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Quantum nature of a nonlinear beam splitter

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Contents

Abstract. This is a review of a very interesting (in the authors' view) phenomenon — the operation of a nonlinear light beam splitter. The beam splitter is a flat interface between two transparent dielectrics, at least one of which exhibits Kerr nonlinearity, i.e., its refractive index depends on the transmitted radiation intensity. Interestingly, quantum and classical theories make directly opposite predictions about the phase fluctuations of the output radiation of this device. In classical theory, the phases remain unchanged; in quantum theory, the phases fluctuate in accordance with the amplitude–phase uncertainty relation. The origin of this difference is established at the fundamental level. A further remarkable point about this quantum paradox is that not only is the source beam split in two but one can also create conditions where the two split parts are respectively dominated by amplitude noise and phase noise, thus allowing the selection of photon fluctuations. Results of original studies are summarized and further developed.

1. Introduction

Quantum paradoxes are a source of intellectual curiosity for two reasons. First, everything that is paradoxical is interesting and usually beautiful. Second, they give a more pragmatic opportunity to mark the boundaries for the model description of physical processes. No matter how long you talk about, for example, wave–particle duality, this will not add anything to the understanding of this phenomenon, except for new terms and words, because there is no illustrative model that would properly describe all the experimental results. Our pursuit of knowledge, however, would never end with a quantitatively ideal theory like quantum physics. Human consciousness irrepressibly strives for comprehending the meaning of phenomena. We do not hope to be able to predict everything, but we nevertheless assume that an adequate interpretation of quantum theory can be found in new concepts of space–time. Pure quantum effects cannot be described in the framework of standard four-dimensional space–time, which is a reason for many fundamental quantum paradoxes. Among these paradoxes, the most important are the ones connected with *quantum nonlocality*: the Einstein–Podolsky–Rosen paradox [1–3]; Bell inequalities [4] (see also, for example, Refs [5–12]); interference of single quantum particles (see, for example, Refs [12–14]); three-photon interference [15–17] (see also Ref. [12]), and the quantum Zeno paradox (see, for example, Refs [13, 18–20]) which can be regarded as violations of the causality principle, revealing themselves in the fact that the subsequent event determines the preceding event. It seems that we need to globally revise the base for our models in order to resolve these fundamental paradoxes.

But there is another category of paradoxes that are not directly connected with the space–time concept. Among them are multiphoton interference, where classical interference minima are replaced with maxima in a quantum description [21]; the recently discovered quantum Bernstein paradox [22] which will not, of course, change our representation of Nature, but is simply very beautiful, and the nonlinear beam-splitter quantum paradox [23–26], where phase fluctuations of light seem to appear out of nothing. We hope that the last paradox will also be interesting to the readers.

To easily explain the problem, we will touch upon the simplest case. The flat surface of a transparent dielectric reflects light. Now let us assume that the dielectric possesses cubic (Kerr) nonlinearity, so that its refractive index decreases as the light intensity increases. This is the so-called *self-defocusing nonlinearity*, because in such a medium a light

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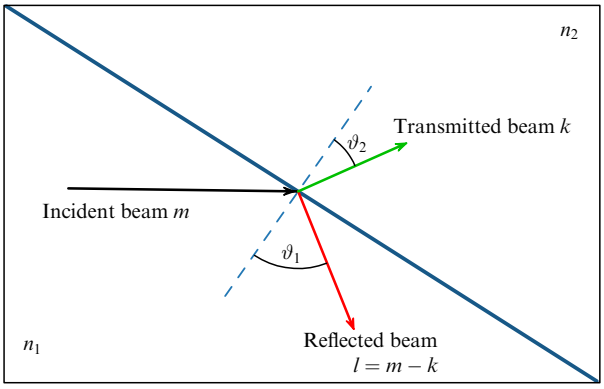

Figure 1. Beam splitter outline: ϑ_1 and ϑ_2 are the angles of incidence and refraction; m photons are incident on the interface, l photons are reflected from it, and $k = m - l$ photons are transmitted.

beam with a flat phase front and Gaussian intensity profile becomes divergent. It is clear that as the light intensity increases, the difference between the refractive indexes of air and the medium decreases and, consequently, the Fresnel reflectivity decreases as well, which stabilizes the intensity fluctuations in the reflected beam with respect to the analogous fluctuations in the incident beam. At the same time, the phase of the wave does not change. If such stabilization occurs not only for classical but also for photon fluctuations, what happens to the uncertainty principle? The amplitude and phase uncertainties are connected, and a decrease in one of them should be followed by an increase in the other.

However, a more interesting object in the sense of the effective fluctuation stabilization is the nonlinear beam splitter, comprising an interface between two transparent dielectrics (Fig. 1). Let us assume that one of them possesses cubic (Kerr) self-focusing nonlinearity, causing an increase in the refractive index as the light intensity increases. The reflectivity and transmittivity of this interface depend on the light intensity in accordance with the Fresnel formulas.

Let us consider the case when the incident beam propagates in a linear medium, and the refractive index of the second dielectric (with the nonlinear correction taken into account) is greater than the refractive index of the first one. In this event, the increase in the beam intensity will cause an increase in the refractive index and, hence, the reflectivity will increase and the transmittivity will decrease. This means that the spontaneous increase in the intensity of the incident beam would be partially compensated by a decrease in the transmittivity. This process, thus, leads to the saturation or a measure of closing of the system, resulting in the stabilization of the transmitted light intensity. Such a tangible stabilization can also take place for a beam reflected from the interface, either if the refractive index decreases as the intensity increases, or if the relation between two refractive indexes is opposite: the first one is greater than the second one.

We will analyze a single mode case of plane monochromatic waves. The phase of both reflected and transmitted plane waves is invariant near the interface between two transparent dielectrics, up to a possible jump by π in the reflected beam. This brings us to a paradoxical situation: amplitude fluctuations of either a transmitted or reflected beam can decrease, while the phase fluctuations should remain unchanged, which will, of course, lead to a violation of the Heisenberg uncertainty principle. In this article, we will mainly discuss the contradictions caused by this paradox.

In order not to take into account the phase incursion in transparent media, we consider the area *in the vicinity* (at distances less than the wavelength) of the interface. Moreover, considering such a thin layer (for instance, by placing it between two plane-parallel plates of linear media) cancels the effect of the plane-wave instability in the medium with self-focusing nonlinearity. This is a classical effect that follows from nonlinear optics, and it can be neglected due to the small propagation length at which the instability does not reveal itself.

Let us also note that the study of the preparation of sub-Poisson states by using, in particular, a nonlinear beam splitter not only is of fundamental interest, but also gives new opportunities for the practical application on the way to the production of devices for extremely precise measurements. Assume that photon fluctuations are suppressed in the light source with respect to those in the coherent state, which is a sum of a vacuum state and a constant component of the ideal nonfluctuating signal. In this event, one can increase the accuracy of the measuring system or the resolution of the image beyond the quantum limit which is determined by quantum noise.

2. Classical description of a beam splitter

It is common to start the consideration of light refraction on the interface between two media with Snell's law

$$n_1 \sin \vartheta_1 = n_2 \sin \vartheta_2 \,, \tag{2.1}$$

and the Fresnel formula for the amplitude reflectivity [27]

$$\rho = -\frac{\sin(\vartheta_1 - \vartheta_2)}{\sin(\vartheta_1 + \vartheta_2)} \,, \tag{2.2}$$

which holds true for plane-polarized light with the electric vector perpendicular to the plane of the drawing (so-called s-polarization). For another, mutually orthogonal, polarization (p-polarization), the sines are replaced by tangents.

If there are no losses, the transmittivity and reflectivity are linked by the simple relation

$$\tau^2 + \rho^2 = 1 \,, \tag{2.3}$$

which follows from the energy conservation law.

Let us note in advance that we only analyze the regimes with light transmittance, without considering the total internal reflection regime, because in the latter case the photonic fluctuations are not suppressed.

The nonlinearity of one of the adjacent media means that its properties depend on the intensity of the penetrating light. For example, Kerr nonlinearity does not change the frequency spectrum of light, but influences the refractive index. If the first medium possesses nonlinearity, then

$$n_1 = n_{10} + \chi\left(\sqrt{m} \pm \sqrt{l}\right)^2 , \qquad n_2 = \text{const} \,, \tag{2.4a}$$

because both incident and reflected waves are propagating in the first medium. These waves add up coherently and the response of the medium contains nonlinear terms for interference. In general, the waves become cnoidal, but this

happens in the bulk, and we consider a thin layer near the interface, where the phase incursion of the reflected wave is either absent or equal to π if it reflects from a denser medium. In the latter case, formula (2.4a) should be written down with the amplitude difference $\sqrt{m} - \sqrt{l}$, while for reflection from a less dense medium with the sum $\sqrt{m} + \sqrt{l}$.

If the second medium is nonlinear, the corresponding relations are simpler:

$$n_2 = n_{20} + \chi k\,, \qquad n_1 = \text{const}\,, \tag{2.4b}$$

where n_{i0} is the refractive index in the dark, and χ is a coefficient which is proportional to the third-order susceptibility $\chi^{(3)}$ of the medium. To easily compare the results with the quantum approach, we conventionally measure the light intensity by means of the number of photons in some quantizing volume. These dimensionless values are, of course, proportional to the intensities of the plane monochromatic waves under consideration.

If we solve the set of equations (2.1)–(2.4) by using obvious relations for the complex amplitudes of the transmitted and reflected waves, namely

$$a_{\rm t} = \tau a_{\rm i}\,, \qquad a_{\rm r} = \rho a_{\rm i}\,, \tag{2.5}$$

we will obtain the relation required between the input and output waves. Subscripts t, r, i correspond to transmitted, reflected, and incident beams, respectively.

