

Vacuum structure in 3D supersymmetric gauge theories

A V Smilga

DOI: 10.3367/UFNe.0184.201402e.0163

Contents

1. Introduction	155
2. Pure $\mathcal{N} = 1$ SYMCS theory: the leading BO analysis	157
2.1 SU(2); 2.2 Higher-rank unitary groups	
3. Loop corrections	159
3.1 Infinite volume; 3.2 Finite volume	
4. Theories with matter	162
4.1 4d theories; 4.2 3d superspace; 4.3 Index calculations	
5. Appendix. Theta functions	165
References	166

Abstract. Based on a talk given at the Pomeranchuk memorial conference at ITEP in June 2013, we review the vacuum dynamics in 3d supersymmetric Yang–Mills–Chern–Simons theories with and without extra matter multiplets. By analyzing the effective Born–Oppenheimer Hamiltonian in a small spatial box, we calculate the number of vacuum states (Witten index) and examine their structure for these theories. The results are identical to those obtained by other methods.

1. Introduction

Probably the best known scientific achievement of Isaak Yakovlevich Pomeranchuk was the concept of the vacuum Regge pole that is nowadays called the *pomeron*. I did not have a chance to meet Pomeranchuk personally — I came to ITEP when he was already gone. But I heard many times from his colleagues and collaborators that Isaak Yakovlevich attributed great significance to studying properties of the vacuum and even used to joke about the urgent need for the ITEP theory group to buy a powerful pump for that purpose.

Pomeranchuk did not know that with the advent of supersymmetry, the issues of vacuum structure and vacuum counting would acquire a special interest. The existence of supersymmetric vacua (ground states of a Hamiltonian annihilated by the action of supercharges and having zero

energy) shows that supersymmetry is unbroken, while the absence of such states signals spontaneous breaking of supersymmetry. The crucial quantity to be studied in this respect is the *Witten index*, the difference between the numbers of bosonic and fermionic vacuum states, which can also be represented as

$$I = \text{tr} [(-1)^F \exp(-\beta H)], \quad (1.1)$$

where H is the Hamiltonian and F is the fermion charge operator. Due to supersymmetry, nonvacuum contributions in the trace cancel. It is important that quantity (1.1) represents an *index*, a close relative of the Atiyah–Singer index and other topological invariants, which is invariant under smooth Hamiltonian deformations. This allows evaluating the Witten index for rather complicated theories: it is sufficient to find a proper simplifying deformation.

This talk (based on three recent studies [1–3]) is devoted exactly to that. We study the vacuum dynamics in a particular class of theories, supersymmetric 3-dimensional gauge theories with a Chern–Simons term. Such theories have recently attracted considerable attention in view of newly discovered dualities between certain $\mathcal{N} = 8$ and $\mathcal{N} = 6$ versions of these theories and the respective string theories on $\text{AdS}_4 \times S^7$ or $\text{AdS}_4 \times \mathbb{CP}^3$ backgrounds [4, 5], where AdS_4 is the 4-dimensional anti-de Sitter space, S^7 is a 7-dimensional sphere, and \mathbb{CP}^3 is the complex projective space.¹ We note, however, that the field theories dual to string theories are conformal and do not involve a mass gap. In such theories, the conventional Witten (i.e., toroidal) index we are interested in here is not well defined, and the proper tool to study them is the so-called superconformal (spherical) index [9, 10].

We calculate the index by deforming the theory, putting it in a small spatial box, and studying the dynamics of the

A V Smilga Laboratoire de Physique Subatomique et des technologies associées (SUBATECH), Université de Nantes, 4 rue Alfred Kastler, BP 20722, Nantes 44307, France
E-mail: smilga@subatech.in2p3.fr
Alikhanov Institute for Theoretical and Experimental Physics, Russian Federation State Scientific Center, ul. B. Cheremushkinskaya 25, 117218 Moscow, Russian Federation

Received 23 November 2013

Uspekhi Fizicheskikh Nauk 184 (2) 163–176 (2014)

DOI: 10.3367/UFNe.0184.201402e.0163

Translated by A V Smilga; edited by A M Semikhatov

¹ Better known is the Maldacena duality between the 4d $\mathcal{N} = 4$ SYM and string theory on $\text{AdS}_5 \times S^5$ [6, 7]. A nice review of this topic was recently published in *Physics–Uspekhi* [8].

Hamiltonian thus obtained in the framework of the Born–Oppenheimer (BO) approximation. The results coincide with those obtained by other methods.

We first discuss the simplest such theory, the $\mathcal{N} = 1$ supersymmetric Yang–Mills–Chern–Simons (SYMCS) theory with the Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \left\langle -\frac{1}{2} F_{\mu\nu}^2 + i\bar{\lambda}\not{D}\lambda \right\rangle + \kappa \left\langle \epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \bar{\lambda}\lambda \right\rangle. \quad (1.2)$$

The conventions are as follows: $\epsilon^{012} = 1$, $D_\mu \mathcal{O} = \partial_\mu \mathcal{O} - i[A_\mu, \mathcal{O}]$ (such that A_μ is Hermitian), λ_α is a 2-component Majorana 3d spinor belonging to the adjoint representation of the gauge group, $\langle \dots \rangle$ stands for the color trace, and $\not{D} = D_\mu \gamma_\mu$. We choose

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^3, \quad (1.3)$$

where σ^a are the Pauli matrices. This is a 3d theory, and the gauge coupling constant g^2 carries the dimension of mass. The physical boson and fermion degrees of freedom in this theory are massive,

$$m = \kappa g^2. \quad (1.4)$$

In three dimensions, a nonzero mass leads to parity breaking. The requirement for $\exp(iS)$ to be invariant under certain *large* (noncontractible) gauge transformations (see, e.g., Ref. [11] for a nice review) leads to the quantization condition

$$\kappa = \frac{k}{4\pi} \quad (1.5)$$

with integer or sometimes (see below) half-integer *level* k .

Index (1.1) was evaluated in [12] with the result

$$I(k, N) = (\text{sgn } k)^{N-1} \binom{|k| + \frac{N}{2} - 1}{N-1} \quad (1.6)$$

for the $SU(N)$ gauge group. This is valid for $|k| \geq N/2$. For $|k| < N/2$, the index vanishes and supersymmetry is broken. In the simplest $SU(2)$ case, the index is just

$$I(k, 2) = k. \quad (1.7)$$

For $SU(3)$, it is

$$I(k, 3) = \frac{k^2 - 1/4}{2}. \quad (1.8)$$

We can now notice that for the index to be an integer, the level k should be a half-integer rather than an integer for $SU(3)$ and for all unitary groups with odd N . The explanation is that in these cases, the large gauge transformation mentioned above not only shifts the classical action but also contributes the extra factor -1 due to the modification of the fermion determinant [13, 14].

Result (1.6) was derived in [12] by the following reasoning. We consider the theory in a *large* spatial volume, $g^2 L \gg 1$, and, in the functional integral for index (1.1), mentally perform the Gaussian integration over fermionic variables. This gives an effective bosonic action that involves the CS term, the Yang–Mills term, and other higher-derivative

gauge-invariant terms. After that, the coefficient of the CS term is renormalized:²

$$k \rightarrow k - \frac{N}{2}. \quad (1.9)$$

At large β , the sum in (1.1) is saturated by the vacuum states of the theory and hence depends on the low-energy dynamics of the corresponding effective Hamiltonian. The vacuum states are determined by the term with the lowest number of derivatives, i.e., the Chern–Simons term; the effects due to the YM term and higher-derivative terms are suppressed at small energies and a large spatial volume. Basically, the spectrum of vacuum states coincides with the *full* spectrum in the topological pure CS theory, which was determined some time ago

- by establishing a relationship between pure 3d CS theories and 2d Wess–Zumino–Novikov–Witten (WZNW) theories [15];

- by canonical quantization of the CS theory and direct determination of the wave functions annihilated by the Gauss law constraints [16, 17].

Index (1.6) is then determined as the number of states in the pure CS theory with shift (1.9). For example, in the $SU(2)$ case, the number of CS states is $k + 1$, which gives (1.7) after the shift.

In Sections 2 and 3, we rederive the result in (1.6) using another method. We choose the spatial box to be *small* rather than large, $g^2 L \ll 1$, and study the dynamics of the corresponding BO Hamiltonian. This method was developed in [18] and applied there to 4d SYM theories. We now explain how it works.

We take the simplest $SU(2)$ theory. With *periodic* boundary conditions for all fields,³ the slow variables in the effective BO Hamiltonian are just the zeroth Fourier modes of the spatial components of the Abelian vector potential and its superpartners,

$$C_j = A_j^{(0)3}, \quad \lambda_\alpha = \lambda_\alpha^{(0)3}. \quad (1.10)$$

(In the 4d case, the spatial index j takes three values, $j = 1, 2, 3$; λ_α is the Weyl 2-component spinor describing the gluino field.) The motion in the field space $\{C_j\}$ is actually finite because the shift

$$C_j \rightarrow C_j + 4\pi \frac{n_j}{L} \quad (1.11)$$

with an integer n_j amounts to a *contractible* (which is the non-Abelian specifics) gauge transformation, under which the wave functions are invariant. To the leading BO order, the effective Hamiltonian is nothing but the Laplacian

$$H^{\text{eff}} = \frac{g^2}{2L^2} P_j^2, \quad (1.12)$$

where P_j is the momentum conjugate to C_j . The vacuum wave function is hence just a constant, which can be multiplied by a function of holomorphic fermionic variables λ_α . We seem to have obtained four vacuum wave functions of fermion

² This is for $k > 0$. In what follows, k is assumed to be positive by default, although the results for negative k are also mentioned.

