# On the multiple internal reflections of particles and photons tunneling in one, two, or three dimensions 

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## Contents

1. Similarity between particles and photons in propagation and tunneling behavior $\mathbf{1 1 3 6}$
2. One-dimensional tunneling 1137
3. Two-dimensional tunneling 1138
4. Spherically symmetric three-dimensional tunneling $\mathbf{1 1 4 0}$
4.1 Three-dimensional tunneling and particle scattering by a hard-core potential barrier with an external Coulomb repulsion barrier; 4.2 $S$-matrix
5. Conclusion 1144
References


#### Abstract

We analyze multiple internal reflections of particles and photons undergoing one-, two-, and three-dimensional tunneling. Results obtained by using the time-dependent Schrödinger equation for nonrelativistic particles and those obtained with the time-dependent Helmholtz equation for electromagnetic waves are presented. The paper closes with conclusions and considerations for future research.


## 1. Similarity between particles and photons in propagation and tunneling behavior

There is a well-recognized formal mathematical analogy between the time-dependent Schrödinger equation for nonrelativistic particles and the time-dependent Helmholtz equation for electromagnetic waves. The fact that the particle wave function and the classical electromagnetic wave packet are both interpreted in probabilistic terms the wave packet is the "wave function of a single photon" according to Refs [1, 2]-is, importantly, a sufficient reason for giving similar definitions of the mean moments and durations of propagation, collision, and tunneling processes involving particles and photons. The only weakness in this analogy lies in the different energy-momentum relations, linear for photons and quadratic for nonrelativistic zero-rest-mass particles, a fact that gives rise to a physical difference between the broadening of a particle wave packet and the absence of broadening of the wave function of a single photon.

Having in mind the waveguide experiments in Refs [3-7], we consider a hollow rectangular waveguide with narrowing

[^0](Fig. 1), which has a cross-sectional area $a b(a<b)$. In the secondary quantized form, the probabilistic one-photon wave function is usually described by a plane wave packet [1, 2]
\[

$$
\begin{equation*}
\mathbf{A}(\mathbf{r}, t)=\int_{k_{0}>0} \frac{\mathrm{~d}^{3} k}{k_{0}} \mathbf{\kappa}(\mathbf{k}) \exp \left(\mathrm{i} \mathbf{k r}-\mathrm{i} k_{0} t\right), \tag{1}
\end{equation*}
$$

\]

where $\quad \mathbf{r}=\{x, y, z\}, \quad \boldsymbol{\kappa}(k)=\sum_{i=1}^{2} \kappa_{i}(\mathbf{k}) \mathbf{e}_{i}(\mathbf{k}), \quad \mathbf{e}_{i} \mathbf{e}_{j}=\delta_{i j}$, $\mathbf{e}_{i}(\mathbf{k}) \mathbf{k}=0, i, j=1,2\left(\right.$ or $y, z$ if $\left.\mathbf{k r}=k_{x} x\right), k_{0}=\omega / c=\varepsilon /(\hbar c)$, $k=|\mathbf{k}|=k_{0}$, and $\kappa_{i}(\mathbf{k})$ is the probability amplitude that the photon has the momentum $\mathbf{k}$ and the polarization corresponding to $\mathbf{e}_{i}$. Then the quantity $\left|\kappa_{i}(\mathbf{k})\right|^{2} \mathrm{~d} \mathbf{k}$ is proportional to the probability of the photon being in the momentum interval between $\mathbf{k}$ and $\mathbf{k}+d \mathbf{k}$ with the polarization $\mathbf{e}_{i}$. Here, $\mathbf{A}$ is the vector potential in the gauge where $\operatorname{div} \mathbf{A}=0$, $\mathbf{E}=-(1 / c) \partial \mathbf{A} / \partial t$ is the electric field strength, and $\mathbf{H}=\operatorname{rot} \mathbf{A}$ is the magnetic field strength. Although a photon cannot be polarized along its direction of motion, if the motion is in one dimension, a space-time probabilistic interpretation can be applied to Eqn (1) along the $x$ axis (which is the direction of motion) [2].


Figure 1. Narrow part of the waveguide with cross section dimensions $a$ and $b$, both smaller than the wavelength, acting as a barrier to photons as a microwave is guided through.


Figure 2. (a) Imperfect total internal reflection and the tunneling of damped waves. (b) Experimental schematic [18] including the GoosHänchen effect.

In the special case of quasimonochromatic photon wave packets, we can use the stationary phase method for particles [8] [with $\left.\left|\kappa_{i}(\mathbf{k})\right|^{2} \mathrm{~d} \mathbf{k} \rightarrow \delta(\mathbf{k}-\overline{\mathbf{k}})\right]$ to obtain a similar expression for the phase tunneling time,

$$
\begin{equation*}
\tau_{\mathrm{tun}, \mathrm{em}}^{\mathrm{ph}}=\frac{2}{c \chi_{\mathrm{em}}}, \quad \chi_{\mathrm{em}} L \gg 1 \tag{2}
\end{equation*}
$$

where $\chi_{\mathrm{em}}$ is the imaginary momentum of the tunneling photon wave packet. From Eqn (2), it readily follows that for $\chi_{\mathrm{em}} L>2$, the effective tunneling rate

$$
\begin{equation*}
v_{\mathrm{tun}}^{\mathrm{eff}}=\frac{L}{\tau_{\mathrm{tun}, \mathrm{em}}^{\mathrm{ph}}} \tag{3}
\end{equation*}
$$

exceeds the speed of light $c$.
The Hartmann effect for the one-dimensional motion of quasimonochromatic particles tunneling through potential barriers was analyzed in [9, 10]. The Hartmann effect, first discovered in 1962 [11], consists in the fact that the phase tunneling time is independent of the barrier width for a sufficiently wide barrier. This result is also consistent with experimental data [3-7] for photons.

Electromagnetic waves and photons tunnel through 'photon barriers' similarly to particles tunneling through potential barriers. The first experimental work [12] on the tunneling of classical damped electromagnetic waves used a two-prism setup similar to that shown in Fig. 2b. Such barriers were created to study the propagation of microwave electromagnetic waves through waveguides, of optical electromagnetic waves through devices with imperfect total internal reflection (Fig. 2), etc. Later work [13-19] produced results on the tunneling of optical photons (we also note a theoretical analysis of photon tunneling in Ref. [17]). Figure 2a is a schematic of the experiment in Ref. [18]. Figure 2b shows a two-prism setup with which the spatial shift of the reflected and transmitted beams can be seen in the framework of geometric optics [19].

We also note that experimental work on the Hartmann effect for two barriers is already available [20, 21], studying off-resonance scattering of electromagnetic waves through waveguides [22] and optical photons in fiber optics [23]. In both cases, tunneling occurs in the region very far from resonance, and hence the total phase tunneling time turns out to be independent of both the barrier width and the barrier separation.

The phenomenon of faster-than-light motion observed in microwave and optical experiments with electromagnetic
waves (Refs $[3-7,15,16,18,19]$ and some later ones) generated an extensive discussion of the idea of relativistic causality (see, e.g., Refs [3-7, 9, 10, 18, 19, 22-36]), with no consensus reached yet. The debates continue, as do the experiments, and we do not discuss this in more detail here.

## 2. One-dimensional tunneling

The analysis of multiple internal reflections for one-dimensional potentials has a long history (see, e.g., Refs [24-33]). While this problem is trivial for attractive potentials and for above-barrier energies in the presence of barriers, this is not so for below-barrier energies in the presence of tunneling. This is where damped and antidamped waves come into play, each of which carries zero flux. Corresponding to nonzero fluxes are linear combinations of damped and antidamped waves.

Multiple successive reflections from the inner walls of the barrier being tunneled through can be effectively studied by applying the tunneling time analysis formalism developed in Ref. [10] using the results in Refs [37-44]. To make the analysis transparent, we limit ourselves to the simplest case of a rectangular barrier of height $V_{0}$ in the interval $(0, a)$ and describe tunneling evolution in terms of a picture of actually traveling wave packets, assuming that the packets are composed of stationary plane waves and that the abovebarrier energy cutoff is given by $g(k) \rightarrow g(k) \Theta\left(E-V_{0}\right)$, where $\Theta\left(E-V_{0}\right)$ is the Heaviside step function. Instead of matching stationary wave functions at the points $x=0$ and $x=a$, as is usually done to find analytic expressions for the reflection $\left(A_{\mathrm{r}}\right)$ and transmission $\left(A_{\mathrm{t}}\right)$ amplitudes and the amplitudes of the damped and antidamped waves ( $\alpha$ and $\beta$, respectively), we turn to the analysis of how the initial wave packet penetrates through the first wall of the potential barrier. In this analysis,
(1) the effect of the second (back) wall of the potential barrier is neglected because the final wave packet has not yet reached the wall due to the finite propagation velocity;
(2) the finite wave packet requirement is not violated for infinitely wide barriers (because the increasing antidamped waves have not yet reached the back wall); and
(3) wave packets are constructed based on the subsequent stages of multiple internal reflections, which are analytic continuations of those expressions corresponding to the traveling waves for above-barrier energies.

