METHODOLOGICAL NOTES

PACS numbers: **47.10.–g**, **47.35.–i**, 47.35.Bb, 47.35.Pq

Phase patterns of dispersive waves from moving localized sources

P N Svirkunov, M V Kalashnik

DOI: 10.3367/UFNe.0184.201401d.0089

Contents

Abstract. **A general approach is proposed within which the phase structure of wave perturbations caused by a moving localized source can be described based on the wave dispersion law alone. Applying this approach, a simple analytical expression for the phase surfaces is obtained. It is used to study the details of phase patterns of gravity-capillary waves, the structure of wave trains in the ocean in the wake of a moving tropical hurricane, and the system of lee waves in Earth's atmosphere.**

1. Introduction

The propagation of dispersive waves in continuous media has peculiarities related to the dependence of the propagation speed on the wavelength. Thus, if a source of perturbations is moving uniformly in such a medium, it generates a wave pattern in its vicinity, the key feature of which is the surfaces (or lines in two dimensions) of constant phase. At distances much exceeding the source size, the structure of this pattern is practically independent of the source shape and is mainly governed by the dispersion law and the speed of source motion. A well-known example of a wave pattern is furnished by the (Kelvin) ship waves left behind a ship moving in deep water (Fig. 1). One more noticeable example (Figs 2 and 3) is the system of lee waves excited in Earth's atmosphere by an air flow around orographic inhomogeneities (isolated mountains or islands in the ocean). This pattern is formed by inertia-gravity waves (IGWs), which owe their existence to the density stratification of the atmosphere and its rotation as the whole [1, 2].

Describing the wave pattern generated by a moving source is among the classical problems of wave theory. In relatively simple cases of gravity or capillary waves on a fluid surface, the wave pattern structure can be obtained by an elementary way by replacing the source with a sequence of instantaneous elementary sources emitting circular waves, and turning to the geometrical analysis of wave front envelopes [3, 4]. In general, a solution of the wave equation with a moving source is used, written as the Fourier transform, and its asymptotic

Figure 1. Pattern of ship waves. Moscow Canal, Iksha Reservoir, 18.08.2013 (photo by O S Kalashnik).

P N Svirkunov Typhoon Research and Production Association, ul. Pobedy 4, 249038 Obninsk, Kaluga region, Russian Federation
E-mail: ps@typhoon.obninsk.ru
M V Kalashnik Typhoon Research and Production Association, ul. Pobedy 4, 249038 Obninsk, Kaluga region, Russian Federation; Obukhov Institute of Atmospheric Physics, Russian Academy of Sciences, Pyzhevskii per. 3, 109017 Moscow, Russian Federation; Obninsk Institute for Nuclear Power Engineering affiliated with the National Research Nuclear University 'MEPhI', Studgorodok 1, 249040 Obninsk, Kaluga region, Russian Federation
E-mail: kalashnik-obn@mail.ru

Received 12 July 2013, revised 19 September 2013
Uspekhi Fizicheskikh Nauk **184** (1) 89 – 100 (2014)
DOI: 10.3367/UFNr.0184.201401d.0089
Translated by S D Danilov; edited by A Radzig

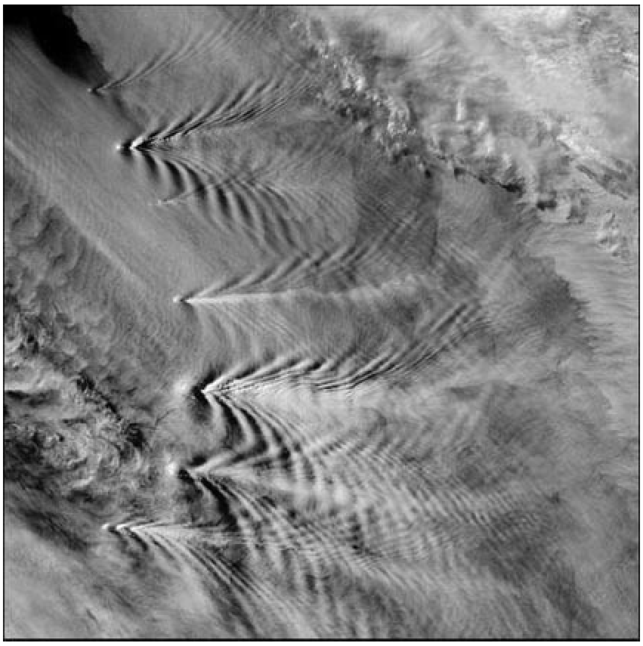

Figure 2. Atmospheric gravity waves downstream from the South Sandwich Islands in the southern Atlantic (58° S, 36° W) captured by the Aqua satellite (02.11.2004, 20:15 UTC) (UTC is the Universal Time Coordinate (Greenwich)). The area of the largest island is 110 km^2, and the maximum altitude over sea level is 1372 m. (Photo from NASA's GSFC, site http://earthobservatory.nasa.gov/IOTD/view.php?id = 651.)

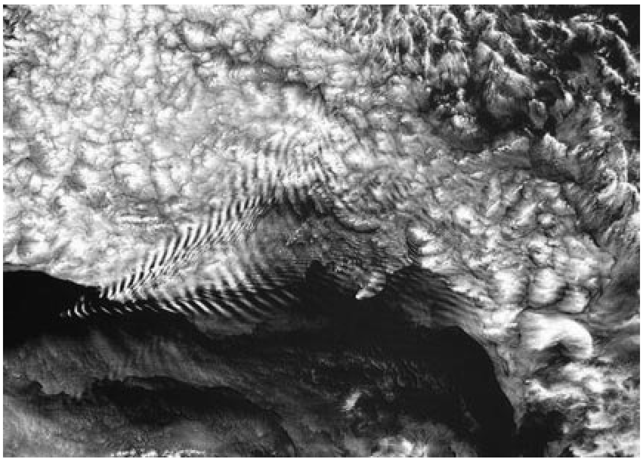

Figure 3. Atmospheric gravity waves downstream from Amsterdam island in the Indian Ocean captured by the Terra satellite (19.12.2005, 04:50 UTC). (Photo from NASA's GSFC, site http://earthobservatory.nasa.gov/IOTD/view. php?id = 4174.)

analysis is carried out by the stationary phase method [5, 6]. This approach offers a reasonably complete description of wave fields, but may prove too cumbersome if the interest lies solely in the phase structure of wave perturbations. Additionally, it makes difficult to account for the inhomogeneity and unsteadiness of the medium.

In this article, we suggest a simple analytical approach to the description of the phase structure of wave perturbations, which does not resort to solving wave equations, but hinges solely on the wave dispersion law. Motivated by methodological reasons, we consider two variants of the approach. The first one, suited for homogeneous and steady media, rests on the general reasoning about the Fourier transform of wave fields and adoption of the stationary phase method. The other, more general, variant, valid for weakly inhomogeneous and unsteady media, leans upon the short-wave (eikonal) approximation and Hamilton's equations for the description of wave packet dynamics [5–7].

The approach proposed is illustrated by the example of gravity-capillary waves and by solutions to two important geophysical problems — the description of a phase pattern of lee waves in planetary atmospheres and the structure of wave wakes in the ocean behind a moving tropical cyclone (hurricane). Extensive literature dealing with these topics can be found, for example, in Refs [8, 9] and [10, 11], respectively.

2. Equations of phase surfaces. The stationary phase method

Consider a stagnant homogeneous medium perturbed by a local source moving with a constant velocity $\mathbf{U}$. It is assumed that the medium supports waves with a given dispersion law

$$\omega = \Omega(\mathbf{k})\,, \tag{1}$$

where ω is the cyclic frequency, and $\mathbf{k}$ is the wave vector. In general, the dispersion law is provided implicitly as $B(\omega, \mathbf{k}) = 0$. This relationship, as Eqn (1), is in essence derived from the solvability condition of homogeneous wave equations for solutions in the form of plane waves, $A \exp{(\mathrm{i}\mathbf{kr} - \omega t)}$.

It is convenient to switch to the reference frame connected to the source which is assumed to be at the coordinate origin. In this reference frame, the medium moves with the velocity $-\mathbf{U}$, and the dispersion relation (1) takes the form [7]

$$\omega = \Omega(\mathbf{k}) - \mathbf{kU}\,. \tag{2}$$

For steady sources, which we will solely consider here, $\omega = 0$, and Eqn (2) yields the condition defining wave vectors that form the phase pattern (the radiation condition):

$$\Omega(\mathbf{k}) = \mathbf{kU}\,, \tag{3}$$

or, in a more general form, $B(\mathbf{kU}, \mathbf{k}) = 0$. The set of wave vectors $\mathbf{k}$ satisfying condition (3) spans a certain surface in the wave vector space (a line in two dimensions), which can be described parametrically as $\mathbf{k} = \mathbf{k}_0(\alpha, \beta)$, where α and β are the parameters. Their role can be played by any two components k_1 and k_2 of the wave vector, the values of which define the third component through Eqn (3). For other parameterizations, we will require that a natural condition is satisfied, namely that the Jacobian of the coordinate transform from (k_1, k_2) to (α, β) be nonzero.

