## METHODOLOGICAL NOTES

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# The relativistic virial theorem and scale invariance

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Abstract. The virial theorem is related to the dilatation properties of bound states. This is realized, in particular, in the Landau-Lifshitz formulation of the relativistic virial theorem, in terms of the trace of the energy-momentum tensor. We construct a Hamiltonian formulation of dilatations in which the relativistic virial theorem naturally arises as the condition of stability under dilatations. A bound state becomes scale invariant in the ultrarelativistic limit, in which its energy vanishes. However, for very relativistic bound states, scale invariance is broken by quantum effects, and the virial theorem must include the energy-momentum tensor trace anomaly. This quantum field theory virial theorem is directly related to the Callan-Symanzik equations. The virial theorem is applied to QED and then to QCD, focusing on the bag model of hadrons. In massless QCD, according to the virial theorem, 3/4 of a hadron mass corresponds to quarks and gluons and 1/4 to the trace anomaly.

# 1. Introduction

The classic virial theorem has been very useful in physics, in particular, in astrophysics, to determine the equilibrium and stability of dynamical systems [1]. The theorem is especially useful for systems interacting via potentials that are homogeneous functions of interparticle distances (e.g., power-law potentials): the theorem then states a simple relation between the long-time averages of the kinetic and potential energies of a system, 2K = nU, where *n* is the homogeneity degree (e.g., the exponent of the power law). Of course, the most relevant potentials to consider are the quadratic potential of harmonic

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Received 7 February 2013 Uspekhi Fizicheskikh Nauk **183** (9) 973–986 (2013) DOI: 10.3367/UFNr.0183.201309f.0973 Translated by S M Apenko; edited by A M Semikhatov oscillations (n = 2) and the inverse distance law of the Newton or Coulomb potentials (n = -1). The virial theorem for homogeneous potentials can be regarded as a consequence of mechanical similarity, in other words, of scale invariance in mechanics [2]. Indeed, several authors have studied the relation of the virial theorem to scale invariance [3–5] and its connection with Noether's theorem [6–8].

The classic virial theorem has been generalized in several ways [1]. The obvious generalizations pertain to relativistic mechanics and quantum mechanics. Whereas the generalization of the virial theorem to nonrelativistic quantum mechanics presents no special problems, the generalization to relativity is nontrivial because the concepts of force and potential are unsuitable for describing relativistic interactions. Nevertheless, there are relativistic formulations of the virial theorem. A virial theorem in electrodynamics that expresses the energy in terms of the energy-momentum tensor trace already appears in the classic textbook by Landau and Lifshitz [9]. This relativistic virial theorem is generalizable, in principle, to other interactions [1].

The relativistic virial theorem has featured in several studies and some papers have been specifically devoted to it, both in classical field theory [10, 11] and in quantum field theory [12]. However, while the relation of the classic virial theorem to scale invariance is well established, the relation to scale invariance of the relativistic virial theorem, especially in the Landau-Lifshitz formulation, is hardly studied. In this paper, we focus on this relation and adopt a fundamental standpoint. Therefore, our approach to the relativistic virial theorem is closer to the one in Ref. [11], and especially to the one in Ref. [12], than to the one in Ref. [10]. Our main purpose is to establish the fundamental role of scale transformations and scale invariance in the relativistic virial theorem. For this, we find it useful to proceed from the classic theorem to the relativistic theorem and finally to the virial theorem in quantum field theory (QFT). In this process, the classic theorem 2K = nU, which allows any homogeneity degree *n*, must be restricted to relativistic interactions mediated by a field that becomes just an inverse-law potential in the nonrelativistic limit, and hence the only allowed value is n = -1. In other words, the virial theorem is restricted to interactions mediated by massless gauge fields, namely, the electromagnetic interaction and the strong interaction described by quantum chromodynamics (QCD). The latter has special features, as is explained in what follows.

The classic virial theorem includes the gravitational interaction, but the relativistic formulation of gravity necessarily leads to General Relativity, which is a sort of interaction mediated by a massless gauge field, although not of the ordinary type. In fact, there have been attempts to formulate the virial theorem in General Relativity [1, p. 27], but they neglect its relation to other gauge theories. Some of the results in this paper can be applied to General Relativity, but this theory has distinct features and, in particular, its concepts of gravitational energy and scale invariance become very subtle and difficult to handle. These problems are beyond the scope of this paper and are left for future work.

We begin in Section 2 with a Hamiltonian formulation of the virial theorem that directly arises from the notion of time-averaged scale invariance. This formulation is applied to the electromagnetic interaction in special relativity, considering, first, the action-at-a-distance interaction of particles and, second, the full field theory, following Landau and Lifshitz [9]. Next, in Section 2.2, we construct a fully general derivation of the field theory virial theorem that is based on scale invariance, by generalizing the Hamiltonian formulation of the virial theorem for a system of particles. Some consequences of the virial theorem for bound states of ultrarelativistic particles appear in Section 2.3. Because these bound states sustain considerable quantum effects, they call for relativistic quantum mechanics. The QFT virial theorem is studied in Section 3, where it is applied, first, to quantum electrodynamics (QED) and, second, to QCD, where the energy-momentum tensor trace anomaly plays a crucial role.

## 2. Relativistic virial theorem

The virial theorem is regarded by Landau and Lifshitz [2] as a consequence of scaling in Lagrangian mechanics. This line of thought has been followed by several authors [3–5, 8], some of whom relate the virial theorem to the Noether theorem. In contrast to Lagrangian scaling, we now introduce the general theory of dilatations in the Hamiltonian formalism, which provides a concise and powerful formulation of the virial theorem.

In Hamiltonian mechanics, canonical transformations are the most general ones that preserve the phase space structure [2]. A canonical transformation of the phase space (q, p) is defined by its infinitesimal generator, which is just any function F(q, p). For simplicity, we use single variables q and p, but each denotes the full set of coordinates or momenta. The transformed phase space variables are

$$Q = q + \varepsilon \delta_F q, \quad \delta_F q = \{q, F\} = \frac{\partial F}{\partial p},$$
$$P = p + \varepsilon \delta_F p, \quad \delta_F p = \{p, F\} = -\frac{\partial F}{\partial q},$$

where  $\varepsilon$  is a small parameter, and we introduce Poisson brackets  $\{f,g\} = \partial_q f \partial_p g - \partial_p f \partial_q g$ . Naturally, the phasespace volume element dq dp is unchanged. Among the canonical transformations, an important role is played by point transformations, namely, canonical transformations induced by transformations of the coordinates q only. They are generated by F = f(q)p, where f(q) is any function. If F = qp, then we have

$$\delta_F q = q \,, \quad \delta_F p = -p \,, \tag{1}$$

i.e., a homogeneous dilatation of q and the corresponding homogeneous contraction of p, which together preserve the phase space volume. But, normally, the dilatation of q is not a symmetry, i.e.,

$$\delta_F H = \{H, F\} = q \frac{\partial H}{\partial q} - p \frac{\partial H}{\partial p} \neq 0,$$

and F is not a constant of motion, that is,  $\dot{F} = \{F, H\} \neq 0$ , unless the Hamiltonian H is very special (e.g., H = F). Nevertheless, if both q and p are bounded, the temporal average of  $\dot{F}$  eventually vanishes, and hence, on the average,

$$\{F, H\} = p\dot{q} - q \,\frac{\partial H}{\partial q} = H + L - q \,\frac{\partial H}{\partial q} = 0.$$
<sup>(2)</sup>

We note that the coordinates q of a system of particles can only be bounded in their rest frame, the frame in which their total momentum is zero.

In many mechanical problems,  $L(q, \dot{q}) = K(\dot{q}) - U(q)$ , i.e., there is a separation between kinetic and potential energies, and *K*, in addition, is quadratic in the velocities. Then H = K + U, whence

$$H + L - q \frac{\partial H}{\partial q} = 2K - q \frac{\partial U}{\partial q} = 0.$$

The term  $q \partial_q U$  is called virial. Furthermore, when U(q) is a homogeneous function of q of degree n, Euler's theorem on homogeneous functions implies that  $q \partial_q U = nU$ , whence follows the standard virial theorem 2K = nU or, in terms of the total energy E, K = nE/(n+2) [1, 2]. This theorem is a consequence of the homogeneity of K in  $\dot{q}$  and the homogeneity of U in q, which also imply mechanical similarity: the equations of motion permit a set of geometrically similar motions, such that the paths are geometrically similar and the times of motion between corresponding points are in a constant ratio [2]. For example, when n = -1, as in the Newton or Coulomb potentials,  $t \to l^{3/2}t$  corresponds to  $q \to lq$  (Kepler's third law) and  $E \to E/l$ .

Unfortunately, the Lagrangian in relativistic mechanics is not of the form  $L(q, \dot{q}) = K(\dot{q}) - U(q)$  with K homogeneous in  $\dot{q}$ . Nevertheless, the classical virial theorem is easily extended to the case of a relativistic particle under external forces [13]. Moreover, the Hamiltonian virial theorem, Eqn (2), is very general and covers the many-body problem in relativistic electrodynamics as a prototype of relativistic interactions. Before considering this problem, we discuss two pertinent extensions of the Hamiltonian virial theorem. The first is the extension to quantum mechanics. It is straightforward because the Hamiltonian formulation of canonical transformations is easily transferred to quantum mechanics, just by transforming Poisson brackets to commutators. But we have to take care of operator ordering problems, for example, by symmetrizing phase space functions with respect to q and p. Any phase space function that can be expanded in powers of q and p can be symmetrized by symmetrizing each term of the expansion (each monomial). In particular, the dilatation generator becomes F = (qp + pq)/2 (naturally,  $p = -i\hbar \partial/\partial q$  in the coordinate representation).