If we assume that the barrier is rectangular, that the tunneling packet has not yet felt the second wall, and that the packet already inside the barrier initially contains only convergent waves, then the matching condition for the wave packet and its derivative with respect to $x$, in the stationary approximation, yields two linear inhomogeneous equations for the unknowns $A_{\mathrm{r}}^{(0)}$ and $\alpha_{0}$. As the wave packet penetrates into the barrier region and then through the second wall, it splits into two parts, one that has tunneled across the barrier and propagates outward and the second that has reflected from the second wall and propagates back into the barrier. The matching condition for the stationary wave function at $x=a$ at the second stage, as at the first stage, gives two linear inhomogeneous equations for the unknowns $A_{\mathrm{t}}^{(0)}$ (the amplitude of the stationary wave that has passed outward through the second wall) and $\beta_{0}$ (the amplitude of the antidamped wave that has been reflected from the second wall and moves inward). The wave packet reflected from the second wall moves inside the barrier toward the first wall and
then splits into two packets: a) one that has passed this wall (in addition to the packet reflected inward at the first stage), and b) one reflected from the first wall forward inside the barrier. Matching the wave function at $x=0$ again, as at the first two stages, yields two linear inhomogeneous equations for the unknowns $A_{\mathrm{r}}^{(1)}$ (the amplitude of the stationary plane wave that has passed through the first wall backward into region I) and $\alpha_{1}$ (the amplitude of the stationary damped wave reflected from the first wall backward to the barrier region II). This third stage, naturally, corresponds to the first internal reflection.

The processes involved in the second and third stages can be iterated by considering successive internal reflections with an ever decreasing wave vector (with particles undergoing more internal collisions with the walls as the wave packet partially escapes outward through the wall). Describing the tunneling process in this way necessarily involves the multiple internal reflection approach [40-44]. It is readily seen that any of the subsequent stages can be reduced to one of the first three.

Thus, requiring that the wave packet and its first derivative with respect to $x$ be continuous, we obtain recursive relations for $\alpha_{n}, \beta_{n}, A_{\mathrm{r}}^{(0)}$, and $A_{\mathrm{t}}^{(n)}$ for all stages of the wave packet tunneling process. Here, $n=0,1,2, \ldots$ labels the stage of the wave packet evolution inside the barrier, with $n=0$ corresponding to the stage at which the packet enters the barrier.

The total evolution of a wave packet tunneling through a barrier is described by summing over all possible stages, giving

$$
A_{\mathrm{t}}=\sum_{n=0}^{\infty} A_{\mathrm{t}}^{(n)}, \quad A_{\mathrm{r}}=\sum_{n=0}^{\infty} A_{\mathrm{r}}^{(n)}, \quad \alpha=\sum_{n=0}^{\infty} \alpha_{n}, \quad \beta=\sum_{n=0}^{\infty} \beta_{n} .
$$

The results for $\alpha, \beta, A_{\mathrm{t}}$, and $A_{\mathrm{r}}$ were found to agree with those obtained by imposing the standard matching conditions on the wave function corresponding to the solution of the Schrödinger equation [44]. Moreover, the replacement $\mathrm{i} \chi \rightarrow k_{1}$, where $k_{1}=\left[2 m\left(E-V_{0}\right)\right]^{1 / 2} / \hbar$ is the wave number in the case of above-barrier energies $\left(E>V_{0}\right)$, transforms all the above expressions for $\alpha, \beta, A_{\mathrm{t}}$, and $A_{\mathrm{r}}$ into ones for the same quantities obtained in terms of multiple internal reflections for the usual motion of particles with abovebarrier energies [44].

Detailed calculations yield the following for the tunneling and reflection phase times and for total tunneling and reflection times [44]:

$$
\begin{align*}
& \tau_{\text {tun }}=\frac{a}{v}+\hbar \sum_{n=0}^{\infty} \frac{\partial\left(\arg A_{\mathrm{t}}^{(n)}\right)}{\partial E} \underset{\chi a \rightarrow \infty}{\longrightarrow} \tau_{\text {tun }}^{(1)}=\frac{2}{v \chi},  \tag{4}\\
& \tau_{\text {refl }}=\sum_{n=0}^{\infty} \frac{\partial\left(\arg A_{\mathrm{r}}^{(n)}\right)}{\partial E}=\tau_{\text {tun }}^{\chi a \rightarrow \infty} \underset{\text { refl }}{\longrightarrow} \tau_{\text {rit }}^{(1)}=\frac{2}{v \chi} .
\end{align*}
$$

It is clear that the Hartmann effect manifests itself not only in $\tau_{\text {tun }}$ but also in all $\tau_{\text {tun }}^{(n)}, n=1,2, \ldots$. This agrees with the theoretical finding [45] that the motion of damped and antidamped waves is always superluminal with a near-zero tunneling time. Thus, calculations with Eqns (4) clearly show that in the approximation as $\chi a \rightarrow \infty, \tau_{\text {tun }}^{(1)}$ at the first tunneling evolution step, $\tau_{\text {tun }}^{(n)}$ at the $n$th tunneling evolution step, and $\tau_{\text {tun }}$ of the total resulting tunneling (i.e., the sum over all steps) are equal to one another and tend to zero.

In view of the analogy between the tunneling of particles and photons that was studied in Refs [9, 10], the obtained results can be extended to tunneling processes involving transmission of particles and tunneling of photons simultaneously.

## 3. Two-dimensional tunneling

We first follow Ref. [46] to briefly describe the motion of a nonrelativistic particle using the quasimonochromatic approximation with the stationary Schrödinger equation in the form

$$
\left[\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{2 m}{\hbar^{2}}(E+V(x, y))\right] \Psi(x, y)=0
$$

where $\Psi(x, y)$ is the stationary wave function, $m$ is the mass of the particle, $V(x, y)$ is the potential (barrier), and $E$ is the total energy.

We define regions I and II as regions of zero potential, $V(x)=V(y)=0$ : I, for $-\infty<x \leqslant 0,-\infty<y<\infty$, II for $a \leqslant x<\infty,-\infty<y<\infty$. Region III contains the barrier $V(x)=V_{0}>0$ and $V(y)=0(0 \leqslant x<\infty,-\infty<y<\infty)$. All three regions extend to infinity along the $y$ axis (parallel to the $\mathrm{I} / \mathrm{II}$ as well as II/III interfaces). The problem has $y$ translational symmetry in all three regions because $V(y)=0$.

In the stationary scheme (see Fig. 3), the initial plane wave $\exp (\mathbf{i k r})$ with $\mathbf{k}=\left\{k_{x}, k_{y}\right\}, \mathbf{r}=\{x, y\},|\mathbf{k}| \equiv k=k_{x}^{2}+k_{y}^{2}$, $\hbar^{2} k_{x}^{2} /(2 m)=E_{x}^{2}, \hbar^{2} k_{y}^{2} /(2 m)=E_{y}^{2}$ and with the total energy $E=E_{x}^{2}+E_{y}^{2}$ (which is kinetic energy in regions I and II) describes a free particle traveling to point $(x=y=0)$ in region I.