Unfortunately, an analytical solution does not seem to exist. Therefore, to study the statistical properties of such a passive nonlinear converter, one should perform a numerical experiment. The most informative representation of the results would be the dependence of the three-dimensional probability distribution of the complex amplitudes for the transmitted and reflected beams on the form of such distribution for the incident beam. Now, let us illuminate our beam splitter with an ideal laser emission. The Wigner distribution $W(X, Y)$ describes the two-dimensional probability density for the real and imaginary parts of the complex amplitude to be equal to the X and Y quadrature components, respectively. In coherent light of an ideal laser, the Wigner distribution has the form of a rotated two-dimensional Gaussoid (Fig. 2), displaced by the mean amplitude value [28]:

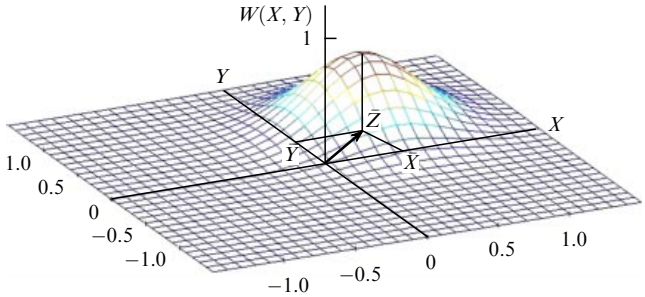

Figure 2. Wigner distribution for a coherent state: vector $\bar{z} = \{\bar{X}, \bar{Y}\}$ is the mean complex amplitude, $\bar{X}$ and $\bar{Y}$ are its mean quadrature components. It is more convenient to use in our consideration precisely quadrature components X and Y, instead of the generalized coordinate and momentum, q and p, that were used in Ref. [28] and are $\sqrt{2}$ times bigger.

$$W_{\rm coh}(X, Y) = \frac{2}{\pi} \exp\left(-2\left[(X - \bar{X})^2 + (Y - \bar{Y})^2\right]\right). \tag{2.6}$$

The normalization of the quadrature components is chosen in such a way that the mean light intensity $\bar{X}^2 + \bar{Y}^2$ would be equal to the mean number of photons in the mode.

But what will be in the outputs? If we know the output complex amplitudes as functions of the input one (and vice versa), then we can find the probability distribution for these amplitudes. Since an analytical description is too cumbersome, we will apply the numerical Monte Carlo method. The results of computer modeling are given in Fig. 3. One can see that the probability distribution for the transmitted beam appears to be squeezed along the real component, which

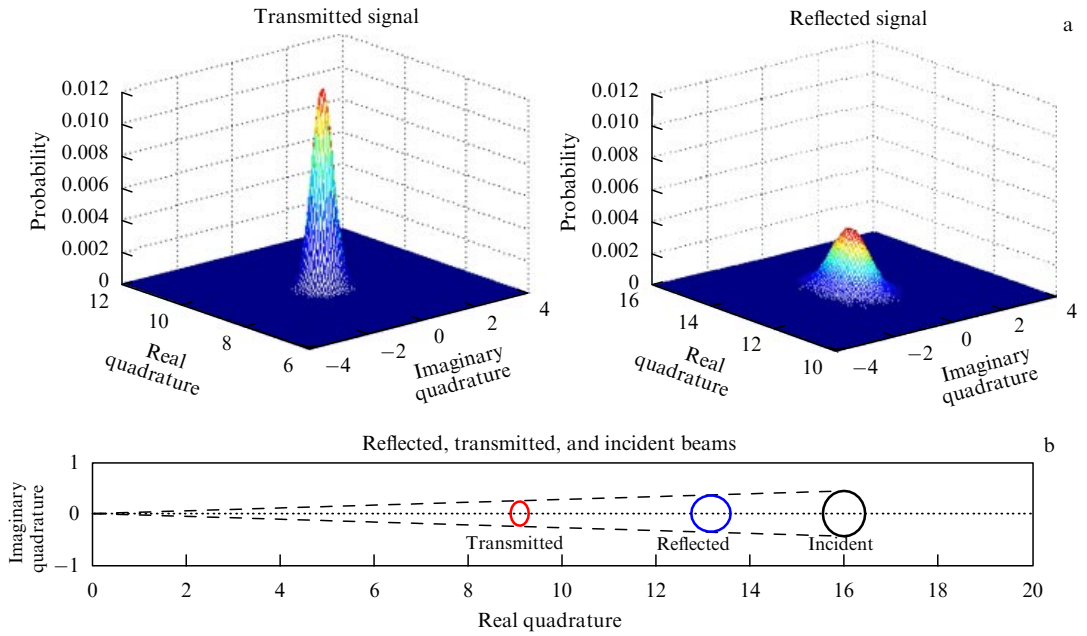

Figure 3. Three-dimensional probability distributions for the complex amplitudes of the output beams (a) and their cross sections (b). Three cross sections are accurately inscribed in one and the same angle that characterizes the phase fluctuations. The first medium is linear, $n_1 = 1.5$. The second medium is nonlinear, $n_2 = 1.51 + 0.001k$. Transmitted beam stabilization is observed. Angle of incidence equals 88°.

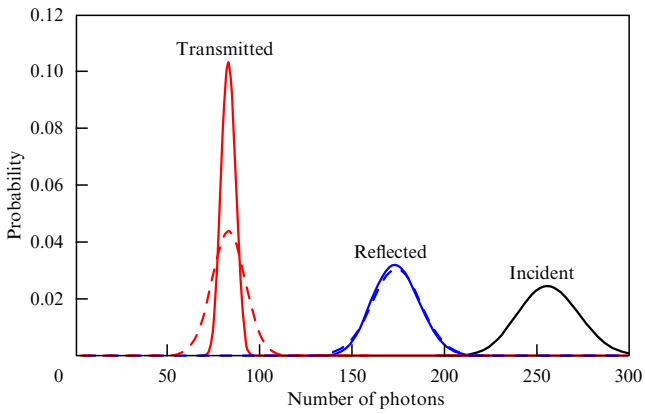

Figure 4. Intensity probability distributions for the reflected, transmitted and incident light beams for the same parameters of the system as in Fig. 3. For comparison, dashed curves show the Poisson distributions, which are common for the coherent state.

determines the light-intensity fluctuations if the mean amplitude is real. This means that the fluctuations are indeed stabilized. In the reflected beam we are faced with the opposite picture, which obviously follows from the energy conservation law, because our system is passive and non-dissipative. For clarity, Fig. 4 shows the intensity distributions for the transmitted $[P(|a_t|^2)]$ and reflected $[P(|a_r|^2)]$ beams.

For a more detailed analysis, one cannot just single out particular cross sections of the three-dimensional probability distributions, but also analyze the so-called *uncertainty regions*. These regions are determined in the following way. The real and the imaginary axes of our distributions represent the quadrature components of the complex amplitudes:

$$X = \frac{a + a^*}{2}\,, \qquad Y = \frac{a - a^*}{\mathrm{i}2}\,. \tag{2.7}$$

When analyzing the noise, we are first of all interested in the fluctuation components of these quadratures:

$$\Delta X = X - \bar{X}\,, \qquad \Delta Y = Y - \bar{Y}\,. \tag{2.8}$$

The mean amplitude is still assumed to be real and equal to the mean quadrature $\bar{X}$. Then, ΔX determines the amplitude fluctuations, and ΔY determines the phase fluctuations. The latter ones can be approximately written out as

$$\Delta\phi \approx \frac{\Delta Y}{\bar{X}} \tag{2.9}$$

for $\bar{X} \gg 1$.

The variances of quadratures $\langle \Delta X^2 \rangle$, $\langle \Delta Y^2 \rangle$ and their standard deviations $\sqrt{\langle \Delta X^2 \rangle}$, $\sqrt{\langle \Delta Y^2 \rangle}$ characterize only fluctuations along the real and imaginary axes of the probability distributions. Fluctuations of a more general type are determined by the generalized quadrature

$$\Delta Q(\theta) = \frac{\Delta a \exp(-\mathrm{i}\theta) + \Delta a^* \exp(\mathrm{i}\theta)}{2}\,, \tag{2.10}$$

which will describe all possible values of the fluctuation quadratures when rotating through the angle θ. For example, if $\theta = 0$, it is simply ΔX, and if $\theta = \pi/2$, it is ΔY. Figure 5 demonstrates uncertainty regions $R(\theta) = \sqrt{\langle \Delta Q^2(\theta) \rangle}$ for the reflected and transmitted light beams. One can see that in the transmitted beam the amplitude fluctuations are suppressed, while in the reflected one they are not. However, phase

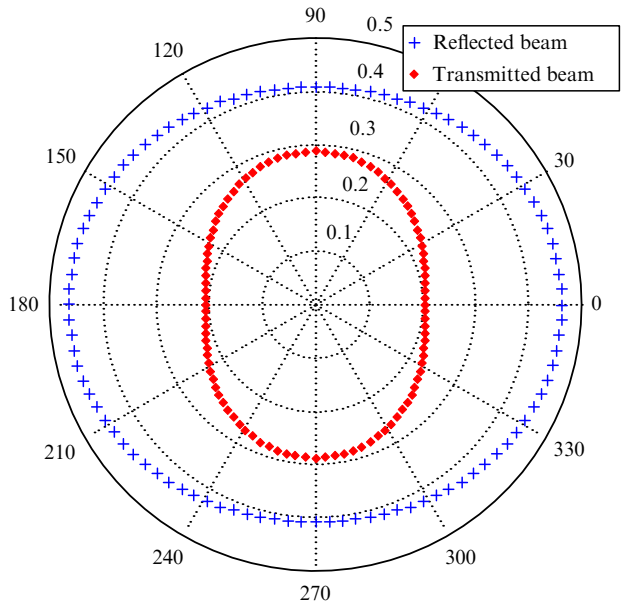

Figure 5. Uncertainty regions of output light beams. These regions would be circles for coherent states.

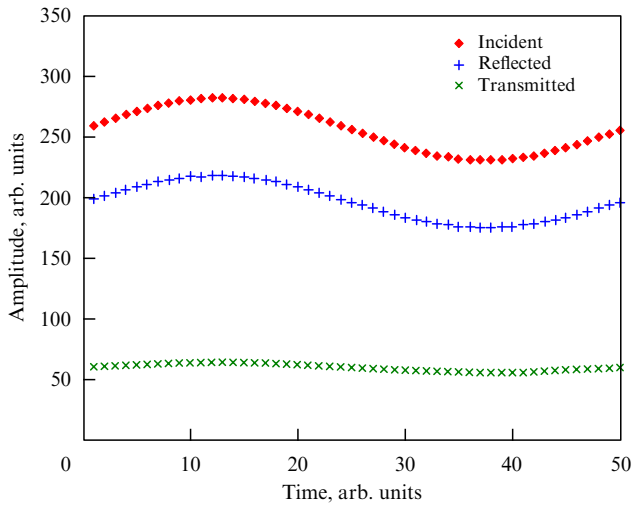

Figure 6. Harmonically modulated signal (upper curve) conversion with a nonlinear beam splitter. The transmitted light beam (lower curve) is stabilized: the ratio between the modulation amplitude and the mean value is 0.068, unlike 0.1 for the incident signal, and the fluctuations were transferred to the reflected beam (middle curve), where this ratio is 1.1. This means that the relative amplification of the signal took place in the passive device. The first medium is linear, $n_1 = 1.3$. The second medium is nonlinear, $n_2 = 1.41 + 0.0028k$. The transmitted beam is stabilized and the angle of incidence equals 88°.

fluctuations due to phase invariance are the same for all three light beams: incident, reflected, and transmitted. This is illustrated in Fig. 3b as well, where all three cross sections are accurately inscribed in one and the same angle which characterizes phase fluctuations of the beams. Before comparing these results with those of the rigorous quantum treatment, we will illustrate the first ones with another demonstrative example.

