concerns the Veneziano limit of QCD when $N_{\rm f}, N_{\rm c} \to \infty$ while the ratio $x = N_{\rm f}/N_{\rm c}$ is fixed. It turns out that this parameter can be regarded as a variable in the RG flow, which resembles finite-dimensional examples. At some value of the RG scale, a tower of condensates is generated with the geometrical Efimov scaling. The period of the RG cycle is given by

$$T \propto \frac{\kappa}{\sqrt{x_{\rm c}-x}}, \tag{12}$$

where $x_{\rm c}$ is the critical value of the x parameter.

Finally, a 4d example with an RG cycle has been found in the $\mathcal{N}=2$ SUSY gauge theory in the Ω-background [9]. In this case, the gauge coupling undergoes an RG cycle whose period is determined by the parameter of the Ω-background,

$$T \propto \epsilon^{-1}. \tag{13}$$

The appearance of the RG cycle in this model can be traced to its relation with spin chain quantum integrable systems.

In this review, we provide the reader with examples of this phenomenon. The list of systems with a finite number of degrees of freedom involves the Calogero model and a relativistic model with the classical conformal symmetry describing an external charge in graphene. Another finite-dimensional example concerns the RD model of the restricted BCS superconductivity. Field theory examples concern the 3d and 4d theories in external fields. We focus on their brane representations and use their connection to finite-dimensional integrable systems.

2. RG cycles in nonrelativistic quantum mechanics

In this section, we consider the example of a limit cycle in the RG in the nonrelativistic system with the inverse-square potential, or the Calogero system:

$$H = \frac{\partial^2}{\partial r^2} - \frac{\mu(\mu-1)}{r^2}. \tag{14}$$

The distinctive feature of the system described by Hamiltonian (14) is its conformality. Namely, the operators (H, D, K), where D is the dilation generator and K is the conformal boost, generate the conformal algebra $\mathbf{so}(2,1)$ (see Section 4).

The finite-energy eigenfunctions of (14) immediately break this symmetry; less trivial is the fact that even the ground state breaks conformal symmetry. Namely, a solution of the equation $H\psi = 0$ is given by

$$\psi_0 = c_+ r^{\mu} + c_- r^{1-\mu}. \tag{15}$$

This solution is scale invariant only if one of the coefficients $c_\pm$ is zero. If both coefficients are present, they define an intrinsic length scale $L = (c_+/c_-)^{1/(-2\mu+1)}$. Requiring that the quantity c_+/c_-, which describes the ground-state solution, be invariant under the change in scale,

$$\frac{c_+}{c_-} = -r_0^{-2\mu+1}\,\frac{\gamma-\mu+1}{\gamma+\mu}, \tag{16}$$

we arrive at the β-function for the γ parameter,

$$\beta_\gamma = \frac{\partial\gamma}{\partial\log r_0} = -(\gamma+\mu)(\gamma-\mu+1) = \left(\mu-\frac{1}{2}\right)^2 - \left(\gamma-\frac{1}{2}\right)^2, \tag{17}$$

where r_0 is the RG scale. We can respectively identify the $\gamma = \mu - 1$ and $\gamma = -\mu$ points, i.e., solutions with $c_+ = 0$ and $c_- = 0$, with UV and IR attractive points of the renormalization group flow [8].

If $\mu = i\nu$ is imaginary, i.e., the potential is attractive, then Eqn (17) allows determining the RG period

$$T = -\int_{-\nu+1}^{\nu} \frac{d\gamma}{\beta_\gamma} = \frac{\pi}{\nu-1/2}. \tag{18}$$

This means that infinitely many scales are generated, differing by the factor $\exp[-\pi/(\nu-1/2)]$. To see this explicitly, we find solutions of the Schrödinger equation at finite energies.

In the attractive potential, solution (15) can be written as

$$\psi_0 \propto \sqrt{r}\sin\left[\left(\nu-\frac{1}{2}\right)\log\frac{r}{r_0} + \alpha\right]. \tag{19}$$

This solution oscillates indeterminately in the vicinity of the origin, and there is no way to fix the α constant. To regularize this behavior, we can break the scale invariance at the level of the Hamiltonian and introduce a regularizing potential. The two most popular regularizations involve the square-well potential [4–6] or the δ-shell potential [3]. One more choice is to introduce a δ-function at the origin [8].

Choosing the square-well regularization,

$$V(r) = \begin{cases} -\dfrac{\nu(\nu-1)}{r^2}, & r > R, \\ -\dfrac{\lambda}{R^2}, & r \leqslant R, \end{cases} \tag{20}$$

we require that the action of the dilatation operator on the wave function inside the well and outside it be equal at $r = R$.

This condition amounts to an equation for λ,

$$\sqrt{\lambda}\cot\sqrt{\lambda} = \frac{1}{2} + \nu\cot\left(\nu\log\frac{R}{r_0}\right). \qquad (21)$$

The multivalued function $\lambda(R)$ can be chosen continuous [5, 6].

The wavefunction regular at infinity is given as a combination of the Bessel functions [5, 6],

$$\psi(r,\kappa_m) = \sqrt{r}\,(-1)^m\left[\mathrm{i}\exp\left(-\mathrm{i}\nu\,\frac{\pi}{2}\right)J_{i\nu}(\kappa_m r) - \mathrm{i}\exp\left(\mathrm{i}\nu\,\frac{\pi}{2}\right)J_{-i\nu}(\kappa_m r)\right], \qquad (22)$$

where κ_m is the energy of the state. The spectrum consists of infinitely many shallow bound states with the adjacent energies differing by an exponential factor:

$$\frac{\kappa_{m+1}}{\kappa_m} = \exp\left(-\frac{\pi}{\nu}\right). \qquad (23)$$

We note that the coordinate enters wave function (22) only in combination with the energy, and the spectrum is generated by the dilation operator:

$$\psi_{m+1} = \exp\left(-\frac{\pi}{\nu}\,r\partial_r\right)\psi_m\,. \qquad (24)$$

We can think of this relation as the action of the dilatation operator shifting the wave function zeros from the domain $r < R$ to the domain with the inverse-square potential, and one step of evolution (24) corresponds to the elimination of a single zero in the domain with the square-well potential. Since the wave function oscillates infinitely at the origin, the elimination of all the zeroes would require an infinite number of steps, and in this way a whole tower of states is generated.