As a second extension, the virial theorem can also be extended to canonical transformations other than dilatations. Indeed, the long-time average of  $\dot{F}$  vanishes for any bounded function F, and so does the average of  $\{F, H\}$ . Therefore, we have an infinite number of (average) relations. However, general canonical transformations, with generators F that depend on p arbitrarily, have little physical significance. By contrast, F = f(q)p, i.e., the set of point transformations, includes rotations and, furthermore, arbitrary deformations of the geometrical "shape" of the mechanical system. These transformations give rise to an extension of the standard virial theorem, namely, the *tensor virial theorem* [1], to be discussed in the next section.

## **2.1 Virial theorem for the electromagnetic interaction** The *N*-particle electromagnetic Lagrangian is [9, 14]

$$L = \sum_{a=1}^{N} \left( -m_a c^2 \sqrt{1 - \frac{\mathbf{v}_a^2}{c^2}} + \frac{e_a}{c} \mathbf{v}_a \mathbf{A}(\mathbf{x}_1, \dots, \mathbf{x}_N) - e_a \Phi(\mathbf{x}_1, \dots, \mathbf{x}_N) \right),$$

where  $\mathbf{v}_a = \dot{\mathbf{x}}_a$ . The corresponding Hamiltonian is

$$H = \sum_{a=1}^{N} \left( \sqrt{(c\mathbf{p}_{a} - e_{a}\mathbf{A})^{2} + m_{a}^{2}c^{4}} + e_{a}\Phi \right).$$
(3)

To allow for interaction between the particles, the electromagnetic potentials must satisfy the D'Alembert wave equation with a source given by the electromagnetic current produced by the particles themselves, and hence the potentials are functions of particle trajectories. If the symmetric Green's function (half-sum of the advanced and retarded functions) is used to solve the wave equation, then the Lagrangian or Hamiltonian correspond to the *action-ata-distance* electrodynamics [15, 16], in which there is no radiation and therefore no need for the electromagnetic field Lagrangian.

Neither the *N*-particle Lagrangian nor the Hamiltonian can be separated into kinetic and potential terms, but we can apply Eqn (2) nonetheless. With

$$\left(\frac{\partial H}{\partial \mathbf{x}_a}\right)_{\mathbf{p}} = -\dot{\mathbf{p}}_a = -\left(\frac{\partial L}{\partial \mathbf{x}_a}\right)_{\mathbf{v}},$$

Eqn (2) is written as

$$H + L + \sum_{a} \mathbf{x}_{a} \frac{\partial}{\partial \mathbf{x}_{a}} \left( \sum_{b} \frac{e_{b}}{c} \mathbf{v}_{b} \mathbf{A} - e_{b} \Phi \right)$$
$$= H + \sum_{a} \left[ -m_{a}c^{2}\sqrt{1 - \frac{\mathbf{v}_{a}^{2}}{c^{2}}} + \left( 1 + \mathbf{x}_{a} \frac{\partial}{\partial \mathbf{x}_{a}} \right) \right.$$
$$\times \left( \sum_{b} \frac{e_{b}}{c} \mathbf{v}_{b} \mathbf{A} - e_{b} \Phi \right) \right] = 0, \qquad (4)$$

where the derivatives are to be taken at constant  $\mathbf{v}_b$ . We note that the action of the operator  $\sum_a (1 + \mathbf{x}_a \partial/\partial \mathbf{x}_a)$  on a homogeneous function of  $\mathbf{x}_1, \dots, \mathbf{x}_N$  of degree n = -1 gives a null result, according to Euler's theorem. In other words, if we assume that the vector and scalar potentials can be written as homogeneous functions of coordinates with the degree n = -1, then each electromagnetic potential and its corresponding virial cancel one another. We thus obtain the energy

$$E = \sum_{a} m_{a} c^{2} \sqrt{1 - \frac{\mathbf{v}_{a}^{2}}{c^{2}}}.$$
 (5)

This is the relativistic virial theorem for a system of particles in classical electrodynamics. If external forces were necessary to confine the particles and, in particular, the forces consisted in a constant pressure *P* exerted on the system surface, then *E* should be replaced with E - 3PV, where *V* is the volume of the system [9, § 35].

Although the electromagnetic parameters are absent from Eqn (5), their effect is implicitly included: we note that  $E < \sum_{a} m_{a}c^{2}$ , which corresponds to a bound state. The bound state becomes nonrelativistic for low velocities, when  $|E - \sum_a m_a c^2| \ll \sum_a m_a c^2$ , and Eqn (5) then reduces to the classical virial theorem  $E - \sum_a m_a c^2 \approx -\sum_a m_a v_a^2/2 = -K$ . But the relativistic dynamics loses the similarity of the classical dynamics under space dilatations and the corresponding time dilatations. This similarity is lost even if  $\Phi$ and A are assumed to be homogeneous functions of degree n = -1. Moreover, in relativistic mechanics, there is no similarity even for free particles ( $e_a = 0$ ) because of the form of the Hamiltonian. (Of course, similarity is recovered for low velocities, but the energy only scales after subtracting the rest energy  $\sum_{a} m_{a}c^{2}$  of the particles.) However, there is a relativistic notion of mechanical similarity [4] in which space and time are equally dilatated, and therefore velocities are unchanged. Under this similarity, masses cannot be held constant and must be contracted such that they transform like energies. Therefore, the nature of the particles changes, and the similarity does not relate different motions of the same system. Nevertheless, this similarity leads to Eqn (5); in addition, it also leads to the classic virial theorem in the nonrelativistic limit [4].

The equations of motion of the action-at-a-distance electrodynamics are not ordinary differential equations but differential-difference equations, and it is difficult to find their solution. A simple solution of the relativistic two-body problem is provided by two opposite charges in circular motion, with calculable radii and angular velocity [15, p. 223]. This solution satisfies Eqn (5) (no temporal average is needed). At any rate, it is natural to consider the electromagnetic field dynamics and the radiation effects. However, because these effects appear-in an expansion in powers of v/c—only in the third order, it is possible to describe the electromagnetic interaction of particles with standard ordinary differential equations based on potentials of the order  $(v/c)^2$  [9, 14, 15]. These potentials, added to the relativistic kinetic term expanded to the same order, give rise to the well-known Darwin Hamiltonian and Lagrangian. In this approximation, both vector and scalar potentials are homogeneous functions of degree n = -1, and hence virial theorem (5), expanded to terms of the second order, holds.

Regarding quantum mechanics, the relativistic virial theorem holds, by canonical quantization of Hamiltonian systems, as stated before. Indeed, quantum relativistic versions of the virial theorem have appeared in the literature [17, 18]. However, they are meant to be applied in nuclear physics, and they are only considered a simple two-quark problem with a phenomenological scalar potential U (we note that the fundamental theory of strong interactions, QCD, includes a *vector* potential, like QED). The scalar potential is the "Cornell potential," which is the sum of a Coulomb term

and a confining linear term. Lucha and Schöberl [17], in particular, obtain

$$\langle \mathbf{x} \nabla U(\mathbf{x}) \rangle = c \left\langle \frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + (m_1 c)^2}} + \frac{\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + (m_2 c)^2}} \right\rangle, \quad (6)$$

whence

$$E = \left\langle \mathbf{x} \nabla U(\mathbf{x}) \right\rangle + \left\langle U(\mathbf{x}) \right\rangle$$
$$+ c^{3} \left\langle \frac{m_{1}^{2}}{\sqrt{\mathbf{p}^{2} + (m_{1}c)^{2}}} + \frac{m_{2}^{2}}{\sqrt{\mathbf{p}^{2} + (m_{2}c)^{2}}} \right\rangle, \tag{7}$$

where the expectation values are understood to be taken with respect to normalized eigenstates. Equation (6) states the equality of the expectation values of the centripetal and centrifugal virials. On the other hand, on account of the relativistic identity

$$1 - \frac{\mathbf{v}^2}{c^2} = \frac{(mc)^2}{\mathbf{p}^2 + (mc)^2} ,$$

Eqn (7) is a quantum version of Eqn (4), such that the temporal expectation values implicit in Eqn (4) are replaced by expectation values in stationary states and such that the system is restricted to two particles with  $\mathbf{A} = 0$ . In Eqn (7), the Coulomb part of the potential and its virial indeed cancel one another, as in Eqn (4). At any rate, the strong-interaction Hamiltonian used in Refs [17, 18] is not fully relativistic and only has relativistic kinematics (the potential *U* can be interpreted as the lowest-order slow-motion approximation of a relativistic interaction). The formulation of a fully relativistic virial theorem for quark bound states requires a QFT framework and is presented in Section 3.

Returning to classical electrodynamics, we consider the full particle-plus-field dynamics and its local conservation laws, namely, the local conservation of energy and momentum. Landau and Lifshitz's virial theorem [9] relies on this conservation law, expressed in terms of the energy-momentum tensor. This conservation law implies that the long-time average of the stress tensor (the spatial part of the energymomentum tensor) is divergence-free:

$$\partial_j T_i^j = 0 \,, \tag{8}$$

where Latin indices denote spatial coordinates and the bar denotes the long-time average. Multiplying Eqn (8) by  $x^i$  and integrating over all space, we obtain (under suitable asymptotic conditions) a space integral of the stress-tensor trace average that vanishes:

$$\overline{T_i^i} \, \mathrm{d}V = 0 \,. \tag{9}$$

Actually, the vanishing of the integral of the *full* stress tensor for a closed (self-contained) and *static* system was proved by Laue [19] in the early days of relativity theory.<sup>1</sup> The vanishing of the integral of the time average of the stress tensor constitutes the *tensor virial theorem*, which holds in classical

<sup>1</sup> Laue's paper studies the energy and momentum of a closed static system and, according to [20], contains the *first* real proof of the mass–energy equivalence  $E = mc^2$ . mechanics as well as in relativistic mechanics (Ref. [1], § II.1 and § II.3). It can be proved by multiplying Eqn (8) by  $x^k$ , that is, by using a generic index k instead of the index i used above, and integrating over all space. Regarding the connection of the tensor virial theorem with arbitrary transformations of spatial coordinates, already mentioned above, this theorem can be understood as just a condition of *dynamical* equilibrium, namely, of the stability of the averaged system shape under deformations, as explained in Section 2.2. Of course, the tensor virial theorem implies the scalar theorem (the vanishing of the trace), which can be understood as a condition of stability of the system against dilatations.