Isolating the amplitude modulation of a classic signal with a constant offset. Let us assume that a signal incident on the beam splitter does not have a constant amplitude, but is harmonically modulated in such a way that the modulation amplitude would be much smaller than the mean one (Fig. 6).

What will we see in the outputs? The transmitted light beam is stabilized and, of course, will be more intensity-stable (lower curve in Fig. 6). But where did the fluctuation energy go? Obviously, it will be transferred to the second, reflected beam, making the fluctuations stronger with respect to the mean value (middle curve). And although there are many methods of separating the signal against the background of the constant pedestal, this one clearly illustrates the amplitude-modulated light conversion with the nonlinear beam splitter.

3. Exact quantum solution in the Schrödinger representation

Problems in the field of nonlinear quantum optics in most cases, with several exceptions (for example, quantum Schrödinger solitons), do not have an analytical solution. Initially, this was the case for the problem considered as well. However, a detailed analysis of the discovered paradox was needed, and after much effort in this area, the solution has been found [25, 26].

Let us assume that the beam splitter (Fig. 7) is illuminated from the left side with a mode in the Fock state $|m\rangle$ with a given number of photons m. But there is also another possible input channel in the beam splitter from the top. Even if the illumination from the top is absent, there will still be the vacuum state $|0\rangle$. For a linear beam splitter, the output state would be described by the vector [28]

$$|\psi\rangle = \sum_{k=0}^{m} \sqrt{C_k^m}\,\tau^k \rho^{m-k}\,|k\rangle|l\rangle\,. \tag{3.1}$$

Here, k is the number of transmitted photons, $l = m - k$ is the number of reflected photons, τ and ρ are the Fresnel amplitude transmittivity and reflectivity, respectively, and C_k^m is the binomial coefficient.

Relation (3.1) describes the exact quantum solution, but it has a simple physical meaning: in this case, photons behave like usual classical and unbound particles. For example, the probability of one photon passing equals τ^2, and the probability of k photons passing is τ^{2k}. The probability of k photons being transmitted and $l = m - k$ photons being reflected equals the product of $(\tau^2)^k\,(\rho^2)^{m-k}$. The binomial coefficient $C_k^m = m!/(k!(m-k)!)$ appears due to the indistinguishability of all possible combinations for transmitted and reflected photons. And the square root has to be applied, because the vector of state determines not the probability itself, but its amplitude.

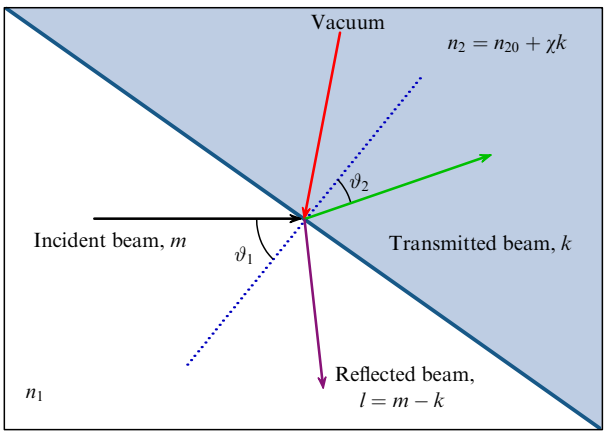

Figure 7. Nonlinear light beam splitter with the second nonlinear medium; ϑ_1 and ϑ_2 are the angles of incidence and refraction. At the first input is the coherent mode with the plane wave front, while at the second one vacuum.

The nonlinearity can be taken into account by introducing the functional dependences of τ and ρ on $m \pm l$ in the case of the first and second media being nonlinear and linear, respectively, and on k in the opposite case. If both adjacent media are nonlinear, two dependences would be introduced simultaneously. In this case, one can use standard expressions (2.2) and (2.3) for τ and ρ, given that the nonlinear additive to the refractive index is proportional to the number of photons in the medium, and χ is proportional in due course to the third-order susceptibility $\chi^{(3)}$.

Let us consider an arbitrary state $|\,\rangle$ at the input, for example, a coherent one $|z\rangle$, which can be expanded in terms of the Fock basis:

$$|\,\rangle = \sum_{m=0}^{\infty} D_m |m\rangle\,. \tag{3.2}$$

Then, one arrives at

$$|\psi\rangle \propto \sum_{m=0}^{\infty} D_m \sum_{k=0}^{m} \sqrt{C_k^m}\,\tau^k \binom{m \pm l}{k} \rho^l \binom{m \pm l}{k} |k\rangle|m-k\rangle$$
$$\equiv \sum_{m=0}^{\infty}\sum_{k=0}^{m} \Lambda_{kl}\,|k\rangle|l\rangle\,. \tag{3.3}$$

The two-row arguments in parentheses after τ and ρ denote two functional dependences on top or bottom rows for two cases specifying the relative position of the nonlinear medium, as mentioned before. Due to the nonlinearity of the problem, one has to perform the renormalization to fulfill the condition $\langle\psi|\psi\rangle = 1$.

There is one subtlety in the last step. If the Fock state is at the input, then, of course, the output states are not Fock-like: the numbers of photons k and l can be different. More precisely, the output states are the superpositions of the Fock states. But which of them should the nonlinearity refer to? Let our reasoning be more operationalistic. If we detect photons at the outputs, for every realization we will obtain quite definite numbers k and l. Exactly these numbers will determine the nonlinearity, as illustrated by Eqn (3.3). This means that a definite number of photons $m = k + l$ were at the input, too.

The physical meaning of the coefficients squared $\Lambda_{kl}^2 = |\langle l|\langle k||\psi\rangle|^2$ is very simple: it is the probability of k photons passing and l photons reflecting at the same time — that is, we have to do with a two-dimensional conditional probability. Now we can easily evaluate the probability distributions for observing some certain number of photons in the reflected and transmitted light beams:

$$P_l = \sum_{k=0}^{\infty} \Lambda_{kl}^2\,, \qquad P_k = \sum_{l=0}^{\infty} \Lambda_{kl}^2\,, \tag{3.4}$$

which are shown in Fig. 8. One can see that the transmitted beam resides in the Poisson state, and the reflected one in the super-Poisson state, which qualitatively corresponds to the classical results (see Fig. 4).

Details about the approximate calculation of the Λ_{kl} coefficients are discussed in Appendix 1.

More detailed information about the quantum states of light fields can be obtained from the Wigner distributions,

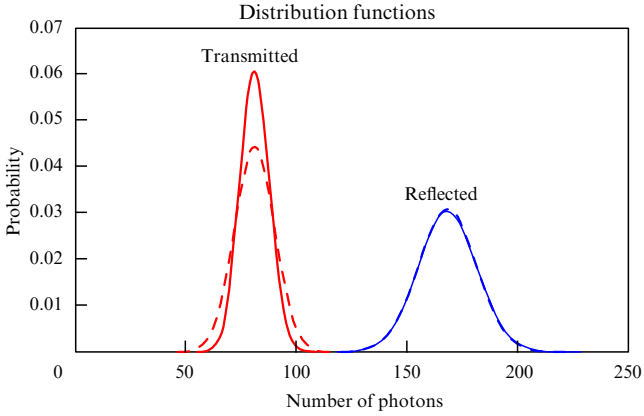

Figure 8. Probability distributions for the number of photons in the output beams. For comparison, dashed curves show Poisson distributions. Initial parameters of the system are the same as in Fig. 3. The most probable number of reflected photons is 169, and the most probable number of transmitted photons is 81 (mean number of input photons is 250).

which are quantum analogs to the classical distributions plotted in Fig. 3. They are determined in the following way.

In order not to lose the information about the phase, one can calculate the projection of our state $|\psi\rangle$ vector onto the coherent state $|z\rangle$ — in other words, calculate the scalar product $\langle\psi|z\rangle$. Its absolute value, if squared, gives the so-called *Q-distribution*:

$$Q(X,Y) = \left|\langle\psi|z\rangle\right|^2 . \tag{3.5}$$

By varying the mean complex amplitude of the coherent state $z = \{X, Y\}$ over the complex plane, we, as odd as it might seem, probe our state $|\psi\rangle$ with a device that has a transfer function

$$W_{\mathrm{coh}}(X,Y) = \frac{2}{\pi}\exp\left\{-2\left[(X-\bar{X})^2 + (Y-\bar{Y})^2\right]\right\}$$

of a coherent state (see Fig. 2). In other words, $Q(X,Y)$ is a convolution of the Wigner distribution $W(X,Y)$ with the coherent state distribution [28]

$$Q(X,Y) = \int_{-\infty}^{+\infty}\int W(X',Y')\,W_{\mathrm{coh}}(X-X',Y-Y')\,\mathrm{d}X'\,\mathrm{d}Y' . \tag{3.6}$$

Distribution (3.6) is plotted in Fig. 9 for the same initial data as in Figs 3 and 8.