³ We stick to this choice here, although in a theory involving only adjoint fields, one could also impose so-called *twisted* boundary conditions. In 4d theories, this gives the same value for the index [18], but in 3d theories, the result turns out to be different [19].

charges 0, 1, and 2:

$$\Psi^{F=0} = 1, \quad \Psi_\alpha^{F=1} = \lambda_\alpha, \quad \Psi^{F=2} = \epsilon^{\alpha\beta} \lambda_\alpha \lambda_\beta. \quad (1.13)$$

However, the fermion wave functions are not allowed in this case; the wave functions in the original theory should be invariant under gauge transformations. For the effective wave functions, this translates into invariance under *Weyl reflections*. In the $SU(2)$ case, these are just a sign flip,

$$C_j \rightarrow -C_j, \quad \lambda_\alpha \rightarrow -\lambda_\alpha. \quad (1.14)$$

The functions $\Psi^{F=1}$ in (1.13) are not invariant under (1.14) and therefore are not allowed. We are left with two bosonic vacuum functions, giving the value $I = 2$ for the index. A somewhat more sophisticated analysis (which is especially nontrivial for orthogonal and exceptional groups [20–22]) allows evaluating the index for other groups. It coincides with the adjoint Casimir eigenvalue c_V (another name for it is the dual Coxeter number h^\vee). For $SU(N)$, $I = N$.

The analysis of 3d SYMCS theories along the same lines turns out to be more complicated:

- the tree-level effective Hamiltonian is not just a free Laplacian, but also involves an extra homogeneous magnetic field;
- the effective wave functions are not invariant under shifts (1.11), but are multiplied by certain phase factors [23];
- it is *not* enough to analyze the effective Hamiltonian to the leading BO order, but one-loop corrections should also be taken into account.

In Section 2, we perform an accurate BO analysis at the tree level. In Section 3, we discuss the loop corrections. Section 4 is devoted to SYMCS theories with matter. We discuss both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories. For the latter, we reproduce the results in [24], but derive them in a more transparent and simple way.

2. Pure $\mathcal{N} = 1$ SYMCS theory: the leading BO analysis

2.1 $SU(2)$

We consider the $SU(2)$ theory first. As was explained above, we impose periodic boundary conditions for all fields. In the 3d case, we are left with two bosonic slow variables $C_{j=1,2} = A_j^{(0)3}$ and one holomorphic fermionic slow variable $\lambda = \lambda_1^{(0)3} - i\lambda_2^{(0)3}$. The tree-level effective BO supercharges and Hamiltonian describe the motion in a homogeneous magnetic field proportional to the Chern–Simons coupling and take the form

$$Q^{\text{eff}} = \frac{g}{L} \lambda (P_- + \mathcal{A}_-), \quad (2.1)$$

$$\bar{Q}^{\text{eff}} = \frac{g}{L} \bar{\lambda} (P_+ + \mathcal{A}_+),$$

$$H^{\text{eff}} = \frac{g^2}{2L^2} [(P_j + \mathcal{A}_j)^2 + \mathcal{B}(\lambda\bar{\lambda} - \bar{\lambda}\lambda)], \quad (2.2)$$

where

$$\mathcal{A}_j = -\frac{\kappa L^2}{2} \epsilon_{jk} C_k, \quad (2.3)$$

$P_\pm = P_1 \pm iP_2$, $\mathcal{A}_\pm = \mathcal{A}_1 \pm i\mathcal{A}_2$, and $\mathcal{B} = \partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1$. The effective vector potential (2.3) depends on the field variables $\{C_1, C_2\}$ and is of course unrelated to $A_j^a(\mathbf{x})$. It is defined up to a gauge transformation,

$$\mathcal{A}_j \rightarrow \mathcal{A}_j + \partial_j f(\mathbf{C}). \quad (2.4)$$

Indeed, the particular form (2.3) follows from the CS terms $\sim \epsilon_{jk} A_j \dot{A}_k$ in Lagrangian (1.2), but we can always add a total time derivative to the Lagrangian, which adds a gradient to the canonical momentum P_j and to the effective vector potential.

Similarly to what we had in the 4d case, the motion in the space $\{C_1, C_2\}$ is finite. However, as was already mentioned, the wave functions are not invariant under shifts along the cycles of the dual torus, but acquire extra phase factors:

$$\Psi(X+1, Y) = \exp(-2\pi i k Y) \Psi(X, Y), \quad (2.5)$$

$$\Psi(X, Y+1) = \exp(2\pi i k X) \Psi(X, Y),$$

where $X = C_1 L / (4\pi)$ and $Y = C_2 L / (4\pi)$.

We explain where these factors come from. As was mentioned, the shifts $X \rightarrow X+1$ and $Y \rightarrow Y+1$ represent contractible gauge transformations. In 4d theories, wave functions are invariant under such transformations. But the YMCS theory is special in this respect. Indeed, the Gauss law constraint in the YMCS theory (and in SYMCS theories) is not just $D_j \Pi_j^a = 0$ but has the form

$$G^a = \frac{\delta \mathcal{L}}{\delta A_0^a} = D_j \Pi_j^a + \frac{\kappa}{2} \epsilon_{jk} \partial_j A_k^a = 0,$$

where $\Pi_j^a = F_{0j}^a / g^2 + (\kappa/2) \epsilon_{jk} A_k^a$ are the canonical momenta. The second term gives rise to a phase factor associated with an infinitesimal gauge transformation $\delta A_j^a(\mathbf{x}) = D_j \alpha^a(\mathbf{x})$ (the spatial coordinates \mathbf{x} are not to be confused with the rescaled vector potentials X, Y):

$$\Psi[A_j^a + D_j \alpha^a] = \exp\left(-\frac{i\kappa}{2} \int d\mathbf{x} \epsilon_{kl} \partial_k \alpha^a A_l^a\right) \Psi[A_j^a]. \quad (2.6)$$

This property also holds for finite contractible gauge transformations $\alpha^a = (4\pi x/L) \delta^{a3}$ or $\alpha^a = (4\pi y/L) \delta^{a3}$ implementing the shifts $C_{1,2} \rightarrow C_{1,2} + 4\pi/L$. The phase factors $\mathcal{E}_{1,2}(X, Y)$ thus obtained coincide with those in Eqn (2.5); they are nothing but the holonomies $\mathcal{E}_1 = \exp(i \int_{\gamma_1} \mathcal{A}_1 dC_1)$ and $\mathcal{E}_2 = \exp(i \int_{\gamma_2} \mathcal{A}_2 dC_2)$, with $\gamma_{1,2}$ being two cycles of the torus attached to the point (X, Y) . The factors $\mathcal{E}_{1,2}$ satisfy the property

$$\begin{aligned} \mathcal{E}_1(X, Y) \mathcal{E}_2(X+1, Y) \mathcal{E}_1^{-1}(X, Y+1) \mathcal{E}_2^{-1}(X, Y) \\ = \exp(4\pi i k) = 1. \end{aligned} \quad (2.7)$$

The phase $4\pi k$ acquired in going around the sequence of two direct and two inverse cycles is nothing other than $2\pi\Phi$, with Φ being the magnetic flux. For the wave functions to be uniquely defined, this flux must be quantized.

We note that if another gauge were chosen for \mathcal{A}_j , the holonomies $\mathcal{E}_{1,2}$ would be different, but property (2.7) would of course be preserved.

The eigenfunctions of Hamiltonian (2.2) satisfying boundary conditions (2.5) are given by elliptic functions—a variety of theta functions. There are $2k$ ground-state wave

functions. For $k > 0$, their explicit form is

$$\Psi_{\text{tree}}^{\text{eff}}(X, Y) \propto \exp(-\pi k \bar{z} z) \exp(\pi k \bar{z}^2) Q_m^{2k}(\bar{z}), \quad (2.8)$$

where $z = X + iY$, $m = 0, \dots, 2k - 1$, and the functions Q_m^q are defined in the Appendix. For negative k , the functions have the same form, but with z and \bar{z} interchanged and with an extra fermionic factor λ .

The index $I = 2k$ of the effective Hamiltonian (2.2) coincides with the flux of the effective magnetic field on the dual torus divided by 2π [25, 26].

We next note that not all the $2|k|$ states are admissible. We have to impose the additional Weyl invariance condition (following from the gauge invariance of the original theory). For $\text{SU}(2)$, this amounts to $\Psi^{\text{eff}}(-C_j) = \Psi^{\text{eff}}(C_j)$,⁴ which singles out $|k| + 1$ vacuum states, bosonic for $k > 0$ and fermionic for $k < 0$.

When $k = 0$, the effective Hamiltonian (2.2) describes free motion on the dual torus. There are two zero-energy ground states, $\Psi^{\text{eff}} = \text{const}$ and $\Psi^{\text{eff}} = \text{const} \times \lambda$ (we need not bother about the Weyl oddness of the factor λ for the same reason as above). The index is zero. We thus derive

$$I_{\text{SU}(2)}^{\text{tree}} = (|k| + 1) \text{sgn } k. \quad (2.9)$$

2.2 Higher-rank unitary groups

The effective Hamiltonian for the $\text{SU}(N)$ group involves $2r = 2(N - 1)$ slow bosonic and $r = N - 1$ slow fermionic variables $\{C_j^a, \lambda^a\}$ belonging to the Cartan subalgebra of $\text{su}(N)$ (r is the rank of the group). It has the form

$$H = \frac{g^2}{2L^2} [(P_j^a + \mathcal{A}_j^a)^2 + \mathcal{B}^{ab}(\lambda^a \bar{\lambda}^b - \bar{\lambda}^b \lambda^a)], \quad (2.10)$$

where

$$\begin{aligned} \mathcal{A}_j^a &= -\frac{\kappa L^2}{2} \epsilon_{jk} C_k^a, \\ \mathcal{B}^{ab} &= \kappa L^2 \delta^{ab} \end{aligned} \quad (2.11)$$

with $a = 1, \dots, r$. By the same token as in the $\text{SU}(2)$ case, the motion is finite and extends over the manifold $\text{T} \times \text{T}$, with T being the maximal torus of the group.