We derive equations for the phase surfaces based on the following considerations. If we seek a solution to the wave equation with a steady source as a series expansion in plane waves $\exp{(\mathrm{i}\mathbf{kr})}$, the result can be cast in the form [6]

$$\psi(\mathbf{r}) = \int \exp{(\mathrm{i}\mathbf{kr})}\, \frac{Q(\mathbf{k})}{B(\mathbf{kU}, \mathbf{k})}\, \mathrm{d}\mathbf{k}\,.$$

Here, $\psi(\mathbf{r})$ is some component of the wave field, the function $Q(\mathbf{k})$ is proportional to the Fourier transform of the source, and the denominator features a function that turns to zero for wave vectors that satisfy the radiation condition (3). Rigorous computation of the integral in the presence of such a singularity relies on the causality principle (the perturbations are absent as $t \to -\infty$), and is realized through adding an infinitely small *positive* imaginary part to the quantity $\mathbf{kU}$. The integration can first be carried out over one component using the residue theorem. For integration over the remaining wave vector components, which will be denoted k_1 and k_2, the stationary phase method is applied (for the mathematical details, see Ref. [6]). The condition of stationarity for the phase $S = \mathbf{kr}$ should now be written with due regard to the vector $\mathbf{k}$ as belonging to the surface (3). Making use of the surface (α, β) parameterization, this condition can be presented as

$$\frac{\partial S}{\partial \alpha} = \mathbf{r}\,\frac{\partial \mathbf{k}}{\partial \alpha} = 0\,, \qquad \frac{\partial S}{\partial \beta} = \mathbf{r}\,\frac{\partial \mathbf{k}}{\partial \beta} = 0\,.$$

Since the vectors $\partial\mathbf{k}/\partial\alpha$ and $\partial\mathbf{k}/\partial\beta$ lie in the plane tangent to the surface of wave vectors (3), it follows that the vector $\mathbf{r}$ is perpendicular to this plane, i.e., it is parallel to the vector of the normal to this surface, or, equivalently, to the vector $\partial(\Omega(\mathbf{k}) - \mathbf{kU})/\partial\mathbf{k} = \mathbf{V}_g - \mathbf{U}$, where $\mathbf{V}_g = \partial\Omega/\partial\mathbf{k}$ is the group velocity vector in the stagnant medium.

We can therefore write down that $\mathbf{r} = \lambda(\mathbf{V}_g - \mathbf{U})$. The parameter λ will be found from the condition that $S = \mathbf{kr} = \text{const}$ at the points of the phase surface. This gives $\lambda = S/(\mathbf{kV}_g - \mathbf{kU})$. Inserting this λ into the expression for $\mathbf{r}$ and using Eqn (3), we finally arrive at the parametric equations for the phase surface:

$$\mathbf{r} = \frac{\mathbf{V}_g(\mathbf{k}) - \mathbf{U}}{\mathbf{kV}_g - \Omega(\mathbf{k})}\,S\,, \qquad \mathbf{k} = \mathbf{k}_0(\alpha, \beta)\,. \tag{4}$$

If the dispersion law is given implicitly, the phase surface is described by the equations

$$\mathbf{r} = \frac{S(\partial B/\partial\mathbf{k})}{\mathbf{k}\,\partial B/\partial\mathbf{k}}\,, \qquad \mathbf{k} = \mathbf{k}_0(\alpha, \beta)\,,$$

where $B = B(\mathbf{kU}, \mathbf{k})$. Since the group velocity is given in this case by the expression

$$\mathbf{V}_g = -\frac{(\partial B/\partial\mathbf{k})_\omega}{(\partial B/\partial\omega)_\mathbf{k}}\,,$$

it can be readily shown that these equations are reduced to equations (4). From the conditions of phase stationarity written above and the definition of the envelope [12], it follows that the phase surfaces are the envelopes of the two-parametric family of surfaces $\mathbf{rk}(\alpha, \beta) = S = \text{const}$.

Note that Eqns (4) can be derived in an alternative way, resorting to two conditions [13]: (a) that the phase $S = \mathbf{kr}$ be constant on the phase surfaces (on the crests and troughs of the wave structure), and (b) that the phase S considered as a function of wave vector $\mathbf{k}$ be stationary under the validity of relation (3). Indeed, in agreement with condition (b), the phase surfaces are obtained by solving the problem in conditional extremum $S = \mathbf{kr} + \lambda(\mathbf{kU} - \Omega(\mathbf{k})) \to \text{extr}$, where λ is the Lagrange multiplier. From here, it immediately follows that $\mathbf{r} = \lambda(\mathbf{V}_g(\mathbf{k}) - \mathbf{U})$. Determining λ from condition (a) and substituting back into the last equation, one arrives at equations (4).

We stress the generality of Eqns (4). For a known dispersion relation, finding the phase surface reduces to the choice of convenient parameterization of the wave vector surface (3). There is, however, a case when equations (4) cannot be applied, namely when $\Omega(\mathbf{k})$ is a homogeneous function of $\mathbf{k}$ in the first degree. In this case, the denominator on the right-hand side in Eqns (4) becomes identically zero by virtue of the Euler theorem on homogeneous functions. An important example is furnished by the dispersion relation for sound waves in a moving medium: $\omega = c_s k + \mathbf{kV}$, where c_s is the speed of sound in a medium at rest, $\mathbf{V}$ is the medium velocity, and $k = |\mathbf{k}|$.

We illustrate the utility of equations (4) by applying them to the description of the phase pattern of surface gravity waves on a fluid surface (ship waves). In this case, the dispersion relation is $\Omega = \sqrt{gk}$, and the radiation condition (3) will be written down as $\sqrt{gk} = kU\cos\varphi$, where g is the acceleration due to gravity, and φ is the angle between the vectors $\mathbf{U}$ and $\mathbf{k}$, which will be used as a parameter. From this condition, we have $k = g/(U^2\cos^2\varphi)$. We choose the coordinate system with the x-axis along the vector $\mathbf{U}$, and the y-axis perpendicular to it. Recalling that $\mathbf{V}_g = 0.5\sqrt{g/k^3}\,\mathbf{k}$, equations (4) become

$$x = -\frac{2S}{\sqrt{gk}}\left(\frac{1}{2}\sqrt{\frac{g}{k}}\cos\varphi - U\right), \qquad y = -\frac{S}{k}\sin\varphi$$

in projections on x-, y-axes. Inserting the expression for k into these equations, we arrive at the parametric equations for the phase lines forming the well-known Kelvin wave wedge [3, 4]:

$$\begin{aligned} x &= -Sg^{-1}U^2\cos\varphi(2 - \cos^2\varphi)\,, \\ y &= -Sg^{-1}U^2\cos^2\varphi\sin\varphi\,. \end{aligned} \tag{5}$$

Concerning these equations, one can be reminded of the classical result by Kelvin on the independence of the wave wedge half-angle θ from the source speed. The tangent of this angle is determined as the maximum of the function

$$F(\varphi) = \frac{y}{x} = \frac{\tan\varphi}{2\tan^2\varphi + 1}\,. \tag{6}$$

From the last formula, it follows that $\tan\theta = F_{\max} = 2^{-3/2}$, and, relatively, $\theta = 19°28'$.

Before moving to further generalizations, let us discuss condition (3), which plays a central role in the question considered. This condition implies that the absolute value of the phase velocity $\Omega(\mathbf{k})/k$ of waves emitted by a moving source is smaller than the source velocity. It also sets the angle between the wave vector of emitted waves and the direction of a source motion. In electrodynamics, the first condition offers the possibility of Cherenkov radiation, and the angle (more precisely, its complement to $\pi/2$) is associated with the name of Mach. In statistical physics, an opposite condition is known, yet written in another notation: $U < \varepsilon(p)/p$, where $\varepsilon = \hbar\Omega$ and $p = \hbar k$ are the energy and momentum of elementary excitations (phonons), and U is the fluid velocity relative to a rigid wall. This inequality is the Landau criterion expressing the occurrence condition of superfluidity [14]. If it holds true, the generation of phonons (waves) by a moving source (the wall of a capillary) is prohibited.

3. Method of Hamilton's equations

Consider now a more general case of a weakly inhomogeneous and unsteady medium [13, 15]. These terms, as usual, imply that the characteristic spatial and time scales of medium inhomogeneity are much larger than the characteristic wavelengths and periods of wave perturbations. In this case, the dispersion relation additionally includes the dependence on the coordinates and time:

$$\omega = \Omega(\mathbf{k}, \mathbf{r}, t)\,, \qquad (7)$$

and the frequency and wave vector are defined as

$$\omega = -\frac{\partial S}{\partial t}\,, \qquad \mathbf{k} = \nabla S\,. \qquad (8)$$

Here, S is the wave phase, which, together with the amplitude A, specifies the representation generalizing plane waves: $\psi = A(\mathbf{r}, t) \exp(\mathrm{i}S(\mathbf{r}, t))$.

The equations for the wave phase and amplitude are obtained by making use of the short-wave (eikonal) approximation [5, 7]. For the phase, a first-order equation in partial derivatives is formulated, which, if resolved for $\partial S/\partial t$, can be cast in the Hamilton–Jacobi form. For the dispersion relation (7), the respective equation follows upon substituting Eqn (8) into (7). The system of equations for the characteristics of this equation is Hamiltonian, with the frequency being the Hamiltonian, and the wave vector the momentum. This approach is analogous to the quasiclassical approximation in quantum mechanics [16], and came to be known as the ray, or kinematic, theory in wave theory.