3. RG cycle in graphene

In this section, we consider a similar problem in $(2+1)$ dimensions, which physically corresponds to an external charge in a planar graphene layer. The problem has the classical conformal symmetry and is a relativistic analogue of the conformal nonrelativistic Calogero-like system. Due to the conformal symmetry, we can expect the occurrence of RG cycles and Efimov-like states in this problem upon imposing a short-distance cutoff [23].

The issue of a charge in the graphene plane has been discussed theoretically [24–26] and experimentally [27, 28]. It was argued that there is indeed a tower of 'quasi-Rydberg' states with exponential scaling [29]. The situation can be interpreted as an atomic collapse phenomenon similar to the instability of $Z > 137$ superheavy atoms in QED [30].

We now turn to the consideration of an electron in graphene interacting with an external charge. The two-dimensional Hamiltonian is

$$H_{\mathrm{D}} = v_{\mathrm{F}}\sigma_i p^i + V(r)\,, \qquad i = 1, 2\,. \qquad (25)$$

The external charge creates the Coulomb potential

$$V(r) = -\frac{\alpha}{r}\,, \qquad r \geqslant R\,. \qquad (26)$$

As we see in what follows, the solution in the presence of potential (26) oscillates indefinitely at the origin and must be regularized by some cutoff R. Hence, close enough to the origin $r \leqslant R$, potential (26) is replaced by some constant potential $V_{\mathrm{reg}}(r,\lambda(R))$. The renormalization condition for the λ parameter is that the zero-energy wave function be independent of the short-distance regularization. This condition is chosen similarly to that for the Calogero system renormalization (see Section 2). Hence, our primary task is to find the zero-energy solution of the Dirac equation

$$H_{\mathrm{D}}\,\psi_0 = 0\,. \qquad (27)$$

Because the Hamiltonian commutes with the J_3 operator,

$$J_3 = \mathrm{i}\,\frac{\partial}{\partial\varphi} + \sigma_3\,, \qquad [H_{\mathrm{D}}, J_3] = 0\,, \qquad (28)$$

we can seek solutions of (27) in the form

$$\psi_0 = \begin{pmatrix} \chi_0(r) \\ \xi_0(r)\exp(\mathrm{i}\varphi) \end{pmatrix}, \qquad J_3\psi_0 = \psi_0\,. \qquad (29)$$

Then, in polar coordinates, Eqn (27) is written as

$$\begin{cases} -\mathrm{i}\hbar v_{\mathrm{F}}\left(\partial_r + \dfrac{1}{r}\right)\xi_0 = -V(r)\chi_0\,, \\ -\mathrm{i}\hbar v_{\mathrm{F}}\,\partial_r\chi_0 = -V(r)\xi_0\,, \end{cases} \qquad (30)$$

which is equivalent to

$$\begin{cases} \xi_0(r) = \mathrm{i}\hbar v_{\mathrm{F}}\big(V(r)\big)^{-1}\partial_r\chi_0\,, \\ \partial_r^2\chi_0 + \left(\dfrac{1}{r} - \dfrac{V'(r)}{V(r)}\right)\partial_r\chi_0 + \dfrac{V^2(r)}{\hbar^2 v_{\mathrm{F}}^2}\,\chi_0 = 0\,. \end{cases} \qquad (31)$$

For the potential $V = -\alpha/r$, we obtain the following equation for $\chi_0(r)$:

$$\partial_r^2\chi_0 + \frac{2}{r}\,\partial_r\chi_0 + \frac{\beta^2}{r^2}\,\chi_0 = 0\,, \qquad \beta = \frac{\alpha}{\hbar v_{\mathrm{F}}}\,. \qquad (32)$$

Assuming that $\beta^2 = 1/4 + \nu^2$, we write the solution as

$$\chi_0 = \sqrt{r}\left(c_-\left(\frac{r}{r_0}\right)^{-i\nu} + c_+\left(\frac{r}{r_0}\right)^{i\nu}\right) \propto \sqrt{r}\sin\left(\nu\log\frac{r}{r_0} + \varphi\right). \qquad (33)$$

We see that this solution shares the properties of the ground-state Calogero wave function (15): at nonzero $c_\pm$, it generates its own intrinsic length scale, and it oscillates indeterminately at the origin.

To fix the φ constant, we need to introduce a cut-off potential. Hence, we consider the solution in the potential

$$V(r) = \begin{cases} -\dfrac{\alpha}{r}\,, & r > R\,, \\ V_{\mathrm{reg}} = -\hbar v_{\mathrm{F}}\,\dfrac{\lambda}{R}\,, & r \leqslant R\,. \end{cases} \qquad (34)$$

The dilatation operator acts on χ as

$$r\,\partial_r\chi_0 = \left(\frac{1}{2} + \nu\cot\left(\nu\log\frac{r}{r_0}\right)\right)\chi_0\,. \qquad (35)$$

For a constant potential V_{reg}, we use (31) to obtain

$$\partial_r^2 \chi_0^{\text{reg}} + \frac{1}{r}\,\partial_r \chi_0^{\text{reg}} + \frac{\lambda^2}{R^2}\,\chi_0^{\text{reg}} = 0\,. \tag{36}$$

Choosing the solution of (36) that is regular at the origin, we obtain

$$\chi_0^{\text{reg}} \propto J_0\left(\lambda\,\frac{r}{R}\right). \tag{37}$$

Computing the action of the dilation operator on the solution in the domain of constant potential and equating it to the action of dilation operator (35), we obtain the equation for the λ regulator parameter:

$$\frac{1}{2} + \nu \cot\left(\nu \log \frac{R}{r_0}\right) = -\lambda\,\frac{J_1(\lambda)}{J_0(\lambda)}\,. \tag{38}$$

Equation (38) defines λ as a multi-valued function of R. The period of the RG flow corresponds to a jump from one branch of $\lambda(R)$ to another.

We now proceed to finding the bound states in potential (26). We again consider the Dirac equation

$$H_{\text{D}}\psi_\kappa = -\hbar v_{\text{F}}\kappa\psi_\kappa\,. \tag{39}$$

Then the equation for χ analogous to (31) is given by

$$\partial_r^2\chi_\kappa + \frac{2\beta - \kappa r}{\beta - \kappa r}\,\frac{1}{r}\,\partial_r\chi_\kappa + \left(\frac{\beta}{r} - \kappa\right)^2 \chi_\kappa = 0\,. \tag{40}$$

Asymptotically, when $r \gg \beta/\kappa$, the solution of (40) that is regular at infinity is given by the Hankel function,

$$\chi_\kappa \propto H_0^{(1)}(\text{i}\kappa r)\,. \tag{41}$$

At small $r \ll \beta/\kappa$, the solution is not regular at the origin,

$$\chi_\kappa \propto \sqrt{r}\,\sin\left(\nu \log \frac{r}{r_0}\right), \tag{42}$$

and we again need a regulator potential.