For $\text{SU}(3)$, the maximal torus is depicted in Fig. 1. Each point in Fig. 1 is a *coweight* $\{w^3, w^8\}$, such that a group element mapped on the maximal torus is $g^{\text{torus}} = \exp[4\pi i(w^3 t^3 + w^8 t^8)]$. The meaning of the dashed lines and special points marked by a square and triangle is to be explained shortly.

The index of the effective Hamiltonian can be evaluated semiclassically by reducing the functional integral for (1.1) to an ordinary one [27] and represents a *generalized* magnetic flux (this is simply the r th Chern class of a $\text{U}(1)$ bundle over $\text{T} \times \text{T}$ with the connection \mathcal{A}_j^a),

$$I = \frac{1}{(2\pi)^r} \int_{\text{T} \times \text{T}} \prod_{ja} dC_j^a \det \|\mathcal{B}^{ab}\|. \quad (2.12)$$

⁴ In contrast to what should be done in 4 dimensions, we did not include the Weyl reflection of the fermion factor λ entering the effective wave function for negative k . The reason is that for negative k , the conveniently defined *fast* wave function (by which the effective wave function depending only on C_j and λ should be multiplied) involves the Weyl-odd factor $C_1 + iC_2$. This oddness compensates the oddness of the factor λ in the effective wave function [1].

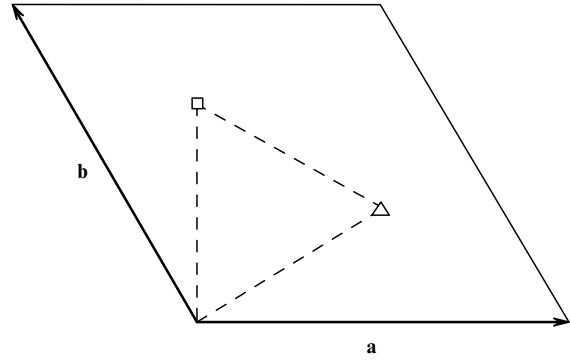


Figure 1. Maximal torus and Weyl alcove for $\text{SU}(3)$: \mathbf{a} and \mathbf{b} are the simple coroots. The points \square and \triangle are fundamental coweights.

For $\text{SU}(N)$,

$$I^{\text{SU}(N)} = Nk^{N-1}. \quad (2.13)$$

We find the explicit expressions for the $3k^2$ ground state wave functions in the case of $\text{SU}(3)$. They are given by generalized theta functions defined on the coroot lattice of $\text{SU}(3)$, and satisfy the boundary conditions

$$\begin{aligned} \Psi(\mathbf{X} + \mathbf{a}, \mathbf{Y}) &= \exp(-2\pi i \mathbf{k} \mathbf{a} \mathbf{Y}) \Psi(\mathbf{X}, \mathbf{Y}), \\ \Psi(\mathbf{X} + \mathbf{b}, \mathbf{Y}) &= \exp(-2\pi i \mathbf{k} \mathbf{b} \mathbf{Y}) \Psi(\mathbf{X}, \mathbf{Y}), \\ \Psi(\mathbf{X}, \mathbf{Y} + \mathbf{a}) &= \exp(2\pi i \mathbf{k} \mathbf{a} \mathbf{X}) \Psi(\mathbf{X}, \mathbf{Y}), \\ \Psi(\mathbf{X}, \mathbf{Y} + \mathbf{b}) &= \exp(2\pi i \mathbf{k} \mathbf{b} \mathbf{X}) \Psi(\mathbf{X}, \mathbf{Y}), \end{aligned} \quad (2.14)$$

where $\mathbf{X} = 4\pi \mathbf{C}_1/L$, $\mathbf{Y} = 4\pi \mathbf{C}_2/L$, and $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (-1/2, \sqrt{3}/2)$ are the simple coroots. When $k = 1$, there are three such states:

$$\begin{aligned} \Psi_0 &= \sum_{\mathbf{n}} \exp[-2\pi(\mathbf{n} + \mathbf{Y})^2 - 2\pi i \mathbf{X} \mathbf{Y} - 4\pi i \mathbf{X} \mathbf{n}], \\ \Psi_{\triangle} &= \sum_{\mathbf{n}} \exp[-2\pi(\mathbf{n} + \mathbf{Y} + \triangle)^2 - 2\pi i \mathbf{X} \mathbf{Y} - 4\pi i \mathbf{X}(\mathbf{n} + \triangle)], \\ \Psi_{\square} &= \sum_{\mathbf{n}} \exp[-2\pi(\mathbf{n} + \mathbf{Y} + \square)^2 - 2\pi i \mathbf{X} \mathbf{Y} - 4\pi i \mathbf{X}(\mathbf{n} + \square)], \end{aligned} \quad (2.15)$$

where the sums range the coroot lattice, $\mathbf{n} = m_a \mathbf{a} + m_b \mathbf{b}$ with integer $m_{a,b}$. Here, \triangle and \square are certain special points on the maximal torus (fundamental coweights), such that

$$\triangle \mathbf{a} = \square \mathbf{b} = \frac{1}{2}, \quad \square \mathbf{a} = \triangle \mathbf{b} = 0.$$

The group elements that correspond to the points 0 , \triangle , and \square belong to the center of the group,

$$\begin{aligned} U_0 &= \text{diag}(1, 1, 1), \\ U_{\square} &= \text{diag}\left(\exp \frac{2i\pi}{3}, \exp \frac{2i\pi}{3}, \exp \frac{2i\pi}{3}\right), \\ U_{\triangle} &= \text{diag}\left(\exp \frac{4i\pi}{3}, \exp \frac{4i\pi}{3}, \exp \frac{4i\pi}{3}\right). \end{aligned} \quad (2.16)$$

They are obviously invariant with respect to the Weyl symmetry, which permutes the eigenvalues.⁵ Thus, all three

⁵ For a generic coweight, the Weyl group elements permuting the eigenvalues $1 \leftrightarrow 2$, $1 \leftrightarrow 3$, and $2 \leftrightarrow 3$ act as reflections with respect to the dashed lines bounding the Weyl alcove (the quotient T/W) in Fig. 1.

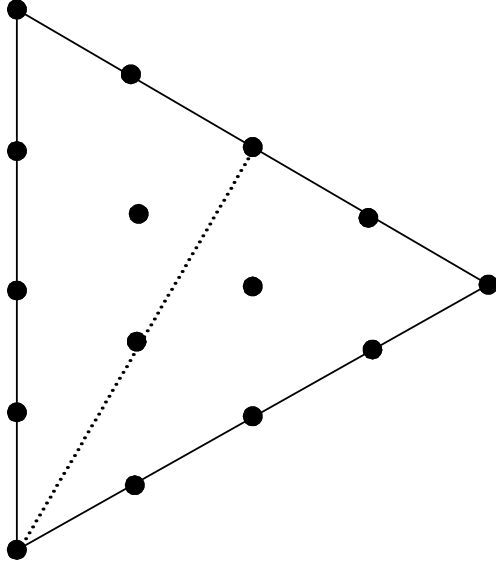


Figure 2. SU(3): 15 vacuum states for $k = 4$. The dotted line marks the boundary of the Weyl alcove for G_2 .

states (2.15) at the level $k = 1$ are Weyl invariant. But for $k > 1$, the number of invariant states is less than $3k^2$. For an arbitrary k , the wave functions of all $3k^2$ eigenstates can be written in the same way as in (2.15),

$$\Psi_n = \sum_{\mathbf{n}} \exp \left[-2\pi i (\mathbf{n} + \mathbf{Y} + \mathbf{w}_n)^2 - 2\pi i \mathbf{X} \mathbf{Y} - 4\pi i \mathbf{X} (\mathbf{n} + \mathbf{w}_n) \right], \quad (2.17)$$

where \mathbf{w}_n are coweights whose projections on the simple coroots \mathbf{a} and \mathbf{b} are integer multiples of $1/(2k)$. Only the functions (2.17) with \mathbf{w}_n lying at the vertices of the Weyl alcove are Weyl invariant. For all other \mathbf{w}_n , we should construct Weyl invariant combinations

$$\Psi = \sum_{\hat{x} \in W} \hat{x} \Psi_{\mathbf{w}_n}. \quad (2.18)$$

As a result, the number of Weyl-invariant states is equal to the number of coweights \mathbf{w}_n lying inside the Weyl alcove (including the boundaries). For example, in the case $k = 4$, there are 15 such coweights shown in Fig. 2 and, accordingly, 15 vacuum states.

For a generic k , the number of states is

$$I_{\text{SU}(3)}^{\text{tree}}(k > 0) = \sum_{m=1}^{k+1} m = \frac{(k+1)(k+2)}{2}. \quad (2.19)$$

The analysis for SU(4) is similar. The Weyl alcove is a tetrahedron with the vertices corresponding to central elements of SU(4). A purely geometric computation gives

$$I_{\text{SU}(4)}^{\text{tree}}(k > 0) = \sum_{m=1}^{k+1} \sum_{p=1}^m p = \frac{(k+1)(k+2)(k+3)}{6}. \quad (2.20)$$

The generalization to an arbitrary N is obvious. It gives the result

$$I^{\text{tree}}(k, N) = \binom{k+N-1}{N-1}. \quad (2.21)$$

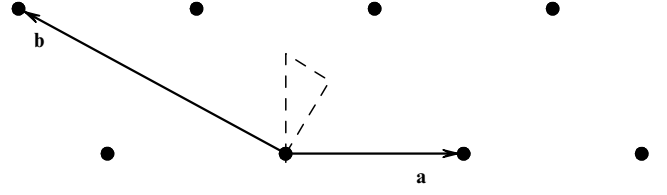


Figure 3. Coroot lattice and Weyl alcove for G_2 .