For a system of N bodies with electromagnetic interaction,

$$T_i^i = \sum_{a=1}^N \frac{m_a \mathbf{v}_a^2}{\sqrt{1 - \mathbf{v}_a^2/c^2}} \,\delta(\mathbf{x} - \mathbf{x}_a) + \frac{1}{2} \left(\mathbf{E}^2 + c^2 \mathbf{B}^2\right). \quad (10)$$

Therefore, the space integral of  $T_i^i$  seems to be strictly positive and the virial theorem, Eqn (9), cannot hold. To interpret Eqn (10) and the corresponding virial theorem, we make the orthogonal decomposition  $\mathbf{E} = \mathbf{E}_{\mathrm{L}} + \mathbf{E}_{\mathrm{T}}$ , where  $\mathbf{E}_{\mathrm{L}} = -\nabla \Phi$ and  $\nabla \mathbf{E}_{\mathrm{T}} = 0$ . Then the electromagnetic energy is

$$\frac{1}{2} \int dV (\mathbf{E}^2 + c^2 \mathbf{B}^2) = \frac{1}{2} \int dV (\mathbf{E}_{\mathrm{L}}^2 + \mathbf{E}_{\mathrm{T}}^2 + c^2 \mathbf{B}^2), \quad (11)$$

where

$$\frac{1}{2} \int dV \mathbf{E}_{\mathrm{L}}^{2} = \frac{1}{8\pi} \int dV dV' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$
(12)

is the Coulomb electrostatic energy. While this electrostatic energy is obviously positive for regular distributions, the electrostatic energy of a system of N positive and negative *point-like* charges can become negative after the subtraction of the (infinite) self-energy of the charges, which amounts to a renormalization of their masses [9, 14]. On the other hand, regarding the contribution of  $\mathbf{E}_{T}$  and  $\mathbf{B}$  to the energy, we must require the absence of incoming or outgoing radiation fields, which would produce contributions to  $\mathbf{E}_{T}^{2} + c^{2}\mathbf{B}^{2}$  of an arbitrary magnitude. Indeed, the proof of Eqn (9) requires the vanishing of a certain surface integral related to radiation fields [9]. As already stated, the absence of radiation is implicit in the action-at-a-distance electrodynamics. These problems have already been noticed by Dudas and Pirjol [12], who emphasize the role of the Wheeler–Feynman time-symmetric formulation of electrodynamics in their solution.

formulation of electrodynamics in their solution. The four-dimensional trace is  $T^{\mu}_{\mu} = -T^{00} + T^{i}_{i}$  [our metric signature convention is (-, +, +, +)]. Therefore, a condition equivalent to (9) that introduces the total energy  $E = \int T^{00} dV$  is [9]

$$E = -\int T^{\mu}_{\mu} \,\mathrm{d}V \tag{13}$$

(as before, the long-time average is implicit when not shown explicitly). For a system of electromagnetically interacting particles, the electromagnetic energy–momentum tensor is traceless and disappears from Eqn (13). This absence of electromagnetic parameters at virial equilibrium is analogous to the cancelation of the electromagnetic potentials with their corresponding virials in Eqn (4). Indeed, Eqn (13) leads again to Eqn (5) [9].

We note that Eqn (13) makes no reference to particles, unlike Eqn (5). Therefore, it can be applied to fields forming a *single* particle, and then it plays a role in the famous problem of modeling the electron or any charged particle in classical electrodynamics [14, Ch. 16]. Indeed, Eqn (13) connects the particle energy with the trace of the energy–momentum tensor of Poincaré stresses, because the electromagnetic energy–momentum tensor is traceless. If we assume, for simplicity, that the energy–momentum tensor of Poincaré stresses has no traceless part, namely, that it is proportional to  $g_{\mu\nu}$ , then the total energy–momentum tensor is  $T^{\mu\nu} = T_{em}^{\mu\nu} + T_{\alpha}^{\alpha} g^{\mu\nu}/4$ , where  $T_{em}^{\mu\nu}$  is the energy–momentum tensor of the electromagnetic field. Therefore, in the rest frame,

$$E = \int \left( T_{\rm em}^{00} + \frac{g^{00}}{4} T_{\alpha}^{\alpha} \right) dV = \int T_{\rm em}^{00} dV + \frac{E}{4} , \qquad (14)$$

and therefore three fourths of the particle rest energy come from its electromagnetic energy and the remaining one fourth from the Poincaré stresses (3/4 is actually the ratio of the electromagnetic mass to electromagnetic inertia, which constitutes the infamous 4/3 problem, solved by the introduction of Poincaré stresses [14, Ch. 16]). If the energy– momentum tensor corresponding to the Poincaré stresses,  $T_p^{\mu\nu}$ , has a nonvanishing traceless part, then this part is on the same footing as the traceless energy–momentum tensor of the electromagnetic field, and hence both together make up the three fourths of the particle total rest energy. The contribution to the energy of the traceless part of the energy– momentum tensor for the Poincaré stresses is the space integral of

$$T_{\rm P}^{00} - T_{\rm P\alpha}^{\ \alpha} \frac{g^{00}}{4} = \frac{3}{4} T_{\rm P}^{00} + \frac{1}{4} T_{\rm Pi}^{\ i}.$$
 (15)

This integral is nonnegative, although  $\int T_{P_i}^i dV < 0$ , as is necessary in order to have cohesive Poincaré stresses, that is, as is necessary for the total stress tensor to satisfy Eqn (9). The non-negativity of the right-hand side of Eqn (15) can be proved by invoking the *null energy condition* (deduced by continuity from the non-negativity of the energy density [21, p. 89]). In conclusion, the fraction of energy due to the Poincaré stresses is at least one fourth, and the fraction of electromagnetic energy can be equal to or less than three fourths.

In a single-particle model, the charge distribution is continuous and can be attributed to a charged field of a fluid. An interesting classical electron model is the Bialynicki-Birula model [22], in which the electron consists of a perfect, charged fluid, with the energy density  $\rho$  and pressure *P*, and an electromagnetic field. The traceless part of the fluid energy-momentum tensor is proportional to  $\rho + P$  and contributes to the energy with  $(3/4) \int (\rho + P) dV$ , which is non-negative, although P < 0 everywhere.

We note that a particle model with a continuous charge distribution makes as much sense for a composite particle as for an elementary particle, because there is no quantization of charge in classical electrodynamics. For consistency with the electron modeling in QED, we assume that matter is described by a Dirac field  $\psi$  with the standard Lagrangian. Then

$$T_{\mu\nu} = \frac{1}{2} c \left( \bar{\psi} \gamma_{(\mu} \mathcal{D}_{\nu)} \psi - \mathcal{D}^*_{(\mu} \bar{\psi} \gamma_{\nu)} \psi \right),$$

where  $D_{\mu} = \hbar \partial_{\mu} + i(e/c) A_{\mu}$  and  $D^*$  is its complex conjugate. Therefore,

$$T^{\mu}_{\mu} = \frac{\mathrm{i}}{2} c \left( \bar{\psi} \gamma^{\mu} \mathcal{D}_{\mu} \psi - \mathcal{D}^{*}_{\mu} \bar{\psi} \gamma^{\mu} \psi \right) = -mc^{2} \bar{\psi} \psi , \qquad (16)$$

and, according to Eqn (13), the energy of a composite or elementary particle is

$$E = mc^2 \int \bar{\psi} \psi \, \mathrm{d}V. \tag{17}$$

This equation is related to Fock's old result for the Dirac equation in an external, central Coulomb field [23] and also to the more general virial theorem of Rose and Welton [24] (also see Ref. [11]).

Because we regard the Dirac field  $\psi$  as a *classical* field constituting a sort of matter fluid,  $n(\mathbf{x}) = \overline{\psi}\psi$  is the total particle number density, with equal weight for particles and antiparticles, and computed in the local reference frame. Equation (17) might seem to imply that the bound-state energy is just the number of particles times the rest energy per particle, as if they were free and at rest, but it does not, because  $n(\mathbf{x})$  must be computed in the *local* reference frame. When  $n(\mathbf{x})$  is computed in the laboratory frame, namely, the bound-state rest frame, then

$$mn(\mathbf{x}) \rightarrow r(\mathbf{x}) \sqrt{1 - \frac{v^2(\mathbf{x})}{c^2}},$$

where  $r(\mathbf{x})$  is the ordinary nonrelativistic mass density and  $v(\mathbf{x})$  is the velocity of the matter-fluid element  $dm = r(\mathbf{x}) dV$ . Thus, we obtain

$$E = c^2 \int \sqrt{1 - \frac{v^2(\mathbf{x})}{c^2}} r(\mathbf{x}) \, \mathrm{d}V, \qquad (18)$$

which is a continuous form of Eqn (5). Naturally, this virial theorem applies only to bound-state solutions of the nonlinear equations of the classical electrodynamics of the Dirac field, which have hardly been explored (see, i.e., Ref. [25]). Because these equations have stable bound-state solutions, the corresponding single-particle models do not need extraneous Poincaré stresses.

