Physics – Uspekhi **56** (9) 883 – 909 (2013)

PHYSICS OF OUR DAYS

PACS numbers: 06.20.Jr, **12.20.–m**, 31.30.J–, 42.50.Xa

Progress in the accuracy of the fundamental physical constants: 2010 CODATA recommended values

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DOI: 10.3367/UFNe.0183.201309d.0935

Contents

Abstract. Every four years, the CODATA Task Group on Fundamental Constants presents tables of recommended values of the fundamental physical constants. Recently, 2010 CODATA recommended values (Mohr P J, Taylor B N, and Newell D B "CODATA recommended values of the fundamental physical constants: 2010" *Rev. Mod. Phys.* 84 1527 (2012)), based on global data up to 31 December 2010, were published. In the present review, we briefly analyze the new recommended values, as well as new original data, on which the determination is based. To facilitate the consideration, the data are subdivided into several groups. New original theoretical and experimental results are discussed for each group separately. Special attention is paid to experimental and theoretical progress in the determination of the Rydberg constant R_∞, the electron–proton mass ratio m_e/m_p, the fine-structure constant α, the Planck constant h, the Boltzmann constant k, the Newtonian constant of gravitation G, and the anomalous magnetic moment of the muon a_μ. In conclusion, the prospects of redefining units of the International System (SI) in terms of fundamental physical constants, which is currently under active discussion by the metrological community, are considered. The very possibility and efficiency of a practical realization of such a scenario with the redefinition directly depends on the status of the determination of the fundamental constants.

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Received 12 September 2012
Uspekhi Fizicheskikh Nauk **183** (9) 935 – 962 (2013)
DOI: 10.3367/UFNr.0183.201309d.0935
Translated by G Pontecorvo; edited by A Radzig

1. Introduction

Determination of the values of the fundamental physical constants always represents a problem that requires utilization of the most advanced experimental and theoretical methods. Studies in this field play an important role in physics and metrology, giving rise to applications of the utmost diversity. Thus, at present, within the metrological community the possibility is actively being discussed of redefining units of the International System (SI) in terms of fundamental constants such as the Planck constant h, the elementary charge e, the Boltzmann constant k, and the Avogadro constant N_A. The very possibility of realizing such a scenario depends directly upon the state of the art

related to determination of the values of the fundamental constants.

In atomic physics, determination of the Rydberg constant R_∞ revealed a considerable discrepancy between the results obtained in atomic spectroscopy of the hydrogen atom and of muonic hydrogen. At the same time, the progress achieved in the measurement of the fine-structure constant α by methods of Raman spectroscopy and of the anomalous magnetic moment of the electron in a Penning trap makes it possible to test calculations of five-loop Feynman diagrams.

The fundamental and practical applications of results of the determination of constants are, apparently, in no way exhausted by the above examples.

Fundamental constants enter into equations in different fields of physics, and for this reason their measurement relies on the existence of different methods, different uncertainties, and different approximations. Often, the results of measurements are not some fundamental constant or another itself, but combinations of them. The joint processing of collected global data is called *adjustment* of the values of fundamental physical constants. This procedure is implemented regularly by the CODATA Task Group on Fundamental Constants[1] [1–7]. The last such adjustment was based on the data available up to the end of 2010. A detailed analysis of the data and results obtained has been published recently [7], so the present review is devoted to this adjustment.

In the Russian language, adjustments of the values of the constants were regularly provided in the form of tables or reviews [8–14]. Thus, for example, the results of the 2006 adjustment [5, 6] were discussed in a review [14], so for this reason, in presenting results of the last adjustment [7], we only deal in detail with the results that appeared in 2007–2010.

Utilization of the term 'adjustment' reflects the fact that the data processing procedure differs significantly from the standard procedure.

Regarding adjustment of the values of the fundamental constants, extremely diverse data are jointly processed. These data are characterized by totally different precisions and are based on measurements or calculations pertaining to different areas of physics, and they are often correlated in quite a complex manner. These correlations are relevant both to the very results of measurements or calculations (for example, the same standards may be utilized) and to values that should appear at the concluding stage of adjustment. Thus, for example, simultaneous measurements of e^2/h, h, e/h, eN_A, hN_A, N_A, and e in different experiments must yield consistent adjusted results.

As to the tables of recommended values of the constants, we note that it is apparently only possible to present a mimimal set of values and to indicate the correlation coefficients. This, however, is not usually done in reference tables, since it complicates their utilization. Thus, the value of e^2/h is known with a higher precision than the individual values of e and h, and without knowledge of the correlation between their uncertainties the calculation of e^2/h by e and h is impossible. As a result, besides the minimal set of independent constants, tables also include a broad set of other, in a sense excess, combinations, the uncertainties of which are found with account of the correlations between the recommended values of fundamental constants from the minimal independent set.

Adjustment implies testing the self-consistency of input data and providing a self-consistent set of output values. For implementing the first task, a detailed discussion of the input data is carried out, and in the case of the second, tables of values are built up that are sufficient for direct application.

In recent years, adjustments of the values of the fundamental constants have been performed every four years by the [International] CODATA Task Group on Fundamental Constants. The actual processing of available data is carried out relatively fast. Thus, all relevant data were accepted until 31 December 2010, and in three months the recommended values were already available on the site of NIST[2] at the following address: http://physics.nist.gov/cuu/Constants/index.html.

Meantime, to write a full-length text (nearly a hundred pages long) with a detailed description of all the data and of the processing procedures applied takes significant time, and for this reason the main publication [7] appeared with a noticeable delay. Since the 2010 adjustment left certain questions related to the poor agreement between some of the input data without answers, not only the details of the adjustment itself are discussed in the present review, but the progress achieved later is also briefly touched upon. To avoid confusion, new, more recent values of the constants are discussed in the text, but they are not presented explicitly in the formulae, tables, or figures.

Taking into account the large volume of accumulated experimental and theoretical data, we only present references to certain recent studies that are the most important. The remaining references can be found in the respective work on the adjustments made in 1998 [3], 2002 [4], 2006 [5, 6], and 2010 [7]. The number of references in each of these studies amounts to several hundred, and in our opinion it would not be expedient to present them completely herein.

2. Structure of global data and the results of their adjustment

The use of data exhibiting significantly differing precisions and of theoretical expressions containing diverse combinations of fundamental constants gives rise to a certain structure permitting analyzing the data by selected groups.

The role of some data does not depend on the method applied in processing them (it might be least-squares simultaneous processing of the data or sequential group-by-group processing), but on the precision with which these data are known. Here, it is not the relative or absolute precision itself that is important, but how this precision relates to the uncertainties of other data.

Thus, often a situation arises when it is necessary to deal with the products of quantities that are known with relative uncertainties that may differ, say, by an order of magnitude. Such products can be dealt with as follows. First, the most precise data are processed without taking into account any such products, since such products exhibit a low precision. Then, the quantities characterized by less precision are established in the course of analyzing the products, and, here, it is actually assumed that the more precise data are already known with absolute precision.

[1] Task Group on Fundamental Constants (TGFC) of the Committee on Data for Science and Technology (CODATA) of the International Council of Science (ICSU).

[2] National Institute of Standards and Technology (NIST).

Often, in the leading approximation various additive corrections to simple formulae arise. In this case, the hierarchy of data is determined not only by the accuracy itself of the quantities involved in them but also by the scale of the terms into which they enter.

Let us clarify the above by a simple example. Thus, it turns out that the values of the Rydberg constant R_∞ and of the fine-structure constant α are determined quite separately, although they enter into the primary expressions in a totally 'mixed-up' form. The following relations are typical:

$$\alpha^2 = R_\infty \frac{2h}{m_e c}\,, \tag{1}$$

$$\left(c_1 + c_2\alpha^2\right) R_\infty c = \nu\,. \tag{2}$$

The expressions in the right-hand parts of the equalities contain measured quantities and, in particular, there are results for the frequencies ν of several transitions in hydrogen and deuterium, associated with different numerical coefficients c_1 and c_2 on the order of unity. When the fine-structure constant is determined, the Rydberg constant is involved multiplicatively, while in the case of determination of the Rydberg constant the fine-structure constant is involved additively. (Real formulae have numerous corrections that do not alter the situation qualitatively and that for this reason are dropped here.)

The most important results for the Rydberg constant are characterized by a relative uncertainty $\sim 10^{-11}$, while the uncertainty of the fine-structure constant is no higher than 3×10^{-10}.

Clearly, the uncertainty of the Rydberg constant must be neglected in relationship (1), and, then, by measuring h/m_e it is possible to find α. At the same time, the fine-structure constant is known sufficiently well for its uncertainty to be neglected in expression (2) in the small term with c_2 proportional to $\alpha^2 \simeq 5 \times 10^{-5}$, which already permits finding R_∞, assuming α to be known with absolute precision.

Thus, data are separated into groups: a group associated with the Rydberg constant, and a group associated with the fine-structure constant. Actually, they are not correlated and are processed independently. (In this section, we repeatedly use the word 'actually' so as to stress that all relevant data, with rare exceptions, are processed together, while their precisions, however, are such that, in the case of independent group-by-group processing of data, the ultimate result remains intact.) Similar mechanisms lead to differentiation of the data into several specific classes, which are actually processed independently of each other (for details see Refs [12–14]).

• There always exist data whose precision may far exceeds that of all others (more correctly, the uncertainties of which may be neglected in evaluating all other constants). Such data are termed *auxiliary*. They can be found before launching the main procedure. In some cases, they are actually adopted from the outside (for instance, as in the case of the masses of several atoms) and are, thus, indeed determined outside the main procedure. In other cases, they are found within the framework of the main procedure, but, being very precise, they are actually determined independently of the main data processing procedure.

• In electrical metrology, there are two groups of data that have historically played a key role for many years. One of these groups (involving the more precise data) is associated with the fine-structure constant α, while the other one is related to the Planck constant h and the elementary charge e. Dealing with these two groups is just what adjustment comprises in the narrow sense of the word. It is precisely here that extremely diverse measurements appear, standards are utilized, and so on. With account of the real precision of available data, the groups are actually processed independently: the first one corresponds to the more precise data, and in processing the second group the value of the fine-structure constant is actually taken from the processed data of the first group.

• There is a series of data of relatively low precision in the case of quantities that can be formally linked by various relationships involving auxiliary constants or constants from the two aforementioned groups. Thus, for example, direct measurements of the elementary charge e are significantly less precise than its values determined from measurements of different combinations, such as e^2/h, e/h, and eN_A, and from measurements of h and N_A.

Since the precision of direct measurements of e and of a number of other similar quantities is extremely low, the results of direct measurements are not taken into account in the adjustment, while the recommended values are calculated on the basis of more precisely determined fundamental constants mutually interrelated with them. Below, we shall not especially single out such constants, but present their values in the course of discussing the respective groups.

• A number of other constants always exist, such as the Newtonian constant of gravitation G, the Boltzmann constant k, or the anomalous magnetic moment of the muon a_μ, that are practically determined absolutely independently of all other quantities and of each other. They are not involved in the actual adjustment, and their recommended values given in tables are calculated independently of the main procedure.

If data were indeed processed applying the least-squares procedure, only one or two of the most precise values would be statistically significant within each one of the groups. However, it is essentially important for the data inside a group to be consistent with each other. As we shall see below, this is not always so; if the data are not consistent, a conclusion provided by experts is required on how to deal with them.

The data of one group or another are corrected on the basis of the conclusion given by experts. A possible version is to discard the data exhibiting poor agreement with all the other ones. As a rule, this is not done. The inconsistency of one measurement with others, in the absence of direct indications of possible errors, cannot be considered a ponderable argument.

The editing of data mainly concerns their uncertainties. In such a case, the uncertainty of input data can be changed, or the uncertainty of values that have already been adjusted can be extended. Extension of the uncertainty implies no arbitrariness: if uncertainties of the data are estimated adequately, the distribution of their scatter should be normal and be characterized by a certain dispersion. If this is not the case, straightforward averaging of the data (and in the case of miscellaneous data the least-squares method serves as a direct analogue of the weighted averaging of data with account of their uncertainties) is insufficient, so extension of the uncertainty reflects the statistically improbable configuration of the input data.

At present, the computational capacity of supercomputers permits the processing of all available data together

Table 1. Recommended values of auxiliary fundamental constants. Precisely known values are presented in the upper part of the table; measured auxiliary constants are in the lower part; u_r represents the relative standard uncertainty.