We also performed a similar analysis for the symplectic groups and for G_2 . We here discuss the G_2 case. The simple coroots for G_2 are $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (-3/2, \sqrt{3}/2)$. The coroot lattice and the maximal torus look exactly the same as those for SU(3) (Fig. 3). Hence, before the Weyl-invariance requirement is imposed, the index is equal to $3k^2$, as for SU(3). The difference is that the Weyl group now involves 12 rather than 6 elements, and the Weyl alcove is half the size of that for SU(3). As a result, for $k = 4$, we have only 9 (rather than 15) Weyl-invariant states (see Fig. 2). The general formula is

$$I_{G_2}^{\text{tree}}(k) = \begin{cases} \frac{(|k|+2)^2}{4} & \text{for even } k, \\ \frac{(|k|+1)(|k|+3)}{4} & \text{for odd } k. \end{cases} \quad (2.22)$$

3. Loop corrections

We mostly discuss the SU(2) theory in this section. For a generalization of all arguments to higher-rank groups, we refer the reader to [49].*

3.1 Infinite volume

It has been known since [28] that the CS coupling κ in a pure YMCS theory is renormalized at the 1-loop level. For $\mathcal{N} = 1, 2, 3$ SYMCS theories, the corresponding calculations were performed in [29]. The effect can be best understood by considering the fermion loop contribution to the renormalization of the $\propto A \partial A$ structure in the Chern–Simons term (Fig. 4). Recalling that κ and k are assumed to be positive by default, we obtain.

$$\Delta\kappa = -m c_V \int \frac{d^3 p_E}{(2\pi)^3} \frac{1}{(p_E^2 + m^2)^2} = -\frac{c_V}{8\pi}. \quad (3.1)$$

There is also a contribution coming from the gluon loop.⁶ It is convenient [2] to choose the Hamilton gauge $A_0 = 0$, in which case the gluon propagator

$$D_{jk}^{ab}(\omega, \mathbf{p}) = \frac{ig^2 \delta^{ab}}{\omega^2 - \mathbf{p}^2 - m^2} \left(\delta_{jk} - \frac{p_j p_k}{\omega^2} - \frac{im}{\omega} \epsilon_{jk} \right) \quad (3.2)$$

involves only transverse degrees of freedom, and there are no ghosts.

An accurate calculation gives

$$k_{\text{ren}} = k + c_V - \frac{c_V}{2}, \quad (3.3)$$

where the first term comes from the gluon loop and the second from the fermion loop.

* Added by the author to English proof.

⁶ On the other hand, spinless scalars (present in SYMCS theories with extended supersymmetries) do not contribute to the renormalization of κ .

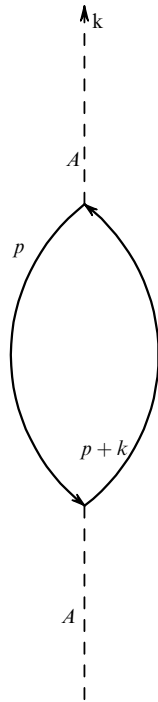


Figure 4. Renormalization of the structure $\propto \epsilon_{\mu\nu\rho} \text{tr} \{A_\mu \partial_\nu A_\rho\}$ by a fermion loop.

A legitimate question is whether the second and higher loops also bring about a renormalization of the level k . The answer is negative. The proof is simple. We consider the case $k \gg c_V$. This is the perturbative regime, where the loop corrections are ordered such that $\Delta k^{(1\text{ loop})} \sim O(1)$, $\Delta k^{(2\text{ loops})} \sim O(1/k)$, etc. But corrections to k of the order $\sim 1/k$ are not allowed. To ensure gauge invariance, k_{ren} must be an integer. Hence, all higher-loop contributions to k_{ren} must vanish, and they do.

We finally note that renormalization (3.3) refers to the supersymmetric Yang–Mills–Chern–Simons theory—a dynamical theory with nontrivial interactions. There is no such renormalization in the topological purely supersymmetric Chern–Simons theory, where the fermions decouple. The number of states in this theory is the same as in the pure CS theory.

3.2 Finite volume

As was mentioned above, the coefficient κ (with the factor L^2) has the meaning of the magnetic field on the dual torus for the effective finite-volume BO Hamiltonian. The renormalization of κ translates into a renormalization of this magnetic field. At the tree level, the magnetic field was constant. The renormalized field is not constant, however, but depends on the slow variables C_j . To find this dependence, we have to evaluate the effective Lagrangian in the slow Abelian background $C_j(t)$. The effective vector potential is extracted from the term $\sim A_j(\mathbf{C})\dot{C}_j$ in this Lagrangian. This term can be evaluated in the background field approach. Up to certain fine technical points [1, 2] that we do not discuss here, the result can be obtained by taking the same Feynman graphs that determine the renormalization of κ in the infinite volume and replacing

$$p_j \rightarrow \frac{2\pi n_j}{L} - C_j, \quad \int \frac{d^2 p}{(2\pi)^2} \rightarrow \frac{1}{L^2} \sum_{n_j} \quad (3.4)$$

in the spatial integrals there. The shift $-C$ in the momentum is due to replacing the usual derivative by the covariant one.

For the effective vector potentials induced by the fermion and the gluon loop, we derive⁷

$$\mathcal{A}_j^F = \frac{\epsilon_{jk}}{2} \sum_{n_j} \frac{(C - 2\pi n/L)_k}{(C - 2\pi n/L)^2} \left[1 - \frac{m}{\sqrt{(C - 2\pi n/L)^2 + m^2}} \right] \\ \xrightarrow{m \rightarrow 0} \frac{\epsilon_{jk}}{2} \sum_{n_j} \frac{(C - 2\pi n/L)_k}{(C - 2\pi n/L)^2}, \quad (3.5)$$

$$\mathcal{A}_j^B = -\frac{\epsilon_{jk}}{2} \sum_{n_j} \frac{(C - 2\pi n/L)_k}{(C - 2\pi n/L)^2} \left[2 - \frac{3m}{\sqrt{(C - 2\pi n/L)^2 + m^2}} \right. \\ \left. + \frac{m^3}{[(C - 2\pi n/L)^2 + m^2]^{3/2}} \right] \xrightarrow{m \rightarrow 0} -\epsilon_{jk} \sum_{n_j} \frac{(C - 2\pi n/L)_k}{(C - 2\pi n/L)^2}. \quad (3.6)$$

The corresponding induced magnetic fields are

$$\Delta \mathcal{B}^F(\mathbf{C}) = -\frac{m}{2} \sum_{n_j} \frac{1}{[(C - 2\pi n/L)^2 + m^2]^{3/2}}, \quad (3.7)$$

$$\Delta \mathcal{B}^B(\mathbf{C}) = \frac{3m}{2} \sum_{n_j} \frac{(C - 2\pi n/L)^2}{[(C - 2\pi n/L)^2 + m^2]^{5/2}}. \quad (3.8)$$

For most values of C_j , the corrections in (3.7) and (3.8) are of the order of $mL^3 = \kappa g^2 L^3$, which is small compared to $\mathcal{B}^{\text{tree}} \sim \kappa L^2$ if $g^2 L \ll 1$, which we assume. There are, however, four special points (the ‘corners’ of the torus),

$$C_j = 0, \quad C_j = \left(\frac{2\pi}{L}, 0 \right), \quad C_j = \left(0, \frac{2\pi}{L} \right), \quad C_j = \left(\frac{2\pi}{L}, \frac{2\pi}{L} \right), \quad (3.9)$$

in the vicinity of which the loop-induced magnetic field is *much larger* than the tree-level magnetic field. This actually means that the ‘Abelian’ BO approximation, based on the assumption that the energy scale associated with the slow variables $\{C_j, \lambda\}$ is small compared to the energy scale of the non-Abelian components and higher Fourier modes, breaks down in this region.

Disregarding this for a while, we can observe that the loop corrections bring about effective *flux lines* similar to Abrikosov vortices located at the corners. The width of these vortices is of the order of m . Gluon corrections generate the lines of unite flux $\Phi^B = 1/(2\pi) \int \Delta \mathcal{B}^B(\mathbf{C}) d^2 C$, while the fermion loops generate the lines of the flux $\Phi^F = -1/2$. In total, we have a line of the flux $\Phi_{\text{line}} = 1/2$ in each corner.

Adding the induced fluxes to the tree-level flux, we obtain the total flux

$$\Phi^{\text{tot}} = 2k + 4 \times 1 - 4 \times \frac{1}{2} = 2k + 2, \quad (3.10)$$

suggesting the presence of $2k + 2$ vacuum states in the effective BO Hamiltonian (that is, before the Weyl invariance requirement is imposed).

Not all these states are admissible, however. The wave functions of *four* such states turn out to be singular at the corners, and they should be dismissed.

⁷ The sums in (3.5) and (3.6) diverge at large $\|\mathbf{n}\|$. Their exact meaning is to be clarified shortly.

Indeed, we find the effective wave functions of all $2k + 2$ states in the Abelian valley far enough from the corners (3.9). The effective vector potential corresponding to one of the loop-induced flux lines can be chosen in the form

$$\mathcal{A}_j = -\frac{\epsilon_{jk} C_k}{2C^2} F(m^2, C^2), \quad (3.11)$$

where the core profile function deduced from (3.5) and (3.6),

$$F(m^2, C^2) = 1 - \frac{2m}{\sqrt{C^2 + m^2}} + \frac{m^3}{(C^2 + m^2)^{3/2}}, \quad (3.12)$$

vanishes at $C_j = 0$ and tends to 1 for large C_j .