### 2.2 Hamiltonian field theory formulation

We can obtain a general field-theory virial theorem within the Hamiltonian formalism by generalizing the derivation of the virial theorem for a finite number of the degrees of freedom in Section 2. We consider a generic field, which we let be denoted by  $\varphi$ , but which can comprise a set of independent fields (it can be a vector field, etc.), and consider its Lagrangian and Hamiltonian densities  $\mathcal{L}$  and  $\mathcal{H}$ . The associated field momentum density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$
.

In particle mechanics, dilatations simply act on the coordinates as  $q \rightarrow lq$  and on the momenta as  $p \rightarrow p/l$ , which in the infinitesimal form are given by Eqn (1). In field theory, dilatations primarily act on space coordinates and, through them, on field coordinates  $\varphi$  and momenta  $\pi$ . Therefore, the infinitesimal generator of a finite dilatation  $\varphi(\mathbf{x}) \rightarrow l^D \varphi(l\mathbf{x})$ , where D is the dimension matrix, is given by the Lie derivative

 $\delta \varphi = D \varphi - x^i \partial_i \varphi \,.$ 

The dimension matrix *D* can be assumed to be diagonal, i.e.,  $\varphi$  can be assumed to be formed by eigenstates of *D*. The Lie derivative  $\delta \varphi$  differs from  $\delta q$  in Eqn (2) in the presence of the transport term, with  $x^i \partial_i$ , and also in the presence of *D*, which generalizes the trivial dimension of *q*. The canonical generator of dilatations *F*, such that  $\delta \varphi = \delta F / \delta \pi$ , is

$$F = \int \pi (D\varphi - x^{i}\partial_{i}\varphi) \,\mathrm{d}V.$$
<sup>(19)</sup>

Therefore,

$$\delta \pi = -\frac{\delta F}{\delta \varphi} = (-3 - D) \pi - x^i \partial_i \pi,$$

up to the vanishing of a surface integral. We deduce that both  $\delta \varphi$  and  $\delta \pi$  constitute infinitesimal generators of Hamiltonian dilatations. We note that under a finite dilatation,  $\pi(\mathbf{x}) \varphi(\mathbf{x}) \rightarrow l^{-3} \pi(l \mathbf{x}) \varphi(l \mathbf{x})$ , and hence *F* is invariant.

The generating function *F* given by Eqn (19) is the integral of the sum of two parts, one corresponding to the intrinsic change in the field and the other corresponding to the spatial transport. The latter can be written as  $x^i T_{C_i^0}$ , where  $T_{C_v^{\mu}}$ comprises the four conserved currents associated with spacetime translations by Noether's theorem [27]; i.e., it is the *canonical* energy-momentum tensor. For example, for a scalar field  $\varphi$  with  $\mathcal{L} = -\partial^{\mu} \varphi \partial_{\mu} \varphi / 2 - \mathcal{V}(\varphi)$ ,

$$\begin{split} T_{\mathrm{C}\,\nu}^{\ \mu} &= \partial^{\mu}\varphi\,\partial_{\nu}\varphi + \delta^{\mu}_{\nu}\mathcal{L}\,,\\ T_{\mathrm{C}\,i}^{\ 0} &= -\dot{\varphi}\,\partial_{i}\varphi = -\pi\,\partial_{i}\,\varphi\,. \end{split}$$

For an electromagnetic field,

$$T_{\mathrm{C}\,\nu}^{\mu} = F^{\mu\rho} \,\partial_{\nu} A_{\rho} + \delta^{\mu}_{\nu} \mathcal{L} \,,$$

and hence, in the Hamiltonian gauge  $A_0 = 0$ ,

$$T_{\mathrm{C}_{i}}^{0} = -\dot{A}_{i}\partial_{i}A^{j} = -\pi_{i}\partial_{i}A^{j}.$$

Therefore, in general,

$$F = \int (\pi D \varphi + x^i T_{\mathcal{C}_i}^0) \,\mathrm{d}V.$$
<sup>(20)</sup>

However,  $T_{C_i}^0$  can be redefined by adding  $\partial_j t_i^j$  to it, where  $t_i^j$  is an arbitrary function of  $\varphi$  and  $\pi$ . With the appropriate choice of  $t_i^j$ , we can cancel the first summand in the right-hand side of Eqn (20) (up to a surface integral). In other words, there is always an "improvement" of the energy–momentum tensor such that the dilatation generator becomes

$$F = \int x^i T_i^0 \,\mathrm{d}V.$$

This connects the Hamiltonian formulation of the virial theorem with Landau and Lifshitz's proof [9]. Indeed, using the conservation law  $\partial_{\mu}T^{\mu\nu} = 0$ , we have

$$\dot{F} = \int T_i^i \,\mathrm{d}V$$

The vanishing of the temporal average of  $\dot{F}$  and therefore of the spatial integral of the temporal average of  $T_i^i$  give rise to the virial theorem, Eqn (9).

We note that instead of dilatations, we can also consider general (anisotropic) coordinate transformations, and thus derive the tensor virial theorem.

The general condition of exact scale invariance is  $\delta_F H = -\dot{F} = 0$ , or

$$\dot{F} = \int T_i^i \,\mathrm{d}V = 0 \tag{21}$$

without averaging. This condition is not satisfied by general field configurations of normal field theories. Naturally,  $\dot{F}$  must vanish for any static field configuration, and hence so does the integral of the stress tensor trace. This is in accordance with Laue's theorem (Section 2.1), applicable to any static relativistic system and, in particular, to any model of an elementary particle, such as the electron. For example, for the Bialynicki-Birula electron model, condition (21) is indeed fundamental for relativistic invariance [22].

We remark on one interesting consequence of Eqn (21) for static field configurations. In the case of a scalar field with  $\mathcal{L} = -\partial^{\mu} \varphi \partial_{\mu} \varphi / 2 - \mathcal{V}(\varphi)$ , the static field has

$$T_{C_i}^{\ i} = \frac{2-d}{2} \left(\nabla\varphi\right)^2 - d\mathcal{V}(\varphi), \qquad (22)$$

where *d* is the space dimension (we simply use the canonical, unimproved energy-momentum tensor, because the space integral of the stress tensor is not altered by the improvement). With full generality, we can take the potential  $\mathcal{V}(\varphi) \ge 0$  and vanishing at its absolute minima; namely, we assume that there are several absolute minima with  $\mathcal{V} = 0$ . Then Eqns (21) and (22) imply, for  $d \ge 2$ , that  $\varphi(\mathbf{x})$  is constant and equal to its value at one of the minima. The absence of localized static solutions of scalar field theories in  $d \ge 2$  is known as the Hobart–Derrick theorem [28, 29] and is usually proved by direct scaling of  $\varphi(\mathbf{x})$ . The generalization of this theorem to other field theories more complicated than the scalar field theory is also given by Eqn (21), although it can be proved by direct scaling of the appropriate field(s), case by case [30, Ch. 6].

In our treatment of scale transformations, we have chosen examples of relativistic fields, but we note that there is no need to impose Lorentz invariance to obtain the virial theorem. When there is Lorentz invariance, the Lagrangian formulation of scale transformations, in terms of  $\mathcal{L}$ , is typically used instead of the Hamiltonian formulation, because it is covariant and hence explicitly relativistic. The Lagrangian formulation of scale transformations is based on *space-time* dilatations, such that

$$\delta \varphi = D \varphi - x^{\alpha} \, \partial_{\alpha} \varphi \, .$$

They coincide with space-only dilatations for static fields. The *local* current associated with space-time dilatations by Noether's theorem can always be expressed as  $j_D^{\mu} = x_v T^{\mu\nu}$ , and scale invariance can be expressed in the local form  $\partial_{\mu} j_D^{\mu} = 0$  [30, 31]. Therefore, in field theory, scale invariance is generally connected with the tracelessness of the energy-momentum tensor. Although the conserved symmetric energy-momentum tensor of a scale-invariant field theory is not necessarily traceless, it is always possible to "improve" it and convert it into one that is traceless, in addition to being

symmetric and conserved [30, 31]. Then the tracelessness of the energy–momentum tensor implies, beyond the Poincaré and scale invariance, the full invariance under the conformal group, which is obtained by joining the discrete inversion to the Poincaré and scale transformations. The sourceless Maxwell equations are of course conformal invariant and, in this case, the energy–momentum tensor that results from symmetrizing its canonical form is already traceless [9], and therefore requires no improvement. But such improvement is necessary in other field theories, e.g., in massless scalar field theory.

#### 2.3 Scale invariance in the ultrarelativistic domain

Regarding the virial theorem for an *N*-particle bound state in electrodynamics, Eqn (5), the possibility arises of bound states of vanishing energy, namely, states such that  $E \ll \sum_a m_a c^2$ , as the bound particles become ultrarelativistic and approach the speed of light. In one such bound state, the kinetic energy tends to infinity, but this is compensated by a potential energy that tends to minus infinity, as the particles approach one another and the system collapses.

The vanishing of the energy of an ultrarelativistic bound state is a consequence of scale invariance. As remarked in Section 2.1, the relativistic form of mechanical similarity involves the scaling of mass, because mass is inextricably linked to energy in relativistic mechanics. Hence, the full similarity demands the absence of masses. In the ultrarelativistic limit  $p_a \gg m_a c$ , electrodynamic Hamiltonian (3) becomes

$$H = \sum_{a} |c\mathbf{p}_{a} - e\mathbf{A}| + e\Phi,$$

which is also obtained just by setting  $m_a = 0$ . The absence of masses suggests scale invariance. Indeed, if **A** and  $\Phi$  are homogeneous functions of degree n = -1, *H* transforms into H/l under the phase-space coordinate scalings  $\mathbf{x} \to l \mathbf{x}$ and  $\mathbf{p} \to \mathbf{p}/l$ . As a consequence, a system of ultrarelativistic particles is such that the energy is proportional to *p*, and therefore, in the limit  $p \to \infty$ , a bound state must have E = 0 exactly and be scale invariant. Otherwise, *E* is a nonvanishing function of  $m_a$ , and the bound state is not scale invariant.