Quantity	Symbol	Value	u_r
Speed of light in vacuum	c	299 792 458 m s^{-1}	(exact)
Magnetic constant	μ_0	$4\pi \times 10^{-7}$ N A^{-2}	(exact)
Electric constant	$\epsilon_0 = 1/(c^2\mu_0)$	$8.854\,187\,817\ldots \times 10^{-12}$ F m^{-1}	(exact)
Mass of ^{12}C atom	$m(^{12}\mathrm{C})$	12 u	(exact)
Rydberg constant	R_∞	10 973 731.568 539(55) m^{-1}	5.0×10^{-12}
Proton–electron mass ratio	m_p/m_e	1836.152 672 45(75)	4.1×10^{-10}
Electron mass	m_e	$5.485\,799\,0946(22) \times 10^{-4}$ u	4.0×10^{-10}
Proton rms charge radius	R_p	$0.8775(51) \times 10^{-15}$ m	5.9×10^{-3}

without dividing them into groups; however, the character of data and their precision reconstruct the group structure owing to statistical weights and correlators. Analysis of the consistency of data and the conclusion concerning the necessity of editing their uncertainties are actually based on a group-by-group analysis. In the present review, we discuss the most important original results relevant to the main data groups.

3. Auxiliary data

3.1 Recommended values

The group of auxiliary data is formed by quantities, the values of which are known with the highest precision. Thus, a number of constants are known exactly (by definition, like the speed of light, c). Several others, such as the Rydberg constant

$$R_\infty = \frac{\alpha^2 m_e c}{2h}\,, \tag{3}$$

or different ratios of particle masses are measured with high precision. Auxiliary constants also include quantities that are required only for taking into account small theoretical corrections to various quantities calculated, as a rule, by means of quantum electrodynamics. The recommended values of auxiliary fundamental constants are presented in Table 1.[3]

When measurements of auxiliary constants are performed, fundamental constants that are known with less precision either do not appear at all or may be involved only in small corrections (see example (2) above). If such corrections do exist, one can assume all the necessary less precise constants to be already known.

We shall now briefly touch upon measurements relevant to such extremely important auxiliary constants as the Rydberg constant and various atomic masses.

3.2 Measurement of the Rydberg constant and the charge radius of the proton

Energy levels in the hydrogen and deuterium atoms are mainly determined by Bohr energy levels, the expressions for which reduce to the Rydberg constant with an accuracy up to the known rational factors. At the same time, real energy levels in these atoms are measured with a high precision and require accounting for numerous corrections to the simplest Bohr expressions. A significant part of these is described by the effects of quantum electrodynamics; however, there is also an extremely important contribution proportional to the square of the proton charge radius.

While quantum-electrodynamical corrections are calculated straightforwardly [with the known value of α, as is explained in the examples with equations (1) and (2)], to calculate corrections involving the proton charge radius it is first necessary to find its value. In this connection, determination of the Rydberg constant is closely related to determination of the proton radius. As we have already mentioned, part of the auxiliary constants is made use of only in calculating small corrections, and such constants are known with a relatively low accuracy. The proton charge radius known with an uncertainty on the order of one percent serves precisely as an example of such a constant.

There being two unknown quantities, in order to determine the Rydberg constant applying only spectroscopic methods, one must measure the frequencies of at least two different optical transitions in the hydrogen atom (for details see Refs [7, 15]). The contribution from the Rydberg constant and the one due to the finite size of the proton depend in different ways on the quantum numbers, so, in measuring two optical transitions, these contributions can always be separated (for details of the dependence of different contributions to the atomic energy levels on the principal quantum number n, see Refs [15–17]).

The Rydberg constant equally determines the electron energy levels in the hydrogen and deuterium atoms, and for its determination it is also possible to take advantage of two transitions in deuterium, for which somewhat fewer good experimental results exist than for hydrogen (see, for instance, the review of measured transition frequencies in Ref. [7]).

Meantime, there are a number of measurements, not related to the spectroscopy of hydrogen and deuterium, that permit determining the charge radius of the proton. Further applying its value, one can also determine the Rydberg constant by a sole transition in hydrogen atom. Clearly, for successful joint processing of all the available data it is necessary that the values of the proton charge radius R_p, obtained by different methods, be consistent with each other. However, this is regretfully not so. The situation is illustrated in Fig. 1.

[3] As an example, here and below we only present the values of several constants of one type or another. The complete set of recommended values is available at the site of NIST at the following address: http://physics.nist.gov/cuu/Constants/index.html, and in Ref. [7].

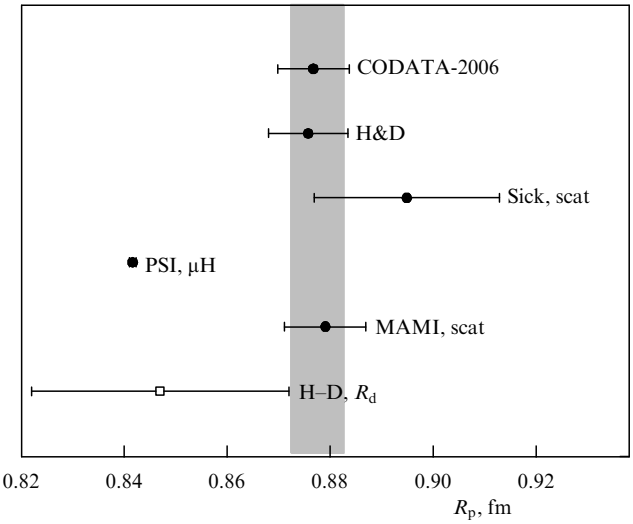

Figure 1. Determination of the root-mean-square charge radius of the proton, R_p. Notation: H&D — hydrogen and deuterium spectroscopic data; scat — results of measurements for scattering cross section, based both on Sick's processing of global data collected before 2003 [18] and on results of the recent experiment at MAMI [19]; PSI, μH represent the result of a measurement of the Lamb shift in muonic hydrogen, performed at PSI [20]. The vertical shaded belt corresponds to the recommended value based on the adjustment [7]. The open square indicates the indirectly measured value obtained by comparison of the isotopic shift in hydrogen and the deuteron charge radius obtained from scattering data (see discussion in Section 5.2).

Besides the hydrogen–deuterium spectroscopic value (H&D), obtained directly within the framework of the adjustment [7], also known are two other values determined from electron scattering on protons (Sick, scat [18] and MAMI, scat [19]), as well as a result obtained from measurements of the Lamb shift in muonic hydrogen (PSI, μH) [20].[4] The latter is in strong contradiction with the three other ones and was discarded from the adjustment. The recommended value of the proton charge radius was obtained from the three remaining values. Correspondingly, the Rydberg constant was determined from the spectroscopic studies of hydrogen and deuterium and the results of electron–proton scattering.

It should be noted that the two values of the proton radius based on electron–proton scattering are assumed to be independent of each other. The result from Ref. [18] was obtained from Sick's processing of the global data on elastic scattering of electrons on protons at small and intermediate momentum transfers collected by 2003. The most important and the only statistically significant (for the determination of R_p) scattering experiment carried out after this processing was the experiment performed at MAMI [19].

Besides the value of the charge radius of the proton, which is consistent with earlier global scattering data and spectroscopic measurements of hydrogen and deuterium, its magnetic radius was also obtained in this work [19]. Its value turned out to be unexpectedly low. Subsequent discussion [21, 22] pointed to its precision being, most likely, underestimated. As a result, one can assert that the real accuracy of the results in Ref. [19] is not quite clear.

While the spectroscopy of atomic hydrogen and electron–proton scattering represent traditional methods of determining the proton charge radius, and the respective experiments have a history of half a century, measurement of the Lamb shift in muonic hydrogen [20] represents an experiment of a totally new type. The accuracy announced in this experiment is many times superior to that of traditional methods. If this result were consistent with the other data, it would doubtless determine the value of the charge radius of the proton.

Strictly speaking, one may also consider the indirectly determined value of the proton charge radius, which can be derived from the result for the isotopic (hydrogen–deuterium) shift for the 1s–2s transition [23, 24], and the value of the deuteron charge radius [25] based on the results of data processing for elastic electron–deuteron scattering (see discussion in Section 5.2). This result is presented in Fig. 1 by the open square; it has a large uncertainty and is consistent with all the contradictory results. Its contribution to the results of adjustment is statistically negligible.

On the whole, the situation with the proton charge radius cannot be considered satisfactory. The results of Refs [20] and [19] appeared in the middle of 2010, i.e., about half a year before the adjustment origin. Since then (both before and after the adjustment), numerous attempts have been undertaken to find an explanation for the contradiction in the results. Specific descriptions of the proton were proposed, which 'satisfied all the results'; allegations were made concerning significant contributions omitted in the theory of muonic hydrogen. But all of them turned out to be unreliable.

Any nontrivial behavior of the charge density or of the charge form factor is based on the existence of a certain broad cloud of virtual charged particles extending far beyond the limits of the proton radius. Only in such a way can one understand how different charge radii are 'seen' at different distances larger than the characteristic proton radius. Meanwhile, the modern understanding of the proton structure is based on (color) confinement and, consequently, only light and colorless virtual particles can go far distances beyond the limits of the proton radius.

This is due to there being a characteristic scale of quantum chromodynamics, $\Lambda_{QCD} \simeq 215$ MeV, and at momenta significantly lower than Λ_{QCD}/c and at distances exceeding hc/Λ_{QCD}, only colorless physics exists. Thus, the proton radius corresponds to characteristic radii at which color charge is neutralized.

Now, the lightest colorless hadron is the pion ($m_\pi \simeq 140$ MeV$/c^2$). Owing to the uncertainty principle, the pion cannot clearly go far beyond the proton radius. At distances beyond scales related to the neutralization of color or to the uncertainty principle for the pion, the decrease in charge density follows an exponential law. We recall that characteristic momenta in hydrogen amount to 2 keV$/c$, while in muonic hydrogen they are 0.4 MeV$/c$. Such reasoning indicates that the proton constitutes quite a compact object, which is subject to no exotic modifications at large distances.

The theory of energy levels of muonic hydrogen, applied in Ref. [20] for determining the proton charge radius from measurements of the Lamb shift, is mainly based on Refs [26–28] and to a significant degree represents a quantum-electrodynamical theory, within the framework of which all contributions can be calculated from first principles.

It is important to stress that, although muonic hydrogen and ordinary hydrogen differ in many aspects, the calculation of energy levels in both atoms is of a perturbative character. The methods of constructing perturbation theory have been

[4] MAMI — Mainzer Mikrotron; PSI — Paul Scherrer Institut.

long well-known, and, therefore, no 'conceptual' news in the theory of muonic hydrogen is to be expected. We refer to the purely practical application of well-known and reliable methods to a problem, in which characteristic physical parameters assume somewhat unusual values.

The reliability of the QED theory of the Lamb shift in muonic hydrogen may be illustrated by the following example: a relativistic correction, caused by recoil, to the contribution of electron vacuum polarization, which was earlier calculated to be -4.1 μeV in Ref. [27], happened to be omitted in Ref. [20]. Recently, it was confirmed by another approach in Ref. [29], but it was soon shown [30] that calculations by the method of Ref. [27] yield the result of -2.1 μeV, which differs from the aforementioned one by a factor of two.

This contradiction was not considered significant in Ref. [30] (since the result of the work was presented as the sum of a well-known relativistic Dirac correction (to the contribution of electron vacuum polarization) [26, 28, 20] and the relativistic recoil to the result, and against the background of a significantly larger Dirac contribution, the disagreement in the results for the recoil effect was not so evident) and was not taken into account. Later on, the origin of the contradiction indicated was explained in Ref. [31], and it was noted that the method of Ref. [29] led in its original form to incomplete results owing to an unreasonable choice of gauge. However, this double discrepancy (like the total omission of the correction in Ref. [20]) only shifts the total result by a mere 2–4 μeV, which does not even exceed the total theoretical uncertainty of 4.5 μeV, indicated in Ref. [20], and is much smaller than the discrepancy which (in terms of comparison between the experimental value of the Lamb shift and the theoretical value obtained using the proton charge radius found for hydrogen) approximately amounts to 310 μeV.

Certain doubts and uncertainties in the theory were also related to the contributions from the light-by-light scattering block. There are three main types of diagrams. Contributions of the first type (related to the so-called Wichmann–Kroll potential) are well known. Corrections of the second type (virtual Delbrück scattering) were calculated in Ref. [29], but the result gave rise to certain doubts (see discussion in review [26]). The contribution of diagrams of the third type ('inverted' Wichmann–Kroll potential) was not known, and only the upper limit for its value was presented in Ref. [20]. Both problems were resolved in Refs [32, 33]: the contribution of the third type was found and it was shown that the approximation applied in Ref. [29] was correct, and at the same time the precision of the respective calculation was improved. All these studies noticeably altered the value of the total contribution of diagrams with the light-by-light scattering block. The old value of 0.32(1.35) μeV was replaced by the new one, $-0.89(2)$ μeV. It is seen that the progress in theory achieved by considerable effort barely manifested itself in the final result.

These examples demonstrate both the degree of reliability of the modern theory and the scale at which unreliable contributions manifest themselves.