We consider the above-barrier transmission $E_{x}>V_{0}$. At the point $(x=y=0)$, the first externally reflected plane


Figure 3. Schematic illustrating multiple two-dimensional collisions, above-barrier transmission, and propagation of nonrelativistic particles.
wave $A_{\mathrm{r}}^{\mathrm{ex}, 1} \exp \left(\mathrm{i}_{\mathrm{r}} \mathbf{r}\right)$ appears, where $A_{\mathrm{r}}^{\mathrm{ex}, 1}$ is the amplitude of the first reflection from the left boundary of region I, $\mathbf{k}_{\mathrm{r}}^{\mathrm{ex}}=\left\{-k_{x}^{\mathrm{ex}}, k_{y}\right\}$, and the first wave that enters region II is $\psi_{\mathrm{II}}^{1}=A_{\text {pen }}^{1} \exp \left(\mathbf{i k}_{\text {pen }} \mathbf{r}\right)$, where $A_{\mathrm{pen}}^{1}$ is the wave amplitude, $\mathbf{k}_{\text {pen }}=\left\{k_{x}^{\text {pen }}, k_{y}\right\}, k_{x}^{\text {pen }}=\left[2 m\left(E_{x}^{\text {pen }}-V_{0}\right)\right]^{1 / 2} / \hbar$, and $E_{x}>V_{0}$. Further, at the first exit point $(x=a, y=\Delta y)$, where $\Delta y$ is the first upward shift in region II (due to motion with the wave vector $k_{y}$ along the $y$ axis), two plane waves appear: $\psi_{\text {III }}^{1}=$ $A_{\mathrm{t}}^{1} \exp (\mathbf{i k r})\left(A_{\mathrm{t}}^{1}\right.$ is the amplitude), the first wave that transverses region III, and $A_{\mathrm{r}}^{\mathrm{in}, 1} \exp \left(i \mathbf{i k}_{\mathrm{r}}^{\mathrm{in}, 1} \mathbf{r}\right)$ (with the amplitude $A_{\mathrm{r}}^{\text {in, } 1}$ and $\mathbf{k}_{\mathrm{r}}^{\text {in }}=\left\{-k_{x}^{\text {pen }}, k_{y}\right\}$ ), the first wave that is reflected (into region II). It is clear that the shift $\Delta y$ is given by

$$
\begin{equation*}
\Delta y=a \tan \theta^{\prime}, \quad \tan \theta^{\prime}=\frac{k_{y}}{k_{x}^{\text {pen }}}, \tag{5a}
\end{equation*}
$$

or

$$
\begin{equation*}
\Delta y=\frac{\hbar k_{y}}{m} \tau_{x}^{\mathrm{ph}, \mathrm{pen}}=a \tan \theta^{\prime} \tag{5b}
\end{equation*}
$$

where $\tau_{x}^{\mathrm{ph}, \text { pen }}=a m /\left(\hbar k_{x}^{\mathrm{pen}}\right)$ is the phase time for a particle traveling the distance $a$ at the velocity $\hbar k_{x} / m$, i.e., the time (calculated by the stationary phase approximation) needed for a quasimonochromatic particle to travel along the $x$ axis in region II from $x=0$ to $x=a$.

By matching the waves and their first derivatives $\partial / \partial x$ at the points $(x=y=0),(x=a, y=\Delta y),(x=0, y=2 \Delta y)$, $(x=a, y=3 \Delta y), \ldots$, we obtain [neglecting the plane wave $\left.\exp \left(\mathrm{i} k_{y} y\right)\right]$ the following amplitudes: for the $n$th wave that has entered region II, $A_{\text {pen }}^{n}$; for the $n$th wave reflected into region II, $A_{\mathrm{r}}^{\mathrm{in}, n}$, for the $n$th wave externally reflected into region I, $A_{\mathrm{r}}^{\mathrm{ex}, n}$; and for the $n$th wave that has tunneled into region III, $A_{\mathrm{t}}^{n}$.

In the case $k=k_{x}$, with the angle $\theta=0$ (see Fig. 3), i.e., when the incident plane wave is perpendicular to the first boundary and $\Delta y=0$, it is easy to see that

$$
\begin{aligned}
& \frac{k_{x}^{\mathrm{pen}}}{k_{x}}\left|A_{\mathrm{pen}}^{1}\right|^{2}+\left|A_{\mathrm{r}}^{\mathrm{ex}, 1}\right|^{2}=1, \\
& k_{x}^{\mathrm{pen}}\left|A_{\mathrm{pen}}^{1}\right|^{2}=k_{x}^{\mathrm{pen}}\left|A_{\mathrm{r}}^{\mathrm{in}, 1}\right|^{2}+k_{x}\left|A_{\mathrm{t}}^{1}\right|^{2}
\end{aligned}
$$

due to the flux conservation in the first passage through the points $(x=y=0)$ and $(x=0, y=a)$.

In the case of one-dimensional transmission (for $\theta=0$, when the incident plane wave is perpendicular to the first boundary and $\Delta y=0$ ), all expressions, including those with $n=1,2, \ldots$, are identical with the corresponding expressions in Ref. [44] obtained by time analysis (for a stationary phase) for one-dimensional tunneling.

We next proceed to analyze sub-barrier tunneling at $E_{x}<V_{0}$. If the angle $\theta$ is sufficiently large, such that $\pi / 2>\theta>\theta_{\text {crit }}=\tan \left(k_{y} / k_{x}^{\text {crit }}\right)$, where $k_{x}^{\text {crit }}$ is determined from the equation $\hbar^{2}\left(k_{x}^{\text {crit }}\right)^{2} /(2 m)=V_{0}$, then $E_{x}<V_{0}$, and the values of $k_{x}^{\text {pen }}$ are imaginary, i.e., $k_{x}^{\text {pen }}=\mathrm{i} \chi$ with $\chi>0$, giving rise to sub-barrier tunneling, $k_{y}^{2}=k^{2}+\chi^{2}$. In this case, we pass from describing sub-barrier tunneling to describing above-barrier tunneling by introducing $\chi$ via the substitution $k_{x}^{\text {pen }}=\mathrm{i} \chi$. Then, instead of traveling waves $\exp \left( \pm \mathrm{i} k_{x}^{\text {pen }} x\right)$ (in region II), damped $\alpha_{n} \exp (-\chi x)$ and antidamped $\beta_{n} \exp (\chi x)$ waves appear, as illustrated in Fig. 4. We here used analytic continuations from the region of real (above-barrier) wave numbers to the region of imaginary (sub-barrier) wave numbers in a way similar to that used in


Figure 4. Schematic illustrating multiple two-dimensional reflections, subbarrier transmission, and propagation of a nonrelativistic particle.

Ref. [44]. The results in Ref. [46] are identical to their onedimensional counterparts obtained in Ref. [44].

The shift along the $y$ axis, denoted by $\Delta y$ in Fig. 4, is given by equations similar to Eqn (5b):

$$
\begin{equation*}
\Delta_{n} y=\frac{\hbar k_{y}}{m} \tau_{\mathrm{t}(\mathrm{r}), x=a(0)}^{\mathrm{ph}(\mathrm{ex}), n}, \tag{6}
\end{equation*}
$$

where

$$
\begin{align*}
& \tau_{\mathrm{t}, x=a}^{\mathrm{ph}, n}=\frac{a}{v_{x}}+\hbar \frac{\partial}{\partial E} \arg A_{\mathrm{t}}^{n}, \quad n=1,2, \ldots,  \tag{7}\\
& \tau_{\mathrm{r}, x=0}^{\mathrm{ph}, \mathrm{ex}, n}=\hbar \frac{\partial}{\partial E} \arg A_{\mathrm{r}}^{\mathrm{ex}, n}, \quad n=1,2, \ldots . \tag{8}
\end{align*}
$$

The quantities $\tau_{\mathrm{t}, x=a}^{\mathrm{ph}, n}$ and $\tau_{\mathrm{r}, x=0}^{\mathrm{ph}, \mathrm{ex}, n}$ are respectively the phase times of travel (the travel times of quasimonochromatic particles calculated in the stationary phase approximation) for the $n$th stage of sub-barrier tunneling through the point $x=a$ and for the $n$th stage of external reflection from the first boundary at the point $x=0[9,10]$. Clearly, the shifts $\Delta_{n} y$ are different for different $n=1,2,3, \ldots\left[\hbar \partial\left(\arg A_{\mathrm{t}}^{n}\right) / \partial E\right.$ and $\hbar \partial\left(\arg A_{\mathrm{r}}^{\mathrm{ex}, n}\right) / \partial E$ slightly increase with $n$ ], but are always proportional to $2 /(v \chi)$ in the limit $\chi a \rightarrow \infty$. As the order $n$ in $A_{\mathrm{t}}^{n}$ and $A_{\mathrm{r}}^{\mathrm{ex}, n}$ increases, the waves that have tunneled and been externally reflected are rapidly suppressed by the factor $\exp (-\chi a)$.