Solving Dirac equation (39) in the presence of a constant potential V_{reg} and computing the action of the dilatation operator,

$$r\,\partial_r\chi_\kappa^{\text{reg}} = -(\lambda - \kappa R)\,\frac{J_1(\lambda - \kappa R)}{J_0(\lambda - \kappa R)}\,\chi_\kappa^{\text{reg}}\,, \tag{43}$$

we can equate (43) to the action of the dilatation operator on (42) and obtain the equation for the spectrum of bound states,

$$\frac{1}{2} + \nu\cot\big(\nu\log(\kappa R)\big) = -(\lambda - \kappa R)\,\frac{J_1(\lambda - \kappa R)}{J_0(\lambda - \kappa R)}\,. \tag{44}$$

This condition gives the spectrum of infinitely many shallow bound states,

$$\kappa_n = \kappa_* \exp\left(-\frac{\pi n}{\nu}\right), \qquad \kappa \to \infty\,. \tag{45}$$

4. Anomaly in the so(2, 1) algebra

We comment on the algebraic counterpart of the phenomenon considered in [10–13]. As we have mentioned, conformal symmetry plays a major role here, because the Hamiltonians under consideration are scale invariant before the regularization. Actually, this group can be considered an example of a spectrum-generating algebra when the Hamiltonian is one of the generators or is expressed in terms of the generators in a simple manner. This is familiar from exactly or quasi-exactly solvable problems when the dimension of the representation selects the size of the algebraic part of the spectrum.

We introduce the generators J_1, J_2, J_3 of the $\mathbf{so}(2,1)$ conformal algebra: the Calogero Hamiltonian

$$J_1 = H = p^2 + V(r)\,, \tag{46}$$

the dilatation generator

$$J_2 = D = tH - \frac{1}{4}(pr + rp)\,, \tag{47}$$

and the generator of special conformal transformations

$$J_3 = K = t^2 H - \frac{t}{2}(pr + rp) + \frac{1}{2}\,r^2\,. \tag{48}$$

They satisfy the $\mathbf{so}(2,1)$ algebra relations

$$[J_2, J_1] = -\text{i}J_1\,, \quad [J_3, J_1] = -2\text{i}J_2\,, \quad [J_2, J_3] = \text{i}J_3\,. \tag{49}$$

The singular behavior of the potential at the origin amounts to a kind of anomaly in the $\mathbf{so}(2,1)$ algebra,

$$A(r) = -\text{i}[D, H] + H\,, \tag{50}$$

which in d spatial dimensions can be represented in the form

$$A(r) = -\frac{d-2}{2}\,V(r) + (r^i\nabla_i)V(r)\,. \tag{51}$$

Simple arguments imply the relation

$$\frac{\text{d}}{\text{d}t}\,\langle D\rangle_{\text{ground}} = E_{\text{ground}}\,, \tag{52}$$

where the matrix element is taken over the ground state.

It turns out that (52) is fulfilled for singular potentials in the Calogero-like model or in models with a contact potential, $V(r) = g\delta(r)$. The expression for the anomaly does not depend on the regularization chosen. Moreover, a more detailed analysis demonstrates that the anomaly is proportional to the β-function of the coupling providing the UV regularization, as could be expected.

A similar calculation of the anomaly for graphene can be performed for an arbitrary state,

$$\left\langle\frac{\text{d}D}{\text{d}t}\right\rangle_\psi = \langle\Xi\rangle_\psi = -\int \text{d}^2x\,\psi^*\big(V(x) + x_i\,\partial_i V(x)\big)\,\psi\,, \tag{53}$$

which, using the square-well regularization, yields

$$\langle\Xi\rangle_\psi = \hbar v_{\text{F}}\,\frac{\lambda(R)}{R}\,\frac{\int_0^R r|\psi|^2\,\text{d}r}{\int_0^\infty r|\psi|^2\,\text{d}r}\,. \tag{54}$$

In (51), it is convenient to use the two-dimensional identity

$$\nabla\,\frac{\mathbf{r}}{r} = 2\pi\delta(\mathbf{r})\,, \tag{55}$$

which simplifies the calculation of the anomaly for any normalized bound state:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle D\rangle_{\Psi} = -g\pi \int \mathrm{d}^2 r\,\delta(r)\big|\Psi(r)\big|^2\,. \tag{56}$$

5. RG cycles in models of superconductivity

In this section, we explain how the cyclic RG flows emerge in truncated models of superconductivity. For this, we first describe the Richardson model and then consider its generalization to the RD model, which is a cyclic RG flow. These models are distinguished by the finiteness of the number of fermionic levels. The relation with integrable many-body systems proves to be quite useful.

5.1 Richardson model versus Gaudin model

We recall the truncated BCS-like Richardson model of superconductivity [32, 33] with some number of doubly degenerate fermionic levels with the energies $\epsilon_{j\sigma}$, $j = 1, \ldots, N$. It describes a system of a fixed number of Cooper pairs. It is assumed that several energy levels are populated by Cooper pairs, while levels with single fermions are blocked. The Hamiltonian is

$$H_{\mathrm{BCS}} = \sum_{j,\,\sigma=\pm}^{N} \epsilon_{j\sigma} c_{j\sigma}^{+} c_{j\sigma} - G \sum_{jk} c_{j+}^{\dagger} c_{j-}^{\dagger} c_{k-} c_{k+}\,, \tag{57}$$

where $c_{j\sigma}$ are fermion operators and G is the coupling constant providing the attraction that leads to the formation of Cooper pairs. In terms of the hard-core boson operators, the Hamiltonian is given by

$$H_{\mathrm{BCS}} = \sum_{j} \epsilon_j b_j^{\dagger} b_j - G \sum_{jk} b_j^{\dagger} b_k\,, \tag{58}$$

where

$$[b_j^{\dagger}, b_k] = \delta_{jk}(2N_j - 1)\,, \qquad b_j = c_{j-} c_{j+}\,, \qquad N_j = b_j^{\dagger} b_j\,. \tag{59}$$