A scale-invariant relativistic state of vanishing energy is a vacuum, which is neutral and has no measurable properties. In fact, we can imagine that a neutral N-particle system loses energy by radiation and traverses a sequence of ultrarelativistic bound states of decreasing energy that ends in the vacuum. However, the study of the final stages of this process requires a QFT treatment. As an example of decay into the vacuum, we can consider the annihilation of positronium (a bound state of an electron and a positron), but positronium is a weakly coupled system, such that its annihilation occurs before it enters the ultrarelativistic domain. In addition, we can consider ultrarelativistic states with a nonzero charge. A simple example of ultrarelativistic dynamics in atomic physics is briefly studied below. This example is useful, in particular, in introducing the phenomenon of vacuum decay, important in QCD (Section 3.2).

Although, apparently, there are no electromagnetically bound particles in ordinary matter that are fully relativistic, the fastest electrons of certain atoms actually are ultrarelativistic. We focus on one of the innermost and fastest electrons of a heavy atom, and for simplicity regard it as if it were not influenced by the other electrons; namely, we consider the one-electron Hamiltonian

$$H = c\sqrt{\mathbf{p}^2 + (mc)^2} - \frac{Ze^2}{4\pi r},$$
 (23)

(the nucleus can be taken at rest and can therefore be neglected). For a circular orbit, the radial equation of motion just states the equality of the centrifugal and Coulomb forces, which can be written in terms of the respective virials as

$$\frac{c\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + (mc)^2}} = \frac{Ze^2}{4\pi r} \,. \tag{24}$$

For a general orbit, the equality of virials holds only on a temporal average. Because

$$\frac{c\mathbf{p}^2}{\sqrt{\mathbf{p}^2 + (mc)^2}} = \mathbf{p}\mathbf{v} = \frac{m\mathbf{v}^2}{\sqrt{1 - \mathbf{v}^2/c^2}},$$

the temporal average of Eqn (24) is a particularly simple case of the virial theorem expressed by Eqns (9)–(12), with  $\mathbf{E}_{\rm T} = \mathbf{B} = 0$ . We also note the connection with Eqn (6). As regards the energy, the virial theorem states that the ratio  $E/(mc^2)$  is equal to the temporal average of

$$\sqrt{1-\frac{\mathbf{v}^2}{c^2}}=\frac{mc}{\sqrt{\mathbf{p}^2+(mc)^2}}\,.$$

As usual, virial relations can be used to draw conclusions on the dynamics without solving the equations of motion. In particular, from Eqn (24) and the inequality

$$\frac{1}{\sqrt{\mathbf{p}^2 + (mc)^2}} \leqslant \frac{1}{p}$$

we deduce that  $cp \ge Ze^2/(4\pi r)$ , on the average. The equality  $cp = Ze^2/(4\pi r)$  occurs in the ultrarelativistic limit, as  $r \to 0$ ,  $p \to \infty$ , and  $E \to 0$ , and scale invariance is approached. For a circular orbit, the angular momentum is M = pr, and we hence have the condition  $M \ge Ze^2/(4\pi c)$ . Actually, when  $M < Ze^2/(4\pi c)$ , no orbit is stable and the electron must fall toward the nucleus, in a spiral trajectory [9, § 39]. In particular, as the electron approaches the nucleus and becomes ultrarelativistic, its trajectory tends to a logarithmic spiral, which is *self-similar*. Nevertheless, there are stable orbits for every E > 0, although as  $E \to 0$ , the only stable orbits are circular ones, and they become just marginally stable. For E < 0, there are no stable orbits.

We now consider the one-electron heavy atom in quantum mechanics, where the uncertainty principle sets a lower limit on the atom size; namely, both *r* and *E* are bounded below and their lowest values correspond to the ground state of the Hamiltonian. This also happens when the electron is not relativistic, but in the relativistic case, namely, for *H* in Eqn (23), a new feature arises: low-*M* states and hence states with the lowest positive energies can be unstable; that is, states with  $M \sim \hbar$  are stable only if  $Z\alpha < 1$ , where  $\alpha = e^2/(4\pi\hbar c) = 1/137$ . This is confirmed by solving the problem with a relativistic wave equation, for example, the Klein-Gordon, Dirac, or Salpeter equations: the ground state is unstable for large  $Z\alpha$ , and the stability bound for  $Z\alpha$ , which depends on the wave equation considered, is always of the

order of unity.<sup>2</sup> This instability can be interpreted as a quantum mechanical collapse in which the standard vacuum decays and gives rise to a new, negatively charged vacuum [33, § 7]. The charged vacuum can be pictured as an electron cloud attached to the nucleus. This cloud corresponds to the classical spiral trajectories that fall on the nucleus and are also asymptotically self-similar. We remark that this vacuum instability occurs due to the presence of a nonelectromagnetic interaction, namely, the strong interaction, which keeps the positive charge Ze within a nucleus sufficiently small so as to produce a very strong electromagnetic field.

By contrast, the electron–positron system does not have negative energy states, owing to the smallness of  $\alpha$ . A proper study of this problem requires QFT methods, but the problem can be reduced, in a certain approximation, to an equation similar to the Schrödinger equation with Hamiltonian (23) [34]. If there were electron–positron negative-energy states, the standard QED vacuum would not be stable against condensation of electron–positron pairs. This vacuum decay does not occur in QED, but the quark–antiquark condensation and vacuum decay do occur in QCD [34]. This property of the QCD vacuum is crucial for hadron physics, as seen in Section 3.2.

Regarding scale invariance, the fundamental effect of quantum mechanics is to introduce the new constant  $\hbar$ , which, together with c, leaves only one physical dimension, say, length. Therefore, particle masses can be associated with length scales, namely, their associated Compton wavelengths. We note that this association is consistent with the Andersen-Baeyer [4] relativistic similarity. The Compton wavelength  $\hbar/(mc)$  marks the scale where the momentum or energy uncertainties are large enough to allow the creation of a particle-antiparticle pair, and therefore the very concept of a particle loses meaning. The relevant wavelength for an atom,  $\hbar/(mv) \sim \hbar/(mcZ\alpha)$ , is definitely larger than the electron Compton wavelength if  $Z\alpha \ll 1$ . In the opposite case  $Z\alpha \gtrsim 1$ , the electron becomes ultrarelativistic and the potential of a (point-like) nucleus is strong enough to induce the creation of electron-positron pairs. Because scale invariance can only occur in the ultrarelativistic domain, it takes place for length scales much smaller than  $\hbar/(mcZ\alpha)$ , where the one-electron description is inadequate. In general, any mass scale breaks scale invariance in relativistic quantum mechanics, and scale invariance can only be recovered in the ultrarelativistic domain, but in this domain the quantum effects associated with particle creation take over. It turns out that in addition to the explicit breaking of scale invariance by any mass, scale invariance is *always* broken by quantum effects on small scales, even in massless systems [30, 31]. The failure of a classical symmetry due to quantum effects is called a quantum anomaly. The scale invariance anomaly is important for bound states, especially in QCD, and features in the formulation of a QFT virial theorem (see next section).

# 3. The quantum field theory virial theorem

In relativistic quantum mechanics, scale invariance is broken on scales of the order of the Compton wavelengths of particles, as noted above. On the other hand, on these scales, a bound system cannot be described in terms of a definite set of particles that interact via a field, because the Schrödinger equation for that set of particles neglects the possible creation of more particles. As is well known, relativistic quantum mechanics leads to QFT, in which particles and fields are on the same footing, and the Schrödinger equation is best expressed in terms of all the fields present. Therefore, the virial theorem for a definite set of elementary particles given by Eqn (5) is naturally replaced by the field theory formulations in Eqn (9) or Eqn (13). The quantum versions of these forms of the virial theorem can be derived by analogy with Landau–Lifshitz's proof [12] or directly from Eqn (21), yielding

$$\int \langle T_i^i \rangle \, \mathrm{d}V = 0 \,, \quad E = -\int \langle T_\mu^\mu \rangle \, \mathrm{d}V \,, \tag{25}$$

where the expectation values are taken with respect to a normalized stationary state representing a bound state in its rest frame.

If a semiclassical (perturbative) expansion is meaningful, then Eqns (25) amount to the classical field theory virial theorem, namely, Eqn (9) or Eqn (13), plus quantum corrections. The simplest quantum correction consists in using the limited Fock-space approximation [34], which is a variational approximation equivalent to the Schrödinger equation in a limited Fock space that does not include renormalization effects. For an electron-positron bound state [12], the result is that its energy E can be expressed either by Eqn (7), without U terms and with  $m_1 = m_2$ , of course, or by Eqn (18). In both equations, the classical motion is replaced by a probability distribution, given by a quantum wave function for the former and by a classical "mass density" for the latter. The limited Fock space approximation for bound states is connected with the standard treatment of bound states in QFT, which involves several approximations that lead to the Bethe–Salpeter equation [33, § 6].