Now, after invoking distribution (3.5) and calculating $Q(X,Y)$, we perform direct and inverse Fourier transforms and obtain $W(X,Y)$:

$$W(X,Y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int \frac{\int_{-\infty}^{+\infty}\int Q(x,y)\exp\left[-\mathrm{i}(\xi x+\eta y)\right]\mathrm{d}x\,\mathrm{d}y}{\int_{-\infty}^{+\infty}\int W_{\mathrm{coh}}(x,y)\exp\left[-\mathrm{i}(\xi x+\eta y)\right]\mathrm{d}x\,\mathrm{d}y} \times \exp\left[\mathrm{i}(\xi X+\eta Y)\right]\mathrm{d}\xi\,\mathrm{d}\eta . \tag{3.7}$$

But how can one calculate $Q(X,Y)$ separately for the transmitted and reflected light beams if the vector of state $|\psi\rangle$ describes them simultaneously? Strictly speaking, the double scalar product $\langle\psi|z_{\mathrm{t}}\rangle|z_{\mathrm{r}}\rangle$ needs to be found, which is a conditional probability for the transmitted and reflected beam amplitudes to take the values z_{t} and z_{r}, respectively. Therefore, we can proceed with the description of only one beam, by summing up the probabilities for all possible values

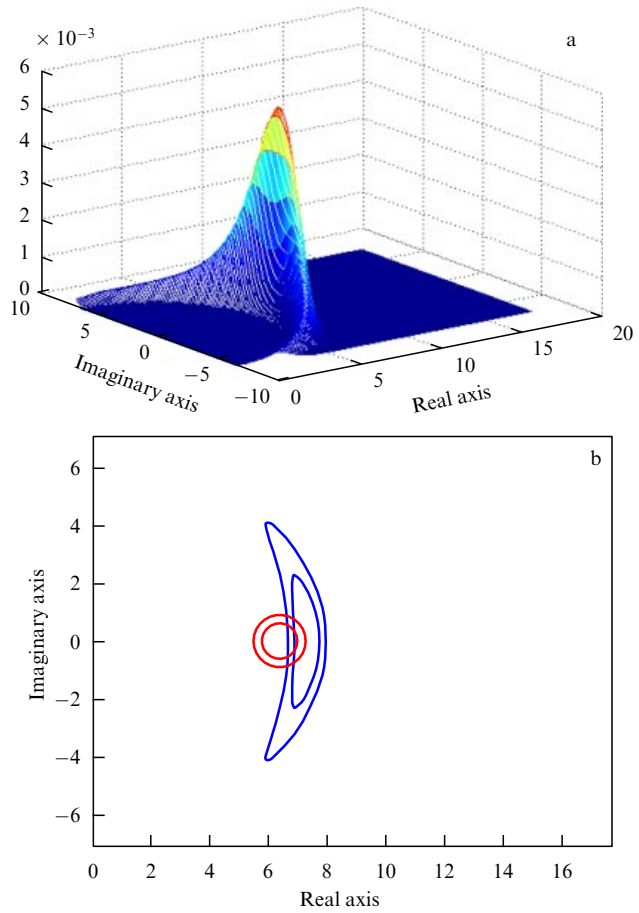

Figure 9. Q-distribution for the transmitted beam (a) and its cross sections (b) on the relative levels of 0.2 and 0.5. For comparison, the circles, which are common for a coherent state, are shown on the cross sections.

of the other beam amplitude:

$$Q_l(X,Y) = \sum_l \left|\langle\psi|z_l\rangle|l\rangle\right|^2 . \tag{3.8}$$

The Wigner distribution is qualitatively similar to the Q-distribution shown in Fig. 9, except for more pronounced outlines. This is a typical sub-Poisson distribution with suppressed amplitude fluctuations (crescent form) and enhanced phase fluctuations (extension along a circular arc). Actually, one would hardly expect a different result: nobody has cancelled yet the Heisenberg uncertainty principle. But what is the mechanism of the phase swing? Which force rocks the phase in our experiment? The uncertainty principle gives a formal mathematical explanation. But we would like to have at least some physical model. The answers to these questions will be discussed in Sections 4 and 5. Meanwhile, let us give a more detailed description of our results.

As in the classical approximation, phase fluctuations can be estimated by using quadrature components X and Y. Let us assume that the mean complex amplitude z of the input state $|z\rangle$ is real: $\bar{z} = \bar{X}$. We express quadrature components through photon creation ($\hat{a}^+$) and annihilation ($\hat{a}$) operators:

$$\hat{X} = \frac{\hat{a}+\hat{a}^+}{2} , \qquad \hat{Y} = \frac{\hat{a}-\hat{a}^+}{\mathrm{i}2} . \tag{3.9}$$

Then, the fluctuation variance of the first component will determine the amplitude fluctuations, and the fluctuation

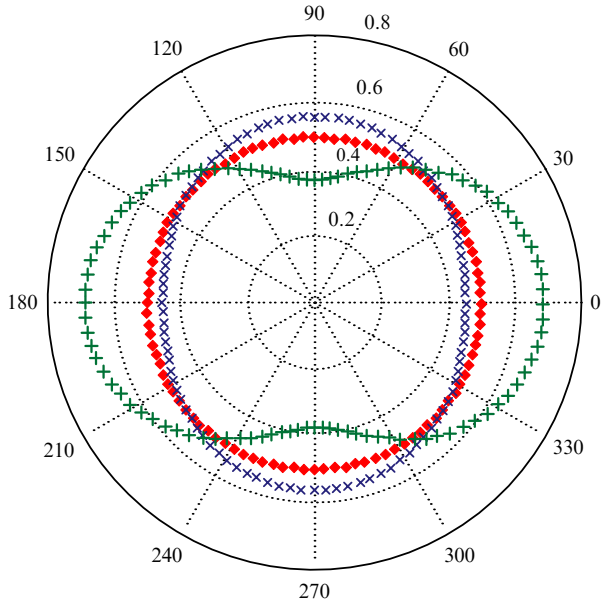

Figure 10. Uncertainty regions for the transmitted (extended along the vertical direction) and reflected (extended along the horizontal direction) beams in the complex plane. A circle, common for a coherent state, is shown for comparison. The first medium is nonlinear, the angle of incidence is $28.7°$, $n_1 = 2.30 + 0.000285\,(\sqrt{m} + \sqrt{l})$, and $n_{20} = 1.30$. Mean number of reflected photons is 155, mean number of transmitted photons is 95, and mean number of incident photons reaches 250.

variance of the second one determines the phase fluctuations. And according to the Heisenberg uncertainty principle, one has

$$\langle \Delta \hat{X}^2 \rangle \langle \Delta \hat{Y}^2 \rangle \geqslant \frac{1}{16}\,. \tag{3.10}$$

By analogy with expression (2.10), we also introduce the generalized quadrature:

$$\Delta \hat{Q}(\theta) = \frac{\Delta \hat{a} \exp(-\mathrm{i}\theta) + \Delta \hat{a}^+ \exp(\mathrm{i}\theta)}{2}\,, \tag{3.11}$$

which can easily be utilized for specifying the uncertainty regions, plotted in Fig. 10. We shall start Section 3.1 with a discussion of this figure. Notice that the derivation of the expressions, needed to calculate this data, can be found in Appendix 2.

3.1 Selecting quantum fluctuations and increasing the accuracy of simultaneous measurements of quantities entering the Heisenberg uncertainty relation

The type of a beam splitter which the uncertainty regions in Fig. 10 refer to is of interest to us, because the X quadrature (amplitude) fluctuations are suppressed in one (transmitted) of the beams, and the Y quadrature fluctuations are suppressed in the other (reflected) beam. This means that amplitude fluctuations are mostly transferred to one of the beams, and phase fluctuations to the other. In this way, one can select either amplitude or phase fluctuations of the input beam: the first ones prevail in the reflected beam, and the second ones in the transmitted beam.

On the other hand, light beams with suppressed fluctuations can be favored to increase the measurement accuracy. Thus, let us assume that we need to simultaneously and as accurately as possible measure the field quadrature components X and Y of the optical monochromatic plane mode. We will send it towards our beam splitter and measure the amplitude component X in the transmitted beam, and the phase component Y in the reflected beam. The measurement accuracy will be higher than in a classical linear beam splitter. But does this mean that we have managed to avoid the Heisenberg uncertainty principle (3.10)? It certainly does not, because after measuring the output quadratures of the beam splitter we need to recalculate them into the input ones, namely, to divide them by the transmittivity and reflectivity, accordingly. But these coefficients are less than unity, and the measurement errors correspondingly increase. Nevertheless, the accuracy of the simultaneous measurements of X and Y will enhance.

3.2 Choosing the optimal arrangement and the beam splitter cascade

The fundamental possibility of preparing sub-Poisson light was shown earlier. The photon fluctuations determining the level of the photodetection shot noise are suppressed in this state of light; therefore, it can be used in high-precision quantum measurements. In order to optimize setups with the beam splitter and its parameters with respect to the maximum effectivity of noise reduction, we will consider all possible arrangements of the nonlinear media. Three cases are possible here: the first medium is nonlinear and the second one is linear; the first medium is linear and the second one is nonlinear, and both media are nonlinear. The optimization criterion is referred to the minimization of the Fano factor, which is introduced as the ratio between the produced beam intensity variance and the variance of the coherent light with the Poisson distribution, or its mean value:

$$F = \frac{\langle \Delta \hat{n}^2 \rangle}{\langle \hat{n} \rangle}\,, \tag{3.12}$$

where $\hat{n} = \hat{a}^+ \hat{a}$ is the photon number operator. For the sub-Poisson distribution, $0 < F < 1$, and the amplitude fluctuation suppression is more efficient if the Fano factor decreases.

The best of our results for one nonlinear medium was obtained with the following values of the initial parameters: angle of incidence $\vartheta_1 = 88°$, $\chi_2 = 0.0028$, $n_1 = n_{20} = 1.3$. For an average number of input photons in the coherent state equal to 250, the amplitude fluctuations in the transmitted light were less than half the fluctuations in the input light: $F \approx 0.47$. The calculated results are given in Fig. 11.

After looking at the parameter values of such a beam splitter, it is reasonable to assume that if, instead of the first linear dielectric, one made use of a nonlinear self-defocusing medium with $\chi < 0$, the efficiency of the photon fluctuation suppression would be even higher. Indeed, for $\chi_1 = -0.004$ and the other parameters just the same as in Fig. 11, the Fano factor reduces to $F \approx 0.44$. An even stronger effect can be achieved with $\chi_1 = 10^{-5}$, $\chi_2 = 0.03$, $\vartheta_1 = 80.2°$, and $n_1 = n_{20} = 1.3$: specifically, $F \approx 0.38$.

Further stabilization of the transmitted beam can be achieved when it sequentially propagates through the cascade of nonlinear beam splitters, as shown, for example, in Fig. 12.

Let k' denote the number of output photons, which were transmitted by the second beam splitter. Then, for m input photons, the probability amplitude for k transmitted and l reflected photons to be at the output of the first beam splitter and k' and l' photons to be at the output of the second beam

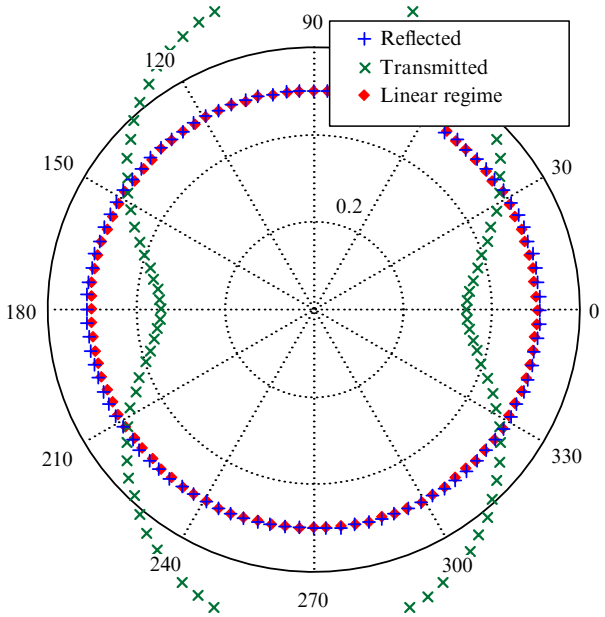

Figure 11. Uncertainty regions for a nonlinear beam splitter with one (second) nonlinear medium, with parameters optimized in the sense of photon fluctuation suppression.