Besides purely quantum-electrodynamical corrections, there are also higher-order contributions taking into account the proton structure. The question concerning the precision with which they are calculated is quite complicated. However, all these contributions are significantly smaller than the discrepancy and, on the whole, assume reasonable values. In the most pessimistic scenario, it would be necessary to double or triple the theoretical uncertainty which according to Ref. [20] amounts to 4.5 μeV, and which, obviously, will not permit resolving the problem of a 300-μeV contradiction.

Evidently, the most delicate aspect of the issue concerning the precision with which the proton charge radius derived from muonic hydrogen is determined consists in adequate estimation of the contribution from the so-called third Zemach moment, or the Friar moment (for details see Ref. [26]):

$$\begin{aligned}\Delta E_3 &= \frac{\alpha^2 mc^2}{24}\left(\frac{\alpha m_{\mathrm{R}} c}{\hbar}\right)^3 \int \mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'\,\rho(r)\,|\mathbf{r}-\mathbf{r}'|^3\rho(r') \\ &= \frac{2\alpha^2 mc^2(\alpha m_{\mathrm{R}} c)^3}{\pi} \\ &\quad\times \int \mathrm{d}q\,\frac{(G(q^2))^2-1+R_{\mathrm{p}}^2 q^2/(3\hbar^2)}{q^4} \approx 25\,\mu\mathrm{eV}\,, \qquad (4)\end{aligned}$$

where $G(q^2)$ is the proton charge form factor, $\rho(r)$ is its Fourier transform, which at large distances from the center reduces to its charge density, m is the muon mass, and m_{R} is the reduced mass of muonic hydrogen. (In the formula presented, the sign of approximate equality is introduced, because the uncertainty of this calculation is not quite evident.) The estimated contribution amounts to less than 10% of the established discrepancy between the result obtained from muonic hydrogen and the remaining available data.

Meanwhile, as is known, no successful quantitative models of the proton exist, yet. A variety of data on the elastic scattering of electrons from protons exist. The contribution of small momenta to the integral in Eqn (4) is not small. This integral can be estimated by rough models of the form factor behavior, but adequate quantitative estimation of both the central value and of its uncertainty represent a nontrivial problem. As we recall, values of the proton charge radius found from scattering data and from calculations of muonic hydrogen disagree with each other and, therefore, utilization of standard methods for processing scattering data makes little sense.

Generally speaking, two ways of resolving the problem are discussed. First, the experimentally measured proton form factor can be integrated in a straightforward manner, and, second, it is possible within the framework of a certain model to relate the third moment with the root-mean-square (rms) proton radius. Since no direct, sufficiently precise, data for the subtracted form factor $(G(q^2))^2-1+R_{\mathrm{p}}^2 q^2/(3\hbar^2)$ at small electron momentum transfers exist, then, in the first case, integration in Eqn (4) implies certain fitting, at least at small transferred momenta. In the second case, a certain behavior of the form factor is described by a model; therefore, the construction of such a 'model' is also nothing but fitting the form factor to experimental data. Thus, it is evidently impossible to do without any data fitting whatsoever: anyhow, the proton charge radius must be and, apparently, is the result of such fitting.

For self-consistent calculation of the contribution of Eqn (4) to the Lamb shift in muonic hydrogen, it is necessary to perform such a joint data processing that includes both the electron scattering on protons and the Lamb shift in muonic hydrogen. Such processing has not yet been implemented, and neither its uncertainty nor its very possibility have been seriously considered. (With account of the discrepancy in the

Table 2. Some recently measured atomic masses. They are used in the adjustment as auxiliary constants; u_r represents the relative standard uncertainty.

Quantity	Symbol	Value	u_r
Mass of ^{16}O atom	$m(^{16}\mathrm{O})$	15.994 914 619 57(18) u	1.1×10^{-11}
Mass of ^{28}Si atom	$m(^{28}\mathrm{Si})$	27.976 926 534 96(62) u	2.2×10^{-11}
Mass of ^{87}Rb atom	$m(^{87}\mathrm{Rb})$	86.909 180 535(10) u	1.2×10^{-10}

values of the charge radius, the possibility of performing successful joint processing of such data does not seem quite obvious.)

It should be stressed that the self-consistent calculation of the discussed contribution of Eqn (4) may significantly increase the uncertainty in the determination of the proton charge radius from muonic hydrogen, but cannot, in any way, eliminate the contradiction. The central value of the contribution, owing to its geometric character (as is seen from expression (4) in the coordinate representation), cannot vary too strongly. Briefly, it is improbable that it will even double over the rough estimates, while in order to resolve the contradiction it must increase tenfold.

It should be noted that among the available experimental spectroscopic data for hydrogen and deuterium (for the second transition, which is necessary in order to separate contributions of the Rydberg constant and of the proton radius) the results of only one group [34–36] dominate. The 1s–2s transition in hydrogen, which plays the part of the reference transition, is somewhat singled out, having been measured with excessive accuracy [37–39]. The uncertainty corresponding to it is negligible in comparison with the uncertainties of all the other transitions.

The most recent progress in QED theory of light hydrogenlike atoms is related to the calculation of two-loop corrections on the order of $\alpha^2(Z\alpha)^6 m_e c^2$ (Z is the charge of the nucleus) to the electron energy levels. These corrections contain terms that are logarithmic in $Z\alpha$. While the cubic logarithm was known previously [40], relatively recent calculations of terms quadratic and linear in the logarithm, as well as of a nonlogarithmic term, were performed only by one theoretical group [41–43]. Independent numerical calculations for multicharged hydrogenlike ions (without expansion in $Z\alpha$) with subsequent extrapolation to the hydrogen value ($Z = 1$) [44, 45] seem, on the whole, promising; however, to confirm (or disprove) convincingly the perturbative (i.e., with expansion in $Z\alpha$) calculations for the case of $Z = 1$ [44, 45], their precision is as yet insufficient. Such calculations must not be considered an alternative to the perturbative approach. With rare exceptions, it is not possible to perform a calculation precise in $Z\alpha$ for $Z = 1$, as, for instance, for two-loop contributions. Extrapolation is not an independent procedure, and it does not exhibit high precision by itself; usually, it is carried out using already known expansion coefficients of terms in the form of $(Z\alpha)^n \ln^m(Z\alpha)\, m_e c^2$ (for example, the coefficient of $\alpha^2(Z\alpha)^5 m_e c^2$ [46, 47]). Taking into account the slow variation of the $\ln(Z\alpha)$ logarithm in the region for which numerical results do exist, no effective extrapolation to $Z = 1$ (performed without taking advantage of coefficients found by expansion in $Z\alpha$) can even be thought of. As a rule, extrapolation permits determining one or two new coefficients.

To conclude this section, we note that certain progress can be emphasized both in partial processing of electron–proton scattering data, aimed at a better understanding of their reliability and accuracy, and in the new independent spectroscopic studies of atomic hydrogen. In both cases, certain satisfaction can be taken owing to the work along these lines becoming more active. However, taking into account the short period that has passed since discrepancies were revealed, no results capable of qualitatively altering our understanding of the situation have been obtained yet.

3.3 Masses of atoms and nuclei in atomic mass units

The masses of various atoms serve as an important example of auxiliary constants. The atomic masses in the most recent adjustment [7] are mainly taken from the previously tabulated results of 2003 atomic mass evaluation [48, 49].[5] Some of the values were later corrected. In such cases, the results were taken directly from original work, such as [50] (^{16}O), [51] (^{28}Si), and [52] (^{87}Rb) (Table 2).

Such atomic masses are by themselves natural constants, but not fundamental. However, some of them play a direct part in the adjustment of the fundamental constants. Thus, speaking of the atomic masses presented in Table 2, we note that the mass of oxygen-16 is important for determining the electron–proton mass ratio (see Section 4), the mass of rubidium-87 for finding the fine-structure constant α (see Section 5), and the atomic mass of silicon-28 for determining the Avogadro constant and the Planck constant (see Section 6).

The values of atomic masses are used in the adjustment in different ways. Thus, for example, the masses of rubidium-87 and of oxygen-16 are considered as adjustable quantities, while the atomic mass of silicon-28 is given before the adjustment. In principle, these natural constants, treated as quantities to be agreed upon, somewhat change in the course of implementing the adjustment. However, these changes can be neglected, so the results for the agreed upon values of ^{16}O and ^{87}Rb masses are not presented.

An exception is represented by the masses of five light atoms which are considered not only as quantities to be agreed upon but also as results of adjustment, which are available in the tables of recommended values [7]. The nuclear masses of the main hydrogen isotopes ($A = 1, 2, 3$) and of helium isotopes ($A = 3, 4$) are given [7] explicitly in the tables of recommended values (Table 3) as results of adjustment. Although recommended values of the respective atomic masses are not given directly in Ref. [7], they can be readily found, since both the mass of atomic electrons ($\sim 10^{-3}$ of the atomic mass) and the mass defect related to their binding energy ($\sim 10^{-8}$ of the atomic mass) are known with quite high accuracy.

4. Determination of the electron–proton mass ratio

One more important constant that is known with high precision is the electron–proton mass ratio. Already in the

[5] Adjustment of the values of atomic masses evaluated in 2003 (2003 Atomic Mass Evaluation, AME2003) was performed at the Atomic Mass Data Center (AMDC) [48, 49].

Table 3. Recommended values of the masses of light nuclei [7]; u_r represents the relative standard uncertainty.

Quantity	Symbol	Value	u_r
Proton mass	m_p	1.007 276 466 812(90) u	8.9×10^{-11}
Deuteron mass	m_d	2.013 553 212 712(77) u	3.8×10^{-11}
Triton mass	m_t	3.015 500 7134(25) u	8.2×10^{-10}
Helion mass	m_h	3.014 932 2468(25) u	8.3×10^{-10}
α particle mass	m_α	4.001 506 179 125(62) u	1.5×10^{-11}

previous adjustment of 2006 [5, 6], its value would indisputably be considered as an auxiliary constant. At present, this quantity plays a transitional role between an auxiliary constant and part of the α-group discussed in Section 5. Below, we shall treat it as an auxiliary constant; however, we shall comment on its status more rigorously in the concluding part of the next section (see Section 5.2).

If one forgets that the precision in determining the fine-structure constant α has become comparable to the precision of m_e/m_p, which may complicate the character of correlations between data, the recent history concerning measurements of the electron–proton mass ratio provides an example of the search for values of a fundamental constant, which ended up quite successfully. While even 10 years ago determination of the value of m_e/m_p was based on a single experiment [53], in which comparison was made of the cyclotron frequencies of electrons and protons in a Penning trap, starting with work [54] an independent method appeared relied on the measurement of the *g*-factor of a bound electron in a hydrogenlike ion with a spinless nucleus. The first result for the electron–proton mass ratio [54] was obtained from measurements for hydrogenlike carbon [55]. Soon, a similar measurement was also performed with hydrogenlike oxygen [56]. The experiments were carried out at the University of Mainz (Germany).

Some time later, studies of antiprotonic helium led to still another independent result [57], which was subsequently somewhat improved [58]. Antiprotonic helium makes up a three-particle system: an antiproton circles about the nucleus on a high circular, or nearly circular, orbit with the principal quantum number $n \sim 35$; somewhat higher there is an electron in its ground state. High circular states possessing large values of n and l forbid annihilation of the antiproton with the nucleus, while the outer electron protects the antiproton orbit from collisions. Depending on its state, the atom lives quite a long time from the standpoint of atomic physics. Such experiments were carried out at CERN by the ASACUSA collaboration.

Thus, at present there are three independent methods for determining the electron–proton mass ratio, and, taking into account both the positive dynamics of experiments towards constant improvement of the precision and the absence of internal contradictions, as well as their agreement with each other, all these methods should be considered reliable. Their reliability is also supported by the fact that in measurements of the *g*-factor of a bound electron two different (C and O) ions are studied, while in the spectroscopy of antiprotonic helium a number of transitions are investigated in two different isotopes (six transitions in helium-3, and nine transitions in helium-4) [58].

The results obtained are presented in Fig. 2, where the values of the *g*-factor of a bound electron in the hydrogenlike ions of carbon and oxygen are given separately, while the spectroscopy of antiprotonic helium is conveyed by a single averaged value.

In all the methods, with the exception of the measurement of cyclotron frequencies, quantum-electrodynamical theory played an essential part, being subject to vigorous development as the respective experimental results became available.

After a certain pause, the theory of the *g*-factor of a bound electron in a hydrogenlike ion started to develop vigorously after 2000, but between 2006 and 2010 (i.e., between the previous [5, 6] and the recent [7] adjustments), no new results appeared. The expressions found have some uncertainty margin, which renders calculations quite reliable.

The theory of antiprotonic helium has also been developing intensively in recent years. References to numerous earlier theoretical studies on the *g*-factor of a bound electron and on energy levels in antiprotonic helium can be found in publications [4–6]. The only important result obtained during the period between adjustments is the one obtained in Ref. [59].