In the limited Fock space approximation for spinor QED, no infinities arise [34, 12], but the infinities that inevitably arise in QFT must be considered in general. Those infinities require regularization with an ultraviolet (UV) cutoff, which necessarily breaks scale invariance. However, we remark that some infinities already arise in classical field theory with point-like particles and, because of this problem, the classical electrodynamics of particles of mass m and charge e are not consistent on scales smaller than the classical radius  $e^2/(mc^2)$ [9, 14]. In fact, as explained in Section 2.1, the classical relativistic virial theorem holds only after subtracting the infinite self-energy of point-like charges. Because the classical radius is smaller than the Compton wavelength  $\hbar/(mc)$ , where the quantum effects take over, the regularization of infinities is essentially a quantum problem.<sup>3</sup> For the virial theorem in the form of Eqn (25), the relevant quantum effects of renormalization manifest themselves in the energy-momentum tensor trace anomaly, as pointed out in [12].

## 3.1 QED virial theorem and the trace anomaly

Before considering the renormalization process for bound states in QFT, we recall why renormalization is necessary for the classical virial theorem to hold (Section 2.1). For a set of

<sup>&</sup>lt;sup>2</sup> The spectrum of *H* in Eqn (23) has been studied by Herbst [32]. He proves that Eqn (24) holds as an equation for the expectation values in eigenstates of *H* and that the spectrum is non-negative for  $Z\alpha \le 2/\pi$ , whereas it is unbounded below for  $Z\alpha > 2/\pi$ .

<sup>&</sup>lt;sup>3</sup> This statement must be qualified: in the exceptional case where the strong interaction holds a charge Ze in a point-like nucleus,  $e^2$  has to be replaced by  $Ze^2$ , and the condition  $Z\alpha > 1$  precisely means that  $Ze^2/(mc^2)$  is larger than  $\hbar/(mc)$ .

electromagnetically bound particles satisfying Eqn (10), the virial theorem in Eqn (9) implies that their positive kinetic "pressure" must be balanced by negative electromagnetic stresses. Therefore, their electrostatic energy, given by Eqn (12), must be negative, although it appears to be positive. In fact, this energy is actually infinite, but it can become negative after suitable subtractions, absorbed by *mass renormalizations*. Mass renormalization is also necessary in QFT because of the appearance of infinite self-energies, but the divergence structure is substantially modified. Moreover, in QFT, the charge must also be renormalized.

In a semiclassical expansion, the calculation of the first quantum correction to some bound-state energy only requires the calculation of small oscillations of the corresponding classical solution. For example, for a heavy atom, this semiclassical approach is equivalent to the old Bloch hydrodynamic treatment of the Thomas–Fermi atom model [35]. However, this model is nonrelativistic. Proper relativistic examples are provided by classical relativistic field theory localized solutions ("classical lumps") [30, Ch. 6]. In general, the calculation of quantum corrections begins with the calculation of the stability matrix determining small oscillations. The total energy is the classical energy plus the contribution of these small oscillations. Once these oscillations are quantized, the first quantum correction to the classical energy is

$$\delta E = \frac{\hbar}{2} \sum_{i} \omega_i \,,$$

where the sum is taken over the oscillation modes and  $\omega_i$  is the angular frequency of the *i*th mode. This sum diverges at high frequencies (in the UV range). The modes can be labeled by three independent numbers that can be assembled, for large values, into a wave-number vector **k**. Therefore, the UV divergence can be segregated by expressing the sum as an integral over **k**, for large *k*, and taking  $\omega_i \approx ck$ , which corresponds to free modes:

$$\delta E = \hbar c V \int^{\Lambda} \left[ k + O(1) \right] \frac{d^{3}k}{(2\pi)^{3}} = \frac{\hbar c}{8\pi^{2}} V \left[ \Lambda^{4} + O(\Lambda^{3}) \right],$$

where V is the system volume, a UV cutoff  $\Lambda$  is introduced, and the sum over two polarizations is taken, assuming that the free modes correspond to photons. We could make  $\delta E$ finite and attribute a physical meaning to it by choosing a physical cutoff  $\Lambda$  such as  $\Lambda \sim mc/\hbar$ , the inverse of the electron Compton wavelength. At any rate, the leading term, which is positive and proportional to  $\Lambda^4$ , is present in the absence of matter and must be subtracted. The subleading divergent terms depend on the detailed spectrum of the  $\omega_i$  and hence on the matter state, and can have either sign. After renormalization (or definition of a finite  $\Lambda$ ), these terms give rise to measurable electromagnetic energies.

The divergent parts of the vacuum energy indeed have to be subtracted in calculations of electromagnetic energies, for example, in the calculation of Casimir or Van der Waals forces [36]. For illustration, we briefly consider the case where the frequencies  $\omega_i$  are easy to calculate, namely, the case of a dilute gas of N atoms per unit volume in a box of volume V [36, § 3.7]. The allowed frequencies are the free-field frequencies modified by the refractive index of the gas  $n(\omega)$ , i.e.,  $\omega_i \approx ck/n$ . If we assume, for simplicity, that there is only one resonant frequency  $\omega_0$ , then we can take

$$n(\omega) = 1 + \frac{Ne^2}{2m(\omega_0^2 - \omega^2)} \,.$$

Therefore,

$$\begin{split} \delta E &= \frac{\hbar c V}{(2\pi)^3} \int \frac{k \,\mathrm{d}^3 k}{n(k)} \\ &= \frac{\hbar c V}{2\pi^2} \left[ \frac{\Lambda^4}{4} + \frac{N e^2 \Lambda^2}{4m c^2} + \frac{N e^2}{4m c^2} \left( k_0^2 + \frac{N e^2}{2m c^2} \right) \right. \\ &\times \left. \ln \left| \frac{\Lambda^2}{k_0^2 + N e^2 / (2m c^2)} - 1 \right| \right], \end{split}$$

where  $k_0 = \omega_0/c$ . The quadratic divergence is proportional to the number of atoms NV, but is independent of  $\omega_0$ , and the proportionality constant only depends on fundamental constants. Indeed, this term corresponds to the energy of *NV free* electrons, and, if we again take  $\Lambda \sim mc/\hbar$ , the energy per electron is of the order of  $e^2/(\hbar/mc)$ , i.e., of the electron quantum self-energy. Therefore, this divergence can be absorbed by mass renormalization. The logarithmic term is smaller in magnitude and depends on  $\omega_0$ . If  $Ne^2/m \ll \omega_0^2$  (the dilution condition), the logarithmic term can be identified with the Lamb shift [36, § 3.7]. We note that after renormalization, the quantum corrections are not only finite but also small (of the order  $\alpha$ ). If, instead of  $\Lambda \sim mc/\hbar$ , we take the much smaller cutoff  $\Lambda = k_0$ , such that n(k) > 1 over the full range of integration, then  $\delta E$  is negative, corresponding to attractive Van der Waals forces.

The result of the preceding calculation of  $\delta E$  and, in particular, its cutoff dependence are typical of one-loop effective potential calculations. Indeed, the energy E can be generally calculated in terms of the minimum of the effective action corresponding to the bound state. The  $\Lambda^4$  term is already present in the vacuum, when a generic regularization method is used, and leads to the well-known cosmological constant problem. In this regard, Ossola and Sirlin [37] study the contribution of fundamental particles to the vacuum energy density, comparing various regularization methods, and conclude that for noninteracting particles, the divergence can be made quadratic rather than quartic (no  $\Lambda^4$  term) and that massless particles do not contribute. This result follows from properly considering relativistic covariance and scale invariance of free-field theories in the massless limit and therefore calculating  $\langle T^{00} \rangle$  or any other component of  $\langle T^{\mu\nu} \rangle$ in terms of the trace  $\langle T_{\lambda}^{\lambda} \rangle$  using the relation  $\langle T^{\mu\nu} \rangle =$  $\langle T_{\lambda}^{\lambda} \rangle g^{\mu\nu}/4$ . The divergences of the trace  $\langle T_{\lambda}^{\lambda} \rangle$  are easily obtained for free fields, but whenever interactions are involved, new divergences appear (new quartic, quadratic, and logarithmic divergences). After renormalization, the energy-momentum tensor trace has quantum corrections, i.e., a trace anomaly appears. Notably, the trace is not zero even in the massless case. Because the virial theorem can be expressed in terms of the energy-momentum tensor trace, we conclude that the virial theorem must include a quantum trace anomaly term [12], since this anomaly appears even in the vacuum.

The general form of the fermionic QED trace anomaly was computed in [38]:

$$T^{\mu}_{\mu} = -K_1 m_0 c^2 \bar{\psi} \psi + K_2 N \left[ F_{\lambda \sigma} F^{\lambda \sigma} \right], \qquad (26)$$

where N indicates the type of normal product (see [38]) and

$$K_1 = 1 + \delta(\alpha) = 1 + \frac{3\alpha}{2\pi} + \dots,$$
  
$$K_2 = \frac{1}{4} \beta(\alpha) = \frac{1}{4} \left( \frac{2\alpha}{3\pi} + \frac{\alpha^2}{2\pi^2} + \dots \right).$$

The functions  $\delta$  and  $\beta$  are respectively associated with the anomalous dimension of the fermion field  $\psi$  and with the coupling constant renormalization. We note that the first term of the trace anomaly consists of the classical part in Eqn (16) and quantum corrections due to mass renormalization, whereas the second term is purely of the quantum origin and is due to charge renormalization.

The QED trace anomaly is indeed small, due to the smallness of  $\alpha$ , which makes the perturbation theory work. This implies that the renormalized value of the quantum correction  $\delta E$  to a classical bound-state energy is also small. Fundamentally, this implies that the vacuum is not altered, that is, there is no vacuum decay. Naturally, the vacuum decay in the strong field of a nucleus is an exception, due to the actual coupling constant  $Z\alpha$  not being small (Section 2.3). At any rate, it is interesting to consider general strongly bound states. A state can be considered strongly bound if the larger part of its energy is due to the interaction of its constituents rather than to their rest masses. This condition is naturally satisfied by hadronic states, namely, the states of quarks bound by the strong interaction described by QCD.