We consider the effective supercharge Q^{eff} given by (2.1) in the vicinity of the origin, but outside the core of the vortex in (3.11),

$$m \ll C_j \ll \frac{4\pi}{L}. \quad (3.13)$$

We can then set $F(m^2, C^2) = 1$ and neglect the contribution of other flux lines as well as the contribution of homogeneous field (2.3). The equation $Q^{\text{eff}} \chi^{\text{eff}} = 0$ for the vacuum effective wave function then acquires the form

$$\left(\frac{\partial}{\partial z} + \frac{1}{4z} \right) \chi^{\text{eff}} = 0 \quad (3.14)$$

[we recall that $z = C_+ L / (4\pi)$]. Its solution is

$$\chi^{\text{eff}}(z, \bar{z}) \sim \frac{F(\bar{z})}{z^{1/4}}. \quad (3.15)$$

The effective wave function on the entire torus can be restored from two conditions:

- it must behave as in (3.15) in the vicinity of each corner;
- it must satisfy the boundary conditions with twist (2.7) corresponding to the total flux (3.10).

This gives the structure

$$\chi_m^{\text{eff}}(z, \bar{z}) \propto \frac{Q_m^{2k+2}(\bar{z})}{(\Pi(z) \Pi(\bar{z}))^{1/4}}, \quad (3.16)$$

where

$$\Pi(\bar{z}) = Q_3^4(\bar{z}) - Q_1^4(\bar{z}) \quad (3.17)$$

is a θ function of level 4 having zeros at the corners (3.9).⁸

We can now return to the sums in (3.5) and (3.6). The divergences can be regularized by subtracting a certain infinite pure-gauge part $\sim \partial_j f(\mathbf{C})$ from \mathcal{A}_j [as a side remark, this regularization breaks the apparent periodicity of (3.5) and (3.6)]. After that, the massless limits of $\mathcal{A}_+^{\text{F,B}}$ and $\mathcal{A}_-^{\text{F,B}}$ are given by meromorphic toric functions $P(\bar{z})$ and $P(z)$ with simple poles at the corners (3.9). They are obviously expressed in terms of $\Pi^{-1}(\bar{z})$ and $\Pi^{-1}(z)$.

The full wave function is the product of the effective wave function (3.16) and the ground-state wave function of the fast Hamiltonian. Near the corner $\mathbf{C} = 0$ in region (3.13), this Hamiltonian behaves as $\Psi^{\text{fast}} \sim 1/\sqrt{|z|}$ (see Eqn (3.16) in

Ref. [2]), which is extended to the behavior

$$\Psi^{\text{fast}} \sim \frac{1}{\sqrt{|\Pi(z)|}} \quad (3.18)$$

in the whole Abelian valley. Therefore, generically, the full wave function thus obtained is singular at the corners:

$$\Psi^{\text{fast}} \chi_m^{\text{eff}}(z, \bar{z}) \sim \frac{1}{|\Pi(z)|}. \quad (3.19)$$

The singularity in \mathcal{A}_j smears out when taking the finite core size into account, suggesting that the singularity in the effective wave function smears out as well. However, we cannot actually go inside the core in the Abelian BO framework: this approximation breaks down there, as we mentioned.

An accurate corner analysis (which is again a Born–Oppenheimer analysis, where we have to treat *all* zero Fourier modes of the fields,⁹ both Abelian and non-Abelian, as slow variables) that involves the matching of the corner wave function with the wave function in the Abelian valley far from the corners was performed in [2]. The result is rather natural. It turns out that the singularity *is not* smeared out when going into the vortex core. In other words, those states whose Abelian BO wave functions exhibit a singularity at the corners in the massless limit, as in (3.19), stay singular there in the exact analysis with finite mass. Such states are not admissible and must be disregarded.

The admissible wave functions still have structure (3.16), but theta functions $Q_m^{2k+2}(\bar{z})$ should have zeros at the corners. In other words, they can be represented as $\Pi(\bar{z})$ times a theta function of level $2k - 2$. This gives

$$\chi_m^{\text{eff}}(z, \bar{z}) \propto Q_m^{2k-2}(\bar{z}) \Pi^{3/4}(\bar{z}) \Pi^{-1/4}(z). \quad (3.20)$$

The parameter m now takes $2k - 2$ values, which gives $2k - 2$ [rather than $2k + 2$, as would follow naively from (3.16)] ‘pre-Weyl’ vacuum states. After imposing the Weyl-invariance condition, we obtain k states, in agreement with (1.7).

The following important remark is in order here. We have obtained $2k - 2$ pre-Weyl states by selecting $2k - 2$ nonsingular states out of $2k + 2$ states in Eqn (3.16). This equation was obtained by taking both gluon-induced and fermion-induced flux lines into account. However, it is possible to eliminate the gluon flux lines altogether.

In the region outside the vortex core, where the BO approximation works, we can translate the effective Lagrangian analysis leading to (3.7) and (3.8) to the effective Hamiltonian analysis. The induced vector potentials are then obtained as Pancharatnam–Berry phases [31, 32],

$$\mathcal{A}_j^{\text{PB}} = -i \frac{\int (\Psi^{\text{fast}})^* (\partial \Psi^{\text{fast}} / \partial C_j) dx^{\text{fast}}}{\int (\Psi^{\text{fast}})^* \Psi^{\text{fast}} dx^{\text{fast}}}. \quad (3.21)$$

The potentials leading to (3.7) and (3.8) correspond to a particular choice of Ψ^{fast} .

But we can as well modify the definition of the fast wave function by multiplying it by any function of slow variables. In particular, we can multiply it by a factor that is singular at

⁸ Function (3.17) is known from the studies of canonical quantization of pure CS theories [16, 17, 30]. It also enters relation (A.7).

⁹ To be precise, zero Fourier modes are relevant at the corner $C_j = 0$. At the other corners in (3.9), the slow modes are characterized by $n_j = (1, 0)$, $n_j = (0, 1)$, and $n_j = (1, 1)$.

the origin (the BO approximation is not applicable there anyway) and define

$$\tilde{\Psi}^{\text{fast}} = \Psi^{\text{fast}} \sqrt{\frac{\Pi(\bar{z})}{\Pi(z)}}. \quad (3.22)$$

We cannot decide between Ψ^{fast} and $\tilde{\Psi}^{\text{fast}}$ in the Abelian BO framework. Evaluating (3.21) with $\tilde{\Psi}^{\text{fast}}$ yields an extra gradient term [cf. (2.4)] giving a negative unit flux line in each corner. It annihilates the fluxes induced by gluon loops.

Now, the equation $Q^{\text{eff}} \tilde{\chi}^{\text{eff}} = 0$ becomes

$$\left(\frac{\partial}{\partial z} - \frac{1}{4z} \right) \chi^{\text{eff}} = 0, \quad (3.23)$$

with the solution $\chi^{\text{eff}} \sim z^{1/4} F(\bar{z})$. Its extension to the entire torus is

$$\tilde{\chi}_m^{\text{eff}}(z, \bar{z}) \propto Q_m^{2k-2}(\bar{z}) \sqrt{|\Pi(z)|}. \quad (3.24)$$

When multiplied by $\tilde{\Psi}^{\text{fast}}$, these functions (*all of which should be taken into account now*) give *exactly the same* full wave functions as before. We can therefore say that gluon-induced flux lines (more generally, any flux line with integer flux) should be disregarded in counting the vacua. Such flux lines (kinds of Dirac strings) are simply not observable. On the other hand, vortices with fractional fluxes affect vacuum counting. Heuristically, four half-integer flux lines in a sense ‘disturb’ this counting, making it ‘more difficult’ for the toric vacuum wave functions to stay uniquely defined (a single half-integer flux line would make it just impossible), such that the number of states decreases.

For all other groups, the gluon loops should also be disregarded (which was recently proved in [49]) and the index is obtained by substituting the value of k renormalized by exclusively fermion loops, $k \rightarrow k - c_V/2$, in the tree-level result.¹⁰

We arrive at the result in (1.6) for $SU(N)$. For G_2 , we obtain

$$I_{N=1}^{\text{SYMCS}}[G_2] = \begin{cases} \frac{k^2}{4} & \text{for even } k, \\ \frac{k^2 - 1}{4} & \text{for odd } k. \end{cases} \quad (3.25)$$

For completeness, we also give the result for the index for symplectic groups. For positive k ,

$$I_{N=1}^{\text{SYMCS}}[\text{sp}(2r)] = \left(k + \frac{r-1}{r} \right). \quad (3.26)$$

For negative k , the index is restored via $I(k) = (-1)^r I(-k)$.

4. Theories with matter

In theories with matter, the index is modified compared to the pure SYMCS theories due to two effects:

- an extra matter-induced renormalization of k ;
- the appearance of extra Higgs vacua due to nontrivial Yukawa interactions.

¹⁰ As we have seen, this renormalization should be understood *cum grano salis* because the renormalized magnetic field is concentrated at the corners, invalidating the Abelian BO approximation.

The first effect seems to be rather transparent: extra fermion loops lead to extra renormalization. There are, however, subtleties to be discussed later. As regards the extra Higgs vacua, their appearance is not limited to three dimensions; they also appear (and modify the index) in 4d supersymmetric gauge theories. We discuss this first.

4.1 4d theories

Historically, it was argued in Ref. [18] that adding *nonchiral* matter to a theory does not change the estimate for the index. Indeed, nonchiral fermions (and their scalar superpartners) can be given a mass. For large masses, they seem to decouple, and the index seems to be the same as in the pure SYM theory.¹¹ However, it was realized later that in *some* cases, massive matter *can* affect the index; the index may change when in addition to the mass term, Yukawa terms that couple different matter multiplets are added.

The simplest example¹² is the $\mathcal{N} = 1$ $SU(2)$ theory involving a couple of fundamental matter multiplets Q_f^j ($j = 1, 2$ being the color and $f = 1, 2$ the subflavor index; the indices are raised and lowered with $\epsilon^{jk} = -\epsilon_{jk}$ and $\epsilon^{fg} = -\epsilon_{fg}$) and an adjoint multiplet $\Phi_j^k = \Phi^a (t^a)_j^k$.