Both theories are constructed within the framework of quantum mechanics and quantum electrodynamics. At the same time, it is to be understood that the theoretical problems concerning the *g*-factor of a bound electron and the energy levels of a three-particle antiprotonic helium (nucleus — antiproton — electron) are essentially different, and the respective calculation uncertainties must be considered to be independent. In the first problem, it is necessary to make use of the relativistic Green's functions of an electron, and two-particle effects can, to a significant degree, be neglected, while in the second case one is dealing with an essentially non-relativistic three-body problem.

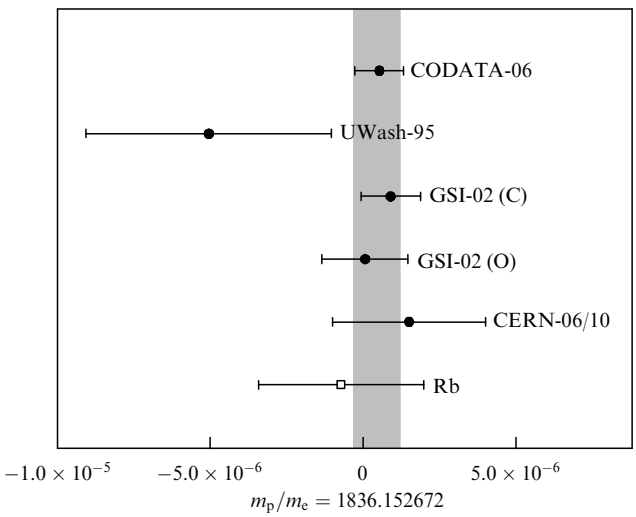

Figure 2. Determination of the proton–electron mass ratio m_p/m_e. The closed circles and their notation correspond to Fig. 5 in Ref. [7], where the necessary references are to be found. The value indicated by the square is discussed in Section 5.2. The vertical shaded belt corresponds to the value recommended in accordance with the adjustment results.

5. Determination of the fine-structure constant α and data related to it

5.1 Main results

As we have already noted above (see Section 3.2), an essential disagreement exists in determining certain auxiliary constants. Thus, if the data on muonic hydrogen were taken into account, the uncertainty in determining the Rydberg constant would become many times larger. However, its value has been found with such an uncertainty margin that neither a possible shift of the result nor a forced increase in the uncertainty (owing to a contradiction in the data on the proton charge radius) will in any way affect the determination of the fine-structure constant.

The fine-structure constant

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \tag{5}$$

contains the electron charge and the Planck constant; therefore, it would be natural to expect that it appears in some form or another in many quantum electric phenomena, which include both macroscopic effects and the very existence of atoms.

Owing to the above, the group of data related to the fine-structure constant α exhibits much diversity. It is formed on the basis of the following relations between the fundamental constants:

• The Rydberg constant is known with a precision significantly superior to the characteristic precision for the group containing α (see the discussion of formulae (1) and (2) in Section 2). On the one hand, this permits determining the quantity R_∞ independently of the determination of the fine-structure constant, while, on the other hand, it establishes a direct relationship between the measurements of α and the electron Compton wavelength $h/(m_e c)$, with their combination representing the Rydberg constant (3).

• The electron–proton mass ratio (see Section 4) is also known with extremely high precision (see Table 1) and, therefore, the electron mass in the ratio h/m_e can be replaced by the proton mass.[6]

• The proton mass, like the masses of a number of other atoms and nuclei (see Tables 1, 2, and 3), is well-known in atomic mass units, so the relationship between measurements of α and h/m can be used for a broad class of objects and, in particular, for rubidium and caesium atoms.

• The value of the electric constant ϵ_0 is adopted in SI by definition, which permits relating the dimensionless fine-structure constant α to the dimensional von Klitzing constant

$$R_K = \frac{h}{e^2}\,. \tag{6}$$

[6] Strictly speaking, as one can see from a comparison of Tables 1 and 4, the ratio m_e/m_p is actually known somewhat *worse* than α. However, the value of α is determined, first of all, by its most precise original value (see below), while the mass ratio enters into the value of α, which is second in precision. The precision of this second value is determined from measurements of the recoil frequency in Raman scattering of light on an atom, and this frequency is known noticeably worse than m_e/m_p. In the case of statistically independent uncertainties, its value found from several sources is known to be determined not by the sum of uncertainties, but by the sum of their squares, so even a threefold difference in relative uncertainties points to an 'essentially' lower uncertainty. For details, see the comment in Section 7.

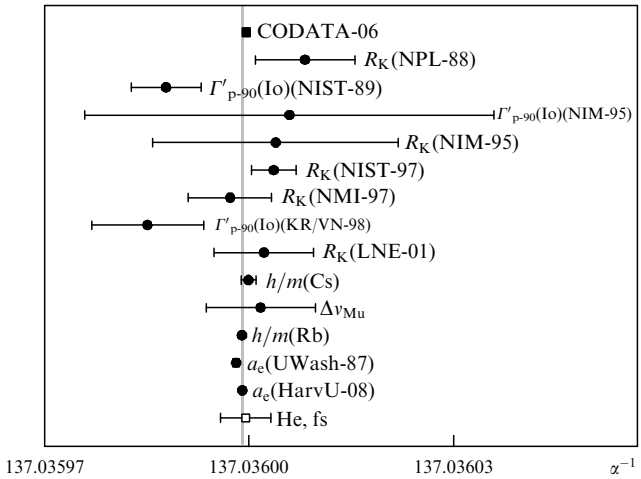

Figure 3. Precision determination of the fine-structure constant α (according to data of the 2010 adjustment [7]). The notation of data represented by closed circles corresponds to Ref. [7], where the necessary references can be found. The vertical shaded belt corresponds to the value recommended in accordance with the adjustment results. The open square indicates the value found from the fine structure of helium and omitted from the main data processing procedure [7].

The von Klitzing constant can be found in various electrical measurements. (In the CGS system, the von Klitzing constant has the dimension of inverse speed and differs from the fine-structure constant by the factor c.)

The results of determining the fine-structure constant by different methods are collected in Fig. 3, where the closed circles indicate the results that are taken into account in adjustment 2010. Presented are 14 independent values obtained by six essentially different methods.

The most precise method for determining the fine-structure constant still remains based on investigation of the anomalous magnetic moment of the electron, a_e, the measurement of which is related to quantum optics of traps for individual particles, while its theory has to do with quantum electrodynamics. The next in precision is Raman spectroscopy on the atoms of rubidium and caesium, which permits determining (by the methods of quantum optics) the value of h/m for the respective atom. To obtain the value of h/m, no theoretical relationships are required other than the conservation laws (for calculating the recoil in the events of absorption and induced emission of a photon), and this quantity is practically measured in a straightforward manner. To go from the quantity h/m to α, it is necessary to know the masses of the atom and of the electron in the same units, for instance, in atomic mass units.

There are also other methods based on a combination of the results of experimental atomic physics and quantum-electrodynamical theory (but already for bound states). Such methods include the investigation of hyperfine splitting in muonium (the result of which in included in the adjustment) and of the fine structure in helium (the result for which, although of a somewhat higher precision, has not been included in the final data processing [7] and is denoted by an open square in Fig. 3, unlike the closed circles which indicate data included in the adjustment). Both values have a relatively low precision, and in both cases further progress can be achieved. References to the respective works can be found in publications [4–7]. Retention in Fig. 3 of values that are not too precise and cannot compete with results obtained

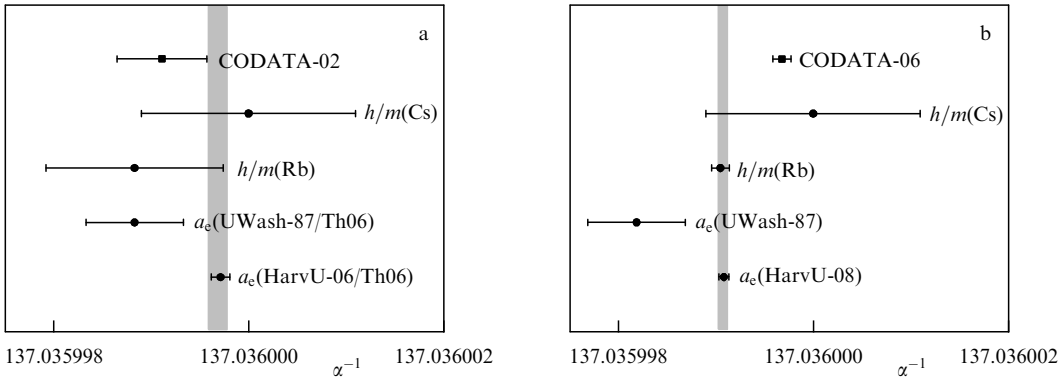

Figure 4. The most precise results of determining the fine-structure constant α (obtained from investigations of the anomalous magnetic moment of the electron and by methods of Raman spectroscopy): (a) from data in the 2006 adjustment [5, 6]; (b) from data in the 2010 adjustment [7]. The vertical shaded belt corresponds to the value recommended in accordance with the results given in the respective adjustment.

by the two aforementioned, more precise, methods is due to their being important for determining certain other fundamental physical constants (for details, see the comment in Section 5.3).

Yet another two approaches are closely related with an application of electrical standards. Their accuracy is greatly inferior to most precise results. However, the agreement between 'electrical' values of the fine-structure constant and those of a 'nonelectrical' nature gains acceptance for our belief that the main electrical standards are understood correctly and the omitted systematic effects are not lacking. Such a statement has not probably too great importance for fundamental physics but its practical significance, however, is impossible to overestimate (see more details in Section 5.3).

A significant part of the points in Fig. 3 were actually determined not by a single measurement, but a chain of measurements of essentially different quantities, the combination of which was necessary in order to find α, so a comparison of different values of α reveals the consistency not of a dozen, but of several dozen experiments of a great variety.

Let us discuss in somewhat greater detail the most important achievements. They are collected in Fig. 4, where the most precise results of the 2006 adjustment [5, 6] and the 2010 adjustment [7] are presented.

In recent decades, the adjusted value of α has practically been fully determined by the contribution of data obtained from investigations of the anomalous magnetic moment of the electron a_e. Taking into account the importance of the fine-structure constant in fundamental physics and electrical metrology, this was due to give rise to certain misgivings, and especially because the result was based on measurements performed by a single experimental group and on calculations by a single theoretical group.

However, the situation has greatly improved recently. In 1998, the precision in determining the value of $\alpha(a_e)$ was significantly superior to the precision of all the other values. In 2002, it was approached (in precision) by the value obtained applying the method of Raman spectroscopy of the caesium atom.

In the 2006 adjustment, the theory of the anomalous magnetic moment of the electron a_e was still represented by the work of T Kinoshita and his coauthors: they succeeded in lowering the relative uncertainty of calculations from $u_r = 9.9 \times 10^{-10}$ in 2002 to $u_r = 2.4 \times 10^{-10}$ in 2006. At the same time, a new experimental result for the anomalous magnetic moment of the electron appeared [60], while the method of Raman spectroscopy (in a significantly altered version) was successfully applied in the case of rubidium atoms, too. Thus, each method had already been realized by two independent experiments.

This progress, however, had not totally resolved the main problem, and there still remained a significant precision gap between the best value and its independent confirmations by the methods of Raman spectroscopy. The situation prevailed in the 2006 adjustment is illustrated in Fig. 4a.

Soon after the 2006 adjustment, it was revealed [61] that the theoretical expression [62] used in reviews [5, 6] was not complete. Two finite terms that were previously left out were found to arise when the divergent contributions of four-loop Feynman diagrams were cancelled out. One of these terms significantly exceeded the claimed theoretical uncertainty. The shift of $\alpha(a_e)$ amounted to 6.4 standard deviation. Meanwhile, both the old and the shifted values were in excellent agreement with the best Raman results in 2006. This means that their precision happened to be insufficient for controling the situation.

Subsequent extremely essential progress consisted in the following. First, significant success was achieved in improving the measurement accuracy for the anomalous magnetic moment of the electron a_e. The new result had a relative uncertainty equal to 2.4×10^{-10} (compare to 7.6×10^{-10} [60]). Second, the precision of Raman spectroscopy of rubidium atom was also significantly improved. The respective uncertainty amounted to 6.6×10^{-10} [64] (compare to 6.7×10^{-9} [65]). The present-day situation is demonstrated in Fig. 4b. The previous recommended value (CODATA 2006) was fully determined by the old value of $\alpha(a_e)$. It is seen that the rubidium result [64] permits distinguishing the old [62] and the corrected [61] theories of the anomalous magnetic moment of the electron quite clearly.

Such accuracy of the second most precise result essentially enhances the reliability in determining the value of the fine-structure constant α. The possibility has now appeared of more subtle tests of the quantum-electrodynamical calculations. We, indeed, do not mean testing QED as a fundamental theory: absolutely nobody is thinking of challenging its Lagrangian. But QED serves as a training ground for mastering different theoretical methods, and now the possibility has appeared to have a look into the fifth order of smallness of the theory, especially as it did not take much time to wait for such calculations to be performed [66].