## 3.2 Scale symmetry breaking,

#### Callan-Symanzik equations, and QCD

QCD is the theory of strong interactions, although effective low-energy theories, such as the meson-nucleon interaction theory, are still useful. The meson-nucleon theory, which only includes the lightest mesons, is somewhat similar to the photon-electron theory in QED, except for the finite range of the mesons. However, the magnitude of the mesonnucleon coupling is  $g^2/(4\pi\hbar c) \simeq 1$ , which should imply that the bound states are ultrarelativistic, and their energies nearly vanish! This does not happen, of course, because scale invariance is badly broken by quantum effects. Coleman [30, Ch. 3] used the meson-nucleon model to illustrate the breaking of scale invariance, by analyzing the perturbative behavior of correlation functions in the deep Euclidean region: in addition to simple powers of the scale (given by the field dimensions), logarithms also appear. Coleman then shows that some series of logarithms can be absorbed into anomalous field dimensions, while others remain, but the series of the latter can be absorbed by a renormalization of the couplings. In general, the scale dependence introduced by the renormalization process can be expressed in terms of a set of simple differential equations for the correlation functions, namely, the renormalization group equations, while the effect of scalesymmetry breaking is expressed by the Callan-Symanzik equations. Naturally, both sets of equations are related [30, Ch. 3]. There is an infinite set of Callan-Symanzik equations, one equation for every correlation function. To apply them to bound states, it is useful to realize that they can all be derived from a master Callan-Symanzik equation, which is in addition connected with the energy-momentum tensor trace anomaly.

To obtain the master Callan–Symanzik equation, we introduce the generating functional

$$Z[\lambda^{j},g_{\mu\nu}] = \int \mathcal{D}\varphi \, \exp\left(\mathrm{i}S\left[\varphi,\lambda^{j}\right]\right),\,$$

namely, the vacuum transition amplitude, where the action S depends on a set of fields  $\varphi$  and coupling constants  $\lambda^{j}$  and is defined on a curved space-time; we also introduce the functional  $W = -i \log Z$ . The effect of a scale transformation can be alternatively realized in terms of the metric, such that

$$l \frac{\mathrm{d}W}{\mathrm{d}l} = 2 \int \mathrm{d}^4 x \, g^{\mu\nu}(x) \, \frac{\delta W}{\delta g^{\mu\nu}(x)} = \int \mathrm{d}^4 x \sqrt{-\det g_{\mu\nu}} \, \langle T^{\mu}_{\mu} \rangle \,,$$

where l is the scale, or in terms of the coupling constants, with

$$l \frac{\mathrm{d}W}{\mathrm{d}l} = \sum_{i} \beta^{i}(\lambda) \int \mathrm{d}^{4}x \sqrt{-\det g_{\mu\nu}} \langle \mathcal{O}_{i} \rangle + \mathcal{A} \,,$$

where  $\beta^{i}(\lambda) = l d\lambda^{i}/dl$  ("beta functions"),  $\{O_{i}\}$  is the set of interaction terms (a subset of "composite fields"), such that

$$\frac{\delta W}{\delta \lambda^i} = \int \mathrm{d}^4 x \sqrt{-\det g_{\mu\nu}} \left\langle \mathcal{O}_i \right\rangle,$$

and A is an extra anomalous term that arises in curved geometry. Therefore, in a flat space-time,

$$\int d^4x \langle T^{\mu}_{\mu} \rangle = \sum_i \beta^i(\lambda) \int d^4x \langle \mathcal{O}_i \rangle .$$
<sup>(27)</sup>

This equation<sup>4</sup> generates an infinite hierarchy of equations for the correlations of the  $O_i$  by taking derivatives with respect to the couplings  $\lambda^i$ . Equations for the correlations of fields other than  $O_i$  and, in particular, of the elementary fields  $\varphi$  can also be obtained by introducing the corresponding sources in W. These equations are, in essence, the standard Callan-Symanzik equations. The master Callan-Symanzik equation (27) shows the general form of the trace anomaly [41]. For example, in QED, Eqn (27) is equivalent to operator equation (26). To see this, we note that the set  $\lambda^i$  of coupling constants includes dimensional constants. However, the dimensionless coupling constants play a special role, because they can contribute to each  $\beta^{i}(\lambda)$ , whereas the dimensional coupling constants (corresponding to super-renormalizable interactions, e.g., mass terms) can only contribute to some of them, as a simple power-counting argument shows. Indeed, in Eqn (6),  $\alpha$  contributes to both terms, but  $m_0$  is only allowed in the first term and with the exponent one. We also note that the "beta functions" of dimensional coupling constants are classically nonvanishing and give rise to the classical value of  $T^{\mu}_{\mu}$ , whereas those of dimensionless coupling constants are purely "anomalous."

Equation (27) can be generalized by replacing the vacuum expectation values with the expectation values in the stationary state corresponding to a bound state, namely, by introducing the appropriate boundary conditions in the generating functional Z. For stationary states, the four-dimensional integrals in Eqn (27) become spatial integrals, and the equation directly yields quantum corrections to the

<sup>&</sup>lt;sup>4</sup> Equation (27) is well known. It plays an important role in 2d field theory [39, 40] and is discussed, in general, in [41].

relativistic virial theorem (13), in terms of the beta functions and the expectation values  $\langle O_i \rangle$ . The beta functions can be calculated in the perturbation theory, but the  $\langle O_i \rangle$  are more difficult to calculate because they are essentially nonperturbative.

To appreciate the importance of quantum corrections to the relativistic virial theorem, we consider the theory of hadrons in QCD. As is well known, the crucial difference between QED and QCD is that their beta functions have opposite signs, such that QCD is *asymptotically free*, namely, the interaction vanishes at high momenta. By contrast, quarks interact strongly at low momenta, and, in fact, hadrons made of light quarks have a size such that the contributions of the interactions to their energy are much larger than the contribution of the quark masses. At the same time, the contribution of the trace anomaly is crucial. To clarify all this, we consider a concrete hadron model.

In the MIT bag model [42], a constant positive potential energy B, per unit volume, is added to a free-field Lagrangian density inside a finite part of space. A semiclassical description of this simple field theory constitutes a sort of statistical model of hadrons. In the MIT bag model, the particles, named "partons" but now identified with quarks and gluons, are free and massless inside the bag, and move with the speed of light. Therefore, hadrons are truly ultrarelativistic bound states, as studied in Section 2.3. The vacuum inside the bag takes place for small distances and high momenta, and therefore corresponds to the perturbative QCD vacuum. As the particles inside the bag separate from one another, they enter the strong-coupling phase, in which the perturbative vacuum is not stable and decays into the standard QCD vacuum (a condensate of quark-antiquark pairs). Vice versa, the QCD vacuum becomes unstable at high momenta and decays into the perturbative vacuum. The transition is assumed to occur over a very small distance on the bag surface. In the statistical model, the confining interaction is just modeled as a vacuum pressure, namely, the difference between the null vacuum pressure outside the bag and the negative "vacuum pressure" inside the bag. The effect of this pressure difference is just to keep the particles inside the bag, as if the difference of pressures were produced by the bag surface.

The application of the virial theorem to the gas of free massless particles of the statistical bag model is straightforward: the theorem just states that the energy density is three times the pressure difference [9, § 35]. On the other hand, the energy-momentum tensor of the perturbative vacuum is constant and proportional to the metric  $g^{\mu\nu}$ , which corresponds to a constant term in the Lagrangian. Therefore, we have the following relations: the virial relation  $E_{\text{quarks+gluons}}/V = 3P_{\text{quarks+gluons}}$  (where V denotes the bag volume), the pressure balance in the bag,  $P_{\text{quarks+gluons}} = -P_{\text{vac}}$ , and the vacuum relation  $E_{\text{vac}} = -P_{\text{vac}}V$ . We remark that the pressure balance is also a virial relation, namely, a particular case of  $\int \overline{T_i}^i dV = 0$ . All the preceding equations together imply that

$$E_{\text{quarks+gluons}} = 3E_{\text{vac}}, \qquad (28)$$
$$E = E_{\text{quarks+gluons}} + E_{\text{vac}} = 4E_{\text{vac}} = 4BV.$$

We note that the confining interaction accounts for one fourth of the total energy, as in the classical model of a particle with an electric charge in Section 2.1 that has the confining Poincaré energy-momentum tensor proportional to  $g^{\mu\nu}$ .

The bag surface has been taken fixed, but this is not necessary: the only change when the bag surface is allowed to move is that V must be replaced with the average  $\overline{V}$ , whence  $E = 4B\overline{V}$ . This equation is also obtained in [42] as a relativistic virial theorem specially tailored to the bag model Lagrangian and proved without using the energy-momentum tensor.