The eigenfunctions of the Hamiltonian can be written as

$$|M\rangle = \prod_{i}^{M} B_i(E_i)|\mathrm{vac}\rangle\,, \qquad B_i = \sum_{j}^{N} \frac{1}{\epsilon_j - E_i}\, b_j^{\dagger}\,, \tag{60}$$

if the Bethe ansatz equations are fulfilled:

$$G^{-1} = -\sum_{j}^{N} \frac{2}{\epsilon_j - E_i} + \sum_{j}^{M} \frac{1}{E_j - E_i}\,. \tag{61}$$

The energy of the corresponding states is

$$E(M) = \sum_{i} E_i\,. \tag{62}$$

It was shown in [33] that the Richardson model is exactly solvable and closely related to a particular generalization of the Gaudin model. To describe this relation, it is convenient to introduce the so-called pseudospin $\mathbf{sl}(2)$ algebra in terms of the creation and annihilation operators for the Cooper pairs:

$$t_j^{-} = b_j, \qquad t_j^{+} = b_j^{\dagger}\,, \qquad t_j^{0} = N_j - \frac{1}{2}\,. \tag{63}$$

The Richardson Hamiltonian commutes with the set of operators

$$R_j = -t_j^0 - 2G \sum_{i\neq j}^{N} \frac{t_i t_j}{\epsilon_i - \epsilon_j}\,, \tag{64}$$

which are identified as the Gaudin Hamiltonians,

$$[H_{\mathrm{BCS}}, R_j] = [R_i, R_j] = 0\,. \tag{65}$$

Moreover, the Richardson Hamiltonian itself can be expressed in terms of the R_i as

$$H_{\mathrm{BCS}} = \sum_{i} \epsilon_i R_i + G\Big(\sum R_i\Big)^2 + \mathrm{const}\,. \tag{66}$$

The number N of the fermionic levels coincides with the number of sites in the Gaudin model, and the coupling constant in the Richardson Hamiltonian corresponds to the ‘twisted boundary condition’ in the Gaudin model. Bethe ansatz equations (61) for the Richardson model exactly coincide with the ones for the generalized Gaudin model. It was argued in [2] that the Bethe roots correspond to excited Cooper pairs. It is natural to think about the solution of the Baxter equation as the wave function of the condensate. In terms of conformal field theory, Cooper pairs correspond to screening operators [34, 35].

With nontrivial degeneracies d_j of the energy levels, the BA equations become

$$G^{-1} = -\sum_{j}^{N} \frac{d_j}{\epsilon_j - E_i} + \sum_{j}^{M} \frac{2}{E_j - E_i}\,. \tag{67}$$

5.2 Russian Doll model of superconductivity and twisted XXX spin chains

An important generalization of the Richardson model describing superconductivity is the so-called RD model [2]. It involves an additional dimensionless parameter α, and the RD Hamiltonian is

$$H_{\mathrm{RD}} = 2\sum_{i}^{N} (\epsilon_i - G) N_i - \bar{G} \sum_{j<k} \big(\exp(\mathrm{i}\alpha) b_k^{+} b_j + \exp(-\mathrm{i}\alpha) b_j^{+} b_k\big)\,, \tag{68}$$

with two dimensional parameters G and η and with $\bar{G} = \sqrt{G^2 + \eta^2}$. In terms of these variables, the dimensionless parameter α has the form

$$\alpha = \arctan \frac{\eta}{G}\,. \tag{69}$$

It is also useful to consider two dimensionless parameters g and θ defined as $G = gd$ and $\eta = \theta d$, where d is the level spacing. The RD model reduces to the Richardson model in the limit $\eta \to 0$.

The RD model also turns out to be integrable. Here, instead of the Gaudin model, the proper counterpart is the generic quantum twisted XXX spin chain [36]. The transfer matrix $t(u)$ of such a spin chain model commutes with H_{RD}, which itself can be expressed in terms of spin chain Hamiltonians.

The equation for the RD model spectrum is

$$\exp(2\mathrm{i}\alpha)\prod_{l=1}^{N}\frac{E_i-\varepsilon_l+\mathrm{i}\eta}{E_i-\varepsilon_l-\mathrm{i}\eta}=\prod_{j\neq i}^{M}\frac{E_i-E_j+2\mathrm{i}\eta}{E_i-E_j-2\mathrm{i}\eta} \quad (70)$$

and coincides with the BA equations for a spin chain. Taking the logarithm of both sides of (70), we obtain

$$\alpha+\pi Q_i+\sum_{l=1}^{N}\arctan\frac{\eta}{E_i-\varepsilon_l}-\sum_{j=1}^{M}\arctan\frac{2\eta}{E_i-E_j}=0\,. \quad (71)$$

We note that we have added an arbitrary integer term to account for the generically multivalued arctangent function.

The RG step amounts to integrating out the Nth degree of freedom in the RD model, or equivalently to integrating out the Nth inhomogeneity in the XXX chain. This results in a renormalization of the twist. From (71), it is easy to see that

$$\arctan\frac{\eta}{G_N}-\arctan\frac{\eta}{G_{N-1}}=\sum_{i=1}^{M}\arctan\frac{2\eta}{E_i-\varepsilon_N}\,. \quad (72)$$

When $M=1$, this implies that

$$G_{N-1}-G_N=\frac{G_N^2+\eta^2}{\varepsilon_N-G_N-E}\,, \quad (73)$$

which is a discrete version of Eqn (1).

Of course, the same relation can be derived from the RD Hamiltonian in (68). If we consider the wave function $\psi=\sum_i^N\psi_i b_i^{\dagger}|0\rangle$, the Schrödinger equation for a state with one Cooper pair amounts to

$$(\varepsilon_i-G-E)\,\psi_i=(G+\mathrm{i}\eta)\sum_{j=1}^{i-1}\psi_j+(G-\mathrm{i}\eta)\sum_{j=i+1}^{N}\psi_j\,. \quad (74)$$

Integrating out the Nth degree of freedom amounts to expressing ψ_N in terms of the other modes,

$$\psi_N=\frac{G+\mathrm{i}\eta}{\varepsilon_N-G-E}\sum_{j=1}^{N-1}\psi_j\,, \quad (75)$$

and substituting it back into Schrödinger equation (74). The G_{N-1} constant in the resulting equation is then different from the initial G_N value in (73).