In the reference frame connected with the source, the equation for the phase can be written as

$$\frac{\partial S}{\partial t} + \Omega'(\nabla S, \mathbf{r}, t) - \mathbf{U}\nabla S = 0\,. \qquad (9)$$

This form corresponds to the dispersion equation

$$\omega = \Omega'(\mathbf{k}, \mathbf{r}, t) - \mathbf{kU}\,, \qquad \Omega'(\mathbf{k}, \mathbf{r}, t) = \Omega(\mathbf{k}, \mathbf{r} + \mathbf{U}t, t)\,. \quad (10)$$

The characteristic system of equations, together with the equation for S, is written as follows [5, 7]

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{V}_{\mathrm{g}} - \mathbf{U}\,, \qquad \frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -\frac{\partial \Omega'}{\partial \mathbf{r}}\,, \qquad \mathbf{V}_{\mathrm{g}} = \frac{\partial \Omega'}{\partial \mathbf{k}}\,, \qquad (11)$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \mathbf{k}\,\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} - \omega = \mathbf{kV}_{\mathrm{g}} - \Omega'\,. \qquad (12)$$

The solution to the characteristic system (11) is sought on some time interval (t_0, t). Its initial conditions are formulated by taking into account that the characteristics must emanate from the perturbation source location, which is at the coordinate origin, whereas initial wave vectors must satisfy the radiation condition $\Omega'(\mathbf{k}, 0, t_0) = \mathbf{kU}$ generalizing condition (3) to the case of weakly inhomogeneous and unsteady medium. We assume that it can be represented in the parametric form: $\mathbf{k} = \mathbf{k}_0(\alpha, \beta, t_0)$. In the end, the initial conditions for system (11) are

$$t = t_0: \quad \mathbf{r} = 0\,, \quad \mathbf{k} = \mathbf{k}_0(\alpha, \beta, t_0)\,.$$

The algorithm for finding the phase surface consists in the following. Let us denote the solution of system (11) with the given initial conditions by $\mathbf{r} = \mathbf{r}(t, t_0, \alpha, \beta)$, $\mathbf{k} = \mathbf{k}(t, t_0, \alpha, \beta)$. Substituting these expressions into the right-hand side of Eqn (12) and integrating with the initial condition $S\big|_{t=t_0} = 0$, we obtain the expression for the phase: $S = S(t, t_0, \alpha, \beta)$. Here, we took into account that the initial value of the wave phase can be set to zero, because all characteristics emanate from the same point (a centered wave [5]), and the phase is acquired as the wave propagates. Having fixed some value of phase S, we determine the respective initial time moment $t_0 = t_0(S, t, \alpha, \beta)$ from the previous relationship, which we then substitute into the expression for $\mathbf{r}$. As a result, we find the representation sought for the phase surfaces:

$$\mathbf{r} = \mathbf{r}(t, S, \alpha, \beta)\,.$$

The phase pattern created by the source will be unsteady ('fluctuating') even in a steady weakly inhomogeneous medium with the dispersion law $\omega = \Omega(\mathbf{k}, \mathbf{r})$. An important special case when the wave pattern is steady in such a medium corresponds to the situation when the frequency $\Omega(\mathbf{k}, \mathbf{r})$ is independent of the coordinate in the direction of source motion (a cyclic coordinate). In this case, $\Omega' = \Omega(\mathbf{k}, \mathbf{r})$ in the characteristic system (11), and the component (momentum) of the wave vector related to the cyclic coordinate does not vary with time.

We illustrate this approach with an example of a homogeneous steady medium with $\Omega' = \Omega(\mathbf{k})$. In this case, Eqns (11) and (12) can be integrated in an elementary way:

$$\mathbf{r} = \big(\mathbf{V}_{\mathrm{g}}(\mathbf{k}) - \mathbf{U}\big)(t - t_0)\,, \quad \mathbf{k} = \mathbf{k}_0(\alpha, \beta)\,,$$
$$S = \big(\mathbf{kV}_{\mathrm{g}} - \Omega(\mathbf{k})\big)(t - t_0)\,. \qquad (13)$$

Having expressed the multiplier $t - t_0$ through S from the last relationship of Eqn (13) and inserted it into the first one, we obtain the parametric equations for the phase surface:

$$\mathbf{r} = \frac{\mathbf{V}_{\mathrm{g}}(\mathbf{k}) - \mathbf{U}}{\mathbf{kV}_{\mathrm{g}} - \Omega(\mathbf{k})}\, S\,, \qquad \mathbf{k} = \mathbf{k}_0(\alpha, \beta)\,, \qquad (14)$$

found earlier from a different reasoning. Notice that from the condition $t - t_0 > 0$ it follows that the coefficient of $\mathbf{V}_{\mathrm{g}}(\mathbf{k}) - \mathbf{U}$ in Eqn (14) is always nonnegative.

We also emphasize the following fact. For some dispersion relations, the denominator in Eqn (14) may become zero for some set of $\mathbf{k}$. For these values of $\mathbf{k}$, the rate of phase change on the trajectory equals zero according to Eqn (12). Excluding the case mentioned earlier of an identically zero denominator, one can assume that this set also spans a surface in the $\mathbf{k}$-space. Its intersection with the surface defining the initial conditions (if the intersection exists) will represent a line $\mathbf{k} = \mathbf{k}_0(\alpha(\gamma), \beta(\gamma)) = \mathbf{k}_0(\gamma)$ (γ is the related parameter). When $\mathbf{k} \to \mathbf{k}_0(\gamma)$ and for $S \neq 0$, as follows from Eqn (13), $t - t_0 \to \infty$ and $|\mathbf{r}| \to \infty$. Fragments of the phase surface in this case will be close to a conical surface with the generatrices $\mathbf{r} = [\mathbf{V}_{\mathrm{g}}(\mathbf{k}_0(\gamma)) - \mathbf{U}]\xi$, $\xi \in (0, \infty)$. For the two-dimensional case, they are straight lines traversing the coordinate origin. The phase as a function of $\mathbf{r}$ will have a singularity at these points, the character of which can only be elucidated on the basis of a more elaborate analysis of wave equations. It is noteworthy that for the dispersion law $\Omega = \Omega(|\mathbf{k}|)$ the condition $\mathbf{kV}_{\mathrm{g}} - \Omega(\mathbf{k}) = 0$ implies the coincidence of group and phase velocities at $\mathbf{k} = \mathbf{k}_0(\gamma)$, and, in general, the equality between the group velocity projection onto the direction of $\mathbf{k}$ and the phase velocity. It deserves mention that the validity of this condition relates to the stationarity of the angle φ between the wave vector of the emitted wave and the source velocity as a function of $|\mathbf{k}|$. For the dispersion law $\Omega = \Omega(|\mathbf{k}|)$, it trivially follows from relationship (3), and the general case is treated in the Appendix.

In the next sections, we will present examples of the utilization of the obtained equations in the analysis of some phase patterns. We employ the notation k_x, k_y, and k_z for the wave vector components along the relative axes in geophysical applications.

4. Gravity-capillary waves on a fluid surface

The dispersion relation for such waves is given by [5, 7]

$$\Omega(k) = \sqrt{gk + \delta k^3}\,, \tag{15}$$

where g is the acceleration due to gravity, and δ is the ratio of the surface tension coefficient to the fluid density.

Using the absolute value of wave vector k as a parameter, we write out the radiation condition (3) as

$$\cos\varphi = \frac{1}{U}\sqrt{\frac{g}{k} + \delta k}\,. \tag{16}$$

The interval where k may vary is found from the condition $\cos\varphi \leqslant 1$. Its boundaries are defined as follows:

$$k_{1,2} = \frac{U^2 \pm \sqrt{U^4 - 4g\delta}}{2\delta}\,.$$

This problem contains two characteristic parameters: the wave number $k_* = \sqrt{g/\delta}$, and velocity $U_* = \sqrt{2}\,(\delta g)^{1/4}$. The first one equals the value of k when $\cos\varphi$ reaches a minimum, i.e., when the angle φ is maximum. Qualitatively, one may say that the interval (k_1, k_*) corresponds to the part of the wave pattern formed largely by gravitational forces, and (k_*, k_2) to the part formed by surface tension forces. The parameter U_* equals the lower bound on the source speed, beginning from which the moving source excites waves ($k_{1,2}$ are real-valued).

For the dispersion relation at hand, there is a peculiarity alluded to above — the group velocity $V_{\rm g} = (g + 3\delta k^2)/2\Omega$ equals the phase velocity $V_{\rm ph} = \Omega/k$ at $k = k_*$, which can easily be established by solving the respective equation or paying attention to the fact that the angle φ is stationary for this value of k Therefore, the phase pattern will involve portions close to rectilinear.

We write down the expressions for phase lines in a coordinate system with the x-axis along $\mathbf{U}$, and the y-axis perpendicular to it. Projecting equations (14) onto (x, y)-axes and performing simple manipulations with account for Eqn (16), we obtain

$$x = \frac{2SU}{\delta k^2 - g}\sqrt{\frac{g}{k} + \delta k}\left[\frac{1}{2U^2}\left(\frac{g}{k} + 3\delta k\right) - 1\right],$$
$$y = \frac{S}{\delta k^2 - g}\left(\frac{g}{k} + 3\delta k\right)\sqrt{1 - \frac{1}{U^2}\left(\frac{g}{k} + \delta k\right)}\,. \tag{17}$$

Hence, it immediately follows that, as $k \to \sqrt{g/\delta}$ $|x| \to \infty$, $|y| \to \infty$. In this case, one has

$$\frac{y}{x} \to \frac{U_*}{\sqrt{U^2 - U_*^2}}\,. \tag{18}$$

The right-hand side of the last formula can also be obtained from the following considerations. The minimum value of $\cos\varphi$ corresponding to the maximum angle φ is reached at $k = k_*$ and equals U_*/U, as follows from Eqn (16). The slope angle of straight lines approached by the phase lines complements φ to $\pi/2$, and, consequently, $y/x \to \cot\varphi = U_*/\sqrt{U^2 - U_*^2}$.