Since the (improved) classical energy-momentum tensor is traceless in the MIT bag model, confinement is due to the quantum energy-momentum tensor trace part (the part proportional to  $g^{\mu\nu}$ ), i.e., to the trace anomaly. In the MIT bag model, this quantity is spatially constant, but the trace anomaly in QCD is given by an equation similar to Eqn (26) that includes suitable color indices; namely, it contains the term  $\bar{\psi}m\psi$ , where *m* is the quark mass matrix, and an  $F^2$  term, with the color indices contracted. According to virial theorem (25) and assuming that the quarks are massless, the energy is given by the spatial integral of  $-\langle T_{\lambda}^{2} \rangle = -\beta(g) \langle F^{2} \rangle/(2g)$  (in what follows, we set  $\hbar = c = 1$ ). Therefore, in the first order in  $\alpha_{\rm s} = g^{2}/(4\pi)$ ,

$$E = -\int \mathrm{d} V \langle T_{\lambda}^{\lambda} \rangle = \frac{9\alpha_{\rm s}}{8\pi} \langle F^2 \rangle$$

(for three quark flavors). This equation is a particular case of Eqn (27), for a stationary state and only one coupling. The required spatial integral

$$\int \langle F^2 \rangle \, \mathrm{d}V = -2 \int (\mathbf{E}^2 - \mathbf{B}^2) \, \mathrm{d}V$$

must be calculated with the bag boundary conditions. For the lowest-energy states, the integral can be calculated in terms of the respective integrals of  $\mathbf{E}^2$  and  $\mathbf{B}^2$ , which also appear in the calculation of gluonic corrections to the ground states of the MIT bag model [43, § 3.12]. These integrals contain divergent terms that correspond to self-interactions that are absorbed by mass renormalizations. The final result is that the electric integral vanishes and the magnetic integral is proportional to 1/R, that is, to the inverse of the spherical bag radius, with the proportionality coefficient of the order of unity, as expected. Indeed, the partons in the lowest excitation states of the bag must have an energy of the order of  $NV^{-1/3}$ , where N is a small integer, as follows from dimensional analysis [42]. The trace anomaly calculation cannot specify N, since  $\alpha_s$  is undetermined.

Taking any one of the lowest-energy hadronic states, the bag relation  $E = 4BV \sim N/R$ , with  $V \sim R^3$ , implies that  $B \sim R^{-4}$ , where R is the hadron radius. Therefore, the fundamental dimensional parameter B can be traded for R. But we instead introduce the fundamental QCD dimensional parameter as the RG-invariant  $\Lambda_{QCD}$ , which naturally arises in massless QCD via the renormalization process constituting what is called *dimensional transmutation*;  $\Lambda_{\text{OCD}}$  can be defined, for example, as the scale at which the QCD running coupling equals unity,  $\alpha_s(\Lambda_{QCD}) = 1$ . The scale  $\Lambda_{QCD}$  determines the size of the lowest excitation states of hadrons,  $R \sim 1/\Lambda_{\rm QCD}$ . Therefore, we have  $B \sim \Lambda_{\rm QCD}^4$  and  $E_{\rm vac} = BV \sim \Lambda_{\rm QCD}$ . Also,  $E \sim \Lambda_{\rm QCD}$ . Naturally, R and E can be expressed in terms of  $\Lambda_{\rm QCD}$  for general hadron models with massless quarks, not only for the MIT bag model, as deduced from dimensional transmutation. Dimensional transmutation is a consequence of the breaking of scale invariance, which also gives rise to the trace anomaly. However, we remark that the virial theorem only allows establishing *one* relation, which is *exact* and states, in essence, that the confining term (the trace anomaly) accounts for one fourth of the total energy, for massless quarks, as in Eqn (28). This exact relation is derived from the obvious equation  $T^{\mu\nu} = T^{\mu\nu}_{quarks+gluons} + T^{\alpha}_{\alpha}g^{\mu\nu}/4$ , where, classically,  $T^{\alpha}_{\alpha} = 0$ , and

$$T^{\mu\nu}_{\text{quarks+gluons}} = rac{1}{2} \left( ar{\psi} \gamma^{(\mu} \mathcal{D}^{
u)} \psi - \mathcal{D}^{*(\mu} ar{\psi} \gamma^{
u)} \psi 
ight) 
onumber \ - rac{g^{\mu
u}}{4} F^2 + F^{\mu\sigma} F^{
u}_{\sigma} \,.$$

Therefore, on account of the QFT virial theorem in Eqn (25),

$$E = \int \langle T_{\text{quarks+gluons}}^{00} \rangle \,\mathrm{d}V + \frac{E}{4} \,, \tag{29}$$

in analogy with Eqn (14). Naturally,  $T_{\alpha}^{\alpha} = 0$  implies E = 0, but the trace anomaly makes *E* nonvanishing. Equation (29) is exact for massless QCD and is approximately valid for hadrons made of light quarks. On the other hand, there is another relation, always approximate, that determines the size of the lowest excitation states of hadrons in terms of a fundamental parameter, say,  $\Lambda_{\rm QCD}$  (or *B*, in the bag model). Obviously, the latter relation cannot hold for high excitation states, whereas the virial theorem does.

In more elaborate hadron models, one can consider quark masses  $m_Q$ , making the classical energy-momentum tensor not traceless. Nevertheless, for light quarks, namely, u and d quarks (and surely the s quark as well),  $m_Q \ll \Lambda_{QCD}$ , and therefore the trace anomaly is still dominated by the  $F^2$  term. If Nature were such that  $\Lambda_{QCD} \ll m_Q$  for all quarks, then QCD, in spite of still having the property of quark confinement and still being asymptotically free, would share some features of QED: all hadrons would actually be nonrelativistic quark bound states.

# 4. Conclusions

The virial theorem expresses the condition of average equilibrium of a bound state, namely, the condition that its average shape stays constant and, in particular, that its average size stays constant. The former condition is expressed as the tensor virial theorem, whereas the latter is expressed as the ordinary scalar virial theorem, Eqn (9), which is actually the trace of the tensor virial theorem. This theorem implies that the positive pressure of particles or fields corresponding to a traceless energy-momentum tensor must be compensated by the negative stresses corresponding to the trace part of the energy-momentum tensor. Equation (9) is valid in both classical and relativistic physics, but in the latter the equivalent form in terms of the energy-momentum tensor trace, Eqn (13), is more convenient. This equation implies that the fraction of the bound-state energy corresponding to a traceless energy-momentum tensor is three fourths. This fraction is connected, in particular, with the classic 4/3problem of electromagnetic inertia.

The trace of the stress tensor and the trace of the energy– momentum tensor are respectively related to the generators of space and space-time dilatations. Naturally, the connection with dilatations arises because these are the transformations that change the size of the system. The virial theorem shows that the average size of the system is determined by its energy *E*. The full space–time scale invariance only takes place when the (average of the) trace of the energy–momentum tensor vanishes, which corresponds to E = 0 and, in principle, to the vacuum.

While nonrelativistic mechanics allows the similarity of motion for kinetic and potential energies that are homogeneous functions of their respective variables, this symmetry is lost in relativistic mechanics, in which the kinetic energy is never a homogeneous function. However, there is a relativistic notion of similarity, which is such that velocities are left invariant, but masses scale as energies do. Therefore, the only situation in which full scale invariance can appear is in the ultrarelativistic domain, when the masses can be set to zero and the virial theorem implies that the energy vanishes. This ultrarelativistic scale invariance appears, for example, in the strong electric field of a heavy nucleus. On the other hand, only in the context of quantum field theory does scale invariance acquire its deepest meaning, since there is naturally only one scale, either length or mass. In a theory with massless fundamental particles, bound systems should have E = 0 and be massless as well, such that there should be no scales and the theory would naturally be scale invariant. However, in quantum field theory the vacuum is nontrivial, and virtual particle-antiparticle pairs arise that make quantum contributions to energies that are actually infinite and, after renormalization, bring about a scale dependence, expressed by the Callan-Symanzik (or renormalization group) equations. This dilatation symmetry breaking can also be expressed in terms of the energy-momentum trace anomaly, which must be included in the quantum field theory virial theorem. Therefore, this theorem is just a generalization of the Callan–Symanzik or trace anomaly equations, which makes them applicable to bound states. Full scale invariance occurs only at renormalization-group fixed points, where the virial theorem becomes trivial.

The QED energy–momentum tensor trace anomaly is proportional to (powers of) the coupling constant  $\alpha = 1/137$ , and is therefore small. In general, quantum corrections to relativistic bound states are small in QED. Because those bound states are weakly coupled, they are weakly relativistic as well. Exceptionally, the strong nuclear interaction within a heavy nucleus creates a concentration of positive charges, such that their electromagnetic field is sufficiently strong to produce important relativistic effects, in particular, a qualitative change in the quantum vacuum around the nucleus: an electron bound to the nucleus can have a *negative* energy, and the bound state actually decays to a new vacuum state with zero energy.

If the QED coupling  $\alpha$  were sufficiently large, vacuum instability would not be exceptional, because positronium could then have a negative-energy state, and the standard QED vacuum would be unstable against production of electron-positron pairs. A phenomenon of this kind occurs in QCD, and hence the QCD vacuum is not a perturbative vacuum but a condensate of quark-antiquark pairs. This phenomenon is associated with the magnitudes of dilatation symmetry breaking and of the trace anomaly in QCD. In fact, the existence of hadrons in massless QCD is conditioned by the trace anomaly, which is an essential ingredient of the virial theorem in this case. The dilatation symmetry breaking can be expressed as a dimensional transmutation, which gives rise to a scale, the renormalization group invariant  $\Lambda_{\rm QCD}$ . This scale is definitely larger than the masses of light quarks, which implies that the essential properties of the hadrons formed by

them are described by massless QCD. Assuming that the quarks are massless, the virial theorem implies that one fourth of the energy (or mass) of a hadron comes from the trace anomaly term, whereas three fourths come from quarks and gluons. The bag model of hadrons is useful for a straightforward application of the virial theorem and the computation of the trace anomaly, as has been shown.

A different type of strong interaction occurs in astrophysics, namely, in compact bodies, where the additive nature of gravity leads to strong gravitational fields, besides strong quantum effects. However, the generalization of the virial theorem to General Relativity presents problems that are beyond the scope of this paper, as cautioned in the introduction. Nevertheless, we can affirm that the master Callan– Symanzik equation (27) that includes the curvature-dependent trace anomaly  $\mathcal{A}$  must play an important role in such a generalization.

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