There is also an alternative approach to two-dimensional tunneling. Rather than using analytic continuations as in Refs [40-44], it is possible [47] to neglect the multiple internal reflections of damped and antidamped waves and to use a single linear combination of waves $\alpha \exp (-\chi x)+\beta \exp (\chi x)$ for the $k_{x}$ component in region II and a single wave $\exp \left(i k_{y} y\right)$ for the $k_{y}$ component in region II, which yields the following expression for a single shift along the $y$ axis at the second (I/II) interface as shown in Fig. 5:

$$
\begin{equation*}
\Delta y=\frac{\hbar k_{y}}{m} \tau_{\mathrm{t}, x=a}^{\mathrm{ph}}, \tag{9}
\end{equation*}
$$



Figure 5. Schematic of two-dimensional tunneling with one reflected and one transmitted wave.
where

$$
\begin{align*}
\tau_{\mathrm{t}, x=a}^{\mathrm{ph}} & =\tau_{\mathrm{tun}}=\frac{a}{v}+\hbar \frac{\partial \arg A_{\mathrm{t}}}{\partial E} \\
& =\frac{1}{v \chi} \frac{k_{0, x}^{2} \sinh (2 \chi a)+2 \chi a k_{x}^{2}\left(\chi^{2}-k^{2}\right)}{4 k_{x}^{2} \chi^{2}+k_{0, x}^{2} \sinh ^{2}(\chi a)},  \tag{10}\\
A_{\mathrm{t}} & =\sum_{n=1}^{\infty} A_{\mathrm{t}}^{n}=4 \mathrm{i} k_{x} \chi \frac{\exp \left(-\chi a-\mathrm{i} k_{x} a\right)}{F_{x}}, \\
F_{x} & =\left(k_{x}^{2}-\chi^{2}\right) D_{-}+2 \mathrm{i} k_{x} \chi D_{+},
\end{align*}
$$

with $\quad D_{ \pm}=1 \pm 4 \exp (-2 \chi a), \quad k_{0, x}^{2}=k_{x}^{2}+\chi^{2}=2 m V_{0} / \hbar^{2}$. This leaves us with a single transmitted two-dimensional wave $A_{\mathrm{t}} \exp \left(\mathrm{i} k_{x} y\right) \exp \left(\mathrm{i} k_{y} y\right)$ (in region III) traveling parallel to the incident wave. From Eqn (10), it is clear that, exactly as in the one-dimensional case, the tunneling time $\tau_{\mathrm{t}, x=a}^{\mathrm{ph}, n}$ at the first stage and the total tunneling time $\tau_{\mathrm{t}, x=a}^{\mathrm{ph}}$ (corresponding to the sum over all stages) are essentially vanishing for superluminal velocities and are equal to each other in the approximation $\chi a \rightarrow \infty$.

Thus, we have two different approaches to describing a two-dimensional tunneling collision of a particle. The first (see Fig. 4) uses Eqns (5a), (5b), (6)-(8) and the substitutions $k_{x}^{\text {pen }} \rightarrow \mathrm{i} \chi, A_{\text {pen }}^{n} \rightarrow \alpha_{n}, A_{\mathrm{r}}^{\text {in, } n} \rightarrow \beta_{n}$ to describe an infinite series of internal reflections and transmitted waves. The second approach (see Fig. 5) fully neglects multiple internal reflections and corresponding transmitted waves and considers a single tunneling shift and a single transmitted wave parallel to the incident wave

Being different, both approaches highlight the fact that sub-barrier tunneling is nonlocal in nature, leading to the Hartmann effect for the tunneling phase time in the limit $\chi a \rightarrow \infty$. It only remains to find which of the two actually describes tunneling. Our approach (see Fig. 4) is supported not only by the methods described in Ref. [44] but also by the experimental data in Refs $[48,49$ ] (which have not most likely been properly processed).

Using the analogy between the motion of a photon and that of a wave, the results obtained above can be extended to the two-dimensional transmission and tunneling of photons. In considering photons propagating through homogeneous glassy media I and III and penetrating and tunneling through a homogeneous air layer, we can again refer to Figs 3-5. In this case,

$$
\begin{equation*}
n=\frac{\sin \theta^{\prime}}{\sin \theta} \tag{11}
\end{equation*}
$$

is the light refractive index for glass (assuming that the same for air is unity). Then Fig. 4 corresponds to photon transmission to layer II for incidence angles less than the critical value $\theta_{\text {crit }}=\tan \left(k_{y} / k_{x}^{\text {crit }}\right)$, which is the angle of total internal reflection for incident photons polarized perpendicular to the light incidence plane $x y$.

Figures 4 and 5 correspond to the total internal reflection, slightly distorted due to the passage through layer II into the glassy medium III, of polarized light tunneling through slit II at an angle of incidence $\theta>\theta_{\text {crit }}$; this is true for both approaches, the two-dimensional particle tunneling with multiple internal reflections - the approach presented above and, in a somewhat different form for light, in Refs [48, 49] (see Fig. 4) - and for the two-dimensional tunneling of a nonrelativistic particle and a photon, an approach presented in Ref. [47] (see Fig. 5). It is hoped that future optical experiments, if properly designed, will provide fascinating insights into multiple internal reflections and multiple transmitted waves, as the preliminary experiments in [48, 49] did.

## 4. Spherically symmetric three-dimensional tunneling

The three-dimensional tunneling problem can naturally be addressed by extending the results for two-dimensional tunneling (for axes $x$ and $y$ ) to the three-dimensional case (for $x, y$, and $z$ axes), assuming that the interfaces are twodimensional (parallel to the $y z$ plane) and that the tunneling is originally along the $x$ axis.

Leaving it to the reader to decide on the usefulness of this approach, we can consider a three-dimensional spherically symmetric tunneling problem, in which the radial coordinate is the most important. Some of the monographs on quantum mechanics and almost all current papers on nuclear physics treat this model within the Wentzel-Kramers-Brillouin (WKB) approximation. Thus far, only a few sufficiently accurate non-WKB results are available [50-52] (see also [53]). Below, we follow Ref. [51] in formally considering threedimensional tunneling for a spherically symmetric problem, for example, for $\alpha$ or proton scattering by spherical nuclei as a result of $\alpha$ decay or the proton decay of spherical nuclei [51].

### 4.1 Three-dimensional tunneling and particle scattering by a hard-core potential barrier with an external Coulomb repulsion barrier

For simplicity, we restrict our attention to the case of a zero orbital quantum number $(l=0)$ and formal stationary functions (in the quasimonochromatic wave packet approximation). We actually return, in a sense, to the onedimensional problem - which, however, becomes distinctly three-dimensional already for $l>0$ or if a nonspherical host nuclei is explicitly considered.

We start by describing the collision of particles as a twostage event. At the first stage, the wave packet of converging waves tunnels through a barrier outside the well, generating outward reflected waves. At the second stage, those waves diverging from the well that appear after the entrance of the original tunneling wave packet tunnel through the barrier outward, generating reflected waves from the potential well through the Coulomb barrier into the well interior. We next proceed to describe low-energy scattering as a whole using the concept of the $S$-matrix and that of probability amplitudes (both for tunneling events and for reflections).


Figure 6. Schematic of a collision between a converging wave outside with a Coulomb repulsion barrier.

Finally, we demonstrate the presence of multiple internal reflections from the inner wall of the Coulomb barrier and from the hard core.
4.1.1 Outside scattering. In the schematic of outside scattering shown in Fig. 6, region I with $r>R_{2}$ is the external region with a slowly decreasing Coulomb potential; region II between $R_{1}$ and $R_{2}$ is the sub-barrier region; region III, $R_{0}<r<R_{1}$, is a potential well; region IV, with $r<R_{0}$, contains a hard core, where the wave function vanishes. The vertical line $r=R_{2}$ separates the outer above-barrier region where the particle kinetic energy $E$ exceeds the Coulomb barrier from the inner sub-barrier region II in which $E$ is below the Coulomb repulsion curve. The vertical line $r=R_{1}$ separates well III from the Coulomb barrier.