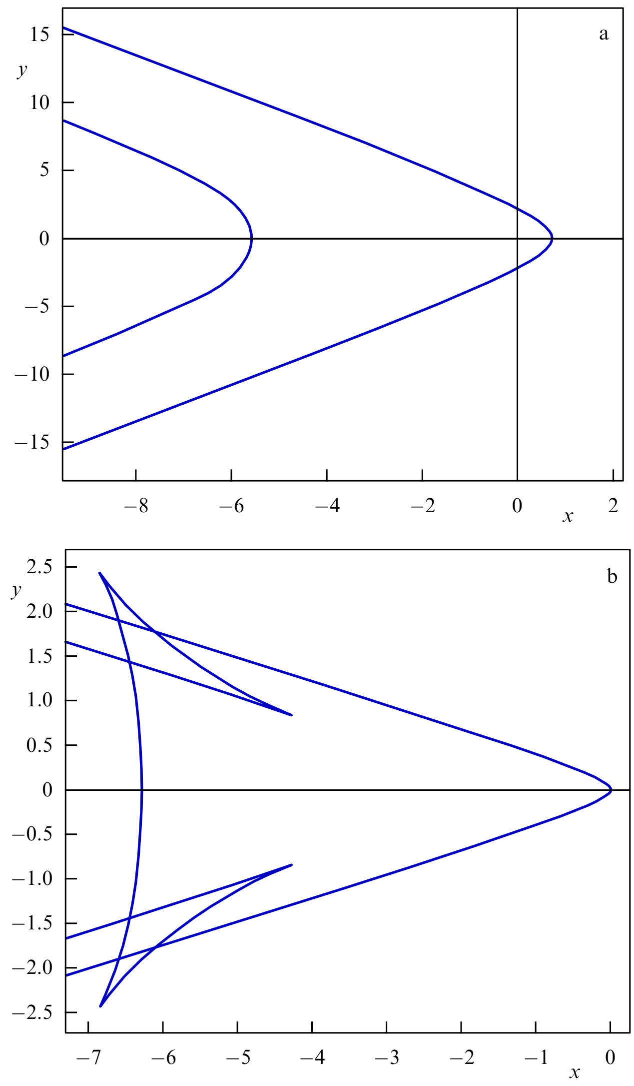

Figure 4. Phase patterns of gravity-capillary waves for $\varepsilon^2 = 0.1$ (a), and $\varepsilon^2 = 0.001$ (b).

One may pass to limits corresponding to gravity and capillary waves in expressions (17) by formally proceeding to $\delta \to 0$ for gravity waves, and $g \to 0$ for capillary waves. Accordingly, we obtain for gravity waves:

$$x = \frac{2SU}{\sqrt{gk}}\left(\frac{g}{2U^2 k} - 1\right), \qquad y = -\frac{S}{k}\sqrt{1 - \frac{g}{U^2 k}}, \tag{19}$$

and for capillary waves

$$x = \frac{2SU}{\sqrt{\delta k^3}}\left(\frac{3\delta k}{2U^2} - 1\right), \qquad y = \frac{3S}{k}\sqrt{1 - \frac{\delta k}{U^2}}\,. \tag{20}$$

In order to switch to the traditional representation with the angle φ as a parameter, we express k as a function of φ from Eqn (16). Doing so, we obtain $k = g/(U^2\cos^2\varphi)$ for gravity waves, and $k = U^2\cos^2\varphi/\delta$ for capillary waves. Substituting these dependences into Eqns (19) and (20), we arrive at well-known expressions [5].

Figure 4 plots phase lines of the waves considered. The quantity $\lambda = U^2/g$ is selected as the length scale, and g/U is selected as the frequency scale. For a given λ, the relative role of surface tension is determined by the dimensionless parameter $\varepsilon = (k_* \lambda)^{-1} = 0.5 U_*^2/U^2$. The results pertain to $\varepsilon^2 = 0.1$ (Fig. 4a), $\varepsilon^2 = 0.001$ (Fig. 4b), and the fixed value of $S = 2\pi$. For $\varepsilon \sim 0.1$, the phase structure is formed in equal degree by capillary (the forward branch in the figure) and gravity (the rear branch) forces. For smaller values of ε, the waves assume a character of gravity waves, and the gravity branch attains a characteristic cusp point of return. One more return point occurs additionally, this time linked to the existence of asymptotes. The second point approaches the coordinate origin as $\varepsilon \to 0$, and the asymptotes approach the abscissa.

We touch on the question of whether it is possible to observe straight-line segments in the phase lines of gravity-capillary waves. One of basic conditions for them to be observed is that the interval of wave numbers of generated waves by far exceed k_*. Since the amplitudes of excited waves are proportional to the Fourier components of the source function [6], decaying for $k > a^{-1}$ (where a is the characteristic source size), the condition $a < k_*^{-1}$ should be valid. For water, one finds $k_*^{-1} \approx 0.27$ cm, so that observing this effect is only possible under specially designed conditions.

5. Wave wake behind a moving hurricane

Tangent wind stress exerted by a moving hurricane forms a wave tail or wake type structure in the ocean [10, 11]. The experimental discovery of this structure was one of impressive achievements in modern oceanography. In the existing numerical models of wave wakes [11, 17], the ocean is, as a rule, conceived of as a system composed of an upper mixed layer with density ρ_1 and thickness H_1, and a lower layer with density $\rho_2 > \rho_1$ and thickness H_2. Typically, H_2 is the thickness of the ocean below the main thermocline. In the presence of background rotation, this two-layer system allows only two classes of wave motions: the barotropic ($n = 0$) and baroclinic ($n = 1$) modes of inertia-gravity waves (IGWs), related, respectively, to vibrations of the free upper surface and the interface between the layers. In the long-wave approximation (the wavelength is much larger than the layer thickness) and under the condition $\delta = (\rho_2 - \rho_1)/\rho_1 \ll 1$, the dispersion relations for the modes are written as [11, 18]

$$\omega^2 = \Omega_n^2(\mathbf{k}) = f^2 + c_n^2(k_x^2 + k_y^2), \quad n = 0, 1. \tag{21}$$

Here, f is the Coriolis parameter (twice the projection of the planet's angular velocity onto the local vertical), and c_n are the phase speeds of the modes in the absence of rotation [18]:

$$c_0 = \sqrt{gH}, \qquad c_1 = \sqrt{\frac{g\delta H_1 H_2}{H}}, \qquad H = H_1 + H_2.$$

For the characteristic oceanic values of $H_1 = 100-150$ m, $H_2 = 4000$ m, and $\delta = 3 \times 10^{-3}$, these speeds amount to $c_0 = 200$ m s^{-1} and $c_1 = 2-3$ m s^{-1}.

We consider a source (hurricane) moving along the x-axis at a speed $\mathbf{U} = (U, 0)$. For the two-dimensional wave vector $\mathbf{k} = (k_x, k_y)$, the radiation condition (3) is transformed to

$$(U^2 - c_n^2)\, k_x^2 - c_n^2 k_y^2 = f^2. \tag{22}$$

The condition of wave wake formation follows directly from Eqn (22):

$$M^2 = \frac{U^2}{c_n^2} > 1, \tag{23}$$

where M is an analog of the Mach number. If this condition is fulfilled, equation (22) describes a hyperbola in the plane of wave numbers. An important point is that the characteristic velocity of hurricane motion, $U = 5$ m s^{-1}, exceeds the speed of the baroclinic mode, but stays much lower than the speed of the barotropic mode: $c_1 < U \ll c_0$. Relatedly, the wave wake behind a moving hurricane is formed solely by the baroclinic mode. We assume further that $n = 1$.

For the two-dimensional group velocity vector, Eqn (21) leads to $\mathbf{V}_g = c_n^2 \mathbf{k}/\omega$. Taking into account this expression and relation (21), general equations (4) for the phase lines acquire the form

$$x = (U^2 - c_n^2)\,\frac{S}{f^2}\,k_x, \qquad y = -c_n^2\,\frac{S}{f^2}\,k_y. \tag{24}$$

In the last equations, the wave numbers k_x and k_y are linked through condition (22), and if $U < 0$, then $k_x < 0$ and $S < 0$. Using parametric hyperbola equations (22) and inserting them into Eqn (24), we get the equations for phase lines. An even simpler approach is to express k_x and k_y through x and y from Eqn (24) and substitute them into Eqn (22). As a result, we obtain the equation

$$\frac{x^2}{U^2 - c_n^2} - \frac{y^2}{c_n^2} = \frac{S^2}{f^2} \tag{25}$$

describing a one-parametric family of hyperbolas with the asymptotes $y = \pm (c_n/\sqrt{U^2 - c_n^2})x$. These latter define a wedge separating the region with wave perturbations from the stagnant fluid (Fig. 5a). The tangent of the angle between the wedge side and source velocity (wedge spreading half-angle) is given by

$$\tan\theta = \frac{c_n}{\sqrt{U^2 - c_n^2}} = \frac{1}{\sqrt{M^2 - 1}}, \tag{26}$$

and, accordingly, $\sin\theta = 1/M$. It should be noted that expression (26), which coincides with the expression for the Mach cone spreading half-angle in gas dynamics [5], was first obtained in Ref. [19] based on a direct solution of the linearized system of hydrodynamical equations. We gave above a purely kinematic derivation.

The results presented here agree well with the data of numerical modeling of the wave wake behind a moving hurricane, which were carried out in the framework of full nonlinear set of equations of fluid dynamics [17]. As an example, Fig. 6 plots the isolines of the velocity field in the upper and lower layers after a lapse of six days following the hurricane started to move, as obtained in Ref. [17].