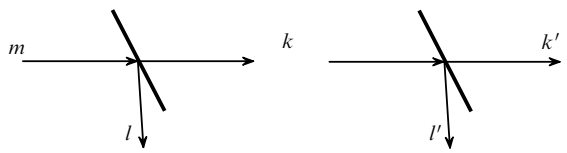

Figure 12. Two-beam splitter cascade.

splitter will be

$$\Lambda_{klk'l'} = D(m) A_{kl} A_{k'l'} \,, \tag{3.13}$$

where $A_{kl} = \sqrt{C_m^k}\,\tau^k \rho^l$, and $A_{k'l'} = \sqrt{C_k^{k'}}\,\tau^{k'} \rho^{l'}$.

The vector of state in this case is expressed as

$$|\psi\rangle \propto \sum_{m=0}^{\infty} \sum_{k=0}^{m} \sum_{k'=0}^{k} \Lambda_{klk'l'} |k\rangle |l\rangle |k'\rangle |l'\rangle \,. \tag{3.14}$$

Due to the nonlinearity of the problem, one has to perform the renormalization to fulfill the condition $\langle\psi|\psi\rangle = 1$.

Obviously, the probability of exactly k' photons being at the output of the second beam splitter will be now

$$P'(k') = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \sum_{l'=0}^{k'} \Lambda_{klk'l'}^2 \,. \tag{3.15}$$

Numerical calculations gave the following result (Fig. 13). After traversing the second beam splitter, the beam became more stable. The Fano factor after passing the first beam splitter turned out to be $F = 0.47$, and after passing the second one was $F = 0.40$. At the same time, the mean number of photons in the beam transmitted twice decreased by ~ 2.7 times.

It should be noted that in the uncertainty diagram for two beam splitters both reflected and transmitted beam amplitude fluctuations are suppressed. This is explained by the sub-

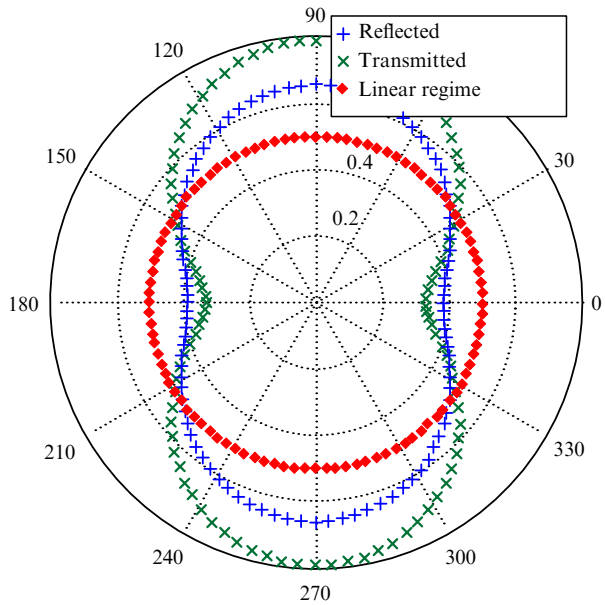

Figure 13. Uncertainty regions for the beam splitter cascade.

Poisson state of light supplied to the input of the second beam splitter.

4. Quasiclassical description

Let us return to our paradox. The first explanation that one can think of after comparing the contradicting classical and quantum results is that the vacuum was not taken into account in the first case. Indeed, the second input from the top (see Fig. 7) contains not emptiness, but fluctuations with zero mean value. Otherwise, the Wigner distribution for the vacuum is the same as for the coherent state: $W_{\text{vac}}(X, Y) = (2/\pi)\exp[-2(X^2 + Y^2)]$. Now we add a plane noise wave to our numerical experiment. The results are presented in Fig. 14. One can see that such refinement of the model gets us closer to the truth: phase fluctuations are no longer invariant. But this is not something extraordinary and common only to our nonlinear beam splitter. Even in the case of a linear beam splitter, due to a vacuum at the second input, the coherent light from the source remains coherent at both outputs with a constant uncertainty region, being a circle with a unit diameter. In the opposite case, the circle diameter would decrease together with the mean amplitude of the output beams. However, if the mean amplitude decreases and the uncertainty region stays the same, the phase fluctuations should increase. We have managed to explain their noninvariance. Moreover, the intensity probability distributions for the quantum and quasiclassical descriptions nearly coincide (Fig. 14c).

But what can we do with the imaginary quadrature variance? From both Fig. 14 and purely qualitative classical considerations, it is absolutely clear that it has to be constant, because in our beam splitter only signal amplitude transformation occurs together with the related real quadrature. The imaginary quadrature variance is only added to the vacuum one and has to remain constant. We will once again prove this by using the linearized approximation. Summing up, the problem of informal explanation of the nonlinear beam splitter paradox remains unsolved, because the Heisenberg uncertainty principle (3.10) is still violated.

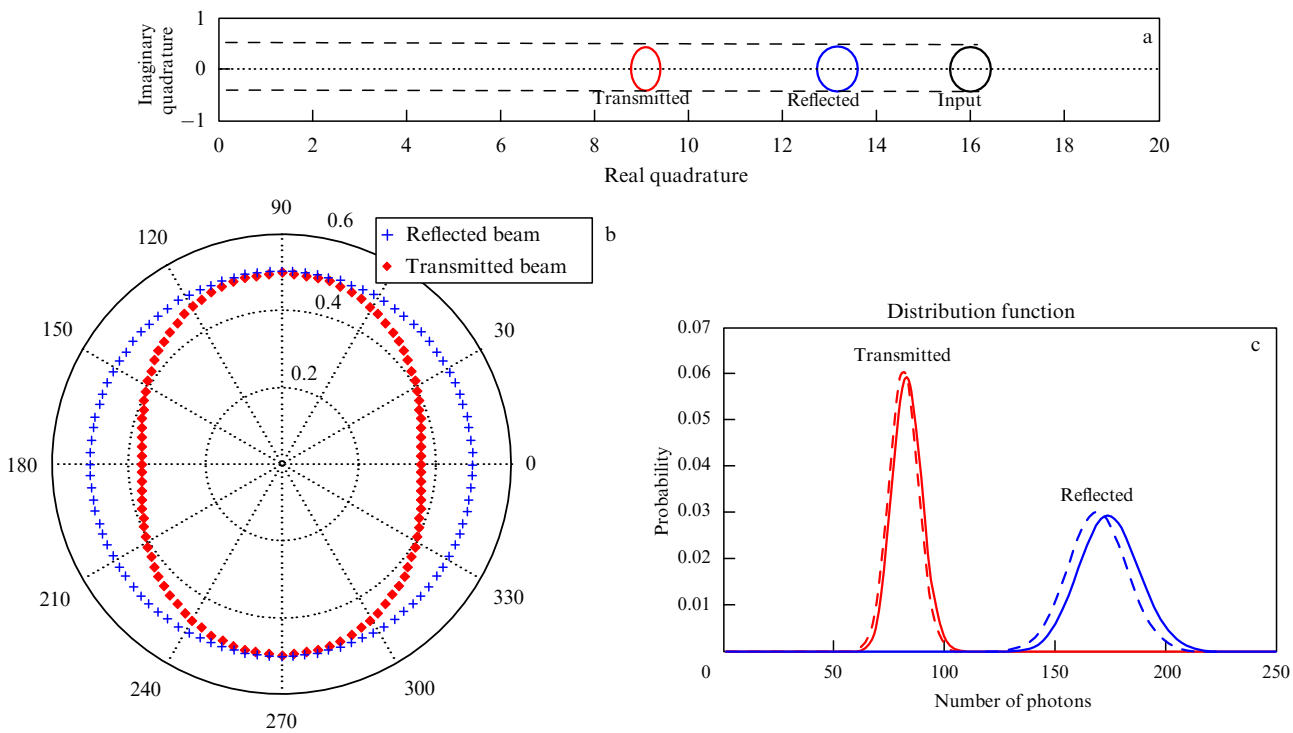

Figure 14. Cross sections of the complex amplitude distributions (a) and uncertainty regions (b) for the input and output beams with vacuum fluctuations taken into account (two inputs). One can see that the imaginary quadrature fluctuations are invariant: they are not inscribed in one angle, as in Fig. 3, but fit between two parallel dashed straight lines. Intensity probability distributions for the reflected and transmitted beams (c) in quasiclassical (solid curve) and quantum (dashed curve) cases nearly coincide.

5. Approximation linearized over fluctuations. Heisenberg picture

The mean amplitude of light should be relatively large for the nonlinear phase self-modulation to be efficient. The fluctuation contributions are small in comparison with the mean amplitude, and they can be properly described by linearizing the problem around the mean values. Now, if $|\bar{z}|^2 \gg 1$, which means that the input mode intensity is large enough and the mean number of photons in the mode is large, then one obtains

$$|\bar{z}|^2 = \langle z|\hat{m}|z\rangle = \langle z|\hat{a}^+\hat{a}|z\rangle \gg \sqrt{\langle z|\Delta\hat{m}^2|z\rangle}\,, \tag{5.1}$$

where $\Delta\hat{m}^2 = \hat{a}^+\hat{a}\hat{a}^+\hat{a} - \langle z|\hat{m}|z\rangle^2$, and the increment of transmittivity and reflectivity for the interface can be linearized over fluctuations:

$$\tau \approx \langle\tau\rangle + \Delta k\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\,, \qquad \rho \approx \langle\rho\rangle + \Delta k\,\frac{\mathrm{d}\rho}{\mathrm{d}k}\,. \tag{5.2}$$

Here, we will consider, for simplicity, the case of the second nonlinear medium, and Δk will be treated as a classical analog of the photon number increment operator $\Delta\hat{k}$, which is an intensity increment normalized in a specific way. Let us note that the similarity of the quantum problem descriptions in the Heisenberg picture and in the classical approximation follows exactly from the fluctuation linearization, when the operators are not multiplied by each other, and their noncommutativity does not count.