Table 4. Recommended values of fundamental constants related to α [7]; u_r is the relative standard uncertainty.

Quantity	Symbol	Value	u_r
Inverse fine-structure constant	α^{-1}	137.035 999 074(44)	3.2×10^{-10}
Molar Planck constant	hN_A	$3.990\,312\,7176(28)\times10^{-10}$ J s mol^{-1}	7.0×10^{-10}
Quantum of circulation	$h/(2m_e)$	$3.636\,947\,5520(24)\times10^{-4}$ m^2 s^{-1}	6.5×10^{-10}
Compton wavelength	$\lambda_C = h/(m_e c)$	$2.426\,310\,2389(16)\times10^{-12}$ m	6.5×10^{-10}
Von Klitzing constant	$R_K = h/e^2$	25 812.807 4434(84) Ω	3.2×10^{-10}
Muon–electron mass ratio	m_μ/m_e	206.768 2843(52)	2.5×10^{-8}

The values of the main constants related to the fine-structure constant α are collected in Table 4. The molar Planck constant hN_A is to be singled out, since it is important for the subsequent discussion. This quantity plays an important role in forming another data group, which is related to the determination of h. Let us briefly explain its relationship to the fine-structure constant. Atomic masses are badly measured in kilograms, but there are a number of microscopic units in which particle masses, on the contrary, are known with high precision. This concerns both the units of frequency (i.e., mc^2/h is determined, not m) and the atomic mass units. The factor relating them is expressed via the molar Planck constant. This can be verified in the following way: the relationship

$$\frac{mc^2}{h} = \frac{1}{(hN_A)}\,\frac{m}{m(^{12}\mathrm{C})/12}\,c^2\left[N_A\,m(^{12}\mathrm{C})/12\right] \tag{7}$$

relates the mass (of an atom), measured in units of frequency, on the left-hand side of the equality to the molar Planck constant and the numerical value of mass in atomic mass units (these are the first two factors on the right-hand side). The last two factors on the right-hand side of the equality are known in SI exactly and, in particular, we have

$$N_A\,m(^{12}\mathrm{C})/12 = 1\ \mathrm{g\ mol^{-1}}\ .$$

As we have already explained above, the measurement of mass m in units of frequency, which is equivalent to measuring the respective Compton wavelength $h/(mc)$, is directly related to the determination of the constant α.

Concluding the discussion on data of the α-group, we note that the precision of data in this group is somewhat more complex. First, the most precise data are comparable in accuracy to m_e/m_p; second, although the data that are inferior in precision by an order of magnitude do not practically affect the results, they are, nevertheless, retained in the data processing, and this has great practical meaning. These issues are discussed in detail below.

5.2 Comment on the status of m_e/m_p as an auxiliary constant

To investigate the issue of how determination of the fine-structure constant can influence determination of the electron–proton mass ratio, we shall find the value of m_e/m_p without taking advantage of the data that were discussed in Section 4, namely, the data that clearly did have to do with measurements of the ratio m_e/m_p. Indeed, an appropriate chain of relations can be built up from other data:

$$\frac{m_p}{m_e}\,[\mathrm{Rb}] = \frac{m_p[\mathrm{u}]}{m_{\mathrm{Rb}}[\mathrm{u}]}\,\frac{1}{h/m_{\mathrm{Rb}}}\,\frac{\alpha^2[a_e]\,c}{2R_\infty} = 1836.152\,6713(27)\,,$$

$$u_r = 1.5\times10^{-9}\,. \tag{8}$$

The quantity in square brackets indicates either the origin of the particular value of the argument (Rb, a_e) or the measurement unit [u for (unified) atomic mass unit] in the ratio $m_p[\mathrm{u}]/m_{\mathrm{Rb}}[\mathrm{u}]$, which is to be understood as the ratio of respective mass values, each presented individually in atomic mass units. The value of m_p/m_e found here is indicated in Fig. 2 by an open square.

It is noticeably inferior in precision to the best results of determining m_e/m_p, although it is far from negligible. It can be seen that the result obtained has an accuracy 3.5 times lower than the recommended value (see Table 1) [7], and is consistent with it. Taking into account the fact that, when averaging is performed, the weights are represented by the respective inverse squares of uncertainties, the influence of such an indirect method for determining the value of m_e/m_p is apparently still insignificant. It is also clear that a two- or threefold change in the uncertainties of certain measurements may lead to a fundamental change in the situation. (In comparing the uncertainties in determining m_e/m_p and α, one must remember that the respective expressions usually contain m_e/m_p and α^2, but not α in itself [see formula (8)]. This means that the uncertainty of the ratio m_e/m_p is to be compared to the doubled uncertainty in the determination of α).

To speak in a more formal language, it can be asserted that the measurement of h/m_{Rb} by Raman spectroscopy of atoms results in determining the value of the quantity

$$\alpha^2\,\frac{m_e}{m_p} = \frac{h}{m_{\mathrm{Rb}}}\,\frac{m_{\mathrm{Rb}}[\mathrm{u}]}{m_p[\mathrm{u}]}\,\frac{2R_\infty}{c}\,,$$

and thus establishes the correlation between the quantities α and m_e/m_p. Since independent experimental values for each of these constants also exist, it is possible to take advantage of this relationship along any line: to find m_e/m_p from the known value of α, or to obtain α from the known ratio m_e/m_p. This relationship is used in the adjustment precisely as the experimentally determined combination $\alpha^2(m_e/m_p)$, while the presented partial value of $\alpha(h/m_{\mathrm{Rb}})$ is actually used only for demonstrable control of data consistency. The presence of such a kind of data exhibits a distinguishing feature of the adjustment.

We must note not only the importance of how the m_e/m_p and α measurement uncertainties are related, but also the excellent agreement existing between the data on m_e/m_p, which is seen from Fig. 2. From this point of view, such agreement is a necessary condition for a reliable determination of α, since without this agreement the value of α obtained from the Raman spectroscopy of rubidium atom cannot be considered precise either.

5.3 Comment concerning the precision of different α values included in the adjustment

On the whole, there is a good agreement among different values obtained for the fine-structure constant (see Fig. 3),

and the recommended value of α is only determined by several of the most precise values (see Fig. 4). Certain data for α, such as the aforementioned result obtained from analysis of the fine structure of the helium atom, were not even included in the adjustment [7],

There are several reasons for involving a number of inaccurate results in the data processing procedure. First, certain values, such as the result for α obtained from hyperfine splitting in muonium, are important within the framework of the main adjustment procedure for obtaining the values of other fundamental constants, like the muon mass and magnetic moment (see, for example, the muon–electron mass ratio in Table 4).

Second, the values of electrical quantities play a decisive role in testing the adjustment of the standards of electrical units. A 'branching' in the adjustment procedure [7] aimed at verifying the relationship between the Hall resistance and the fine-structure constant is also related to electrical measurements.

The metrological community is seriously preoccupied by the absence of any reliable theory of the quantum Hall effect, since this effect constitutes one of the foundations of the modern way of establishing units of the main electrical quantities. The words *theory of the quantum Hall effect* should be placed within quotation marks. The point is that the quantum Hall effect represents a complex phenomenon, and it exhibits a number of nontrivial qualitative aspects. On the other hand, metrology is only interested in the pragmatic side of the story, characterized by the following assertions:

• the transverse resistance (the magnetic and electric fields are perpendicular to each other; in the plane perpendicular to the magnetic field, the conductivity can be considered a two-dimensional matrix; it is the ratio of the currents perpendicular to the electric and to the magnetic fields, i.e., to the voltage and to the magnetic field, which are of interest in metrology) is quantized, so when the magnetic field changes, discrete steps arise in the resistance R_n;

• the ratio between resistance values equals the ratio of integer numbers: $R_n/R_m = m/n$ or $R_n = R_1/n$;

• the 'quantum of resistance' R_1 does not depend on the way in which the experiment is performed, and is a universal constant of nature: $R_1 = R_{\rm H}$, which characterizes the quantum Hall effect;

• this constant is expressed via a combination of fundamental physical constants, which is called the von Klitzing constant: $R_{\rm H} = R_{\rm K} \equiv h/e^2$.

To verify all the assertions but the last, no adjustments are required; such tests have been carried out successfully, and a high precision was achieved [67–73].

A certain peculiarity of the tests of the relationship $R_1 = R_{\rm H}$ consists in the fact that, generally speaking, certain parameters may undergo not too strong changes in different realizations of the standards (owing to constructive restrictions), so only the application of qualitatively novel technologies can broaden the parameter space and provide a positive answer to the question of whether the relationship $R_1 = R_{\rm H}$ is satisfied with the required precision. In other words, the possibility remains for a correction to the universal constant $R_{\rm H}$ to be found, but this correction will only depend on such a combination of parameters which has hitherto changed insignificantly from one realization to another.

As to verification of the equality $R_{\rm H} = R_{\rm K} = h/e^2$, this can be done with the help of adjusted values of fundamental constants, and such a procedure has been performed (see reviews [4–7]). Assuming that there are no corrections depending on realization of the quantum device, verification is performed of the existence of a universal correction factor $(1+\epsilon_{\rm K})$ [7].

On the whole, the correction is consistent with zero: $\epsilon_{\rm K} = 2.2(1.8)\times 10^{-8}$ or $\epsilon_{\rm K} = 2.6(1.8)\times 10^{-8}$ [7]. The restriction alters somewhat, depending on whether all the data are used or only the data covering the comparison of the capacity of a calculable capacitor and the Hall resistance (see Refs [74–76, 7] and references cited therein).

It is not difficult to understand that the constant $R_{\rm H}$, independently of whether it coincides with $R_{\rm K} = h/e^2$ or not, is a purely electrical constant. With account of the absence of other dimensionless parameters, besides α, and following Ref. [77], we might have expected expansions of the form

$$R_{\rm H} = (1+\epsilon_{\rm K})\,\frac{h}{e^2} = \left[C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + \ldots\right]\frac{h}{e^2}\,. \quad (9)$$

The expansion parameter is $\alpha/\pi \simeq 2.3\times 10^{-3}$, and its cube is $(\alpha/\pi)^3 \simeq 1.3\times 10^{-8}$. There are simply no other universal electrical parameters in the problem, and the use of α instead of α/π leads, in this case, to a stronger, but less reliable, estimate.

If one assumes the expansion coefficients to be on the order of unity, it would be natural to expect that

$$C_1(R_{\rm K}) = 0\,,\quad C_2(R_{\rm K}) = 0\,,\quad C_3(R_{\rm K}) \simeq 2(2)\,.$$

The origin of the series starting from the third term (and, also, like the numerically small coefficients of lower orders, $C_1 \sim 10^{-5}$ or $C_2 \sim 10^{-2}$) seems quite improbable from the point of view of the structure of the series [77]. It is more natural to expect the relationship $R_{\rm H} = R_{\rm K} = h/e^2$ to be exact in an ideal two-dimensional system or to be violated by exponentially small corrections, while the corrections due to the nonideality may turn out to be quite noticeable at the experimental uncertainty level. These corrections, however, must depend on the parameters characterizing nonideality and must not be universal; such corrections can and must be searched for by continuing experiments [67–73] and by parametrizing their results in an appropriate manner.

6. Determination of the values of the Planck constant *h*, the elementary charge *e*, and the Avogadro constant $N_{\rm A}$

As in the case of determination of the fine-structure constant α, the data group that is next highest in precision includes the results of numerous electrical measurements. This group is composed of data related to the Planck constant h, the elementary charge e, and the Avogadro constant $N_{\rm A}$.

It should be noted that the Planck constant on its own rarely enters into relationships that can be investigated with high precision. The most well-known equality relating the energy and frequency of a photon is quite typical in this sense. While the frequency of photons makes up one of the most precisely measured quantities, their energy permits no accurate measurements.

The investigation of quantum effects, such as the Josephson effect and the quantum Hall effect, which are essentially electrical effects, leads to situations in which two important combinations of the elementary charge and the Planck constant play an essential role. These are the von

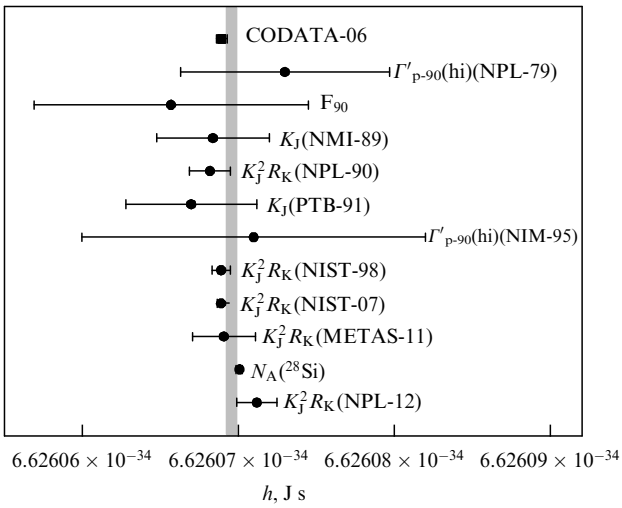

Figure 5. Determination of the Planck constant h in the 2010 adjustment [7]. The notation follows Ref. [7]; the vertical shaded belt corresponds to the recommended value.