6. Lee waves

Orographic obstacles flown past a stream of continuous medium move relative to the latter with the speed of the stream. To find the pattern of accompanying lee waves, we turn to the model of a rotating continuously stratified atmosphere with a constant buoyancy (or Brunt–Väisälä)

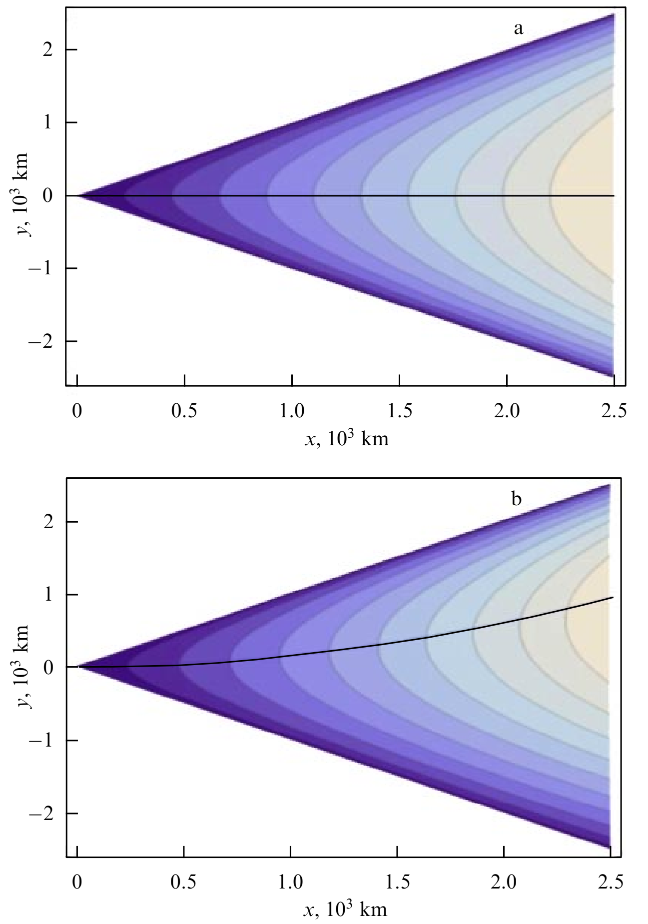

Figure 5. (Color, in the online version.) Phase lines for a constant Coriolis parameter (a), and in the presence of the β-effect (b). The lines correspond to the wake axis position: the straight line $y = 0$ (a), and the parabola (56) (b).

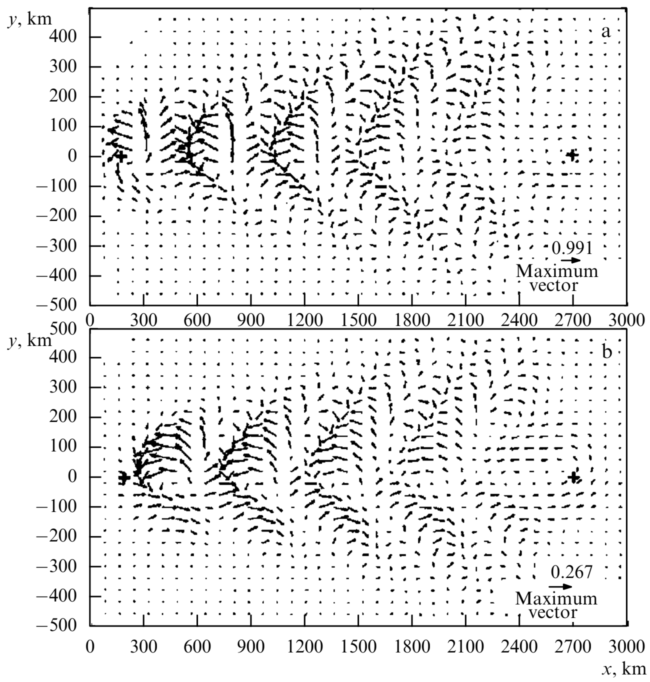

Figure 6. Wave wake behind a hurricane in a numerical model of Ref. [17].

frequency N. We assume the atmosphere is confined between two horizontal boundaries, $z = 0, H$, which correspond, respectively, to the underlying surface and the tropopause. In this case, there is a discrete (countable) set of IGW baroclinic modes with the dispersion relations [1, 2]

$$\omega^2 = \Omega_n^2(\mathbf{k}) = \frac{f^2 m^2 + N^2 k^2}{m^2 + k^2}\,, \qquad k^2 = k_x^2 + k_y^2\,, \qquad (27)$$

where $\mathbf{k} = (k_x, k_y)$, and $k_z \equiv m = \pi n/H$, and $n = 1, 2, \ldots$ is the vertical wave number. For Earth's atmosphere at midlatitudes, the characteristic values of the buoyancy frequency N and inertial frequency f (the Coriolis parameter) differ by two orders of magnitude: $N = 10^{-2}$ s^{-1} and $f = 10^{-4}$ s^{-1}.

In the long-wave approximation, $k^2 \ll m^2$ and, subject to the condition $N^2 \gg f^2$, Eqn (27) leads to dispersion relations for so-called Poincaré waves [1]:

$$\omega^2 = f^2 + c_n^2 (k_x^2 + k_y^2)\,, \qquad c_n = \frac{N}{m} = \frac{NH}{\pi n}\,, \qquad (28)$$

analogous to relationships (21). Hence, it immediately follows that, if condition (23) is valid (for a fixed mode), a wedge-like wake forms downstream from the obstacle with a spreading half-angle (26) and phase lines (25). This result is apparently valid only for obstacles with a horizontal size much larger than the thickness of the atmosphere. We will explore further the structure of phase lines, abandoning the long-wave approximation and turning to the exact dispersion relation (27). This will enable us to generalize the results to obstacles of modest horizontal scale (small-sized islands; see Figs 2 and 3). Note that for $H = 10$ km the phase speed of the first baroclinic mode (28) in the absence of rotation makes up $c_1 = 20$ m s^{-1}. Since, as n increases, the velocity $c_n \to 0$, condition (23) will be valid always beginning from some n.

For a flow directed along the x-axis and wave modes (27), radiation condition (3) can be written out as

$$U^2 k_x^2 = \frac{f^2 m^2 + N^2 k^2}{m^2 + k^2}\,. \qquad (29)$$

Taking into account the explicit expression for the two-dimensional (horizontal) vector of group velocity, namely

$$\mathbf{V}_{\mathrm{g}} = \frac{(N^2 - f^2)\, m^2}{\omega \kappa^4}\,\mathbf{k}\,, \qquad \kappa^2 = m^2 + k^2\,,$$

equations (4) for the phase lines become

$$x = \frac{U^2 \kappa^4 - (N^2 - f^2)\, m^2}{(N^2 - f^2)\, k^4 + f^2 \kappa^4}\, k_x S\,, \qquad y = -\frac{(N^2 - f^2)\, m^2}{(N^2 - f^2)\, k^4 + f^2 \kappa^4}\, k_y S\,. \qquad (30)$$

Analyzing Eqns (29) and (30), it is convenient to utilize dimensionless variables normalized to a fixed vertical wave number: $\bar{k}_x = k_x/m$, $\bar{k}_y = k_y/m$, $\bar{x} = mx$, and $\bar{y} = my$. Additionally, we assume $U > 0$ and adopt the representation of dimensionless wave vector in polar coordinates: $\bar{k}_x = -\bar{k}\cos\varphi$ and $\bar{k}_y = \bar{k}\sin\varphi$. Condition (29), in this case, can be rewritten as

$$\cos^2\varphi = \frac{1}{M^2}\,\frac{\varepsilon^2 + \alpha}{\alpha(1 + \alpha)}\,, \qquad \alpha = \bar{k}^2 = \bar{k}_x^2 + \bar{k}_y^2\,, \qquad (31)$$

where $M = U/c_n$, $\varepsilon^2 = f^2/N^2$, and α is the modulus squared of the dimensionless wave vector, which is used further as a parameter. From Eqn (31) and the condition $\cos^2\varphi \leqslant 1$, it follows that the system of lee waves is formed by waves with $\alpha \geqslant \alpha_*$, where

$$\alpha_* = \frac{1 - M^2 + \sqrt{(1-M^2)^2 + 4\varepsilon^2 M^2}}{2M^2} \,. \tag{32}$$

Expressing $\bar{k}_x$ and $\bar{k}_y$ in terms of α with account for Eqn (31) and substituting them into Eqn (30), we obtain in dimensionless variables a one-parametric representation for the phase lines with the parameter α:

$$\bar{x} = \frac{M^2(1+\alpha)^2 - \eta}{\varepsilon^2(1+2\alpha)+\alpha^2}\sqrt{\frac{\varepsilon^2+\alpha}{1+\alpha}}\,\frac{S}{M}\,, \quad \eta = 1 - \varepsilon^2\,,$$
$$\bar{y} = \pm\frac{\eta}{\varepsilon^2(1+2\alpha)+\alpha^2}\sqrt{\frac{M^2\alpha^2 + (M^2-1)\alpha - \varepsilon^2}{1+\alpha}}\,\frac{S}{M}\,. \tag{33}$$

For $U > 0$, the wave number $k_x < 0$ and the value of the phase is taken by the absolute value in formulas (33). If the parameter α is expressed in terms of φ from Eqn (31), one can get an alternative representation of phase lines with the parameter φ. The respective representation is rather cumbersome. The structure of phase lines (33) will be explored further in the characteristic case of $\varepsilon \ll 1$.