The key feature of the RD model is the existence of multiple solutions of the gap equation. The gaps are parameterized as

$$\Delta_n=\frac{\omega}{\sinh t_n}\,,\qquad t_n=t_0+\frac{\pi n}{\theta}\,,\qquad n=0,1,\ldots\,, \quad (76)$$

where t_0 is the solution of the equation

$$\tan(\theta t_0)=\frac{\theta}{g}\,,\qquad 0<t_0<\frac{\pi}{\theta}\,, \quad (77)$$

and $\omega=dN$ for equal level spacing. Here, $E^2=\varepsilon^2+|\Delta|^2$. This behavior can be derived in the mean field approximation [18]. The gap with the minimal energy defines the ground state, and other values of the gap describe excitations. In the limit $\theta\to 0$, the gaps $\Delta_{n>0}\to 0$ and

$$t_0=\frac{1}{g}\,,\qquad \Delta_0=2\omega\exp\left(-\frac{1}{g}\right); \quad (78)$$

the standard BCS expression for the gap is thus recovered. In the weak-coupling limit, the gaps behave as

$$\Delta_n\propto\Delta_0\exp\left(-\frac{n\pi}{\theta}\right). \quad (79)$$

In terms of the solutions of the BA equations, the multiple gaps correspond to the choices of different branches of the logarithms, i.e., to different choices of the integer Q parameter in (70).

If the degeneracy of the levels is d_n, the RD model is somewhat modified and is related to a higher-spin XXX chain. The local spins s_i are determined by the corresponding higher pair degeneracy d_i of the ith level,

$$s_i=\frac{d_i}{2}\,, \quad (80)$$

and the corresponding BA equations is written as

$$\exp(2\mathrm{i}\alpha)\prod_{l=1}^{N}\frac{E_i-\varepsilon_l+\mathrm{i}d_l+\mathrm{i}\eta}{E_i-\varepsilon_l-\mathrm{i}d_l-\mathrm{i}\eta}=\prod_{j\neq i}^{M}\frac{E_i-E_j+2\mathrm{i}\eta}{E_i-E_j-2\mathrm{i}\eta}\,. \quad (81)$$

5.3 Cyclic RG flows in the RD model

The RD model of truncated superconductivity exhibits a cyclic RG behavior [2]. The RG flows can be treated by integrating out the highest fermionic level with an appropriate scaling of the parameters using the procedure developed in [1]. The RG equations are (73),

$$g_{N-1}=g_N+\frac{1}{N}(g_N^2+\theta^2)\,,\qquad \theta_{N-1}=\theta_N\,. \quad (82)$$

In the a large-N limit, the natural RG variable is identified with $s=\log(N/N_0)$, and the solution of the RG equation is

$$g(s)=\theta\tan\left(\theta s+\tan^{-1}\frac{g_0}{\theta}\right). \quad (83)$$

Hence, the running coupling is cyclic,

$$g(s+\lambda)=g(s)\,,\qquad g\big(\exp(-\lambda)N\big)=g(N)\,, \quad (84)$$

with the RG period

$$\lambda=\frac{\pi}{\theta}\,, \quad (85)$$

and the total number of independent gaps in the model is

$$N_{\mathrm{cond}}\propto\frac{\theta}{\pi}\log N\,. \quad (86)$$

The multiple gaps are manifestations of Efimov-like states. The sizes of Cooper pairs in the Nth condensates also have the RD scaling. The cyclic RG can be derived even for a single Cooper pair.

What happens to the spectrum of the model during the period? It was shown in [18] that it is reorganized. The RG flow experiences discontinuities from $g=+\infty$ to $g=-\infty$ when a new cycle is started. At each jump, the lowest condensate disappears from the spectrum,

$$\Delta_{N+1}(g=+\infty)=\Delta_N(g=-\infty)\,, \quad (87)$$

indicating that the $(N+1)$th-state wave function plays the role of the Nth-state wave function at the next cycle [see (75)].

The same behavior can be derived from the BA equation [18]. To identify multiple gaps, we must remember that the solutions of the BA equations are classified by the integers Q_i, $i = 1, \ldots, M$, parameterizing the branches of the logarithm. If we assume that $Q_i = Q$ for all Bethe roots, this quantum number is shifted by one at each RG cycle and is identified with the integer parameterizing the solution of the gap equations,

$$\Delta_Q \propto \Delta_0 \exp(-\lambda Q) \,. \tag{88}$$

In the large-N limit, the BA equations of the RD model reduce to the BA equation of the Richardson–Gaudin model with the rescaled coupling

$$G_Q^{-1} = \eta^{-1}(\alpha + \pi Q) \,, \tag{89}$$

which can be treated as shifted boundary conditions in the generalized Gaudin model.

We emphasize that the unusual cyclic RG behavior is due to the presence of two couplings in the RD model.

6. Triality in integrable models and RG cycles

In this section, we summarize several dualities between integrable models and consider the realizations of cyclic RG flows in these systems. The problem is motivated by a close relationship between the restricted BCS models and spin chains. Actually, there are three different families of models related to each other by the particular identifications of phase spaces and parameters. The first family concerns the system of fermions (Richardson–Russian Doll) that develop a superconducting gap. The second family involves twisted inhomogeneous Gaudin–XXX–XXZ spin systems and their generalizations. The third family involves the Calogero–Ruijsenaars (CR) chain of integrable many-body systems (Fig. 1).

We seek the answers to the following questions:

• What is the condition yielding the RG equation for some coupling in each family?

• What is the RG variable?

• What determines the period of the cycle?

In the superconducting system at the RG step, the highest energy level decouples and the Cooper-pair coupling constant is renormalized. The RG time is identified with the number of energy levels $t = \log N$. The RG period is defined by the T-asymmetric parameter of the RD model.

In the spin chain model, the RG step corresponds to 'integrating out' one 'highest' inhomogeneity with the corresponding renormalization of the twist. The period of the RG flow is fixed by the Planck constant in the quantum spin chain. In the bispectral dual spin chain [38, 39], we 'integrate out' one of the twists and 'renormalize' the inhomogeneity. Because the Planck constant is inverted under bispectrality, $\hbar_{\text{spin}} \to \hbar_{\text{spin}}^{-1}$, the period of the RG cycle is also inverted. We note that the RG equation in the superconducting model can be mapped into the BAE in a spin chain [18]. The condition yielding the RG equation corresponds to the independence of the Bethe root in the RG step.

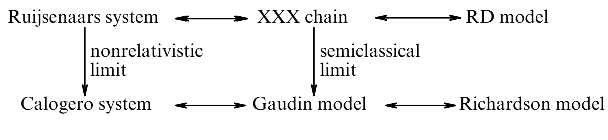

Figure 1. Three different families of models: fermion systems (the Richardson and Russian Doll models), twisted inhomogeneous Gaudin chains, XXX and XXZ models and their generalizations, and many-body systems of the Calogero–Ruijsenaars models. Besides the triality shown in the picture, a bispectral duality acts on RS/Calogero and XXX/Gaudin sides of the correspondence. Because it originates from three-dimensional mirror-symmetry [37], this duality interchanges coordinates with Lax eigenvalues in classical systems and inhomogeneities with twists in quantum ones.