Klitzing constant R_K (6) and the Josephson constant

$$K_J = \frac{2e}{h} . \tag{10}$$

These constants do not necessarily have to be measured directly. The measurement implies comparison of these electrical constants with certain quantities (resistance or voltage) known in SI units. Thus, for example, the fine-structure constant α is determined with the help of the farad standard realized on a calculable capacitor (the respective values in Fig. 3 are indicated by R_K).

Another version of utilization of these values is related to the rejection of SI units in electrical measurements; instead, quantum standards are applied, their characteristics being expressed via R_K and K_J. Then, the quantities measured directly in SI units represent combinations of quantities that are measured with the help of quantum standards and of 'calibration' constants R_K and K_J.

One of the combinations of these constants, namely

$$\frac{1}{R_K K_J^2} = \frac{h}{4} , \tag{11}$$

is an exclusive mechanical quantity and can be measured quite apart from which electrical units are used. This equality is the base of the most precise 'electrical' value of the Planck constant.

As we have already noted above, the value of the molar Planck constant hN_A is known with extremely high precision. The respective relative uncertainty amounts to 7.0×10^{-10} (see Table 4), which is significantly lower than the uncertainties of data related to the Planck constant proper. This will also permit finding the value of h by nonelectrical methods — by measuring the Avogadro constant.

On the whole, the group includes data relating to such quantities as the Planck constant h, the electron charge (or elementary charge) e, the Josephson constant K_J, the Avogadro and Faraday constants N_A and $F = eN_A$, respectively, as well as their various combinations that involve the charge and mass of the electron, other constants, and, in particular, the ratio e/m_e, the Bohr magneton μ_B, and the nuclear magneton μ_N (in SI units). Part of them are measured directly, part in combinations with auxiliary and more precisely known constants from the α-group (see Section 5.2).

The structure of the data group under discussion is determined by the fact that the fine-structure constant α, the molar Planck constant hN_A, and the Compton wavelength $\lambda_C = h/(m_e c)$ (see Table 4) are known with a higher precision than the characteristic precision in the h-group.

The results of determining the Planck constant by different methods are given in Fig. 5.

The eleven points shown in Fig. 5 were arrived at by five essentially different methods. The dominant results are the ones obtained by two methods. One of them is based on application of the so-called watt balance, and the second relies on measurement of the Avogadro constant with the help of an enriched silicon monocrystal of a mass equal (approximately) to one kilogram. The results of the most precise measurements used for the 2006 adjustment and for the 2010 adjustment are presented in Fig. 6, and it must be noted that the two adjustments manifest significant changes.

The dominance of the watt balance method over other electrical methods is related to the following circumstance. In electrical measurements, whether utilizing one electrical balance or another, or the measurement of the gyromagnetic ratio of a particle in a calculated field, one unknown quantity exists, namely, the very constant that is to be determined.

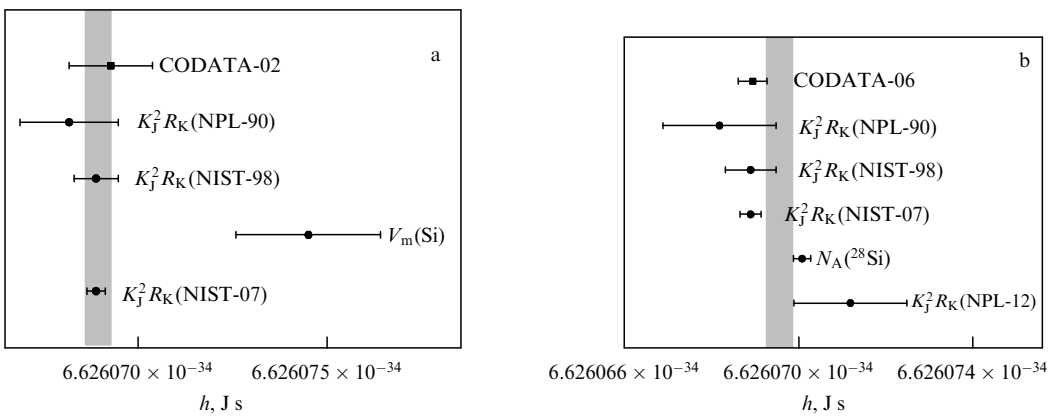

Figure 6. The most precise results obtained with the aid of a watt balance and the project for determination of the Avogadro constant are presented on an enlarged scale: (a) on the basis of the 2006 adjustment [5, 6], and (b) from the results of the 2010 adjustment [7]; V_m(Si) is the value obtained with a natural silicon composition included in the 2006 adjustment [5, 6], and $N_A(^{28}Si)$ is the result obtained with an enriched silicon used in the 2010 adjustment [7]; the vertical shaded belt corresponds to the respective recommended value.

At the same time, there is also another quantity, which is calculated but not quite precisely. The matter is that in determining the Planck constant, comparison between a certain electromagnetic quantity and a certain mechanical quantity is performed locally, i.e., at a definite point in space. When applying a balance method, the electrostatic or magnetostatic force is compared at a certain point with the weight of a probe body.

As a result, for calculation of the magnetic (or electric) field induced by a known source at a given point in space, it is necessary to find a certain effective geometrical factor with high precision. Namely such a calculation has limited the precision of traditional electromagnetic methods for determining the Planck constant.

In the case of a watt balance method, the problem of a geometrical factor is resolved by two measurements ('statical' and 'dynamical'), instead of one, and in the same spatial configuration [78], which permits excluding in the course of the experiment the geometrical factor from the equation for the sought constant (the Planck constant in the case being considered). Directly measured are the current, required in the static mode for counterpoising the weight of the probe body, and the voltage (more correctly, the induced electromotive force) in the dynamic mode, which, in turn, are calibrated by the quantum standards, i.e. in terms of R_K and K_J. The product of the current and voltage represents power, which is what determined the title of the method discussed.

Till recently, there were only two successful realizations of the watt balance in NIST and NPL.[7] (Strictly speaking, the balances in NIST were nearly completely reconstructed several times.) The watt balances from NIST [79, 80] in the 2006 and 2010 adjustments are still significantly superior to the other watt balances; however, another new operational watt balance has appeared in METAS [81]. The watt balance of NPL yielded a new result [82] with high precision; however, its uncertainty was shortly increased [83] owing to the revelation of systematic uncertainties previously not taken into account. This is the result with the extended uncertainty that was included in the 2010 adjustment.

The 2002 [4] and 2006 [5, 6] adjustments were characterized by a long-time discrepancy between the electrical result for the Planck constant and the result based on the Avogadro constant measured in a silicon monocrystal with a natural isotopic composition [84] (Fig. 6a) for the 2006 adjustment. Although the precision of the watt balance was noticeably superior to the precision of this value, the discrepancy required special consideration and was the reason for extension of the uncertainty of the recommended value, as opposed to the direct application of the least-squares method in data processing. As is readily seen from the figure, the uncertainty of the recommended value [5, 6] is greater than the uncertainty of the most precise original result [79, 80].

With time, it became clear that the main source of uncertainty was due to the isotopic composition of silicon. A new project was launched for determining the Avogadro constant in artificially enriched crystal. While natural silicon is approximately composed of 92% of silicon-28, 5% of silicon-29, and 3% of silicon-30, the silicon used in the new project [85] contains 99.985% of silicon-28. Such a weighty decrease in the content of the silicon-29 and silicon-30 isotopes significantly improves the measurement accuracy of the molar mass of the utilized silicon and, subsequently, of the Avogadro constant, as well. The result [86, 87] turns out to be comparable in precision to the best watt-balance results [79, 80] (Fig. 6b); however, it disagrees with them (by about triple the value of the combined standard deviation). That are these two results (Refs [79, 80] and [86, 87]) that determine, with account of their uncertainties and their discrepancy, the present-day recommended value of the Planck constant [7].

It should be noted that the result of measurement with the natural composition of silicon isotopes [84] was not included in the 2010 adjustment, since the authors of Ref. [84] in a number of private communications have indicated that the new determination of the isotopic composition of silicon essentially shifts the result and, on the whole, it is now consistent with the most precise results [79, 80] and [86, 87]; however, no final publication on this subject has been presented, so the eventual corrected value is still unknown. Meantime, since such a value is approximately tenfold inferior to the best values [79, 80] and [86, 87], yet is also consistent with both of them, accounting for it could not noticeably affect the mean value.

Bearing in mind the particular importance of the Planck constant value for reproducing units of contemporary SI [now electrical standards are realized on the basis of quantum effects, and the units output by them are determined by the precision with which we know R_K (6) and K_J (10)] and from the point of view of reproducing the future version of the SI units (see Section 11) [88–90], it is hard to consider the situation satisfactory.

Certain changes have taken place since the 2010 adjustment. First, the British watt balance (NPL) was sold some time ago to Canada (NRC[8]), and realization of the project continues there. The watt balance has been noticeably reconstructed, and the systematic effects that previously led to an increase in the uncertainty [82] have been thoroughly investigated [83]. The first result has been published [91]. Its precision is insignificantly inferior to the precision of the best values [79, 80] and [86, 87] (Fig. 6b) and is in excellent agreement with the result of Ref. [86], based on measurement of the Avogadro constant.

One more operational watt balance has appeared (at BIPM[9]) [92], and it is to be expected that in the next several years work with the new watt balance at METAS and BIPM will lead to results with a precision comparable to that of the best available results [79, 80, 86, 87, 91]. There are also other projects that are being intensively advanced, but they have not yielded any significant results, yet.

The results presented in review [7] for the fundamental constants related to the Planck constant are collected in Table 5.

7. Fundamental constants and the units for elementary particle masses

The masses of elementary particles, nuclei, and atoms can be measured in extremely diverse units. Some results for the proton mass are presented in Table 6, which is divided into

[7] NPL — National Physical Laboratory; METAS — Bundesamt für Metrologie.

[8] NRC — National Research Council.

[9] BIPM — Bureau International des Poids et Mesures.

Table 5. Recommended values of the fundamental constants from the h-group [7]; u_r is the relative standard uncertainty.

Quantity	Symbol	Value	u_r
Planck constant	h	$6.626\,069\,57(29)\times10^{-34}$ J s	4.4×10^{-8}
Elementary charge	e	$1.602\,176\,565(35)\times10^{-19}$ C	2.2×10^{-8}
Avogadro constant	N_A	$6.022\,141\,29(27)\times10^{23}$ mol^{-1}	4.4×10^{-8}
Faraday constant	$F = eN_A$	96,485.3365(21) C mol^{-1}	2.2×10^{-8}
Electron charge to mass quotient	e/m_e	$1.758\,820\,088(39)\times10^{11}$ C kg^{-1}	2.2×10^{-8}
Electron gyromagnetic ratio	$\gamma_e = 2\mu_e/\hbar$	$1.760\,859\,708(39)\times10^{11}$ s^{-1} T^{-1}	2.2×10^{-8}
Electron mass	m_e	$9.109\,382\,91(40)\times10^{-31}$ kg 0.510 998 928(11) МэВ/c^2	4.4×10^{-8} 2.2×10^{-8}
Proton mass	m_p	$1.672\,621\,777(74)\times10^{-27}$ kg 938.272 046(21) MeV/c^2	4.4×10^{-8} 2.2×10^{-8}
Bohr magneton	$\mu_B = e\hbar/2m_e$	$927.400\,968(20)\times10^{-26}$ J T^{-1}	2.2×10^{-8}
Nuclear magneton	$\mu_N = e\hbar/2m_p$	$5.050\,783\,53(11)\times10^{-27}$ J T^{-1}	2.2×10^{-8}
Josephson constant	$K_J = 2e/h$	$483{,}597.870(11)\times10^{9}$ Hz V^{-1}	2.2×10^{-8}

Table. 6. Recommended values of the proton mass and its equivalents (such as m_pc^2) [7] presented in different units and arranged according to the decrease in their precision; u_r is the relative standard uncertainty. The value of m_pc/h is not given directly in review [7] and has been found here from the proton Compton wavelength m_pc/h set out in Ref. [7].

Quantity	Value	u_r
m_p m_p	1.007 276 466 812(90) u 1,836.152 672 45(75) m_e	8.9×10^{-11} 4.1×10^{-10}
m_pc^2/h	$2.268\,731\,8139(16)\times10^{23}$ Hz	7.1×10^{-10}
m_pc^2 m_p	938.272 046(21) MeV $1.672\,621\,777(74)\times10^{-27}$ kg	2.2×10^{-8} 4.4×10^{-8}

three parts. In the upper part, results are presented that are related to auxiliary constants. They are the most precise ones. The central part presents the result related to the α-group. The lower part contains the two least precise results from the h-group.