We begin the analysis of curves (33) by finding the wedge spreading half-angle θ confining wave perturbations. In polar coordinates $\bar{x} = r\cos\psi$, $\bar{y} = r\sin\psi$, Eqn (33) gives for the tangent of the angle ψ (the first quarter) the following expression

$$\tan\psi = \frac{y}{x} = \frac{\eta}{M^2(1+\alpha)^2 - \eta}\sqrt{\frac{M^2\alpha^2 + (M^2-1)\alpha - \varepsilon^2}{1+\alpha}} \equiv F(\alpha)\,. \tag{34}$$

The nonnegative function $F(\alpha)$ in Eqn (34) is defined on the semiaxis $\alpha > \alpha_*$, with $F(\alpha_*) = 0$ and $F(\alpha) \to 0$ as $\alpha \to \infty$. This implies the existence of a point α_{max} where the function attains its maximum F_{max}. The respective value, dependent on M^2 and ε, equals the tangent of the wedge spreading half-angle: $\tan\theta = F_{max}(M^2, \varepsilon)$. For $\varepsilon \ll 1$, the asymptotic form is given by

$$\tan\theta \approx \frac{1}{\sqrt{M^2-1}}\,, \quad M^2 - 1 \gg O(\varepsilon)\,,$$
$$\tan\theta \approx \frac{\sqrt[4]{3}}{3\sqrt{2}}\,\frac{1}{\sqrt{\varepsilon}}\,, \quad M^2 = 1\,, \quad \tan\theta \approx \frac{3\sqrt{3}}{16}\,M^2\,, \quad M^2 \ll 1\,. \tag{35}$$

As can be seen, the wave pattern is excited by a flow past an obstacle for arbitrary Mach numbers. The maximum value of the wedge spreading half-angle is attained at $M = 1$. The respective value of $\theta \approx \arctan(0.31\sqrt{N/f})$ at $N/f = 10^2$ makes $\theta \approx 72°$. The numerically computed dependence of $\tan\theta$ on the Mach number squared for $\varepsilon = 0.01$ is presented in Fig. 7. This dependence can be utilized when estimating the velocity of an impinging stream based on field observations. For example, in the upper part of Fig. 2, one finds $\theta \approx 26°$ and $\tan\theta = 0.49$. Assuming that the pattern is formed by the first mode with the phase speed $c_1 = 20$ m s^{-1} [6] and $M < 1$, we find $M^2 \approx 0.65$ from the plot and, accordingly, $U = 16$ m s^{-1}.

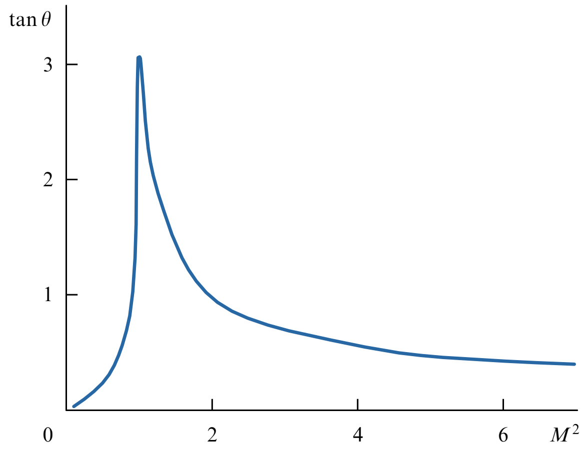

Figure 7. Dependence of the tangent of the wave wedge spreading half-angle on the Mach number squared.

We turn now to the structure of phase lines $S = \text{const}$. For definiteness, consider the case of $M^2 - 1 \gg O(\varepsilon)$. Here, $\alpha_* \approx \varepsilon^2/(M^2-1)$ and $\alpha_{max} \approx \varepsilon/2$. From Eqn (33), it follows that, on varying α from α_* to α_{max}, the coordinate $\bar{x}$ increases from $\bar{x}_* \approx \varepsilon^{-1}\sqrt{M^2-1}\,S$ to $\bar{x}_{max} \approx M^{-1}\varepsilon^{-3/2}(\sqrt{2}/3) \times (M^2-1)S$. Accordingly, $\bar{y}$ increases from zero to $\bar{y}_{max} \approx \bar{x}_{max}/\sqrt{M^2-1}$ in the first quarter. This range of α variation is associated with the curve segment AB in Fig. 8. Further, with the growth of α from α_{max} to infinity, the coordinates $\bar{x}$ and $\bar{y}$ decrease, so that $\bar{y} \to 0$, and $\bar{x} \to \bar{x}_{min} \approx MS$. These values of α are associated with the curve segment BC. Thus, the phase curve in the first quarter consists of two branches with turning points B and C. Because of the symmetry of the curve with respect to the horizontal axis, the lower half-plane contains similar branches. The curves for other values of M look similar.

The pattern of phase lines in Fig. 8 is qualitatively similar to that of ship waves behind a moving ship [3, 4]. Here, oblique and transverse waves, which correspond to different branches of the phase curve, are present, too. The main difference consists in the fact that, for the ship waves, the wedge spreading half-angle does not depend on the source velocity. This dependence is rather strong for lee waves in the stratified rotating atmosphere of Earth. Lee wave patters were also discovered in the atmospheres of other planets. A photograph of a lee wave pattern in the atmosphere of Mars, formed by a flow past a crater, is presented in monograph [20].

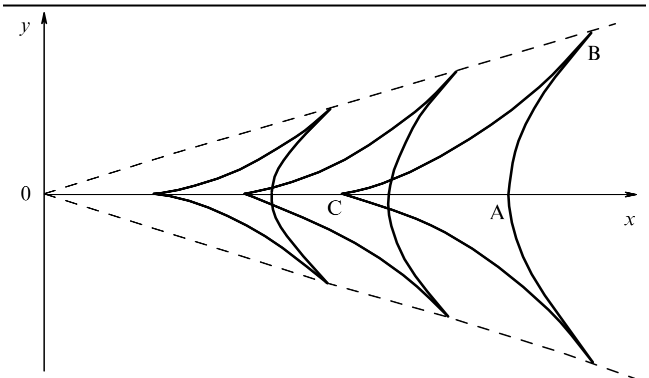

Figure 8. Structure of phase lines for lee waves in the presence of background rotation.

7. Flow past a moving source in the absence of rotation

The case with no background rotation, $\varepsilon = 0$, acquires independent interest as applied to the conditions of laboratory experiments or to atmospheric motions in the proximity of the equator. In this case, the behavior of phase curves changes qualitatively on crossing the value of $M = 1$. Thus, there is no uniform limiting process as $\varepsilon \to 0$. Notice that wave motions in stratified fluid are referred to as internal gravity waves in the absence of rotation.

Thus, radiation condition (31) in the absence of rotation takes the form

$$\cos^2 \varphi = \frac{1}{M^2} \frac{1}{1+\alpha} \,. \qquad (36)$$

Equation (36) constrains the absolute value squared of the wave vector: $\alpha \geqslant \alpha_*$, where $\alpha_* = 0$ for $M^2 \geqslant 1$ and $\alpha_* = M^{-2} - 1$ for $M^2 < 1$. The parametric equations for phase curves (33) can be written out as

$$\bar{x} = \frac{M^2(1+\alpha)^2 - 1}{\alpha\sqrt{\alpha(1+\alpha)}} \frac{S}{M} \,, \qquad \bar{y} = \pm \frac{1}{\alpha} \sqrt{\frac{M^2(1+\alpha)-1}{\alpha(1+\alpha)}} \frac{S}{M} \,. \qquad (37)$$

Expressing α through φ in Eqn (36), one can obtain an alternative representation for the phase lines parameterized by the angle φ:

$$\bar{x} = \frac{M(1 - M^2\cos^4\varphi)}{(1 - M^2\cos^2\varphi)^{3/2}} S \,, \quad \bar{y} = \frac{M^3 \sin\varphi \cos^3\varphi}{(1 - M^2\cos^2\varphi)^{3/2}} S \,. \quad (38)$$

In equations (38), one has $|\varphi| \leqslant \varphi_*$, where $\varphi_* = \pi/2$ for $M^2 \leqslant 1$, and $\varphi_* = \arccos M^{-1}$ for $M^2 > 1$. Although both the parameterizations are equivalent, we will use formulas (37) below.

With account for Eqn (37), the tangent of the wedge spreading half-angle θ is the maximum of the function

$$\tan\psi = \frac{y}{x} = \frac{\sqrt{M^2(1+\alpha)-1}}{M^2(1+\alpha)^2 - 1} \equiv F(\alpha) \qquad (39)$$

on the semiaxis $\alpha > \alpha_*$. Equation (39) leads to the exact result

$$\tan\theta = F_{\max} = \begin{cases} \dfrac{1}{\sqrt{M^2-1}} \,, & M^2 \geqslant 1 \,, \\[2ex] \dfrac{3\sqrt{3}\, M^2 \sqrt{\sqrt{4-3M^2}-1}}{\left(2+\sqrt{4-3M^2}\right)^2 - 9M^2} \,, & M^2 < 1 \,. \end{cases} \qquad (40)$$

According to the last formulas, $\tan\theta \approx 3\sqrt{3}M^2/16$ if $M^2 \ll 1$, and $\tan\theta \approx M^{-1}$ if $M^2 \gg 1$. For $M^2 = 1$, one finds $\tan\theta = \infty$ ($\theta = \pi/2$), and the phase curves fill the entire half-plane downstream from the obstacle. This draws an important distinction from the case with background rotation, when the tangent of the wedge spreading half-angle is always finite at $M^2 = 1$. A plot of the dependence of $\tan\theta$ on the Mach number squared is presented in Fig. 9.

If parameterization (38) is used, the tangent of the wedge spreading half-angle is defined as the maximum of the function

$$\tan\psi = \frac{M^2 \tan\varphi}{(1+\tan^2\varphi)^2 - M^2} \equiv \Phi(\varphi)$$

on the interval $|\varphi| \leqslant \varphi_*$. Finding the respective extremum also leads to expression (40).