For a two-body system with an attractive rational potential, the RG equations can be derived from the continuity of the ground-state wave function with respect to the cutoff parameter. This condition imposes the RG equation on the cutoff UV coupling constant. This RG equation has a cycle with the period

$$T_{\text{Cal}} = \frac{\pi}{\nu - 1/2} \,, \tag{90}$$

as was shown in Section 2.

The quantum–classical (QC) duality [37, 40] connects quantum spin chain systems and classical Calogero-type systems. Via the QC correspondence, the rational Gaudin model can be linked to the rational Calogero system spin chain inhomogeneities that are the Calogero coordinates, the twist in the spin chain (which is a single variable in our case) being the Lax matrix eigenvalue. It is also possible to make a bispectrality transformation of the rational Calogero model such that Lax eigenvalues are interchanged with coordinates. This means that Calogero coordinates correspond to twists on the spin chain side. In this case, the Calogero coupling is inverted, which means that the period of the RG cycle is also inverted.

To consider the mapping of RG cycles in the Calogero system and the spin chain, we need a generalization of the QC duality to the quantum–quantum (QQ) case. The spectral problem in the Calogero model has been identified with the Knizhnik–Zamolodchikov (KZ) equation [41] involving the Gaudin Hamiltonian H_{Gaud} in Eqn (64),

$$\frac{\mathrm{d}}{\mathrm{d}z_i} \Psi = H_{\text{Gaud}} \Psi + \lambda \Psi \,. \tag{91}$$

Because we formulate the RG condition on the Calogero side for the $E = 0$ state, the inhomogeneous term in the KZ equation is absent. The simplest test of the mapping of RG cycles under the QQ duality concerns the identifications of periods. On the spin chain side, the period is identified with the Planck constant, while on the Calogero side, it is defined by the coupling constant. The following identification holds for the QC duality [40]:

$$\hbar_{\text{spin}} = \nu \,, \tag{92}$$

which implies that the periods of the cycles on the Calogero and spin chain sides match.

The Efimov-like towers in these families have the following interpretations. In the superconducting system, they correspond to a family of gaps Δ_n, with the Efimov scaling responsible for the scale symmetry broken down to the discrete subgroup. In the spin chain, they correspond to different branches of solutions of the BAE, which can also be interpreted in terms of the allowed set of twists. Finally, in the CR family, they correspond to the family of shallow bound states near the continuum spectrum threshold.

7. RG cycles in Ω-deformed SUSY gauge theories

In this section, we explain how the RG flows in Ω-deformed SUSY gauge theories can be reformulated in terms of brane moves. Why can the RG cycles be expected in deformed gauge theories? The answer is based on the identification of quantum spin chains in one context or another in SUSY gauge theories [42, 43]. One such quantum spin chain has been found in the preceding sections, where the place of the RG cycles in the spin chain framework has been clarified.

First, we briefly review the Ω-deformation of SUSY gauge theories. Then we make some general comments concerning the realization of gauge theories as worldvolume theories on D-branes to explain how the gauge theory parameters are identified with brane coordinates.

7.1 Four-dimensional Ω-deformed gauge theory

The Bethe-ansatz equations can be encountered not only in models of superconductivity but also in gauge theories. A quantum XXX spin chain governs the moduli space of vacua of an Ω-deformed four-dimensional theory in the Nekrasov–Shatashvili limit, i.e., when one of the deformations is chosen to be zero: $\epsilon_2 = 0, \epsilon_1 = \epsilon$ [42, 43]. Because the quantum XXX spin chain displays a cyclic RG behavior, as we have seen in Section 5, it is interesting to identify this phenomenon in the four-dimensional gauge theory.

We consider a four-dimensional $\mathcal{N} = 2$ theory with a matter hypermultiplet that has a vanishing β-function, i.e., with $N_{\mathrm{f}} = 2N_{\mathrm{c}}$. This theory is dual to a classical inhomogeneous twisted XXX chain, in the sense that the Seiberg–Witten curve for the gauge theory coincides with the spectral curve for the spin chain. The twist of the spin chain is identified with the modular parameter of the curve and with the complexified coupling of the gauge theory; the inhomogeneities of the spin chain are mapped into masses of hypermultiplets. For more information on the correspondence between classical integrable systems and gauge theories, the reader can consult [44].

The Ω-deformation is introduced to regularize the instanton divergence in the partition function of gauge theory [45]. We can consider the four-dimensional theory as a reduction of the six-dimensional $\mathcal{N} = 1$ theory with the metric

$$\mathrm{d}s^2 = 2\,\mathrm{d}z\,\mathrm{d}\bar{z} + (\mathrm{d}x^m + \Omega^{mn}x_n\,\mathrm{d}\bar{z} + \bar{\Omega}^{mn}x_n\,\mathrm{d}z)^2, \quad m = 1,\ldots,4\,, \tag{93}$$

i.e., we can consider the theory on a four-dimensional space fibered over a two-dimensional torus. We can regard the $\epsilon_{1,2}$ deformation parameters as chemical potentials for the rotations in two orthogonal planes in the four-dimensional Euclidean space. We can also imagine that the Euclidean $\mathbb{R}^4$ space is replaced with a sphere S^4 with finite volume.

A nontrivial Ω-deformation modifies the correspondence between gauge theories and integrable systems. Namely, in the Nekrasov–Shatashvili limit, the Ω-deformed gauge theory corresponds to a quantum XXX spin chain, with ϵ playing the role of the Planck constant [42, 43]. This deformed gauge theory also appears to be dual to the two-dimensional effective theory on a worldsheet of a non-Abelian string [46, 47].