The stationary radial wave function that satisfies the radial Schrödinger equation is, for the situation shown in Fig. 6, given by

$$
\begin{array}{ll}
\Phi_{\mathrm{I}}^{\mathrm{ex}}=G_{0}\left(k_{1}, \eta, r\right)-\mathrm{i} F_{0}\left(k_{1}, \eta, r\right) \\
\quad+A_{\mathrm{r}}^{\mathrm{ex}}\left(G_{0}\left(k_{1}, \eta, r\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, r\right)\right), & R_{2} \leqslant r<\infty, \\
\Phi_{\mathrm{II}}^{\mathrm{ex}} \text { is the same formally as } \Phi_{\mathrm{I}}^{\mathrm{ex}}, & R_{1} \leqslant r<R_{2},  \tag{12}\\
\Phi_{\mathrm{III}}^{\mathrm{ex}}=A_{\mathrm{t}}^{\mathrm{in}} \exp \left(-\mathrm{i} k_{2} \rho\right), \quad \rho=r-R_{0}, & R_{0}<r<R_{1} .
\end{array}
$$

Here, $k_{1}$ and $E=\hbar^{2} k_{1}^{2} /(2 m)$ are the wave number and the particle kinetic energy,

$$
\eta=\frac{z_{1} z_{2} e^{2} m}{\hbar^{2} k_{1}}
$$

is the Sommerfeld parameter, and $k_{2}=\left[2 m\left(V_{0}+E\right)\right]^{1 / 2} / \hbar$; for a repulsive Coulomb barrier,

$$
\begin{equation*}
V_{\mathrm{C}}=\frac{z_{1} z_{2} e^{2}}{r}, \quad R_{1} \leqslant r<\infty, \tag{13}
\end{equation*}
$$

where $z_{1} e$ and $z_{2} e$ are the electric charges of the interacting particles (nuclei); the Coulomb functions $G_{0}\left(k_{1}, \eta, r\right)$ and $F_{0}\left(k_{1}, \eta, r\right)$ have the asymptotic forms

$$
\begin{align*}
& G_{0}\left(k_{1}, \eta, r\right) \underset{r \rightarrow \infty}{\longrightarrow} \cos \left(k_{1} r-\eta \ln \left(2 k_{1} r\right)+\sigma\right),  \tag{14}\\
& F_{0}\left(k_{1}, \eta, r\right) \underset{r \rightarrow \infty}{\longrightarrow} \sin \left(k_{1} r-\eta \ln \left(2 k_{1} r\right)+\sigma\right),
\end{align*}
$$

$\sigma=\arg \Gamma(1+\mathrm{i} \eta) ; A_{\mathrm{r}}^{\text {ex }}$ and $A_{\mathrm{t}}^{\text {in }}$ are the amplitudes of external reflection and of penetration inside the potential for the first
tunneling, the analytic expressions for which can be found from the continuity conditions for the amplitudes and their first derivatives at $r=R_{2}$ and $r=R_{1}$ :

$$
\begin{align*}
A_{\mathrm{r}}^{\mathrm{ex}}= & -\left[\left(G_{0}\left(k_{1}, \eta, R_{1}\right)-\mathrm{i} F_{0}\left(k_{1}, \eta, R_{1}\right)\right) \mathrm{i} k_{2}\right. \\
& \left.+\left(G_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)-\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)\right) k_{1}\right] \\
& \times\left[\left(G_{0}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, R_{1}\right)\right) \mathrm{i} k_{2}\right. \\
& \left.+\left(G_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)\right) k_{1}\right]^{-1},  \tag{15}\\
A_{\mathrm{t}}^{\mathrm{in}}= & 2 \mathrm{i} k_{1} \exp \left[\mathrm{i} k_{2}\left(R_{1}-R_{0}\right)\right] \\
& \times\left[\left(G_{0}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, R_{1}\right)\right) \mathrm{i} k_{2}\right. \\
& \left.+\left(G_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)\right) k_{1}\right]^{-1} . \tag{16}
\end{align*}
$$

It is easily seen that

$$
\begin{equation*}
\left|A_{\mathrm{r}}^{\mathrm{ex}}\right|^{2}+\frac{k_{2}}{k_{1}}\left|A_{\mathrm{t}}^{\mathrm{in}}\right|^{2}=1 \tag{17}
\end{equation*}
$$

where we have used the well-known Wronskian relation $F_{0} G_{0}^{\prime}-G_{0} F_{0}^{\prime}=1$, where $G_{0}^{\prime}$ and $F_{0}^{\prime}$ are the derivatives of $G_{0}$ and $F_{0}$ with respect to $k_{1} R_{1}$. Equality (17) is a consequence of the conservation of probability flow density.

For vanishingly small values of $k_{1}$ (or more precisely, for $2 \eta \gg k_{1} R_{1}$ ),

$$
\begin{aligned}
& G_{0} \rightarrow 2\left(\frac{k_{1} R_{1}}{\pi}\right)^{1 / 2} I_{0}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \exp (\pi \eta) \\
& \text { as } I_{0}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \rightarrow 1, \\
& G_{0}^{\prime} \rightarrow-2\left(\frac{2 \eta}{\pi}\right)^{1 / 2} K_{0}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \exp (\pi \eta) \\
& \text { as } K_{0}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \rightarrow \ln \left(\frac{1}{\gamma}\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right),
\end{aligned}
$$

where $\gamma=1.781 \ldots$ is the Euler constant, and

$$
\begin{aligned}
& F_{0} \rightarrow\left(\pi k_{1} R_{1}\right)^{1 / 2} I_{1}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \exp (-\pi \eta) \\
& \text { as } I_{1}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \rightarrow\left(2 \pi k_{1} R_{1}\right)^{1 / 2}, \\
& F_{0}^{\prime} \rightarrow\left(2 \pi k_{1} R_{1}\right)^{1 / 2} I_{0}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \exp (-\pi \eta) \\
& \text { as } I_{0}\left(2\left(2 \pi k_{1} R_{1}\right)^{1 / 2}\right) \rightarrow 1 .
\end{aligned}
$$

If $\left[\left(2 k_{1}^{2} / k_{2}^{2}\right) \eta /\left(k_{1} R_{1}\right)\right]\left[(\ln \gamma)^{-1}\left(2 \pi k_{1} R_{1}\right)^{-1 / 2}\right] \ll 1$, then the probability $\left|A_{\mathrm{t}}^{\text {in }}\right|^{2}$ of transmission from the outside through the Coulomb barrier into the rectangular potential well takes the form

$$
\begin{equation*}
\left|A_{\mathrm{t}}^{\mathrm{in}}\right|^{2} \rightarrow \frac{\pi k_{1}}{k_{2}^{2} R_{1}} \exp (-2 \pi \eta) . \tag{18}
\end{equation*}
$$

In three dimensions, we note that for very small $k_{1}$, it is necessary to take into account not only the exponential $\exp (-2 \pi \eta)$ but also its prefactor $\pi k_{1} /\left(k_{2}^{2} R_{1}\right)$, both of which are neglected in the one-dimensional WKB approximation often used in the study of high-energy nuclear collisions in nuclear astrophysics.
4.1.2 Emission from the inside of the barrier. Figure 7 is a schematic of emission outward from the barrier. The


Figure 7. Emission of an internal divergent wave through the repulsion barrier (schematic).
stationary radial wave function takes the form
$\Phi_{\text {III }}^{\text {in }}=\exp \left(\mathrm{i} k_{2} \rho\right)+A_{\mathrm{r}}^{\text {in }} \exp \left(-\mathrm{i} k_{2} \rho\right), \rho=r-R_{0}, R_{0}<r<R_{1}$,
$\Phi_{\mathrm{II}}^{\mathrm{ex}}$ is formally the same as $\Phi_{\mathrm{I}}^{\mathrm{in}}, \quad R_{1} \leqslant r<R_{2}$,
$\Phi_{\mathrm{I}}^{\mathrm{ex}}=A_{\mathrm{t}}^{\mathrm{ex}}\left(G_{0}\left(k_{1}, \eta, r\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, r\right)\right), \quad R_{1} \leqslant r<\infty$.
Using the continuity of the stationary wave function and its derivatives at $r=R_{2}$ and $r=R_{1}$, it is straightforward to obtain analytic expressions for the amplitudes $A_{\mathrm{r}}^{\mathrm{in}}$ and $A_{\mathrm{t}}^{\mathrm{ex}}$ :

$$
\begin{align*}
A_{\mathrm{r}}^{\mathrm{in}}= & \exp \left[2 \mathrm{i} k_{2}\left(R_{1}-R_{0}\right)\right] \\
& \times\left[\left(G_{0}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, R_{1}\right)\right) \mathrm{i} k_{2}\right. \\
& \left.-\left(G_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)\right) k_{1}\right] \\
& \times\left[\left(G_{0}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, R_{1}\right)\right) \mathrm{i} k_{2}\right. \\
& \left.+\left(G_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)\right) k_{1}\right]^{-1},  \tag{20}\\
A_{\mathrm{t}}^{\mathrm{ex}}= & 2 \mathrm{i} k_{2} \exp \left[\mathrm{i} k_{2}\left(R_{1}-R_{0}\right)\right] \\
& \times\left[\left(G_{0}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, R_{1}\right)\right) \mathrm{i} k_{2}\right. \\
& \left.+\left(G_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)+\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, R_{1}\right)\right) k_{1}\right]^{-1} . \tag{21}
\end{align*}
$$