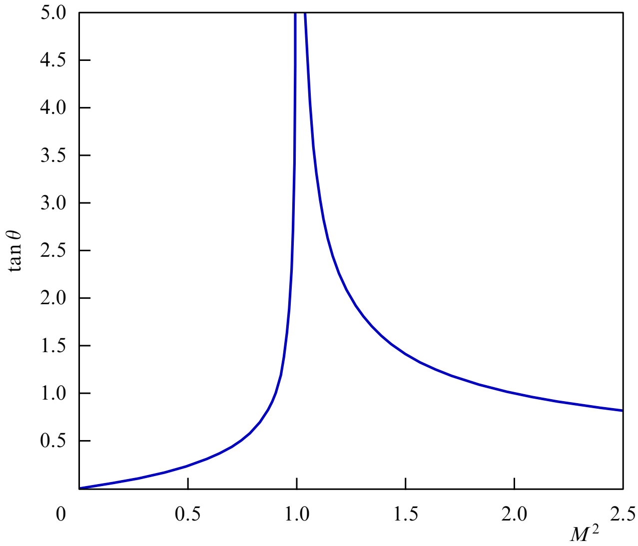

Figure 9. Dependence of the tangent of the wave wedge spreading half-angle on the Mach number squared in the absence of background rotation.

We explore here the behavior of phase curves (37). Let us begin with the case of $M^2 > 1$. As in the case with rotation, for $\alpha \to \infty$ the phase curve is similar to a semicubical parabola with the vertex at the point $\bar{y} = 0$, $\bar{x} = MS$. For $\alpha \to \alpha_* = 0$, each phase curve approaches the straight lines $\bar{y} = \pm\bar{x}/\sqrt{M^2-1}$ (the wedge sides). The behavior of the phase curves, in this case, is illustrated in Fig. 10b.

At $M^2 = 1$, phase curves (37) fill the entire half-plane $x > 0$ downstream from the obstacle. For $\alpha \to \infty$ and $\alpha \to 0$, the asymptotic form of each phase curve is given by the

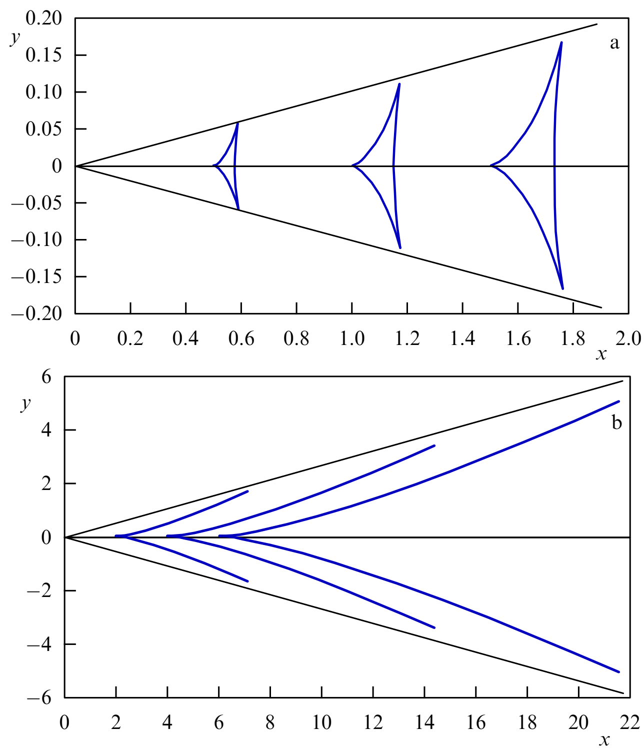

Figure 10. Structure of phase lines for lee waves in the absence of rotation for $M < 1$ (a), and $M > 1$ (b).

equations $8(\bar{y}/S)^2 = (\bar{x}/S - 1)^3$ and $4\bar{y}/S = (\bar{x}/S)^2$, which correspond, respectively, to semicubic and ordinary parabolas.

For $M^2 < 1$, the phase curves are closed and exhibit a structure characteristic of the case with rotation (that of a 'cusp'). On varying α from $\alpha_* = M^{-2} - 1$ to $\alpha_{\max} = [(2 + \sqrt{4 - 3M^2})/3M^2] - 1$, we obtain from Eqn (37) a branch of the phase curve which corresponds to transverse waves. Further, for α varied from $\alpha_{\max}$ to infinity, we get a side branch (Fig. 10a). Thus, the behavior of phase curves may be qualitatively different, depending on the values of M^2.

8. Asymmetry of the wave wake behind a hurricane in the presence of the β-effect

For a medium with fixed parameters, the wave wake studied in Section 5 is symmetric with respect to the source (hurricane) direction (Fig. 5a). Rather unexpectedly, field observations [21, 22] revealed asymmetry in the wake structure, consisting in the displacement of the wake axis to the right of the hurricane trajectory. A qualitative explanation proposed in these studies attributed the asymmetry to that in the wind tangent stress field created by the moving hurricane. In this section, we show that yet another source of asymmetry can be the so-called β-effect [1, 18], namely the dependence of the Coriolis parameter on the latitude. The characteristic horizontal scale of a wave wake (about 1000 km) is also the scale at which this dependence becomes significant. An example considered below illustrates how the general theory can be applied to find the structure of phase surfaces in a steady inhomogeneous medium.

As in Section 5, let us consider a source moving along the x-axis with the velocity $\mathbf{V} = (U, 0)$ and creating a wave wake in a two-layer system modeling the ocean. This wake is formed by the first baroclinic mode with the phase velocity $c = c_1$ and the dispersion relation

$$\omega^2 = \Omega^2(\mathbf{k}, \mathbf{r}) = f^2(y) + c^2(k_x^2 + k_y^2)\,. \tag{41}$$

In the framework of the traditional β-plane approximation, the Coriolis parameter in the vicinity of latitude ϑ_0 is expanded as $f = f(y) = f_0 + \beta y$, where $f_0 = 2\Omega_* \sin\vartheta_0$ is the constant central value, and $\beta = (2\Omega_*/R)\cos\vartheta_0$, with Ω_* and R, respectively, being Earth's angular velocity and radius [1]. This approximation is valid in the latitude band of $|y| \ll L_* = f_0/\beta = R\tan\vartheta_0$. Notice that at the latitude $\vartheta_0 = 30°$ we have $f_0 = 0.8\times10^{-4}$ s and $\beta = 2\times10^{-11}$ m^{-1} s^{-1}, so that the length scale is $L_* = f_0/\beta = 4000$ km.

For waves with dispersion relationship (41), the radiation condition $\Omega(\mathbf{k}, 0) = \mathbf{kU}$ is transformed to

$$U^2k_x^2 = f_0^2 + c^2(k_x^2 + k_y^2)\,. \tag{42}$$

In the plane of wave numbers, it describes a hyperbola (22) with real and imaginary semiaxes $a = f_0/\sqrt{U^2 - c^2}$ and $b = f_0/c$. Here, we assume the condition of wake existence, $M^2 > 1$, to be fulfilled. If $U < 0$ (the source moves westward), the wave number $k_x < 0$ and the parametric equations of the hyperbola (the left branch) are $k_x = -a\cosh\alpha$, $k_y = b\sinh\alpha$, where α is a parameter.

Since dispersion relation (41) does not depend on the coordinate x along the source trajectory, the structure of phase lines forming a stationary wake can be found from the solutions of equations (11) and (12), where $\Omega' = \Omega(\mathbf{k}, \mathbf{r})$. With account for the explicit expression for the group velocity, these equations can be written out as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{c^2 - U^2}{U}\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{c^2}{U}\frac{k_y}{k_x}\,, \quad \frac{\mathrm{d}k_x}{\mathrm{d}t} = 0\,, \quad \frac{\mathrm{d}k_y}{\mathrm{d}t} = -\frac{\beta(f_0 + \beta y)}{Uk_x}\,, \tag{43}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\frac{(f_0 + \beta y)^2}{Uk_x}\,. \tag{44}$$

Equations (43) and (44) are augmented by the initial conditions

$$t = 0: \quad x = y = S = 0\,, \quad k_x = -a\cosh\alpha\,, \quad k_y = b\sinh\alpha \tag{45}$$

(without loss of generality, one may take $t_0 = 0$ for steady media). The characteristic equations (43) form a closed system to be resolved for the coordinates and wave vector components.

System (43) immediately gives $k_x = -a\cosh\alpha = \text{const}$ (momentum preservation for the cyclic coordinate) and

$$x = \frac{(c^2 - U^2)\,t}{U}\,. \tag{46}$$

The second and fourth equations in set (43) lead to a second-order linear equation for the meridional coordinate y:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \lambda^2 y + \frac{f_0\lambda^2}{\beta} = 0\,, \qquad \lambda = \frac{c\beta}{|Uk_x|}\,, \tag{47}$$

with the initial conditions

$$t = 0: \quad y = 0\,, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{c^2}{U}\frac{b}{a}\tanh\alpha\,. \tag{48}$$

Having integrated equations (48) and (49), from equation (44) we find the phase S. It is convenient to express t in terms of x in the respective solutions by making use of relationship (46). Denoting $\bar{\beta} = \beta/f_0\sqrt{M^2 - 1}$, we obtain

$$y = \frac{1}{\bar{\beta}\sqrt{M^2 - 1}}(\sinh\alpha\sin X + \cos X - 1)\,, \quad X = \frac{\bar{\beta}x}{\cosh\alpha}\,, \tag{49}$$

$$S = \frac{f_0}{2c\sqrt{M^2 - 1}}\left[-x\cosh\alpha + \frac{\sinh^2\alpha - 1}{2\bar{\beta}}\sin(2X) + \frac{\sinh\alpha}{\bar{\beta}}\big(\cos(2X) - 1\big)\right]. \tag{50}$$

Expressions (49) and (50) define the dependences $y = y(x, \alpha)$, $S = S(x, \alpha)$, and the elimination of the parameter α from them yields the equation for phase curves in the explicit $(y = y(x, S))$ or implicit $(S = S(x, y))$ form.