We consider Ω-deformed $\mathcal{N} = 2$ SQCD with the $\mathrm{SU}(L)$ gauge group, L fundamental hypermultiplets with masses m_i^{f}, and L antifundamental hypermultiplets with masses m_i^{af}. We let $\mathbf{a}$ denote the set of eigenvalues of the adjoint scalar in the vector multiplet. We can expand the deformed partition function around $\epsilon = 0$ to identify the prepotential and effective twisted superpotential:

$$\log \mathcal{Z}(\mathbf{a}, \epsilon_1, \epsilon_2) \sim \frac{1}{\epsilon_1\epsilon_2}\,\mathcal{F}(\mathbf{a}, \epsilon) + \frac{1}{\epsilon_2}\,\mathcal{W}(\mathbf{a}, \epsilon)\,. \tag{94}$$

The effective twisted superpotential is a multivalued function, with the branch fixed by a set of integers $\mathbf{k}$:

$$\mathcal{W}(\mathbf{a}, \epsilon) = \frac{1}{\epsilon}\,\mathcal{F}(\mathbf{a}, \epsilon) - 2\pi\mathrm{i}\mathbf{k}\mathbf{a}\,, \qquad \mathbf{k} \in \mathbb{Z}^L\,. \tag{95}$$

The equation for vacua,

$$\frac{\partial \mathcal{W}(\mathbf{a}, \epsilon)}{\partial \mathbf{a}} = \mathbf{n}\,, \qquad \mathbf{n} \in \mathbb{Z}^L\,, \tag{96}$$

provides the condition on $\mathbf{a}$,

$$\mathbf{a} = \mathbf{m}^{\mathrm{f}} - \epsilon\mathbf{n}\,. \tag{97}$$

This theory permits the existence of non-Abelian strings probing the four-dimensional space–time. The two-dimensional worldsheet theory of a non-Abelian string involves L fundamental chiral multiplets with twisted masses M_i^{F} and L antifundamental multiplets with twisted masses M_i^{AF}, which are identified as

$$M_i^{\mathrm{F}} = m_i^{\mathrm{f}} - \frac{3}{2}\,\epsilon\,, \qquad M_i^{\mathrm{AF}} = m_i^{\mathrm{af}} + \frac{1}{2}\,\epsilon\,. \tag{98}$$

The two-dimensional theory also contains an adjoint chiral multiplet with the mass ϵ. The rank of the gauge group N (or equivalently the number of non-Abelian strings) is given in terms of the vector $\mathbf{n}$ by the relation

$$N + L = \sum_{l=1}^{L} n_l\,. \tag{99}$$

The modular parameters of the four-dimensional and two-dimensional theories are related as

$$\tau_{2d} = \tau_{4d} + \frac{1}{2}(N+1)\,. \tag{100}$$

The effective twisted worldsheet superpotential is given in terms of the four-dimensional superpotential:

$$\mathcal{W}_{4d}(a_i = m_i^{\mathrm{f}} - \epsilon n_i, \epsilon) - \mathcal{W}_{4d}(a_i = m_i^{\mathrm{f}} - \epsilon, \epsilon) = \mathcal{W}_{2d}\big(\{n_i\}\big)\,. \tag{101}$$

The two-dimensional superpotential depends on the set of eigenvalues λ_i, $i = 1,\ldots,N$, of the adjoint scalar in the vector representation. The set of equations $\partial W_{2d}/\partial\lambda = 0$ coincides with the Bethe ansatz equations for the XXX spin chain:

$$\prod_{l=1}^{L}\left(\frac{\lambda_j - M_l^{\mathrm{F}}}{\lambda_j - M_l^{\mathrm{AF}}}\right) = \exp\,(2\pi\mathrm{i}\tau_{4d}) \prod_{k\neq j}^{N}\left(\frac{\lambda_j - \lambda_k - \epsilon}{\lambda_j - \lambda_k + \epsilon}\right). \tag{102}$$

The Planck constant in the spin chain is identified with the ϵ deformation parameter. The complexified coupling parameter plays the role of twist in the spin chain. The renormalization of the spin chain amounts to the decoupling of one fundamental and one anti-fundamental chiral multiplet. In the four-dimensional theory, it corresponds to a decrease in the number of flavors, $N_f \to N_f - 2$, simultaneously with a reduction in the gauge group rank, $N_c \to N_c - 1$. Therefore, the theory remains conformal. The renormalization of the coupling constant analogous to (73) derived from relation (102) for $N = 1$ is

$$\exp\left[2\pi i(\tau_L - \tau_{L-1})\right] = \frac{\lambda - M_L^{F}}{\lambda - M_L^{AF}} . \qquad (103)$$

If we choose the masses to be equidistant with a spacing δm, the change in the coupling constant under one step of the RG flow is

$$\exp\left[2\pi i(\tau_L - \tau_{L-1})\right] \propto \frac{\epsilon}{\delta m} . \qquad (104)$$

Hence, a number of nonperturbative scales emerge in the theory, analogously to the generation of the Efimov scaling in the Calogero model. These scales correspond to multiple gaps in the superconducting model:

$$\Delta_n \propto \Delta_0 \exp\left(-\frac{\pi n \delta m}{\epsilon}\right) . \qquad (105)$$

We note that the emergence of cyclic RG evolution is a feature caused by the Ω-deformation, since the decoupling of a heavy flavor in a nondeformed theory does not lead to any cyclic dynamics.

7.2 3d gauge theories and theories on brane worldvolumes

We briefly explain the main points concerning the geometric engineering of gauge theories on D-branes, suggesting that the reader address review paper [48] for the details. A Dp brane is a $(p+1)$-dimensional hypersurface in the ten-dimensional space–time that supports a U(1) gauge field. This feature provides the possibility of building gauge theories with the desired properties. We summarize the key elements of the 'building procedure'.

• A stack of N coinciding D-branes supports the $U(N)$ gauge theory with the maximal supersymmetry.

• Displacing some branes from the stack in the transverse direction corresponds to the Higgs mechanism in the $U(N)$ gauge theory and the distance between branes corresponds to the Higgs vev.

• To reduce the SUSY, some boundary conditions at some coordinates are imposed using other types of branes or some branes are rotated.

• All geometrical characteristics of the brane configurations have the meaning of gauge theory parameters, like couplings or vevs of some operators in the gauge theory on worldvolumes of branes.

• If we move some brane through another one, a brane of a smaller dimensions can be created. The Hanany–Witten move is the simplest example (Fig. 2).

• Generically, we have branes of different dimensions in the configuration, for example, N D2 branes and M D4 branes, and hence we simultaneously have a $U(N)$ $(2+1)$-dimensional gauge theory and a $U(M)$ $(4+1)$-dimensional theory on brane worldvolumes. These theories coexist with each other; hence, there is a highly nontrivial interplay between two gauge theories.