Let us express the complex amplitude of the light mode that passed the interface in the following way:

$$a = \big(\langle a\rangle + \Delta a\big)\exp\left(-\mathrm{i}\Delta\Phi\right). \tag{5.3}$$

Here, Δa and $\Delta\Phi$ are fluctuation components of the amplitude and phase. The phase of the amplitude constant component is taken to be zero. In this case, $\Delta a = \Delta X$.

On the other hand, the complex amplitude a of the transmitted light can be expressed as

$$a = \tau a_1 + \rho a_\mathrm{v}\,, \tag{5.4}$$

where a_1 is the amplitude of the input beam mode, and a_v is the amplitude of the second input mode coming from the top and imitating the vacuum fluctuations.

We now equate right-hand parts of formulas (5.3) and (5.4):

$$\big(\langle a\rangle + \Delta a\big)\exp\left(-\mathrm{i}\Delta\Phi\right) = \tau a_1 + \rho a_\mathrm{v}\,. \tag{5.5}$$

Let us express τ in the form (5.2): $\tau \approx \langle\tau\rangle + \Delta k(\mathrm{d}\tau/\mathrm{d}k)$, express a_1 in the form $a_1 = \langle a_1\rangle + \Delta a_1$, and substitute these expansions into expression (5.5):

$$\big(\langle a\rangle + \Delta a\big)\exp\left(-\mathrm{i}\Delta\Phi\right) = \left(\langle\tau\rangle + \Delta k\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(\langle a_1\rangle + \Delta a_1\big) + \rho a_\mathrm{v}\,. \tag{5.6}$$

Here, Δa_1 and a_v are complex, all other values being real.

Taking into account the smallness of fluctuations, $\exp\left(-\mathrm{i}\Delta\Phi\right) \approx 1 - \mathrm{i}\Delta\Phi$, we obtain

$$\big(\langle a\rangle + \Delta a\big)(1 - \mathrm{i}\Delta\Phi) = \left(\langle\tau\rangle + \Delta k\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(\langle a_1\rangle + \Delta a_1\big) + \rho a_\mathrm{v}\,, \tag{5.7}$$

$$\big(\langle a\rangle + \Delta a\big)(1 + \mathrm{i}\Delta\Phi) = \left(\langle\tau\rangle + \Delta k\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(\langle a_1\rangle + \Delta a_1^*\big) + \rho a_\mathrm{v}^*\,. \tag{5.8}$$

Relation (5.8) is the complex conjugate to relation (5.8).

Due to the linearity of the problem over fluctuations, we can replace complex amplitudes of fluctuation components in relations (5.7), (5.8) with the corresponding photon creation and annihilation operators. Now we add Eqns (5.7) and (5.8), thus obtaining

$$2\big(\langle\hat{a}\rangle+\Delta\hat{a}\big)=\left(\langle\tau\rangle+\Delta\hat{k}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(2\langle\hat{a}_1\rangle+\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)\,. \tag{5.9}$$

Given that

$$\langle\hat{a}\rangle=\langle\tau\rangle\langle\hat{a}_1\rangle\,, \tag{5.10}$$

we have

$$2\big(\langle\tau\rangle\langle\hat{a}_1\rangle+\Delta\hat{a}\big)=\left(\langle\tau\rangle+\Delta\hat{k}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(2\langle\hat{a}_1\rangle+\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)\,. \tag{5.11}$$

Now, we can easily find $2\Delta\hat{a}$:

$$\begin{aligned}2\Delta\hat{a}&=\left(\langle\tau\rangle+\Delta\hat{k}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(2\langle\hat{a}_1\rangle+\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)\\&\quad+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)-2\langle\tau\rangle\langle\hat{a}_1\rangle=\langle\tau\rangle\big(\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\Delta\hat{k}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\\&\quad\times\big(2\langle\hat{a}_1\rangle+\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)\,.\end{aligned} \tag{5.12}$$

Since the phase of the constant component $\langle\hat{a}\rangle$ is zero, we have

$$\Delta\hat{k}\approx 2\langle\hat{a}\rangle\Delta\hat{a}\,, \tag{5.13}$$

then

$$\begin{aligned}2\Delta\hat{a}&=\langle\tau\rangle\big(\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+2\langle\hat{a}\rangle\Delta\hat{a}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\\&\quad\times\big(2\langle\hat{a}_1\rangle+\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)\,.\end{aligned} \tag{5.14}$$

Now we neglect the fluctuation terms of the second order:

$$2\Delta\hat{a}=\langle\tau\rangle\big(\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+4\langle\hat{a}\rangle\Delta\hat{a}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\,\langle\hat{a}_1\rangle+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)\,, \tag{5.15}$$

and subsequently arrive at

$$\Delta\hat{a}=\frac{\langle\tau\rangle\big(\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)}{2\big(1-2\langle\hat{a}\rangle(\mathrm{d}\tau/\mathrm{d}k)\langle\hat{a}_1\rangle\big)}\,. \tag{5.16}$$

Let us now represent the derivatives in expressions (5.2) and (5.16) in the following way:

$$\frac{\mathrm{d}\tau}{\mathrm{d}k}=\frac{\mathrm{d}\tau}{\mathrm{d}n_2}\,\frac{\mathrm{d}n_2}{\mathrm{d}k}=\chi\,\frac{\mathrm{d}\tau}{\mathrm{d}n_2}\,,\qquad\frac{\mathrm{d}\rho}{\mathrm{d}k}=\frac{\mathrm{d}\rho}{\mathrm{d}n_2}\,\frac{\mathrm{d}n_2}{\mathrm{d}k}=\chi\,\frac{\mathrm{d}\rho}{\mathrm{d}n_2}\,, \tag{5.17}$$

where the coefficient χ, as earlier, is proportional to the third-order susceptibility $\chi^{(3)}$ of the second dielectric. Now, given relationship (5.10), we obtain

$$\Delta\hat{a}=\frac{\langle\tau\rangle\big(\Delta\hat{a}_1+\Delta\hat{a}_1^{+}\big)+\langle\rho\rangle\big(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+}\big)}{2}\left(1-2\chi\,\frac{\langle\hat{k}\rangle}{\langle\tau\rangle}\,\frac{\mathrm{d}\tau}{\mathrm{d}n_2}\right)^{-1}. \tag{5.18}$$

Assuming the amplitude transmittivity to be real, according to the energy conservation law, one finds

$$\begin{aligned}\tau^2=1-\rho^2&=\frac{\sin^2(\vartheta_1+\vartheta_2)-\sin^2(\vartheta_1-\vartheta_2)}{\sin^2(\vartheta_1+\vartheta_2)}\\&=\frac{\sin(2\vartheta_1)\sin(2\vartheta_2)}{\sin^2(\vartheta_1+\vartheta_2)}\,.\end{aligned} \tag{5.19}$$

Since

$$\tau\,\mathrm{d}\tau=-\rho\,\mathrm{d}\rho\,, \tag{5.20}$$

we may write down that

$$\frac{\mathrm{d}\tau}{\tau}=-\frac{\rho}{\tau^2}\,\mathrm{d}\rho=-\frac{\sin^2(\vartheta_1+\vartheta_2)}{\sin(2\vartheta_1)\sin(2\vartheta_2)}\,\rho\,\mathrm{d}\rho\,. \tag{5.21}$$

According to the refraction law (2.1), the following relation is valid:

$$\frac{\mathrm{d}\vartheta_2}{\mathrm{d}n_2}=-\frac{\sin\vartheta_2}{n_2\cos\vartheta_2}\,. \tag{5.22}$$

And now we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}n_2}=\frac{\mathrm{d}\rho}{\mathrm{d}\vartheta_2}\,\frac{\mathrm{d}\vartheta_2}{\mathrm{d}n_2}\,. \tag{5.23}$$

Because, in accordance with Fresnel formula (2.2), one finds

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}\vartheta_2}&=\frac{\sin(\vartheta_1+\vartheta_2)\cos(\vartheta_1-\vartheta_2)+\sin(\vartheta_1-\vartheta_2)\cos(\vartheta_1+\vartheta_2)}{\sin^2(\vartheta_1+\vartheta_2)}\\&=\frac{\sin(2\vartheta_1)}{\sin^2(\vartheta_1+\vartheta_2)}\,,\end{aligned} \tag{5.24}$$

we have

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}n_2}&=\frac{\mathrm{d}\rho}{\mathrm{d}\vartheta_2}\,\frac{\mathrm{d}\vartheta_2}{\mathrm{d}n_2}=-\frac{\sin(2\vartheta_1)}{\sin^2(\vartheta_1+\vartheta_2)}\,\frac{\sin\vartheta_2}{n_2\cos\vartheta_2}\\&=-\frac{\sin\vartheta_2\sin(2\vartheta_1)}{n_2\cos\vartheta_2\sin^2(\vartheta_1+\vartheta_2)}\,;\end{aligned} \tag{5.25}$$

subsequently, given relation (5.19), we arrive at

$$\begin{aligned}\frac{2}{\tau^2}\,\frac{\mathrm{d}\rho}{\mathrm{d}n_2}&=-\frac{2\sin^2(\vartheta_1+\vartheta_2)}{\sin(2\vartheta_1)\sin(2\vartheta_2)}\,\frac{\sin\vartheta_2\sin(2\vartheta_1)}{n_2\cos\vartheta_2\sin^2(\vartheta_1+\vartheta_2)}\\&=-\frac{1}{n_2\cos^2\vartheta_2}\,.\end{aligned} \tag{5.26}$$

Now, using the known relation for ρ (2.2), we can write down

$$\begin{aligned}\frac{2}{\tau}\,\frac{\mathrm{d}\tau}{\mathrm{d}n_2}&=\frac{2}{\tau}\,\frac{\mathrm{d}\tau}{\mathrm{d}\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}n_2}=-\rho\,\frac{2}{\tau^2}\,\frac{\mathrm{d}\rho}{\mathrm{d}n_2}\\&=\frac{\sin(\vartheta_1-\vartheta_2)}{\sin(\vartheta_1+\vartheta_2)}\left(-\frac{1}{n_2\cos^2\vartheta_2}\right)=-\frac{\sin(\vartheta_1-\vartheta_2)}{n_2\cos^2\vartheta_2\sin(\vartheta_1+\vartheta_2)}\,.\end{aligned} \tag{5.27}$$

Let us substitute the result (5.27) into expression (5.18):

$$\Delta\hat{a}=\frac{1}{2}\,G\big[\langle\tau\rangle(\Delta\hat{a}_1+\Delta\hat{a}_1^{+})+\langle\rho\rangle(\hat{a}_{\mathrm{v}}+\hat{a}_{\mathrm{v}}^{+})\big]\,, \tag{5.28}$$

where

$$G = \left[1 + \frac{2\chi\langle\hat{k}\rangle \sin(\vartheta_1 - \vartheta_2)}{n_2 \cos^2\vartheta_2 \sin(\vartheta_1 + \vartheta_2)}\right]^{-1}. \tag{5.29}$$