Let the tree superpotential be

$$\mathcal{W}^{\text{tree}} = \mu \Phi_k^j \Phi_j^k + \frac{m}{2} Q_f^j Q_j^f + \frac{h}{\sqrt{2}} Q_{jf} \Phi_k^j Q^{kf}, \quad (4.1)$$

where μ and m are adjoint and fundamental masses, and h is the Yukawa constant.

There is also the instanton-generated superpotential [37]

$$\mathcal{W}^{\text{inst}} = \frac{\Lambda^5}{V}, \quad (4.2)$$

where Λ is a constant with the dimension of mass and $V = Q_f^j Q_j^f / 2$ is the gauge-invariant modulus. Eliminating Φ , we obtain the effective superpotential

$$\mathcal{W}^{\text{eff}} = mV - \frac{h^2 V^2}{4\mu} + \frac{\Lambda^5}{V}. \quad (4.3)$$

The vacua are given by solutions of the equation $\partial \mathcal{W}^{\text{eff}} / \partial V = 0$. This equation is cubic, and hence there are *three* roots and *three* vacua.¹³

We now note that when h is very small, *one* of these vacua is characterized by a very large value, $\langle V \rangle \approx 2\mu m / h^2$ (and the instanton term in the superpotential plays no role here). In the limit $h \rightarrow 0$, it tends to infinity, and we are left with only *two* vacua, the same as in the pure SYM $SU(2)$ theory. Another way to see this is to observe that the equation $\partial \mathcal{W}^{\text{eff}} / \partial V = 0$ becomes quadratic for $h = 0$, having only two solutions.

The same phenomenon shows up in the theory with the G_2 gauge group studied in [39]. This theory involves three 7-plets S_f^j . The index of a pure SYM with the G_2 group is known to coincide with the adjoint Casimir eigenvalue c_V of G_2 . It is equal to 4. However, if we include the Yukawa term

$$\mathcal{W}^{\text{Yukawa}} = h \epsilon^{fgh} f^{jkl} S_{fj} S_{gk} S_{hl} \quad (4.4)$$

¹¹ This does not work for chiral multiplets, which are always massless and always affect the index [33, 34].

¹² It was very briefly considered in [35] and analyzed in detail in [36].

¹³ These three vacua are intimately related to three *singularities* in the moduli space of the associated $\mathcal{N} = 2$ supersymmetric theory with a single matter hypermultiplet, studied in [38].

in the superpotential (f^{jkl} being the Fano antisymmetric tensor), then two new vacua appear. They tend to infinity in the limit $h \rightarrow 0$.

The appearance of new vacua when Yukawa terms are added should by no means come as a surprise. This basically occurs because the Yukawa term has a higher dimension than the mass term.

4.2 3d superspace

We use a variant of the $\mathcal{N} = 1$ 3d superspace formalism developed in [40]. The superspace (x^μ, θ^α) involves a real 2-component spinor θ^α . Indices are lowered and raised with the antisymmetric $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta}$, $\theta^2 \equiv \theta^\alpha \theta_\alpha$, $\int d^2\theta \theta^2 = -2$. The 3d γ -matrices $(\gamma^\mu)^\alpha_\beta$, chosen as in (1.3), satisfy the identity

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\rho} \gamma_\rho. \quad (4.5)$$

We note that $(\gamma^\mu)^\alpha_\beta$ are all imaginary and symmetric.

Gauge theories are described in terms of the real spinorial superfield Γ_α . For non-Abelian theories, the Γ_α are Hermitian matrices. As in 4d, we can choose the Wess–Zumino gauge, reducing the number of components of Γ_α . In this gauge,

$$\Gamma_\alpha = i(\gamma^\mu)_{\alpha\beta} \theta^\beta A_\mu + i\theta^2 \lambda_\alpha. \quad (4.6)$$

The covariant superfield strength is then

$$\begin{aligned} W_\alpha &= \frac{1}{2} \mathcal{D}^\beta \mathcal{D}_\alpha \Gamma_\beta - \frac{1}{2} [\Gamma^\beta, \mathcal{D}_\beta \Gamma_\alpha] \\ &= -i\lambda_\alpha + \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu} (\gamma_\rho)_{\alpha\beta} \theta^\beta + \frac{i\theta^2}{2} (\gamma^\mu)^\beta_\alpha \nabla_\mu \lambda_\beta. \end{aligned} \quad (4.7)$$

In the superfield language, Lagrangian (1.2) is written as

$$\mathcal{L} = \int d^2\theta \left\langle \frac{1}{2g^2} W_\alpha W^\alpha + \frac{i\kappa}{2} \left(W_\alpha \Gamma^\alpha + \frac{1}{3} \{ \Gamma^\alpha, \Gamma^\beta \} \mathcal{D}_\beta \Gamma_\alpha \right) \right\rangle. \quad (4.8)$$

We now add matter multiplets. In this talk, we consider only real adjoint multiplets. (In Ref. [3], we also treat theories with complex fundamental multiplets.) Let there be only one such multiplet,

$$\Sigma = \sigma + i\psi_\alpha \theta^\alpha + i\theta^2 D. \quad (4.9)$$

The gauge invariant kinetic term has the form

$$\mathcal{L}^{\text{kin}} = -\frac{1}{2g^2} \int d^2\theta \langle \nabla_\alpha \Sigma \nabla^\alpha \Sigma \rangle, \quad (4.10)$$

where $\nabla_\alpha \Sigma = \mathcal{D}_\alpha \Sigma - [\Gamma_\alpha, \Sigma]$. We can also add the mass term¹⁴

$$\mathcal{L}_M = -i\zeta \int d^2\theta \langle \Sigma^2 \rangle. \quad (4.11)$$

Adding (4.8), (4.10), and (4.11), expressing the Lagrangian in components, and eliminating the auxiliary field D , we

¹⁴ This is a so-called *real* mass term, which can be expressed in terms of $\mathcal{N} = 2$ 3d superfields only by adding an explicit θ -dependence in the integrand in $S = \int d^4\theta \dots$ [41–43]. In a theory with two real multiplets (4.9) (which form a chiral $\mathcal{N} = 2$ multiplet), we can also write the complex mass term in the same way as in 4d theories. Such terms do not renormalize k , however, and we do not consider them here.

obtain

$$\begin{aligned} \mathcal{L} &= \frac{1}{g^2} \left\langle -\frac{1}{2} F_{\mu\nu}^2 + \nabla_\mu \sigma \nabla^\mu \sigma + \lambda \not{\nabla} \lambda + \psi \not{\nabla} \psi \right\rangle \\ &\quad + \kappa \left\langle \epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + i\lambda^2 \right\rangle \\ &\quad + i\zeta \langle \psi^2 \rangle - \zeta^2 g^2 \langle \sigma^2 \rangle. \end{aligned} \quad (4.12)$$

Besides the gauge field, the Lagrangian involves the adjoint fermion λ with the mass $m_\lambda = \kappa g^2$, the adjoint fermion ψ with the mass $m_\psi = \zeta g^2$, and the adjoint scalar σ with the same mass. The point $\zeta = \kappa$ is special. In this case, Lagrangian (4.12) enjoys the $\mathcal{N} = 2$ supersymmetry.

4.3 Index calculations

We consider the theory defined by (4.12). First, let $\zeta > 0$. Then the mass of the matter fermions is positive. To be more precise, it has the same sign as the gluino mass for $k > 0$. The matter loops lead to an extra renormalization of k .

We note that the status of this renormalization is different from the one due to the gluino loop. We have seen that for the latter, the induced magnetic field on the dual torus is concentrated in the corners (3.9), which follows from the equality $m_i L \ll 1$. On the other hand, the mass of the matter fields $m_\psi = \zeta g^2$ is an independent parameter. It is convenient to make it *large*, $m_\psi L \gg 1$. For a finite mass, the induced magnetic field has the form as in Eqn (3.7). For small $m_\psi L$, it is concentrated at the corners. But in the opposite limit, the induced flux density becomes constant, as the tree flux density. Thus, sufficiently massive matter brings about a true renormalization of k without any qualifications (*sine sale*, if you will).

For positive ζ , the renormalization is negative, $k \rightarrow k - 1$. The index coincides with the index of the $\mathcal{N} = 1$ SYMCS theory with a renormalized k ,

$$I_{\zeta>0} = k - 1. \quad (4.13)$$

For $k = 1$, the index is zero and supersymmetry is spontaneously broken.

For negative ζ , two things happen.

- The fermion matter mass has the opposite sign and so does the renormalization of k due to the matter loop. We seem to obtain $I_{\zeta<0} = k + 1$.

- This is wrong, however, due to another effect. For a positive ζ , the ground-state wave function in the matter sector is bosonic. But for a negative ζ , it is fermionic, $\Psi \propto \prod_a \psi^a$, changing the sign of the index.

We obtain

$$I_{\zeta<0} = -k - 1. \quad (4.14)$$

Supersymmetry is broken here for $k = -1$.

As was mentioned, Lagrangian (4.12) with $\zeta = \kappa$ has the $\mathcal{N} = 2$ extended supersymmetry. That means, in particular, that ζ changes sign together with κ , and the result is given by

$$I_{\mathcal{N}=2}^{\text{SYMCS}} = |k| - 1, \quad (4.15)$$

in agreement with [44, 45]. In contrast to (4.13) and (4.14), this expression is not analytic at $k = 0$, the nonanalyticity being exactly due to the sign flip of the matter fermion mass.

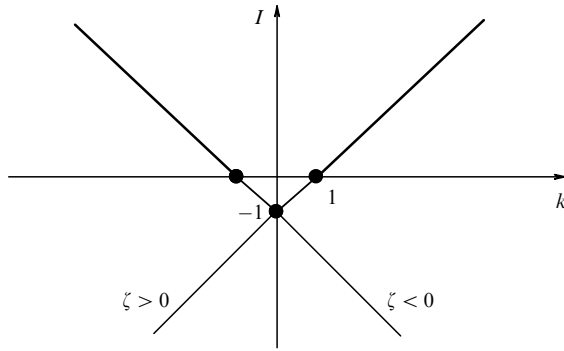


Figure 5. The indices in the theory (4.12) with $\zeta > 0$, $\zeta < 0$, and $\zeta = \kappa$ (bold lines).