This data structure can be readily understood. The proton mass in atomic mass units is measured with exceptional precision. To find the proton mass in units of the electron mass, it is necessary either to measure the ratio between these masses directly or to find the electron mass in atomic mass units. This quantity is discussed in Sections 4 and 5.2. The conversion factor from atomic mass units to frequency units (7) is expressed via the molar Planck constant hN_A and is closely related to the fine-structure constant α (see Section 5).

In the conversion from frequency units to kilograms, we only need the Planck constant h and the speed of light c, which is known exactly, while for the conversion to electron-volts it is necessary to take advantage of the ratio h/e and the speed of light c. The search for the quantities h and h/e are related directly, but the precisions in their determination differ by two (see Section 6). This relationship shows that from the point of view of the experimental accuracy the electron-volts, in spite of their visibility, are in no way better than the kilograms. The most precise data must be expressed in terms of atomic mass units, or of the mass of one particle or another, or, in the extreme case, in terms of an equivalent frequency (for details, see Ref. [93]).

The situation with the measurement of the energies of different radiative transitions is similar. Relative determination methods (those manifesting high precision) deal with ratios between frequencies or between wavelengths (depending on the range), while absolute methods (including, where necessary, the calibration of reference lines) also have to do with frequencies (the radio frequency and visible spectra and, to a significant degree, the infrared and ultraviolet spectra) and wavelengths (visible and X-ray spectra, gamma rays). The description of the most precise data for transitions in terms of their energies leads to a loss of precision. In such a case, it is necessary to use frequency or wave number (inverse wavelength). A problem may also arise with the evolution of the conversion factor (from frequency to energy) with time. Sometimes, jumps in the recommended values may go beyond the limits of the uncertainties of measurements.

8. Independent constants: G, k, and others

8.1 General comments: precision measurements of gravity

The fundamental physical constants appear in the most diverse problems. Thus, we saw that each quantity from the set that includes the Rydberg constant R_∞ and the proton charge radius R_p (see Section 3.2), the electron–proton mass ratio m_e/m_p (see Section 4), the fine-structure constant α (see Section 5), and the Planck constant h (see Section 6) is determined by different methods, and in a number of cases by methods from different fields of physics.

Nevertheless, determination of the values of certain fundamental constants playing the most important roles in very diverse phenomena turns out to be a separate problem.

Newtonian constant of gravitation G is just such a quantity. In particular, it determines:

• the motion of planets of the Solar System, however, only via a product involving the solar mass, namely

$$GM_\odot = 1.327\,124\,4210(1)\times10^{20}\ \mathrm{m^3\,s^{-2}}\,; \tag{12}$$

this product (the heliocentric gravitational constant) is known well, but the solar mass $M_\odot$ in SI units is known poorly;

• the motion of the Moon and of numerous spacecraft around Earth, which is described by the geocentric gravitational constant

$$GM_\oplus = 3.986\,004\,418(8)\times10^{14}\ \mathrm{m^3\,s^{-2}}\,, \tag{13}$$

where $M_\oplus$ is the mass of Earth, which in kilograms is known poorly;

• the free fall of a body near the surface of Earth (gravitational acceleration) is measured very well at a given site and at a given instant of time (up to the ninth decimal place); however, it does not reduce to the geocentric gravitation constant, since it is also necessary to take into account the altitude above the equipotential surface of a mean sea level, the geoid, the complex shape of which is determined by the distribution of masses below the surface of Earth (in the case of a 'naive' description of Earth's gravity by the Newtonian gravitation with a pointlike source, the geoid is shaped to a sphere, which is, obviously, wrong; one must also not forget the tidal forces depending on time, and Earth's rotation which is much more essential);

• characteristics of stars (the values of which are either poorly known themselves or are observed only in combination with other poorly known parameters);

• cosmological parameters such as the critical density of matter in the Universe (which are known with low precision and in combination with such quantities as the Hubble constant, also known with not quite metrological accuracy);

• the Planck scale of distances, times, masses, and energies (which is important rather as a scale, since it enters into no precise relationships of a practical character).

The list of fundamental phenomena described with the help of the constant G can be continued: there are many applications, but either they assume a low experimental or theoretical accuracy from the very beginning, or, in the case of high precision of the very applications, the Newtonian constant of gravitation enters into expressions only in combination with poorly known quantities (such as the solar mass).

The high-precision values of the heliocentric (12) and geocentric (13) gravitational constants are taken from the recommendations of IERS[10] [94]. We also note that, making use of the values of these constants [94] together with the recommended value of G (see Table 7) [7], it is possible to determine the masses of the celestial bodies closest to us in kilograms. Thus, the respective masses of the Sun and Earth happen to be

$$M_\odot = 1.988\,55(24) \times 10^{30}\ \text{kg},$$
$$M_\oplus = 5.972\,58(72) \times 10^{24}\ \text{kg}\,. \tag{14}$$

The relative uncertainty in determining the solar mass and Earth's mass turns out to be equal to the relative measurement uncertainty of the Newtonian constant of gravitation, namely to 1.2×10^{-4}. Is this a good or bad precision for the masses of celestial bodies? Of course, the mass ratios of the Sun, planets, small planets, and some moons are known with a precision that is superior by several orders of magnitude [94].

A certain absurdity of the situation lies in the fact that the masses (in kilograms) of certain distant stars are known with an uncertainty comparable with 1.2×10^{-4}. Let us consider, for example, the masses of components in the double pulsar PSR J0737-3039/A/B [95], which is composed of a millisecond pulsar and an ordinary pulsar. Their masses are, respectively, equal to

$$M_\text{m} = 1.3381(7)\ M_\odot = 2.6609(14) \times 10^{30}\ \text{kg},$$
$$M_\text{p} = 1.2489(7)\ M_\odot = 2.4835(14) \times 10^{30}\ \text{kg}\,, \tag{15}$$

which (in kilograms) is only 4.5 times worse in precision than the similar value for the solar mass or Earth's mass (14). Meantime, the total mass of the double system,

$$M_\text{tot} = M_\text{m} + M_\text{p} = 2.587\,08(16)\ M_\odot = 5.144\,53(69) \times 10^{30}\ \text{kg}\,,$$

is known in units of the solar mass with a relative uncertainty of 6×10^{-5}, which is better than the uncertainty of determining the solar mass in kilograms. This means the mass of the binary system is known in kilograms with the same precision as the solar mass! Thus, utilization of the kilogram as a mass unit limits the measurement accuracy not only of the masses of celestial bodies belonging to the Solar System, but of quite distant stars, too.

All this reasoning shows that in those problems where the gravity of a given gravitating source plays a fundamental role and where the Newtonian constant of gravitation G should appear in respective expressions, application of SI units and, in particular, the measurement of mass in kilograms turns out to be inappropriate. It does not reflect the physical meaning of the problem, and leads to a loss of accuracy in a number of cases: these are problems beyond the laboratory scale. On the contrary, in laboratory experiments, in which the measurement of mass in kilograms is quite natural, the gravity effects themselves represent, to a certain degree, an exotic phenomenon.

As a result, it turns out that, even given a variety of physical phenomena in fundamental nature, the experiments performed for determining G are of a quite limited type. Clearly, gravitation between bodies plays no essential role in laboratory conditions. At present, laboratory experiments are aimed at measuring the gravitational forces between two massive bodies at a laboratory scale (see a review of results in Section 8.4). The atomic-interferometric method [96] related to the behavior of trajectories of atomic beams in the vicinity of a massive body (on a laboratory scale) is also being mastered.

One more comment must be made concerning the Newtonian constant of gravitation determined by astronomical methods. In metrological literature, it is conventionally assumed that a physical quantity can always be uniquely represented as the product of a numerical value and a measurement unit. The gravity constant is a vivid example of a situation, where this is not quite correct. SI, in its traditional sense, is intended for local measurements, i.e., measurements in which the standard and the body investigated execute no motion relative to each other and are both at the same point (and, consequently, in the same gravitational potential). If gravitational effects show their worth on a nonlaboratory scale, the influence of the effects of special and general relativity will be felt. Thus, the result for the period of rotation of Earth about its axis or the period of its revolution around the Sun, measured with a caesium clock (i.e. in 'SI units'!), will depend on the location of the clock. The speed of its rotation about Earth's center will vary with latitude, and the gravitational potential will also depend on its height above (or below) the surface of the geoid. As a result,

[10] IERS — International Earth Rotation and Reference Systems Service.

different caesium clocks will show different times. For such measurements, it is necessary to fix the frame of reference as it is understood in the general relativity. The value of the heliocentric gravitation constant (12) is given for Barycentric Coordinate Time,[11] and that of the geocentric gravitation constant (13) is given for Geocentric Coordinate Time[12]. Different choices of coordinates and, in particular, of the time scale lead to results that differ noticeably from each other. The respective reference systems and time scales were defined by IAU[13] (references can be found, for example, in Refs [94, 97]), and for details concerning the application of the general relativity to metrology see paper [98]. We stress that the issue does not concern the fundamental constant itself, but its numerical value obtained with the help of clocks located in a certain frame of reference.

8.2 General comments: precision thermometry

The situation with the Boltzmann constant is, in many aspects, similar to the one dealt with above. This quantity has the meaning of a conversion factor from temperature to energy per degree of freedom. It is extremely difficult to measure such energy, because we have to do with a mean energy, while real measurements yield complex profiles. A characteristic example is provided by studies of the Doppler effect, with the molecular velocities following a Maxwellian distribution. It is very difficult to find an average value with high precision in such experiments; however, at present such experiments are being implemented [99, 100].

Another difficulty consists in the intensive character of temperature. Extensive quantities, such as mass or electric resistance, are easy to add up, subtract, multiply by an integer number, etc. This significantly facilitates the comparison of different values. In the case of temperature, this cannot be done. In practice, the measurement of temperature means the creation of a certain conventional temperature scale on the basis of reference points.

Such a scale is required to show a one-to-one correspondence between points of the scale and the absolute thermodynamic temperature, the continuity of this correspondence, and, apparently, the experimental reproducibility of the scale. The reference points are based on reproducible phenomena at a rigorously fixed temperature, for which the triple point of a substance serves as a typical example.

The realization of one temperature scale or another and measurement in its units are the subject of secondary thermometry. Determination of the absolute thermodynamic temperature of the reference points is performed by comparing them with the triple point of water using a gas thermometer, which is a matter of primary thermometry.

From a technical point of view, primary and secondary thermometries represent two fields of applied physics that are absolutely different and practically independent of each other. Measurement of the Boltzmann constant also belongs to primary thermometry and, in a certain sense, is extraneous to most measurements of temperature (within the framework of an effective temperature scale).

The basis of modern practical thermometry is formed by ITS-90[14] [101]. The uncertainties of its realization are discussed in detail in article [102]. The scale, generally speaking, is not linear, and the ITS-90 temperature constitutes a complex function of absolute thermodynamic temperature. In this case, thermodynamic methods are operational, as before, but the conversion to energy hardly makes sense.

Examples of the realization of such scales are represented by platinum thermometers, in which the resistance is a function of temperature and the scale is obtained by interpolation and extrapolation over several reference points.

The uncertainties of a temperature scale are related both to the uncertainty in the interpolation and to the precision with which the reference points themselves are known. It is, however, important to understand that, for example, temperature reproducibility as a condition for one physical experiment or another does not require knowledge of the correct values of reference points. It suffices to know the location of some temperature on the scale with respect to the reference points — more or less like when one orients oneself in a locality by correlating the position of the sought point with known reference points.

Practical thermodynamic methods making use of measurements in ITS-90 terms are, as a rule, contact methods and cannot be applied in measurements at a distance. Nevertheless, it must be recalled that certain special classes of phenomena exist, where high-precision measurements of the absolute thermodynamic temperature are performed just at a distance. A good example here is represented by the survey of anisotropy of the cosmic microwave background radiation. The most precise of these surveys was made with the help of the WMAP mission: its sensitivity $\Delta T/T$ amounted to approximately 2×10^{-5} [103]. At present, the data are being processed, that were obtained by all-sky mapping with the HFI[15] instrument aboard the Planck satellite [104], and from which a comparison of temperatures is expected with an uncertainty at the level of 2×10^{-6}.

Naturally, relative measurements do not require the Boltzmann constant k to be utilized, while the absolute astrophysical measurement of temperature,

$$T_{\mathrm{CMB}} = 2.725\,48(57)\ \mathrm{K}\,,$$

by the blackbody frequency spectrum of the cosmic microwave background [105], based on the processing of data from the FIRAS instrument on the COBE[16] satellite [106, 107] actually only expresses frequency in terms of temperature using the known value of the Boltzmann constant.

Quite probably, from the point of view of fundamental physics, that it is the relative measurements of the CMB temperature that are the most important temperature measurements of the last decade.