Since $\Delta\hat{a} = \Delta\hat{X}$, the variance of the real quadrature is

$$\langle\Delta\hat{X}^2\rangle = 4\left[1 + \frac{2\chi\langle\hat{k}\rangle \sin(\vartheta_1 - \vartheta_2)}{n_2 \cos^2\vartheta_2 \sin(\vartheta_1 + \vartheta_2)}\right]^{-2} = \frac{G^2}{4}. \tag{5.30}$$

Let us perform similar transformations by subtracting equation (5.7) from complex conjugate equation (5.8):

$$2\mathrm{i}\Delta\hat{\Phi}\big(\langle\hat{a}\rangle + \Delta\hat{a}\big) = \left(\langle\tau\rangle + \Delta\hat{k}\,\frac{\mathrm{d}\tau}{\mathrm{d}k}\right)\big(\Delta\hat{a}_1^+ - \Delta\hat{a}_1\big) + \langle\rho\rangle\big(\hat{a}_\mathrm{v} - \hat{a}_\mathrm{v}^+\big). \tag{5.31}$$

We neglect the second-order fluctuation terms:

$$2\mathrm{i}\Delta\hat{\Phi}\langle\hat{a}\rangle = \langle\tau\rangle\big(\Delta\hat{a}_1^+ - \Delta\hat{a}_1\big) + \langle\rho\rangle\big(\hat{a}_\mathrm{v} - \hat{a}_\mathrm{v}^+\big). \tag{5.32}$$

Since $\Delta\hat{\Phi}\langle\hat{a}\rangle = \Delta\hat{Y}$, we finally obtain

$$\langle\Delta\hat{Y}\rangle^2 = \frac{1}{4}. \tag{5.33}$$

Now we see that the variance of the imaginary quadrature fluctuations indeed remains constant after passing the nonlinear beam splitter, as we have already derived from general considerations, and the uncertainty principle for quadrature components (3.10) is violated. But what is the matter?

While deriving relations for $\Delta\hat{X}$ and $\Delta\hat{Y}$, we did not check the commutation relation

$$[\Delta\hat{a}, \Delta\hat{a}^+] = \hat{I}, \tag{5.34}$$

where $\hat{I}$ is the unit operator. But relation (5.34) is not fulfilled due to the nonlinearity of the system. Let us perform the renormalization, using a constant coefficient C, and expand the expression for the commutation:

$$\begin{aligned} C[\Delta\hat{a}, \Delta\hat{a}^+] &= C(\Delta\hat{a}\Delta\hat{a}^+ - \Delta\hat{a}^+\Delta\hat{a}) \\ &= C(\Delta\hat{X}^2 + \Delta\hat{Y}^2 + \mathrm{i}\Delta\hat{Y}\Delta\hat{X} - \mathrm{i}\Delta\hat{X}\Delta\hat{Y} - \Delta\hat{X}^2 - \Delta\hat{Y}^2 \\ &+ \mathrm{i}\Delta\hat{Y}\Delta\hat{X} - \mathrm{i}\Delta\hat{X}\Delta\hat{Y}) = 2\mathrm{i}C(\Delta\hat{Y}\Delta\hat{X} - \Delta\hat{X}\Delta\hat{Y}) = \hat{I}. \end{aligned} \tag{5.35}$$

Since without the renormalization the following relation is valid:

$$\begin{aligned} \Delta\hat{Y}\Delta\hat{X} - \Delta\hat{X}\Delta\hat{Y} = \frac{\hat{G}}{2\mathrm{i}}\big[\langle\tau\rangle^2\Delta\hat{a}_1^+\Delta\hat{a}_1 - \langle\tau\rangle^2 + \langle\rho\rangle^2\hat{a}_\mathrm{v}^+\hat{a}_\mathrm{v} - \langle\rho\rangle^2 \\ + \langle\tau\rangle\langle\rho\rangle(\Delta\hat{a}_1^+\hat{a}_\mathrm{v} - \hat{a}_\mathrm{v}\Delta\hat{a}_1^+ + \hat{a}_\mathrm{v}^+\Delta\hat{a}_1 - \hat{a}_\mathrm{v}\Delta\hat{a}_1^+)\big] = -\frac{G\hat{I}}{2\mathrm{i}}, \end{aligned} \tag{5.36}$$

we obtain

$$C = -\frac{1}{G} = -\left[1 + \frac{2\chi\langle\hat{k}\rangle \sin(\vartheta_1 - \vartheta_2)}{n_2 \cos^2\vartheta_2 \sin(\vartheta_1 + \vartheta_2)}\right]. \tag{5.37}$$

And now everything appears to be correct (Fig. 15), at least from a qualitative point of view. Small quantitative discrepancies with the exact quantum solution are explained, apparently, by the fact that we have used the linearized approximation. It is easy to show that the current radius of the fluctuation uncertainty region in the polar coordinate system has the form

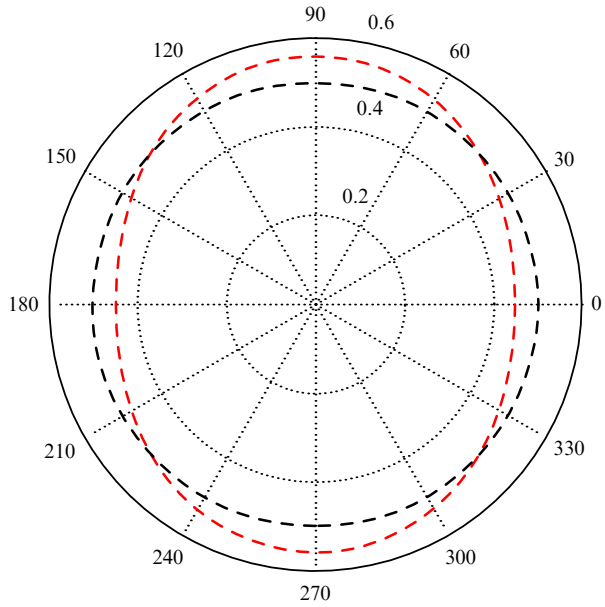

Figure 15. Fluctuation uncertainty region for transmitted light after renormalization. A circle, common for a coherent state, is shown for comparison.

$$R(\theta) = \sqrt{\frac{G}{4}\cos^2\theta + \frac{1}{4G}\sin^2\theta}. \tag{5.38}$$

At last we have eliminated the contradiction with the Heisenberg uncertainty principle. Indeed, the phase fluctuations in the transmitted beam have increased, and the amplitude ones have decreased. The key point here was the renormalization of the quadratures in order to satisfy the commutation relation $[\Delta\hat{a}, \Delta\hat{a}^+] = \hat{I}$, which cannot be any different, due to the *indivisibility of the photon*. Precisely this factor causes the swing of the phase and fluctuations of the imaginary quadrature $\Delta\hat{Y}$ without any visible reason, and it is this factor that underlies the paradox occurred.

Let us also note that we have presented expressions (5.2) and (5.17) for both transmittivity and reflectivity not accidentally. The fact is that by using similar transformations one can also describe the field of the reflected beam. The result will differ only quantitatively, and the main fundamental conclusions will be the same.

6. Conclusions

This article presents quantum theory for boson field transformations with a nonlinear beam splitter. The nonlinear beam splitter is a flat interface between two transparent media, and either one or both of them have cubic (Kerr) nonlinearity: the refractive index depends on the intensity of light passing through the medium. By choosing the proper set of parameters of the adjacent media, one can achieve intensity stabilization for the reflected or transmitted beams, and not only for classical, but also for quantum fields, which gives an opportunity of increasing the informativity of optical devices where light is used as the information carrier. After comparing with the classical approach, it turned out that the latter makes predictions about phase fluctuation behavior, which

contradicts the quantum theory. This finding, which is interesting from the fundamental point of view, formed the basis for the nonlinear beam-splitter quantum paradox.

Let us return to the formulation of this paradox and analyze ways to resolve it. Amplitude fluctuation stabilization takes place in a nonlinear beam splitter for both classical and quantum descriptions. It is expressed, particularly, in fluctuation suppression for the quadrature component X directed along the vector of the mean real amplitude in the complex plane. At the same time, the variance of the other quadrature Y, directed along the imaginary axis, remains invariant in the classical description, even with vacuum fluctuations taken into account. Indeed, there are no reasons for it to change in term of the physical model. But this violates the Heisenberg uncertainty relation. Only quantum theory predicts an increase in the fluctuation variance for the quadrature component Y. Which factor is responsible for that? The answer is given by the linearized quantum theory in the Heisenberg picture, which is the closest to the classical description. The direct solution does not satisfy the commutation relation $[\Delta\hat{a}, \Delta\hat{a}^+] \neq \hat{I}$ due to the nonlinearity of the system. But can the commutation relation $[\Delta\hat{a}, \Delta\hat{a}^+] = \hat{I}$ be violated? If the photons are indivisible, this relation cannot be violated. The commutation relation can be satisfied by performing the renormalization of the creation and annihilation operators. Then, the variance over Y increases and the uncertainty principle, naturally, holds true.

The fundamental reason for the paradox lies in the indivisibility of the photon.

In addition, we also come to a practically convenient conclusion that photon fluctuations can be selected: amplitude fluctuations are transferred either to the transmitted beam or to the reflected beam, depending on the arrangement of the adjacent media, and the phase fluctuations are transferred to the reflected or transmitted beams, respectively. This means that we can separate amplitude and phase fluctuations of the signal by sending the first ones to one of the channels, and the second ones to the other one. This is very attractive for systems of optical transfer or information processing, and also for reducing noise in communication systems, for instance. For the classical description, this sort of nonlinear filtration turns out to be defective, because it works only for amplitude fluctuations, while the phase ones remain unchanged. Nevertheless, even for the latter case, the layout solutions are possible, which result in the separation of the useful amplitude-modulated signal from the constant noise component.