By the flux density conservation,

$$
\begin{equation*}
\left|A_{\mathrm{r}}^{\mathrm{in}}\right|^{2}+\frac{k_{1}}{k_{2}}\left|A_{\mathrm{t}}^{\mathrm{ex}}\right|^{2}=1 \tag{22}
\end{equation*}
$$

By repeating the reasoning used for small $k_{1}\left(k_{1} \rightarrow 0\right)$ in deriving Eqn (18) from Eqn (17), we find that

$$
\begin{equation*}
\left|A_{\mathrm{t}}^{\mathrm{ex}}\right|^{2} \rightarrow \frac{\pi}{k_{2} R_{1}} \exp (-2 \pi \eta) \tag{23}
\end{equation*}
$$

As in the derivation of Eqn (18), in the three-dimensional case of small $k_{1}$, it is necessary to include not only the exponential $\exp (-2 \pi \eta)$ but also its prefactor $\pi /\left(k_{2} R_{1}\right)$, which is not done in the one-dimensional WKB approximation typically applied to low-energy nuclear collisions.

In the simplest case of a rectangular barrier with a zero potential (instead of the Coulomb potential) for $R_{1} \leqslant r<R_{2}$, the calculation of the phase times for reflection into the well from the inner wall of the barrier and for tunneling through the barrier confirms the presence of the Hartmann effect. In other words, we have an elementary confirmation of this effect for three-dimensional tunneling [50].


Figure 8. Schematic of scattering as a whole.

## 4.2 $S$-matrix

We can now unify these two collision stages into a single scattering event by introducing the $S$-matrix and taking multiple reflections inside the potential well into account. For this, we represent scattering schematically as shown in Fig. 8 and write the stationary radial wave function as

$$
\begin{aligned}
\Psi_{\mathrm{I}} & =\left(G_{0}\left(k_{1}, \eta, r\right)-\mathrm{i} F_{0}\left(k_{1}, \eta, r\right)\right) \\
& -S\left(G_{0}\left(k_{1}, \eta, r\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, r\right)\right), \quad r \geqslant R_{2},
\end{aligned}
$$

$\Psi_{\text {II }}$ has the same form as $\Psi_{\mathrm{I}}, \quad R_{1} \leqslant r<R_{2}$,

$$
\begin{array}{lrl}
\Psi_{\mathrm{III}}=A\left[\exp \left(-\mathrm{i} k_{2} r\right)-\exp \left(\mathrm{i} k_{2} r\right)\right], & R_{0}<r<R_{1},  \tag{24}\\
\Psi_{\mathrm{IV}} \equiv 0, & r \leqslant R_{0} .
\end{array}
$$

Imposing the continuity of the stationary wave function and its derivative at $r=R_{2}$ and $r=R_{1}$, we obtain analytic expressions for the $S$-matrix and the amplitude $A$ :

$$
S=\frac{H_{1}\left[k_{2} \cos k_{2}\left(R_{1}-R_{0}\right)\right]-H_{2}\left[k_{1} \sin k_{2}\left(R_{1}-R_{0}\right)\right]}{H_{3}\left[k_{2} \cos k_{2}\left(R_{1}-R_{0}\right)\right]-H_{4}\left[k_{1} \sin k_{2}\left(R_{1}-R_{0}\right)\right]},
$$

$$
\begin{align*}
A= & 2 \mathrm{i} \exp \left[\mathrm{i} k_{2}\left(R_{2}-R_{0}\right)\right] k_{1}  \tag{25a}\\
& \times\left(H_{4} k_{1}\left\{1-\exp \left[2 \mathrm{i} k_{2}\left(R_{1}-R_{0}\right)\right]\right\}\right. \\
& \left.+H_{3} \mathrm{i} k_{2}\left\{1+\exp \left[2 \mathrm{i} k_{2}\left(R_{1}-R_{0}\right)\right]\right\}\right)^{-1}, \tag{25b}
\end{align*}
$$

where

$$
\begin{aligned}
& H_{1}=G_{0}\left(k_{1}, \eta, r\right)-\mathrm{i} F_{0}\left(k_{1}, \eta, r\right), \\
& H_{2}=G_{0}^{\prime}\left(k_{1}, \eta, r\right)-\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, r\right), \\
& H_{3}=G_{0}\left(k_{1}, \eta, r\right)+\mathrm{i} F_{0}\left(k_{1}, \eta, r\right), \\
& H_{4}=G_{0}^{\prime}\left(k_{1}, \eta, r\right)+\mathrm{i} F_{0}^{\prime}\left(k_{1}, \eta, r\right) .
\end{aligned}
$$

It is clear that $|S|=1$, and just by comparing $A$ and $S$ with $A_{\mathrm{r}}^{\text {ex }}, A_{\mathrm{t}}^{\mathrm{ex}}, A_{\mathrm{r}}^{\mathrm{in}}$, and $A_{\mathrm{t}}^{\text {in }}$, we find that

$$
\begin{align*}
& A=\frac{A_{\mathrm{t}}^{\mathrm{in}}}{1+A_{\mathrm{r}}^{\mathrm{in}}},  \tag{26}\\
& S=-A_{\mathrm{r}}^{\mathrm{ex}}+A A_{\mathrm{t}}^{\mathrm{ex}}=-A_{\mathrm{r}}^{\mathrm{ex}}+\frac{A_{\mathrm{t}}^{\mathrm{ex}} A_{\mathrm{t}}^{\mathrm{in}}}{1+A_{\mathrm{r}}^{\text {in }}} . \tag{27}
\end{align*}
$$

Physically, the term $1 /\left(1+A_{\mathrm{r}}^{\mathrm{in}}\right)$ implies the existence of an infinite series of coherent multiple internal reflections
described by the stationary wave functions:

$$
\begin{align*}
& A_{\mathrm{t}}^{\mathrm{in}}\left(1-A_{\mathrm{r}}^{\text {in }}+\left(A_{\mathrm{r}}^{\mathrm{in}}\right)^{2}-\left(A_{\mathrm{r}}^{\text {in }}\right)^{3}+\ldots\right) \exp (-\mathrm{i} k r) \\
& \quad=\frac{A_{\mathrm{t}}^{\text {in }}}{1+A_{\mathrm{r}}^{\text {in }}} \exp (-\mathrm{i} k r),  \tag{28}\\
& A_{\mathrm{t}}^{\text {in }} \\
&  \tag{29}\\
& \left.\quad=\frac{A_{\mathrm{t}}^{\text {in }}}{1+A_{\mathrm{r}}^{\text {in }}}+\left(A_{\mathrm{r}}^{\text {in }}\right)^{2}-\left(A_{\mathrm{r}}^{\text {in }}\right)^{3}+\ldots\right) \exp (\mathrm{i} k r) \\
&
\end{align*}
$$

For the incoming and outgoing waves, there are multiple internal reflections between the points of the hard core and their diametrically opposite counterparts on the inner wall of the spherically symmetric Coulomb barrier. Clearly, the natural way to physically explore their time sequence is to apply a direct time-dependent approach by simply extending the approach described above for one-dimensional tunneling. In the case of spherically symmetric nuclei, it is assumed that a packet of waves diverging (uniformly contracting) within a nucleus that moves radially toward the inner surface of the Coulomb barrier is reflected inward and then moves radially further inward already as a packet of converging waves, toward the spherical wall of the hard core, and after that is reflected inward into the potential well, where it again travels as a packet of diverging waves, toward the inner wall of the Coulomb barrier, all this repeating itself until, step by step, the wave packet escapes completely.

Such coherent internal reflections usually occur for charged elementary particles (protons, positrons, $\pi^{+}$mesons, etc.), for both nonresonant and resonant scattering.