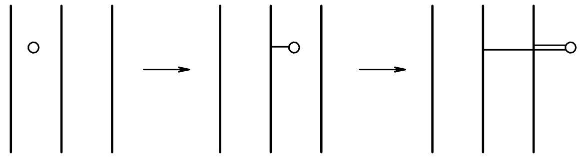

Figure 2. Hanany–Witten move. The vertical lines are NS5 branes, horizontal lines are D3 branes, and circles are D5 branes. When a D5 brane is moved through a sequence of NS5 branes, the linking number between them is preserved; hence, additional D3 branes appear.

	0	1	2	3	4	5	6	7	8	9
D3	×	×	×	×						
NS5	×	×	×		×	×	×			
D5	×	×	×					×	×	×

Figure 3. Brane construction of the 3d quiver theory.

We now explain how these brane rules can be used to engineer the gauge theories related to quantum spin chains. Our main example is a 3d $\mathcal{N} = 2$ quiver gauge theory.

The brane configuration relevant for this theory is built as follows. We have M parallel NS5 branes extended in (012456), N_i D3 branes extended in (0123) between the ith and $(i+1)$th NS5 branes, and M_i D5 branes extended in (012789) between the ith and $(i+1)$th NS5 branes (Fig. 3). From this brane configuration, we obtain the $\prod_i^M U(N_i)$ gauge group on the D3 branes worldvolume with M_i fundamentals for the ith gauge group. The distances between the ith and $(i+1)$th NS5 branes yield the complexified gauge coupling for the $U(N_i)$ gauge group, while the coordinates of the D5 branes in the (45) plane correspond to the masses of fundamentals. The positions of the D3 branes on the (45) plane correspond to the coordinates on the Coulomb branch in the quiver theory. The additional Ω deformation reduces the theory with $\mathcal{N} = 4$ SUSY to the $\mathcal{N} = 2^*$ theory, i.e., an $\mathcal{N} = 2$ theory with a massive adjoint. It is identified as a 3d gauge theory when the distance between NS5 is assumed to be small enough. We assume that one coordinate is compact, that is, the theory lives on $\mathbb{R}^2 \times S^1$.

The mapping of gauge theory data into the integrability framework goes as follows. In the NS limit of the Ω-deformation, the twisted superpotential in the 3d gauge theory on the D3 branes is mapped into the Yang–Yang function for the XXZ chain [49]. The minimization of the superpotential yields equations describing the supersymmetric vacua and at the same time they are the Bethe ansatz equations for the XXZ spin chain (generally speaking, the nested Bethe ansatz equations). That is, D3 branes are identified with the Bethe roots, which are distributed according to the ranks of the gauge groups at each of M steps of nesting, $\prod_i^M U(N_i)$.

The positions of the D5 branes in the (45) plane correspond to inhomogeneities in the XXZ spin chain. The anisotropy of the XXZ chain is defined by the radius of the compact dimensions, while the parameter of the Ω deformation plays the role of the Planck constant in the XXZ spin chain. At a small radius, the XXZ spin chain turns into the XXX spin chain. The twists in the spin chain correspond to the coordinates of the NS5 branes in the (78) plane and the Fayet–Iliopoulos parameters in the three-dimensional theory [37].

One step of the RG flow corresponds to the elimination of one inhomogeneity in the spin chain, resulting in a renormalization of the twists. In the three-dimensional theory, this means that the integration of one massive flavor leads to a renormalization of the FI parameters. In terms of the transformations of brane configurations, this process acquires a transparent geometrical interpretation:

• The RG step is the removal of one D5 brane, which amounts to a renormalization of the position of NS5 branes or twists.

• The period of the RG cycle is fixed by the number of NS5 branes [49], since it was identified with the Planck constant in the spin chain.

• At some scale, the twists flow from $+\infty$ to $-\infty$.

8. Conclusion

Are there any general lessons that we can learn for the quantum field theory from the very existence of cyclic RG flows? The most important point is that there is some fine structure on the UV scale, which is reflected in the Efimov tower with the BKT scaling behavior. Moreover, cyclic flows imply an interplay between the UV and IR cutoffs in the theory, which was usually attributed to noncommutative theory. This mixing could presumably shed additional light on the dimensional transmutation phenomenon in field theory and provide examples for simultaneous generation of multiple scales.

The presence of two parameters in the RG is quite common; however, some additional properties of these parameters are probably required. In particular, in many (although not all) examples, the period of the cycle is fixed by the 'filling fraction' in some external field, which could be a magnetic field or a parameter of the Ω background. This parameter has the meaning of a Planck constant in an auxiliary finite dimensional integrable model. This could suggest that the entire issue can be formulated purely in terms of a quantum phase space, since the Planck constant can be interpreted as an external field applied to the classical phase space.

Actually, we could expect a relation of RG cycles with some refinement of the path integral in quantum mechanics. As an aside, we note that the attempt to obtain a rigorous mathematical formulation of renormalization in QFT led to the motive generalization of the path integral. It corresponds to some fine structure on the regulator scale, which has some similarities with the discussion above. The RG cycle in the quantum rational Calogero model implies an intimate relation with knot theory, because knot invariants are characteristics of the Calogero spectrum with a rational Calogero coupling (see, e.g., [42]).

As we have already mentioned, the cyclic renormalization dynamics are connected with the BKT pairing of partons in a two-dimensional model. One could wonder whether this connection is universal. One four-dimensional example of such a pairing has to be mentioned: bion condensation in $(3+1)$ dimensions. The RG analysis of a model involving the gas of bions and electrically charged W bosons was considered in [50], where the RG flow involves fugacities for electric and magnetic components and the coupling constant. A coupled set of RG equations has been solved explicitly in the self-dual case, and the solution of the RG equations for the fugacities obtained in [50] is identical to the solution for the coupling in the RD model upon analytic continuation. The RG period in the solution above is fixed by the RG invariant that has been identified with the product of UV values of the electric and magnetic fugacities $\mu_e + \mu_m$. The similarity with the RG behavior is not accidental: a relation between the gauge theory and the perturbed XY model was found in [50].

We emphasize that the investigation of various aspects of limit cycles in RG dynamics still remains at an early stage, and there are numerous open questions. The RG cycles can have numerous applications to different aspects of mathematical physics. In this case, the RG dynamics is regarded as an example of a nontrivial dynamical system.

The work of A G and K B was supported in part by the grants RFBR-12-02-00284 and PICS-12-02-91052. The work of K B was also supported by a Dynasty fellowship. A G thanks FTPI at the University of Minnesota, where part of this work was done, for the hospitality and support. We thank N A Nekrasov and F K Popov for the useful discussions and comments.

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