Strictly speaking, formula (4.15) does not work for $k = 0$. In this case, we should also keep $\zeta = 0$, the matter is massless, massless scalars make the motion infinite, and the index is ill-defined. However, bearing in mind that the regularized theory with $\zeta \neq 0$ gives the result $I_{\mathcal{N}=2 \text{ deformed}}^{\text{SYMCS}}(0) = -1$, irrespective of the sign of ζ , we can attribute this value of the index also to $I_{\mathcal{N}=2}^{\text{SYMCS}}(0)$.

The three index formulas (4.13)–(4.15) are represented together in Fig. 5.

We next consider the theory involving gauge multiplet (4.6) and *three* $\mathcal{N} = 1$ real adjoint matter multiplets (4.9). If all the masses are equal, the theory has the extended $\mathcal{N} = 2$ supersymmetry if an extra Yukawa term is added to the Lagrangian [46]. If we call one of the matter multiplets (the one that together with (4.6) forms an $\mathcal{N} = 2$ superfield) Σ and combine the other two real multiplets into a complex $\mathcal{N} = 2$ multiplet Φ , then the Yukawa term acquires the form

$$\mathcal{L}_{\text{Yukawa}} = -\frac{2i}{g^2} \int d^2\theta \langle \Sigma \Phi \bar{\Phi} \rangle. \quad (4.16)$$

To calculate the index, however, we consider the *deformed* $\mathcal{N} = 1$ Lagrangian with different masses M (for the field Σ) and m (for the complex field Φ). We assume M to be large, but, to make the transition to the $\mathcal{N} = 2$ theory smooth, choose its sign to coincide with that of k . There are four different cases:¹⁵

$$\begin{aligned} 1. \quad m > 0, \quad k > 0 &\Rightarrow M > 0. \\ k \rightarrow k - 1_\Sigma - 2_\Phi &= k - 3. \end{aligned} \quad (4.17)$$

This contributes $k - 3$ to the index. We note that for $k = 1, 2$, this contribution is negative.

$$\begin{aligned} 2. \quad m > 0, \quad k, M < 0. \\ k \rightarrow k + 1_\Sigma - 2_\Phi &= k - 1. \end{aligned} \quad (4.18)$$

Multiplying it by -1 to account for the fermionic nature of the wave function [in this case, it involves a fermionic factor associated with the real adjoint matter multiplet Σ ; see the

¹⁵ When comparing with [24], note that the mass sign convention for the matter fermions there is *opposite* to the one in our convention. We call the mass positive if it has the same sign as the masses of fermions in the gauge multiplet for positive k (and hence positive ζ). In other words, for positive k ζ , the shifts of k due to both the gluino loop and the adjoint matter fermion loop have the negative sign.

discussion before Eqn (4.14)], we obtain $I = -k + 1$.

$$\begin{aligned} 3. \quad m < 0, \quad k, M > 0. \\ k \rightarrow k - 1_\Sigma + 2_\Phi &= k + 1, \end{aligned} \quad (4.19)$$

giving the contribution $I = k + 1$.

$$\begin{aligned} 4. \quad m < 0, \quad k, M < 0. \\ k \rightarrow k + 1_\Sigma + 2_\Phi &= k + 3. \end{aligned} \quad (4.20)$$

The contribution to the index is $-k - 3$.

In contrast to the model with only one real adjoint multiplet, this is not the full answer yet. There are also additional states on the Higgs branch that contribute to the index. Indeed, the superpotential is

$$\mathcal{W} = -\frac{i}{g^2} \left(\frac{M}{2} \Sigma^a \Sigma^a + m \bar{\Phi}^a \Phi^a + i\epsilon^{abc} \Sigma^a \Phi^b \bar{\Phi}^c \right). \quad (4.21)$$

The bosonic potential vanishes if

$$\begin{aligned} m\phi^b &= i\epsilon^{abc} \sigma^a \phi^c, \\ M\sigma^a &= i\epsilon^{abc} \bar{\phi}^b \phi^c. \end{aligned} \quad (4.22)$$

These equations have nontrivial solutions when both M and m are positive or when both M and m are negative. Let them be positive. Then (4.22) has a unique solution up to a gauge rotation:

$$\sigma^a = m \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \phi^a = \sqrt{\frac{Mm}{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}. \quad (4.23)$$

The corresponding contribution to the index is not just equal to 1, however, due to a new important effect that did not take place in the 4d theory with superpotential (4.1) considered above and would also be absent in a 3d theory with a fundamental $\mathcal{N} = 2$ matter multiplet.

Indeed, besides solution (4.23), there are also the solutions obtained from this one by gauge transformations. These are not necessarily global; they might depend on the spatial coordinates x, y . We note that for the theory defined on a torus, certain transformations can be applied to (4.23) that look like gauge transformations, but are not contractible due to the nontrivial group $\pi_1[\text{SO}(3)] = \mathbb{Z}_2$. (Here, $\text{SO}(3)$ should be understood not as the orthogonal group itself but as the adjoint representation space; cf. the discussion of higher isospins.) An example of such a quasi-gauge transformation is

$$\Omega_1 : O^{ab}(x) = \begin{pmatrix} \cos \frac{2\pi x}{L} & \sin \frac{2\pi x}{L} & 0 \\ -\sin \frac{2\pi x}{L} & \cos \frac{2\pi x}{L} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.24)$$

where L is the length of our box. Transformation (4.24) does not affect $\sigma^a = \sigma \delta^{a3}$ and keeps the fields $\phi^a(\mathbf{x})$ periodic.¹⁶ There is a similar transformation Ω_2 along the second cycle of the torus.

¹⁶ For matter in the fundamental representation, transformation (4.24) is inadmissible: when lifted to $\text{SU}(2)$, it would make a constant solution, the fundamental analog of (4.22), antiperiodic.

In 4d theories, wave functions are invariant under contractible gauge transformations. In 3d SYMCS theories, they are invariant up to a possible phase factor, as in (2.5). But nothing dictates the behavior of the wave functions under the transformations $\Omega_{1,2}$, which are actually *not* gauge symmetries, but rather some global symmetries of the theory living on a torus. We thus obtain four different wave functions, even or odd under the action of $\Omega_{1,2}$.¹⁷ The final result for the index of this theory is

$$I^{\mathcal{N}=2} = |k| + 1 \quad (4.25)$$

universally for positive and negative k . Extra Higgs states contribute only to positive k .

Result (4.25) was derived among others in [24] following a different logic. Intriligator and Seiberg did not deform $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$, but kept the fields in the real adjoint matter multiplet Σ light. Then the light matter fields $\{\sigma, \psi\}$ enter the effective BO Hamiltonian on the same ground as the Abelian components of the gluon and gluino fields. As was mentioned, the fluxes induced by the light fields are not homogeneous, being concentrated at the corners. This makes an accurate analysis substantially more difficult. Index (4.25) was obtained in [24] as a sum of *three* rather than just two contributions,¹⁸ and it is still not quite clear how this works in the particular case $k = 2$, where k_{eff} , defined in Ref. [24] and including only renormalizations due to the complex matter multiplet, $k_{\text{eff}} = k - 2$, vanishes.

Our method is simpler.

We can also add $\mathcal{N} = 2$ multiplets with higher isospins. Then the counting of Higgs vacuum states becomes more complicated. For example, for $I = 3/2$, there are 10 such states. This number is obtained as a sum of a single state with the isospin projection $1/2$ and $3^2 = 9$ states with the isospin projection $3/2$ (in the latter case, there is a constant solution supplemented by eight \mathbf{x} -dependent quasi-gauge copies). The generic result for the index in the theory involving several $\mathcal{N} = 2$ matter multiplets with different isospins is

$$I = |k| - 1 + \frac{1}{2} \sum_f T_2(I_f), \quad (4.26)$$

where

$$T_2(I) = \frac{2I(I+1)(2I+1)}{3} \quad (4.27)$$

is the Dynkin index of the corresponding representation normalized to $T_2(\text{fund}) = 1$.

When deriving (4.26), it was assumed that the matter-induced shift of the index is the sum of the individual shifts due to individual multiplets. This is true if the Lagrangian does not involve extra cubic $\mathcal{N} = 2$ invariant superpotentials, which can bring about extra Higgs vacuum states.

We can observe that the index does not depend on the sign of k , although this universal result is obtained by adding the contributions that look completely different for $k > 0$ and $k < 0$. For an individual multiplet contribution, the Higgs states contribute only for one sign of k (positive or negative,

depending on the sign of the mass). An interesting explanation of the symmetry under the mass sign flip with a given k (and hence under the sign flip of k with a given m) was suggested in [24]. Basically, the authors argued that one can add to the mass the size of one of the dual torus cycles times i to obtain a complex holomorphic parameter on which the index of an $\mathcal{N} = 2$ theory should not depend. Hence, it should not depend on the real part of this parameter (the mass). We believe that it is still dangerous to pass the point $m = 0$, where the index is not defined, and this argument therefore lacks rigor. Anyway, an explicit $\text{SU}(2)$ calculation shows that the symmetry with respect to mass sign flip is indeed maintained.

The reasoning above can be generalized to higher-rank unitary groups. Intriligator and Seiberg conjectured the following generalization of (4.26):

$$I^{\text{SU}(N)} = \frac{1}{(N-1)!} \prod_{j=-N/2+1}^{N/2-1} \left(|k| + \frac{1}{2} \sum_f T_2(R_f) - \frac{N}{2} - j \right), \quad (4.28)$$

implying that the overall shift of k is represented as the sum of the individual shifts due to individual multiplets. For an individual contribution to the shift, this formula can be derived for different signs of k and m when the extra Higgs states do not contribute. It can be extended to k, m of the same sign using the symmetry discussed above.