However, if the projectile nuclear particle is a cluster (for example, an $\alpha$ particle) and if the scattering is resonant and produces an $\alpha$-radioactive composite nucleus, a more complicated noncoherent process may take place: the $\alpha$ particle disappears ('is dissolved') within the composite (host) nucleus and then, within a certain virtual and real time interval (stay time) $\tau_{\text {resid }}$ (which includes the travel time under the surface until disappearance and subsequent reemergence), the $\alpha$ particle reappears near the nucleus surface and passes through the Coulomb barrier outward (see, e.g., Refs [54, 55]).

Assuming that as the infinite series of multiple reflections proceeds, any outgoing 'portion' of the $\alpha$ particle probabilistic wave packet decreases in each collision with the inner wall of the barrier by the factor $\left|A_{\mathrm{r}}^{\text {in }}\right|^{2}$ compared with its counterpart at the previous collision, the total (equal to unity) probability of the $\alpha$ decay can be presented as an infinite sum of a decreasing geometric series,
$\frac{k_{1}}{k_{2}}\left|A_{\mathrm{t}}^{\mathrm{ex}}\right|^{2}\left(1+\left|A_{\mathrm{r}}^{\text {in }}\right|^{2}+\left|A_{\mathrm{r}}^{\text {in }}\right|^{2}\left|A_{\mathrm{r}}^{\text {in }}\right|^{2}+\ldots\right)=\frac{\left(k_{1} / k_{2}\right)\left|A_{\mathrm{t}}^{\mathrm{ex}}\right|^{2}}{1-\left|A_{\mathrm{r}}^{\mathrm{in}}\right|^{2}}=1$.
Furthermore, it is also assumed that the multiple successive internal reflections inside the host nucleus that occur between the formation of the $\alpha$ particle near the nucleus surface and its further disappearance within the nucleus are not all coherent between themselves due to the independence of the successive disappearance processes, implying that it is not the probability amplitudes but the probabilities themselves that should be summed in the chain of multiple internal reflections. A further natural assumption is that at each stage of the noncoherent multiple internal reflections of the $\alpha$ particle, the total average duration $\tau_{\text {resid }}$ of the virtual and real stay of the particle inside the host nucleus between two successive
internal reflections is the sum of the average disappearance time within the nucleus, the average time of the subsequent formation of the particle, its average time of reflection into the nucleus, and the average time of its kinematic travel after the reflection inward and then toward the surface after formation. It is also assumed that $\tau_{\text {resid }}$ is the same for each pair of the successive impacts of the $\alpha$ particle against the nuclear surface.

The effective, or average, time of virtual or real stay of an $\alpha$ particle within a host $\alpha$-radioactive nucleus between two successive, noncoherent, multiple internal reflections during a long $\alpha$ decay can be estimated phenomenologically by simply using the exponential decay law of the $\alpha$ particle with the lifetime $\tau$ :

$$
\begin{equation*}
L(t)=\exp \left(-\frac{t}{\tau}\right) \tag{31}
\end{equation*}
$$

The exponential law (31) for the decay of a resonant state with a Lorentzian energy distribution (Breit-Wigner curve) has been verified theoretically to a high degree of accuracy except for very small times $t<t_{0}\left(\Gamma / E_{\text {res }}\right)$ and very large times $t>t_{0} \ln \left(E_{\text {res }} / \Gamma\right)$ [56] (see also Ref. [57]), where $\Gamma$ and $E_{\text {res }}$ are the width and energy of the resonance.

If $\tau_{\text {resid }} \ll \tau$, then in the time $\tau_{\text {resid }}$, the decay probability decreases by

$$
\begin{equation*}
\Delta L=1-\left|A_{\mathrm{r}}^{\mathrm{in}}\right|^{2}=\frac{\tau_{\text {resid }}}{\tau} \tag{32}
\end{equation*}
$$

Using Eqn (30), we then obtain

$$
\begin{equation*}
\frac{\tau_{\text {resid }}}{\tau}=\frac{k_{1}}{k_{2}}\left|A_{\mathrm{t}}^{\mathrm{ex}}\right|^{2} \quad \text { and } \quad \tau_{\text {resid }}=\frac{k_{1}}{k_{2}}\left|A_{\mathrm{t}}^{\mathrm{ex}}\right|^{2} \tau=P \tau \tag{33}
\end{equation*}
$$

where $P=\left(k_{1} / k_{2}\right)\left|A_{t}\right|^{2}$.
Equation (33) with the substitutions $v=1 / \tau_{\text {resid }}$ and $\tau=1 / \lambda$ yields the well-known formula (see, e.g., Refs [58, 59] and also [60])

$$
\begin{equation*}
\lambda=v P \tag{34}
\end{equation*}
$$

and essentially describes a new phenomenological approach to the physical meaning of the exponential prefactor.

As an example, we calculate $\tau_{\text {resid }}$ for the ${ }^{210} \mathrm{Po}$ nucleus, with $E=5.407 \mathrm{MeV}, V_{0}=16.7 \mathrm{MeV}, R_{1}=8.76 \mathrm{fm}$ and $R_{1}=8.975 \mathrm{fm}[51]$ and $\tau=138.376$ days $=11,955,686.4 \mathrm{~s}$. We then obtain the values $\tau_{\text {resid }}=2.434 \times 10^{-20} \mathrm{~s}$ and $\tau_{\text {resid }}=$ $5.740 \times 10^{-20}$ s and hence $v=1 / \tau_{\text {resid }}=4.108 \times 10^{19} \mathrm{~s}^{-1}$ and $1.742 \times 10^{19} \mathrm{~s}^{-1}$. Our calculated values of $v=1 / \tau_{\text {resid }}$ differ from the Gamow exponential prefactor $v_{0} / 2 R_{1} \equiv$ $\left[2\left(E+V_{0}\right) / m\right]^{1 / 2} / 2 R_{1}$ obtained simply as a classical number of the $\alpha$ particle's purely kinematic impacts against the nuclear surface per unit time, and also from Landau's [61] prefactor estimate $D /(2 \pi \hbar)$, where $D$ is the average energy level separation in the host nucleus in the energy range of interest.

There is a large numerical difference between our results and Gamow's estimate: for the parameters assumed in Ref. [51], $E=5.407 \mathrm{MeV}, V_{0}=16.7 \mathrm{MeV}$, and $R_{1}=8.76 \mathrm{fm}$ and $R_{1}=8.975 \mathrm{fm}$, we respectively obtain $v_{0} /\left(2 R_{1}\right)=$ $1.881 \times 10^{21} \mathrm{~s}^{-1}$ and $1.836 \times 10^{21} \mathrm{~s}^{-1}$. This difference can be explained physically by the fact that the time travel within the well is very small compared with the disappearance and formation times of the $\alpha$ particle within the nucleus. More-
over, it can be seen that the value of $R_{0}$ has no effect on the values of $A_{\mathrm{r}}, A_{\mathrm{t}}$, and $\tau_{\text {resid }}$.

For $D=100 \mathrm{keV}, D /(2 \pi \hbar)=2.418 \times 10^{19} \mathrm{~s}^{-1}$, and our result does not differ appreciably from that of Landau in Ref. [61], possibly because both values are related to the fundamental properties of the host nucleus (indeed, the value of $D /(2 \pi \hbar)$ is quantitatively close to the inverse value of the Poincaré cycle for the time operator in a quasidiscrete energy spectrum).

## 5. Conclusion

We summarize the above analysis of theoretical results on two- and three-dimensional tunneling of particles and photons and on multiple internal reflections.
(1) Multiple internal reflections of tunneling particles and photons are the subject of some difference of opinion in the literature, in particular between the authors of Refs [31-44, 46, 51] and the authors of Ref. [47]. The preliminary experimental data $[48,49]$ seem to support the standpoint advanced in the former set of references. A proposal with the potential to settle the question is to follow Refs [48, 49] by carrying out correct optical experiments (with real processing of experimental data) on total internal reflections.
(2) The three-dimensional results obtained here can be used as the initial step in the time-dependent approach to the description of nuclear reactions and decays for any value of $l$ and for distinctly nonspherical shapes, as well as to the study of sub-barrier fusion reactions in astrophysics. In the second case, it is important to include not only penetration factor (18) but also the factor $1 /\left(1+A_{\mathrm{r}}^{\mathrm{in}}\right)$, which arises due to the multiple internal collisions between the hard core and the diametrically opposite points of the inner wall of the threedimensional Coulomb barrier. Both these factors are absent in the one-dimensional approximation, which is still commonly used in nuclear astrophysics (see, e.g., Refs [62-66] and the references therein).
(3) In further work, it would be expedient to recalculate the formula $\lambda=v P$ for all $\alpha$-radioactive nuclei for $v=1 / \tau_{\text {resid }}$.