There is another important practical conclusion, which just refers to the theoretical description of nonlinear quantum systems. It is absolutely clear that the linear quantum model can be described classically by using, for instance, complex amplitudes, only in the final expression replacing them with the corresponding operators in the Heisenberg picture, then performing averaging, etc. This significant simplification is fair, because in the linear problem operators are neither multiplied by each other nor by themselves, and subsequently their noncommutation does not influence the result. As an example, one can consider here an ordinary linear beam splitter without losses. And what about the approximation linearized over fluctuations? It seems that it should be the same. The variables are introduced in the description linearly and are not multiplied by each other. Indeed, quantum and classical descriptions of the multibeam interferometer with Kerr nonlinearity [13, 29–32] are identical as long as the linearization is an adequate approximation. But in that case, the commutation relation $[\Delta\hat{a}, \Delta\hat{a}^+] = \hat{I}$ is not violated! However, through the example of the nonlinear beam splitter we have learned that this is not always the case. Therefore, one should be extremely careful, even when describing quantum systems linearized over small fluctuations, and should not hurry when using the results obtained from classical model descriptive treatments.

The practical aspect of the phenomenon considered was also discussed: all possible combinations of linear and nonlinear media are analyzed with respect to the maximum possible photon fluctuation suppression and corresponding overcoming of the quantum limit for the data capacity of optical devices. The exact quantum theory provided estimates which show that photon noise variance reduction by a factor of more than two with respect to the coherent state can be achieved.

Finally, we will note that Ya A Fofanov managed to experimentally observe the sub-Poisson photon-count distribution by using a nonlinear beam splitter. The detected amplitude fluctuation variance was approximately 1% less than amplitude fluctuation variance for the Poisson distribution [33, 34].

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7. Appendix 1

Two types of factors are present in expression (3.3). The first one (D_m) is the Poisson expansion coefficient of the input coherent state in terms of Fock states $|m\rangle$:

$$D_m^2 = \exp(-\bar{m})\,\frac{|\bar{m}|^m}{m!}\,,$$

where $\bar{m}$ is the mean number of photons. For $\bar{m} > 20$, one can use with good accuracy the Gaussian approximation for D_m^2:

$$D_m^2 \approx \frac{1}{\sqrt{2\pi\bar{m}}}\exp\left[-\frac{(m-\bar{m})^2}{2\bar{m}}\right].$$

The second type of factors is the binomial term of the form $A_{kl} = \sqrt{C_m^k}\,\tau^k\rho^l$. It determines the probability of k photons passing and $l = m - k$ reflecting for the fixed number m of incident photons. This probability equals A_{kl}^2. If the conditions $\bar{m} > 100$ and $\bar{m}\tau^2\rho^2 > 20$ are fulfilled, then, given the Moivre–Laplace theorem, one can apply with good accuracy an approximation for A_{kl}^2:

$$A_{kl}^2 \approx \frac{1}{\sqrt{2\pi\tau^2 m(1-\tau^2 m)}}\exp\left[-\frac{(k-\tau^2 m)^2}{2\tau^2 m(1-\tau^2 m)}\right].$$

The sum in expression (3.3), up to infinity over m in the case of the input coherent state, can be practically performed only until the upper limit $\bar{m} + 5\sqrt{\bar{m}}$, because the terms deviating from the mean value by more than 5σ are negligibly small.

8. Appendix 2

Let us derive the relationships which determine the fluctuation variance of the quadrature components for transmitted and reflected light beams.

First, we will express the mean generalized quadrature of the reflected beam:

$$
\begin{aligned}
2\langle\psi|\hat{Q}(\theta)|\psi\rangle &= \langle\psi|\hat{a}|\psi\rangle\exp(-\mathrm{i}\theta) + \langle\psi|\hat{a}^{+}|\psi\rangle\exp(\mathrm{i}\theta) \\
&= \exp(-\mathrm{i}\theta)\sum_{k,l}\langle k|\langle l|\Lambda_{kl}\sum_{k'l'}\Lambda_{k'l'}\sqrt{l'}\,|l'-1\rangle|k'\rangle \\
&+ \exp(\mathrm{i}\theta)\sum_{k,l}\langle k|\langle l|\Lambda_{kl}\sum_{k'l'}\Lambda_{k'l'}\sqrt{l'+1}\,|l'+1\rangle|k'\rangle \\
&= \exp(-\mathrm{i}\theta)\sum_{kl}\Lambda_{kl}\Lambda_{kl+1}\sqrt{l+1} + \text{c.c.},
\end{aligned}
$$

where c.c. are complex conjugate terms. Subsequently, one finds

$$
\langle\psi|\hat{Q}(\theta)|\psi\rangle = \cos\theta\sum_{kl}\Lambda_{kl}\Lambda_{kl+1}\sqrt{l+1}\,.
$$

Similarly:

$$
\begin{aligned}
4\langle\psi|\hat{Q}^{2}(\theta)|\psi\rangle &= \langle\psi|\hat{a}\hat{a}|\psi\rangle\exp(-\mathrm{i}2\theta) \\
&+ \langle\psi|\hat{a}^{+}\hat{a}^{+}|\psi\rangle\exp(\mathrm{i}2\theta) + 2\langle\psi|\hat{a}^{+}\hat{a}|\psi\rangle + 1 \\
&= \exp(-\mathrm{i}2\theta)\sum_{k,l}\langle k|\langle l|\Lambda_{kl} \\
&\times\sum_{k'l'}\Lambda_{k'l'}\sqrt{l'(l'-1)}\,|l'-2\rangle|k'\rangle + \text{c.c.} + 2\sum_{k,l}l\Lambda_{kl}^{2} + 1 \\
&= 2\cos(2\theta)\sum_{kl}\Lambda_{kl}\Lambda_{kl+2}\sqrt{(l+1)(l+2)} + 2\sum_{k,l}l\Lambda_{kl}^{2} + 1\,.
\end{aligned}
$$

This means that the quadrature variance

$$
\langle\psi|\Delta\hat{Q}^{2}(\theta)|\psi\rangle = \langle\psi|\hat{Q}^{2}(\theta)|\psi\rangle - \langle\psi|\hat{Q}(\theta)|\psi\rangle^{2}
$$

can be calculated by using the corresponding substitution.

For the transmitted light beam, k and l should be interchanged with positions.

References

1. Einstein A, Podolsky B, Rosen N *Phys. Rev.* **47** 777 (1935)
2. Bohr N *Phys. Rev.* **48** 696 (1935)
3. Fock V A, Einstein A, Podolsky B, Rozen N, Bohr N *Usp. Fiz. Nauk* **16** 436 (1936)
4. Bell J S *Physics* **1** 195 (1964)
5. Belinskii A V, Klyshko D N *Phys. Usp.* **36** 653 (1993); *Usp. Fiz. Nauk* **163** (8) 1 (1993)
6. Belinskii A V *Phys. Usp.* **40** 305 (1997); *Usp. Fiz. Nauk* **167** 323 (1997)
7. Klyshko D N *Sov. Phys. Usp.* **31** 74 (1988); *Usp. Fiz. Nauk* **154** 133 (1988)
8. Klyshko D N *Sov. Phys. Usp.* **32** 555 (1989); *Usp. Fiz. Nauk* **158** 327 (1989)
9. Kadomtsev B B *Phys. Usp.* **37** 425 (1994); *Usp. Fiz. Nauk* **164** 449 (1994)
10. Kadomtsev B B *Dinamika i Informatsiya* (Dynamics and Information) 2nd ed. (Moscow: Redaktsiya Zhurnala "Uspekhi Fizicheskikh Nauk", 1999)
11. Kadomtsev B B *Phys. Usp.* **46** 1183 (2003); *Usp. Fiz. Nauk* **173** 1221 (2003)
12. Belinskii A V *Phys. Usp.* **46** 877 (2003); *Usp. Fiz. Nauk* **173** 905 (2003)
13. Belinsky A V *Kvantovye Izmereniya* (Quantum Measurements) (Moscow: BINOM. Laboratoriya Znanii, 2012)
14. Belinsky A V *Metafizika* (2) 72 (2011)
15. Belinsky A V, Klyshko D N *Laser Phys.* **6** 1082 (1996)
16. Belinsky A V, Klyshko D N *Laser Phys.* **2** 112 (1992)
17. Belinskii A V *JETP Lett.* **54** 11 (1991); *Pis'ma Zh. Eksp. Teor. Fiz.* **54** 13 (1991)
18. Menskii M B *Phys. Usp.* **41** 923 (1998); *Usp. Fiz. Nauk* **168** 1017 (1998)
19. Mensky M B *Phys. Usp.* **54** 519 (2011); *Usp. Fiz. Nauk* **181** 543 (2011)
20. Mensky M B *Quantum Measurements and Decoherence: Models and Phenomenology* (Dordrecht: Kluwer Acad. Publ., 2000); Translated into Russian: *Kvantovye Izmereniya i Dekogerentsiya. Modeli i Fenomenologiya* (Moscow: Fizmatlit, 2001)
21. Belinskii A V, Klyshko D N *Sov. Phys. JETP* **75** 606 (1992); *Zh. Eksp. Teor. Fiz.* **102** 1116 (1992)
22. Belinsky A V, Chirkin A S *Phys. Usp.* **56** 1126 (2013); *Usp. Fiz. Nauk* **183** 1231 (2013)
23. Belinskii A V *JETP Lett.* **51** 387 (1990); *Pis'ma Zh. Eksp. Teor. Fiz.* **51** 341 (1990)
24. Belinskii A V *Sov. J. Quantum Electron.* **21** 75 (1991); *Kvantovaya Elektron.* **18** 84 (1991)
25. Belinskii A V, Granovskii A A *JETP Lett.* **93** 495 (2011); *Pis'ma Zh. Eksp. Teor. Fiz.* **93** 552 (2011)
26. Belinsky A V, Volkov D V, Dmitriev A V, Shulman M Kh *JETP* **117** 771 (2013); *Zh. Eksp. Teor. Fiz.* **144** 891 (2013)
27. Born M, Wolf E *Principles of Optics* (Oxford: Pergamon Press, 1969); Translated into Russian: *Osnovy Optiki* (Moscow: Nauka, 1970)
28. Leonhardt U *Measuring the Quantum State of Light* (Cambridge: Cambridge Univ. Press, 1997)
29. Reynaud S, Giacobino E *J. Physique* **49** C2-477 (1988)
30. Belinskii A V, Chirkin A S *Opt. Spectrosc.* **67** 723 (1989); *Opt. Spektrosk.* **67** 1226 (1989)
31. Belinskii A V, Rozanov N N *Opt. Spectrosc.* **73** 89 (1992); *Opt. Spektrosk.* **73** 153 (1992)
32. Belinskii A V, Rosanov N N *Proc. SPIE* **1840** 291 (1991)
33. Fofanov Ya A *Opt. Spectrosc.* **94** 802 (2003); *Opt. Spektrosk.* **94** 861 (2003)
34. Fofanov Ya A *Proc. SPIE* **7993** 799300 (2011)