We checked that this works for all $\text{SU}(N)$ groups with fundamental matter and for $\text{SU}(3)$ with adjoint matter.¹⁹ It would be interesting to construct a rigorous proof of this fact.

5. Appendix. Theta functions

We here recall certain mathematical facts concerning the properties of analytic functions on a torus. They are mostly taken from textbook [48], but we are using a different notation, which we find clearer and more appropriate for our purposes.

Theta functions play the same role for the torus as ordinary polynomials for the Riemann sphere. They are analytic, but satisfy certain nontrivial quasiperiodic boundary conditions with respect to shifts along the cycles of the torus. A generic torus is characterized by a complex modular parameter τ , but we stick to the simplest choice $\tau = i$, such that the torus represents the square $x, y \in [0, 1]$ ($z = x + iy$) glued along the opposite sides.

The simplest θ -function satisfies the boundary conditions

$$\theta(z+1) = \theta(z), \quad (A.1)$$

$$\theta(z+i) = \exp[\pi(1-2iz)]\theta(z).$$

This defines a *unique* (up to a constant complex factor) analytic function. Its explicit form is

$$\theta(z) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 + 2\pi i n z). \quad (A.2)$$

This function [we call it the theta function of level 1 and introduce an alternative notation $\theta(z) \equiv \mathcal{Q}^1(z)$] has only one zero in the square $x, y \in [0, 1]$, exactly in its middle, $\theta((1+i)/2) = 0$.

¹⁷ The oddness of a wave function under transformation (4.24) means a nonzero *electric flux* in the language of Ref. [47].

¹⁸ On top of the usual vacua with $\phi = \sigma = 0$ and the Higgs vacua with $\phi, \sigma \neq 0$, they also had ‘topological vacua’ with $\phi = 0, \sigma \neq 0$. These do not appear in our approach.

¹⁹ We emphasize that this is all specific to $\mathcal{N} = 2$. For $\mathcal{N} = 1$ theories, there is no such symmetry (see, e.g., Fig. 2).

For any integer $q > 0$, theta functions of level q can be defined such that

$$Q^q(z+1) = Q^q(z), \quad (\text{A.3})$$

$$Q^q(z+i) = \exp[\pi i(1-2iz)] Q^q(z).$$

Boundary conditions (A.3) involve a twist [the exponent in the right-hand side of (2.7)], $-\pi q$, corresponding to a negative *magnetic flux*.²⁰ The functions $Q^q(\bar{z})$ have positive fluxes $2\pi q$. Multiplying $Q^{2k}(\bar{z})$ and $Q^{2k}(z)$ by proper exponentials yields functions (2.8) (no longer analytic) satisfying boundary conditions (2.5).

The functions satisfying (A.3) lie in a vector space of dimension q . The basis in this vector space can be chosen as

$$Q_m^q(z) = \sum_{n=-\infty}^{\infty} \exp \left[-\pi q \left(n + \frac{m}{q} \right)^2 + 2\pi i q z \left(n + \frac{m}{q} \right) \right],$$

$$m = 0, \dots, q-1. \quad (\text{A.4})$$

In Mumford's notation [48], $Q_m^q(z)$ can be expressed as

$$Q_m^q(z) = \theta_{m/q,0}(qz, iq), \quad (\text{A.5})$$

where $\theta_{a,b}(z, \tau)$ are theta functions of rational characteristics.

The $Q_m^q(z)$ can be called 'elliptic polynomials' of degree q . Indeed, each $Q_m^q(z)$ has q simple zeros at

$$z_s^{(m)} = \frac{2s+1}{2q} + i \left(\frac{1}{2} - \frac{m}{q} \right), \quad s = 0, \dots, q-1 \quad (\text{A.6})$$

(add i to bring it onto the fundamental domain $x, y \in [0, 1]$ when necessary). The product $Q^q(z)Q^{q'}(z)$ of two such 'polynomials' of degrees q and q' gives a polynomial of degree $q+q'$. There are many relations between the theta functions of different levels and their products. We can amuse the reader with the relation

$$\frac{Q_5^6(z) - Q_1^6(z)}{(Q_3^4(z) - Q_1^4(z))Q_0^2(z)} = \frac{1}{\eta(i)} = \frac{2\pi^{3/4}}{\Gamma(1/4)}, \quad (\text{A.7})$$

where $\Gamma(1/4)$ is the gamma function. The ratios of different elliptic functions of the same level give double periodic meromorphic elliptic functions. For example, the ratio of a properly chosen linear combination $\alpha Q_1^2(z) + \beta Q_2^2(z)$ and $\theta^2(z)$ is the Weierstrass function.

References

- Smilga A V *JHEP* (01) 086 (2010); arXiv:0910.0803
- Smilga A V *JHEP* (05) 103 (2012); arXiv:1202.6566
- Smilga A V, arXiv:1308.5951
- Bagger J, Lambert N *Phys. Rev. D* **77** 065008 (2008); arXiv:0711.0955
- Aharony O et al. *JHEP* (10) 091 (2008); arXiv:0806.1218
- Maldacena J M *Adv. Theor. Math. Phys.* **2** 231 (1998)
- Gubser S S, Klebanov I R, Polyakov A M *Phys. Lett. B* **428** 105 (1998)
- Gorsky A S *Phys. Usp.* **48** 1093 (2005); *Usp. Fiz. Nauk* **175** 1145 (2005)
- Römelberger C *Nucl. Phys. B* **747** 329 (2006); hep-th/0510060
- Spiridonov V P, Vartanov G S *Nucl. Phys. B* **824** 192 (2010); arXiv:0811.1909
- Dunne G V, in *Topological Aspects of Low Dimensional Systems: Ecole de Physique des Houches, Les Houches, Session LXIX, 7–31 July 1998* (Eds A Comtet et al.) (Berlin: Springer, 1999) p. 177; hep-th/9902115
- Witten E, in *The Many Faces of the Superworld* (Ed. M A Shifman) (Singapore: World Scientific, 2000) p. 156; hep-th/9903005
- Niemi A J, Semenoff G W *Phys. Rev. Lett.* **51** 2077 (1983)
- Redlich A N *Phys. Rev. D* **29** 2366 (1984)
- Witten E *Commun. Math. Phys.* **121** 351 (1989)
- Elitzur S et al. *Nucl. Phys. B* **326** 108 (1989)
- Labastida J M F, Ramallo A V *Phys. Lett. B* **227** 92 (1989)
- Witten E *Nucl. Phys. B* **202** 253 (1982)
- Henningson M *JHEP* (11) 013 (2012); arXiv:1209.1798
- Kac V G, Smilga A V, in *The Many Faces of the Superworld* (Ed. M A Shifman) (Singapore: World Scientific, 2000) p. 185; hep-th/9902029
- Keurentjes A *JHEP* (05) 001 (1999); hep-th/9901154
- Keurentjes A *JHEP* (05) 014 (1999); hep-th/9902186
- Deser S, Jackiw R, Templeton S *Ann. Physics* **140** 372 (1982)
- Intriligator K, Seiberg N, arXiv:1305.1633
- Dubrovnik B A, Krichever I M, Novikov S P *Sov. Math. Dokl.* **17** 947 (1977); *Dokl. Akad. Nauk SSSR* **229** 15 (1976)
- Dubrovnik B A, Novikov S P *Sov. Phys. JETP* **52** 511 (1980); *Zh. Eksp. Teor. Fiz.* **79** 1006 (1980)
- Cecotti S, Girardello L *Phys. Lett. B* **110** 39 (1982)
- Pisarski R D, Rao S *Phys. Rev. D* **32** 2081 (1985)
- Kao H-C, Lee K, Lee T *Phys. Lett. B* **373** 94 (1996); hep-th/9506170
- Gawedzki K, Kupiainen A *Nucl. Phys. B* **320** 625 (1989)
- Pancharatnam S *Proc. Indian Acad. Sci. A* **44** 247 (1956)
- Berry M V *Proc. R. Soc. Lond. A* **392** 45 (1984)
- Smilga A V *Sov. Phys. JETP* **64** 8 (1986); *Zh. Eksp. Teor. Fiz.* **91** 14 (1986)
- Blok B Yu, Smilga A V *Nucl. Phys. B* **287** 589 (1987)
- Intriligator K, Seiberg N *Nucl. Phys. B* **431** 551 (1994); hep-th/9408155
- Gorsky A, Vainshtein A, Yung A *Nucl. Phys. B* **584** 197 (2000); hep-th/0004087
- Affleck I, Dine M, Seiberg N *Phys. Lett. B* **137** 187 (1984)
- Seiberg N, Witten E *Nucl. Phys. B* **431** 484 (1994); hep-th/9408099
- Smilga A V *Phys. Rev. D* **58** 105014 (1998); hep-th/9801078
- Gates S J (Jr.) et al. *Front. Phys.* **58** 1 (1983); hep-th/0108200
- Nishino H, Gates S J (Jr.) *Int. J. Mod. Phys. A* **8** 3371 (1993)
- Aharony O et al. *Nucl. Phys. B* **499** 67 (1997); hep-th/9703110
- de Boer J, Hori K, Oz Y *Nucl. Phys. B* **500** 163 (1997); hep-th/9703100
- Ohta K *JHEP* (10) 006 (1999); hep-th/9908120
- Bergman O et al. *JHEP* (10) 036 (1999); hep-th/9908075
- Ivanov E A *Phys. Lett. B* **268** 203 (1991)
- 't Hooft G *Nucl. Phys. B* **153** 141 (1979)
- Mumford D *Tata Lectures on Theta* (Boston: Birkhäuser, 1983–1991)
- Smilga A V, arXiv:1404.2779

²⁰ This is a physical interpretation. Mathematicians would call it *monodromy*.