## References

1. Akhiezer A I, Berestetskii V B Quantum Electrodynamics (New York: Interscience Publ., 1965); Translated from Russian: Kvantovaya Elektrodinamika (Moscow: Fizmatgiz, 1959)
2. Schweber S S An Introduction to Relativistic Quantum Field Theory (Evanston, Ill.: Row, Peterson, 1961); Translated into Russian: Vvedenie v Relyativistskuyu Kvantovuyu Teoriyu Polya (Moscow: IL, 1963) Ch. 5.3
3. Enders A, Nimtz G J. Physique I 21693 (1992)
4. Enders A, Nimtz G J. Physique I 31089 (1993)
5. Enders A, Nimtz G Phys. Rev. B 479605 (1993)
6. Enders A, Nimtz G Phys. Rev. E 48632 (1993)
7. Nimtz G, in Tunneling and Its Implications (Eds A Ranfagni, L S Schulman) (Singapore: World Scientific, 1997) p. 223
8. Wigner E P Phys. Rev. 98145 (1955)
9. Olkhovsky V S, Recami E Phys. Rep. 214339 (1992)
10. Olkhovsky V S, Recami E, Jakiel J Phys. Rep. 398133 (2004)
11. Hartman T E J. Appl. Phys. 333427 (1962)
12. Bose J Ch Bose Inst. Trans. 42 (1927)
13. Steinberg A M, Kwiat P G, Chiao R Y Phys. Rev. Lett. 71708 (1993)
14. Chiao R Y, Kwiat P G, Steinberg A M Sci. Am. 269 (2) 52 (1993)
15. Spielmann Ch et al. Phys. Rev. Lett. 732308 (1994)
16. Balcou Ph, Dutriaux L Phys. Rev. Lett. 78851 (1997)
17. Jakiel J, Olkhovsky V S, Recami E Phys. Lett. A 248156 (1998)
18. Laude V, Tournois P J. Opt. Soc. Am. B 16194 (1999)
19. Haibel A, Nimtz G, Stahlhofen A A Phys. Rev. E 63047601 (2001)
20. Olkhovsky V S, Recami E, Salesi G Europhys. Lett. 57879 (2002)
21. Olkhovsky V S, Recami E, Zaichenko A K Europhys. Lett. 70712 (2005)
22. Nimtz G, Enders A, Spieker H J. Physique I 4565 (1994)
23. Longhi S et al. Phys. Rev. E 65046610 (2002)
24. Chiao R Y AIP Conf. Proc. 4613 (1999)
25. Nimtz G AIP Conf. Proc. 46114 (1999)
26. Recami E AIP Conf. Proc. 46132 (1999)
27. Steinberg A AIP Conf. Proc. 46136 (1999)
28. Recami E, in Time's Arrows, Quantum Measurement, and Superluminal Behavior: Intern. Conf. TAQMSB, Palazzo Serra di Cassano, Napoli, October 3-5, 2000 (Eds D Mugnai, A Ranfagni, L S Schulman) (Roma: Consiglio Nazionale delle Ricerche, 2001) p. 17
29. Nimtz G, Haibel A, Vetter R-M, in Time's Arrows, Quantum Measurement, and Superluminal Behavior: Intern. Conf. TAQMSB, Palazzo Serra di Cassano, Napoli, October 3-5, 2000 (Eds D Mugnai, A Ranfagni, L S Schulman) (Roma: Consiglio Nazionale delle Ricerche, 2001) p. 125
30. Olkhovsky V S, in Time's Arrows, Quantum Measurement, and Superluminal Behavior: Intern. Conf. TAQMSB, Palazzo Serra di Cassano, Napoli, October 3-5, 2000 (Eds D Mugnai, A Ranfagni, L S Schulman) (Roma: Consiglio Nazionale delle Ricerche, 2001) p. 173
31. Nimtz G Prog. Quantum Electron. 27417 (2003)
32. Winful H G Phys. Rep. 4361 (2006)
33. Recami E J. Phys. Conf. Ser. 196012020 (2009)
34. Nimtz G Found. Phys. 391346 (2009)
35. Shvartsburg A B, Petite G, Zuev M J. Opt. Soc. Am. B 282271 (2011)
36. Aharonov Y, Erez N, Reznik B Phys. Rev. A 65052124 (2002)
37. Cardone F, Mignani R, Olkhovsky V S Phys. Lett. 289279 (2001)
38. Cardone F, Mignani R, Olkhovsky V S Mod. Phys. Lett. B 14109 (2000)
39. Cardone F, Mignani R, Olkhovsky V S J. Physique I 71211 (1997)
40. Fermor J H Am. J. Phys. 341168 (1966)
41. McVoy K W, Heller L, Bolsterli M Rev. Mod. Phys. 39245 (1967)
42. Anderson A Am. J. Phys. 57230 (1989)
43. Maidanyuk S P, Olkhovsky V S, Zaichenko A K J. Phys. Studies 6 (1) 24 (2002)
44. Cardone F, Maidanyuk S P, Mignani R, Olkhovsky V S Found. Phys. Lett. 19441 (2006)
45. Barbero A P L, Hernández-Figueroa H E, Recami E Phys. Rev. E 62 8628 (2000)
46. Olkhovsky V S, Romaniuk M V J. Mod. Phys. 21166 (2011)
47. Steinberg A M, Chiao R Y Phys. Rev. A 493283 (1994)
48. Carniglia C K, Mandel L J. Opt. Soc. Am. 611035 (1971)
49. Zhu S et al. Am. J. Phys. 54601 (1986)
50. Olkhovsky V S et al. Central Eur. J. Phys. 6122 (2008)
51. Olkhovsky V S, Romaniuk M V Nucl. Phys. Atom. Energy 10273 (2009)
52. Olkhovsky V S Int. J. Mod. Phys. E 231460006 (2014); "About the similarity of particle and photon tunneling and multiple internal reflections in 1-dimensional, 2 -dimensional and 3-dimensional photon tunneling", in Materiály IX Mezinárodni Vëdecko-praktická Konf. "Zprávy Vědecké Ideje", 27 Řínň̆á-05 Listopadu 2013 Roku, Fyzika, Praha (Praha: Publ. House Education and Science, 2013) p. 43
53. Davidovskii V V, Zaichenko A K, Olkhovsky V S Yadernaya Fiz. Sb. Nauch. Tr. Inst. Yadernykh Issled. (1(14)) 28 (2005)
54. Preston M A Physics of the Nucleus (Reading, Mass.: AddisonWesley Publ. Co., 1962)
55. Kadmenskii S G, Furman V I Al'fa-Raspad i Rodstvennye Yadernye Reaktsii (Alpha Decay and Related Nuclear Reactions) (Moscow: Energoatomizdat, 1985)
56. Baz' A I, Zel'dovich Ya B, Perelomov A M Scattering, Reactions and Decay in Nonrelativistic Quantum Mechanics (Jerusalem: Israel Program for Scientific Translations, 1969); Translated from Rus-
sian: Rasseyanie, Reaktsii i Raspady v Nerelyativistskoi Kvantovoi Mekhanike (Moscow: Nauka, 1971)
57. Baz' A I, Gol'danskii V I, Zel'dovich Ya B Sov. Phys. Usp. 8177 (1965); Usp. Fiz. Nauk 85445 (1965)
58. Gamow G Z. Phys. 51204 (1928)
59. Gurney R W, Condon E D W U Nature 122439 (1928)
60. Gamov G A Phys. Usp. 36267 (1993); Usp. Fiz. Nauk 163 (4) 51 (1993); Usp. Fiz. Nauk 10531 (1930); Usp. Fiz. Nauk 1346 (1933)
61. Landau L D, Smorodinsky Ya Lectures on Nuclear Theory (New York: Dover Publ., 1993); Translated from Russian: Lektsii po Teorii Atomnogo Yadra (Moscow: Gostekhizdat, 1955)
62. Bonetti R et al. Phys. Rev. Lett. 825205 (1999)
63. Spitaleri C et al. Phys. Rev. C 63055801 (2001)
64. Junghans A R et al. Phys. Rev. Lett. 88041101 (2002)
65. Imbriani G et al. Eur. Phys. J. A 25455 (2005)
66. Lemut A et al. (LUNA Collab.) Phys. Lett. B 634483 (2006)

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