We find phase curve asymptotics for $\bar{\beta}x \ll 1$ or $x \ll \bar{\beta}^{-1} = L_*\sqrt{M^2 - 1}$. For these values, one has $X \ll 1$ and, keeping only the principle terms in expansions of trigonometric functions in powers of X, from formulas (49) and (50) we find

$$y = \frac{1}{\sqrt{M^2 - 1}}\left[x\tanh\alpha - 0.5\bar{\beta}x^2(1 - \tanh^2\alpha)\right], \tag{51}$$

$$S = -\frac{f_0}{c\sqrt{M^2 - 1}\cosh\alpha}(x + \bar{\beta}x^2\tanh\alpha)\,. \tag{52}$$

For $\bar{\beta}x \ll 1$, from formula (52) it asymptotically follows that

$$\tanh\alpha = \frac{\bar{y}}{x} - 0.5\bar{\beta}x\left[\left(\frac{\bar{y}}{x}\right)^2 - 1\right], \qquad (53)$$

where we denoted $\bar{y} = y\sqrt{M^2-1}$ for brevity. Making use of Eqn (53), one can exclude the parameter α from expression (52). Taking the square of Eqn (52), recalling the identity $1/\cosh^2\alpha = 1 - \tanh^2\alpha$, and making use of Eqn (53), we arrive at the expressions sought for the phase curves:

$$x^2 - \bar{y}^2 + \bar{\beta}\bar{y}(x^2 - \bar{y}^2) = c^2(M^2-1)\left(\frac{S}{f_0}\right)^2. \qquad (54)$$

Here, we kept the principal term in the expansion in powers of $\bar{\beta}$.

At $\bar{\beta} = 0$, Eqn (54) gives $x^2 - \bar{y}^2 = c^2(M^2-1)(S/f_0)^2$, which is the alternative form of presenting the family of hyperbolas (25). Their vertices lie on the line $y = 0$ (the symmetry axis of the wave wake). The plots of curves $S = \text{const}$ (54), computed numerically for the values of f_0 and β given above and $M = \sqrt{2}$, are displayed in Fig. 5b. Apparently, the symmetry is broken in the presence of the β-effect. Close to the coordinate origin, the phase curves are analogous to hyperbolas with vertices displaced to the region of $y > 0$. Simple asymptotic equations for these curves can be obtained for $\bar{\beta}\bar{y} \ll 1$. Denoting the right-hand side of Eqn (54) as A^2, we write down Eqn (54) in the form $x^2 - \bar{y}^2 = A^2/(1+\bar{\beta}\bar{y}) \approx A^2(1-\bar{\beta}\bar{y})$. Forming a complete square, we obtain the equations for hyperbolas

$$x^2 - (\bar{y} - 0.5A^2\bar{\beta})^2 = A^2 - 0.25A^4\bar{\beta}^2$$

with vertices at the points

$$x = \sqrt{A^2 - 0.25A^4\bar{\beta}^2}, \qquad \bar{y} = 0.5A^2\bar{\beta}. \qquad (55)$$

Exclusion of the parameter A^2 (the values of the phase) from Eqn (55) gives the equation of the curve that is the loci of the vertices: $0.5\bar{\beta}(x^2 + \bar{y}^2) - \bar{y} = 0$ (the asymptotic equation for the wake axis). For $\bar{\beta}\bar{y} \ll 1$, the curve at hand (Fig. 5b) is the parabola $\bar{y} = 0.5\bar{\beta}x^2$ or

$$y = \frac{\beta}{2f_0(M^2-1)}\,x^2. \qquad (56)$$

It should be noted that typical trajectories of hurricanes in the northern Atlantic head west or northwest. It can be shown that for all such trajectories (not only zonal) the wave wake structure is asymmetric.

9. Conclusion

In this paper, we proposed an approach enabling a description of the phase structure of wave perturbations without solving wave equations and based only on the given dispersion law for waves. In the framework of the proposed approach, an analytical expression is obtained for phase surfaces (lines) making a wave pattern generated by a moving localized source. This expression is further used to explore the details of a phase pattern of gravity-capillary waves, the structure of a wave wake behind a moving tropical cyclone (hurricane), and the system of lee waves in Earth's atmosphere.

Acknowledgments

The authors are indebted to G S Golitsyn for valuable comments, and L Kh Ingel and S V Kozlov for their help with this work. Our special thanks go to M S Aksent'eva for her encouraging optimism and support in preparing the final version of this paper. The work was carried out with a partial support from the RFBR (project No. 12-05-00400-a).

10. Appendix

We demonstrate that the stationarity of the angle φ between the vectors $\mathbf{k}$ and $\mathbf{U}$ as a function of k implies the equality $\mathbf{k}\mathbf{V}_{\text{g}} - \Omega(\mathbf{k}) = 0$. Let us use a spherical coordinate system in the $\mathbf{k}$-space with the polar axis aligned with the vector $\mathbf{U}$ and with the azimuthal angle ψ. In this case, the Cartesian components of the vector $\mathbf{k}$ are, respectively, $k\cos\varphi$, $k\sin\varphi\cos\psi$, and $k\sin\varphi\sin\psi$. We write out the radiation condition (3) as

$$U\cos\varphi = \frac{\Omega(k\cos\varphi, k\sin\varphi\cos\psi, k\sin\varphi\sin\psi)}{k}.$$

Considering φ as a function of k and ψ and taking the partial derivative with respect to k from both sides of this equality, we rearrange the result to

$$\frac{\partial\varphi}{\partial k}(-U\sin\varphi + \Omega_1'\sin\varphi - \Omega_2'\cos\varphi\cos\psi - \Omega_3'\cos\varphi\sin\psi) = \frac{1}{k^2}\big(\mathbf{k}\mathbf{V}_{\text{g}} - \Omega(\mathbf{k})\big),$$

where Ω_i' ($i = 1, 2, 3$) are the partial derivatives of $\Omega(\mathbf{k})$ over the Cartesian components k_x, k_y, k_z. This equality immediately proves the above statement.

References

1. Gill A *Atmosphere–Ocean Dynamics* (New York: Academic Press, 1982) [Translated into Russian (Moscow: Mir, 1986)]
2. Monin A S *Theoretical Geophysical Fluid Dynamics* (Dordrecht: Kluwer Acad. Publ., 1990) [Translated from Russian: *Teoreticheskie Osnovy Geofizicheskoi Gidrodinamiki* (Leningrad: Gidrometeoizdat, 1988)]
3. Lamb H (Sir) *Hydrodynamics* (New York: Dover Publ., 1945) [Translated into Russian (Moscow – Leningrad: Gostekhizdat, 1947)]
4. Kochin N E, Kibel' I A, Roze N V *Theoretical Hydromechanics* (New York: Interscience Publ., 1964) [Translated from Russian: *Teoreticheskaya Gidromekhanika* Pt. 1 (Moscow: Fizmatgiz, 1963)]
5. Whitham G B *Linear and Nonlinear Waves* (New York: Wiley, 1974) [Translated into Russian (Moscow: Mir, 1977)]
6. Lighthill J *Waves in Fluids* (Cambridge: Cambridge Univ. Press, 1978) [Translated into Russian (Moscow: Mir, 1981)]
7. Landau L D, Lifshitz E M *Fluid Mechanics* (Oxford: Pergamon Press, 1987) [Translated from Russian: *Gidrodinamika* (Moscow: Nauka, 1986)]
8. Scorer R S *Environmental Aerodynamics* (New York: Halsted Press, 1978) [Translated into Russian (Moscow: Mir, 1980)]
9. Baines P G *Topographic Effects in Stratified Flows* (Cambridge: Cambridge Univ. Press, 1995)
10. Khain A P, Sutyrin G G *Tropicheskie Tsiklony i Ikh Vzaimodeistvie s Okeanom* (Tropical Cyclones and Their Interaction with the Ocean) (Leningrad: Gidrometeoizdat, 1983)
11. Ginis I *Adv. Fluid Mech.* **33** 83 (2002)
12. Zalgaller V A *Teoriya Ogibayushchikh* (The Theory of Envelops) (Moscow: Nauka, 1975)
13. Svirkunov P N, in *Trudy Instituta Eksperimental'noi Meteorologii* Issue 27(162) (St. Petersburg: Gidrometeoizdat, 1996)

14. Lifshitz E M, Pitaevskii L P *Statistical Physics* Vol. 2 *Theory of the Condensed State* (Oxford: Pergamon Press, 1980) [Translated from Russian: *Statisticheskaya Fizika* Pt. 2 *Teoriya Kondensirovannogo Sostoyaniya* (Moscow: Fizmatlit, 2002)]
15. Svirkunov P N, Kalashnik M V *Dokl. Phys.* **57** 474 (2012) [*Dokl. Ross. Akad. Nauk* **447** 396 (2012)]
16. Blokhintsev D I *Osnovy Kvantovoi Mekhaniki* (Fundamentals of Quantum Mechanics) (Moscow: Nauka, 1976)
17. Ginis I, Sutyrin G *J. Phys. Oceanogr.* **25** 1218 (1995)
18. LeBlond P H, Mysak L A *Waves in the Ocean* (Amsterdam: Elsevier, 1978) [Translated into Russian (Moscow: Mir, 1981)]
19. Geisler J E *Geophys. Fluid Dyn.* **1** 249 (1970)
20. Golitsyn G S *Vvedenie v Dinamiku Planetnykh Atmosfer* (The Introduction into Dynamics of Planetary Atmospheres) (Leningrad: Gidrometeoizdat, 1973)
21. Pudov V D, Varfolomeev A A, Fedorov K N *Okeanologiya* **18** 218 (1978)
22. Pudov V D, Petrichenko S A *Izv. Atmos. Ocean. Phys.* **36** 645 (2000) [*Izv. Ross. Akad. Nauk Fiz. Atmos. Okeana* **36** 700 (2000)]