8.3 Recommended values

Below we shall briefly touch upon the determination of three fundamental constants: Newtonian constant of gravitation G, the Boltzmann constant k, and the anomalous magnetic moment of the muon a_μ, the values of which are collected in Table 7 together with certain of their derivatives.

Taking into account the independence of these 'independent constants' from each other, we shall consider the determination of their values separately.

[11] Temps-coordonnée barycentrique (TCB).
[12] Temps-coordonnée géocentrique (TCG).
[13] International Astronomical Union (IAU).
[14] ITS-90 — International Temperature Scale of 1990.
[15] WMAP — Wilkinson Microwave Anisotropy Probe; HFI — High Frequency Instrument.
[16] FIRAS — Far InfraRed Absolute Spectrophotometer; COBE — COsmic Background Explorer.

Table 7. Recommended values of some independent fundamental constants and of their derivatives [7]; u_r is the relative standard uncertainty.

Quantity	Symbol	Value	u_r
Newtonian constant of gravitation Planck mass	G $m_P = \sqrt{\hbar c/G}$	$6.673\,84(80) \times 10^{-11}$ m^3 kg^{-1} s^{-2} $2.176\,51(13) \times 10^{-8}$ kg	1.2×10^{-4} 6.0×10^{-5}
Boltzmann constant Molar gas constant Stefan–Boltzmann constant	k $R = kN_A$ $\sigma = (\pi^2/60)(k^4/\hbar^3 c^2)$	$1.380\,6488(13) \times 10^{-23}$ J K^{-1} $8.314\,4621(75)$ J K^{-1} mol^{-1} $5.670\,373(21) \times 10^{-8}$ W m^{-2} K^{-4}	9.1×10^{-7} 9.1×10^{-7} 3.6×10^{-6}
Anomalous magnetic moment of the muon	a_μ	$1.165\,920\,91(63) \times 10^{-3}$	5.4×10^{-7}

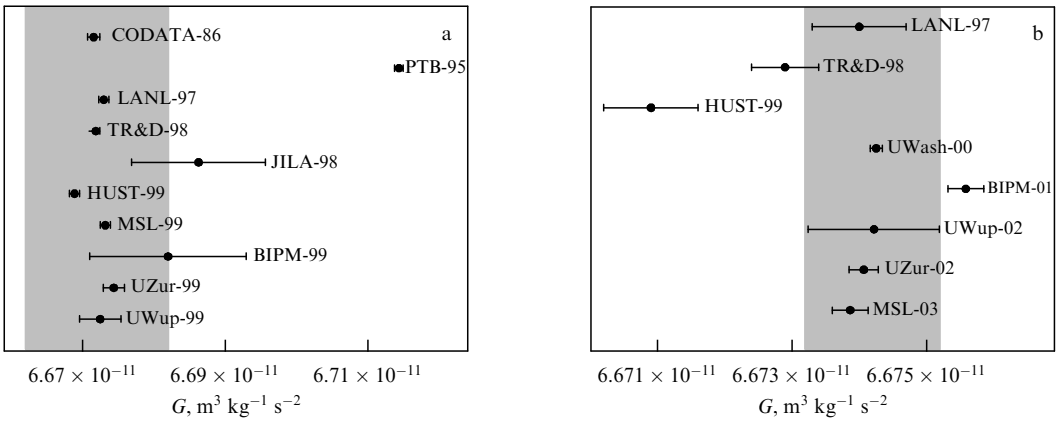

Figure 7. Measurements of gravitation constant G in the 1998 adjustment [3] (a) and in the 2002 adjustment [4] (b). Vertical shaded belts correspond to recommended values of the respective adjustments.

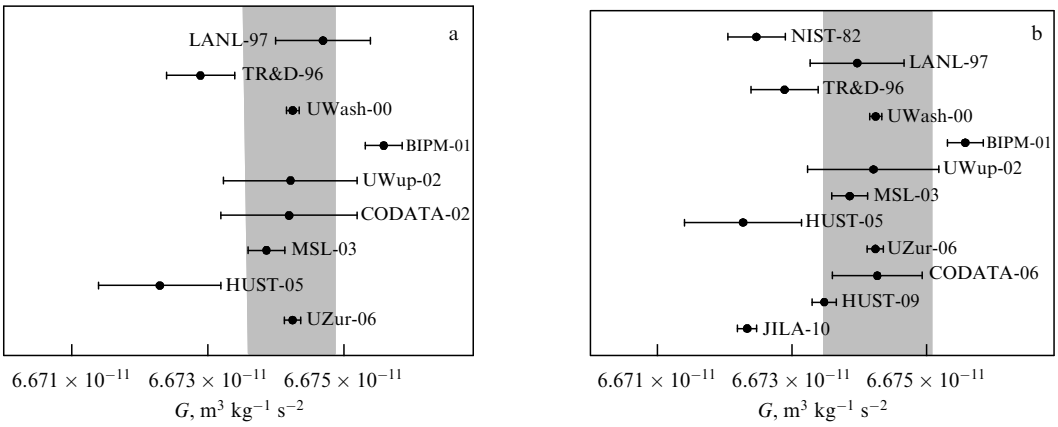

Figure 8. Measurements of gravitation constant G in the 2006 adjustment [5, 6] (a) and in the 2010 adjustment [7] (b). Vertical shaded belts correspond to recommended values of the respective adjustments.

8.4 Measurement of the Newtonian constant of gravitation

The situations concerning the fundamental constants k and G, given in Table 7, are quite diverse. As to the gravitation constant G, the main circumstance in its measurement is the scatter of experimental data, which is caused by a certain artificiality in conditions of the experiment: the gravity effects are necessarily studied in conditions where they are far from dominant.

Measurements of G are done in the course of special dedicated experiments, where, on the one hand, there are classical macroscopic objects, and, on the other hand, it is necessary to perform precise measurements of quite small forces. Such experiments are always complicated and exhibit numerous systematic uncertainties.

The gravity effects being small compared to electromagnetic effects can result in scattered charges or currents giving rise to relatively strong forces. Various deformations are also of no small importance, when work is performed with massive macroscopic objects.

The scatter of data in Figs 7 and 8, where original measurement results for G are presented, reveals the difficulty of dealing with such effects. The figures present the results of the last five adjustments (the recommended 1986 value [2] is represented by a dot in Fig. 7a). The main change in the situation consists in the following.

The 1986 result exhibited a relatively high precision, but in 1998 the uncertainty underwent revision and was extended owing to the serious contradiction between the data that served as a basis for the 1986 adjustment and the PTB[17] value. In 2002, the decision was taken after a thorough analysis [4] to

[17] PTB — Physikalisch-Technische Bundesanstalt.

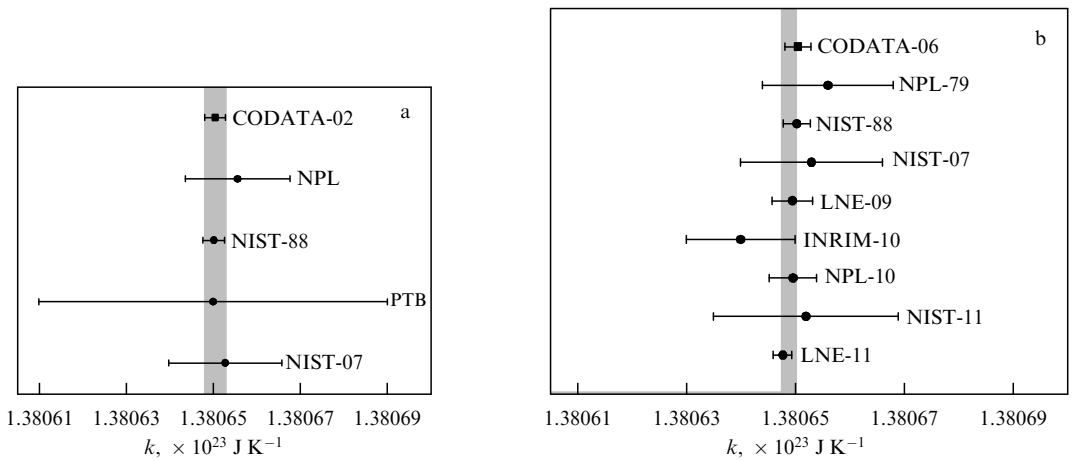

Figure 9. Determination of the Boltzmann constant k in 2006 adjustment [5, 6] (a) and in 2010 adjustment [7] (b). Vertical shaded belts correspond to respective recommended values.

exclude the PTB data (with the authors' agreement) from the processing, and the uncertainty in determining the constant once again dropped to approximately the precision level of the 1986 adjustment.

The uncertainty of the result sought after is still determined by the scatter of the data. In the 2002, 2006, and 2010 adjustments, it changed insignificantly, although new original results have appeared, and, on the whole, their declared precision has improved.

Thus, the most precise result (in accordance with the declared relative standard uncertainty $u_r = 1.4 \times 10^{-5}$), obtained in 2000 at the University of Washington, was published [108]: it is indicated as UWash-00 in Fig. 8. Only three results are inferior to it in their declared precision by a factor of two or less: they are shown in Fig. 8b as UZur-06 (the result from the University of Zürich with $u_r = 1.9 \times 10^{-5}$) [109], HUST-09 ($u_r = 2.1 \times 10^{-5}$) [110][18], and JILA-10 ($u_r = 2.7 \times 10^{-5}$) [111]. They were all achieved quite recently, while the last two appeared just between the 2006 and 2010 adjustments.

However, the precision of the recommended value changed, in this respect, insignificantly and remained at a level of $\sim 10^{-4}$. In spite of the high declared precision of individual results, the process does not tend toward any single value, and the problem of systematic effects, which have hitherto been unaccounted for, has evidently not been resolved.

As we have previously noted (see Section 8.1), in no way does the situation with the gravitation constant being 'unsettled' affect various precision applications. There are several fundamental constants, the values of which are related to G; these include the Planck mass (see Table 7) and the Einstein's gravitation constant

$$\kappa = \frac{8\pi G}{c^2} = 1.866\,27(22) \times 10^{-26}\ \mathrm{m\ kg^{-1}}\,.$$

However, their values obtained with high precision are also of no practical interest.

The general situation with determination of the Newtonian constant of gravitation G can be characterized as follows: it is unacceptable from a technical standpoint owing to significant contradictions among data, but it is quite acceptable from the point of view of a 'user', who really does not need to know the value of G (in SI units) with greater precision.

8.5 Progress in measurement of the Boltzmann constant

For a long period of time, the recommended value of the Boltzmann constant k (like of the molar gas constant R) was fully determined by a single experiment [112]. The situation changed drastically when the proposal was made to redefine the kelvin on the basis of the fixed value of the Boltzmann constant [89]. The metrological community, or to be more precise, the part of the community that deals with primary thermometry, perceived this proposal on the whole positively. With time, however, it became clear that one must not rely on the value of a constant measured by a single method and only by a single team.

The necessity arose, if not for a more precise, certainly for a more reliable determination of the Boltzmann constant. This necessity ultimately led to a sharp increase in experimental activity. In part, this necessity was related both to the development of absolute methods of measuring energy temperatures and to the necessity for appropriate knowledge of the conversion factor.

There is no need, however, to overestimate the influence the progress in absolute measurements had in raising the issue of the Boltzmann constant as the conversion factor from thermodynamic to energy temperatures. It was rather general reasoning (see Section 11 and publications [89, 90]), while the progress in absolute measurements (see, for instance, Refs [99, 100]) to the same extent serves as both the cause and the consequence of a possible transition to a new definition of the kelvin.

The progress made in determining the Boltzmann constant is illustrated in Fig. 9. The difference between the situations in 2006 and in 2010 is quite striking. In no way are they related to some momentary revolutionary breakthrough in the measurement methods of the Boltzmann constant. Incidentally, the precision in determination of the constant has not also improved — more measurements have been taken, and they have become more diverse, but not more precise than the 1988 measurement [112]. All the measurements have been carried out at national metrological centers, and, in this regard, the peculiarity of metrology is felt in that it is not a domain of science, but a high-technology area of a certain practical activity, which is quite sensitive to the needs of society.

[18] HUST — Huazhong University of Science and Technology.

It should be noted that, at present, together with the 1988 result [112], diverse results [113–116] that were obtained by the method of acoustic gas thermometry, i.e., by the same method as in Ref. [112], dominate. These results are presented in Fig. 9 and marked as LNE-09, NPL-10, INRIM-10, and LNE-11 [19] (see also article [117]). However, other methods are also undergoing noticeable development, although the respective results (NIST-07 and NIST-11) [118, 119] have exhibited so far somewhat lower precision (see also Refs [120, 121]). Enhanced activity continues in this field.

9. Determination of the anomalous magnetic moment of the muon

The anomalous magnetic moment of the muon a_μ stands somewhat apart. Its numerical value is a more or less separate constant, and, in principle, may be considered an auxiliary constant used for interpreting data related to the muon. These data, however, play an insignificant role in determining the most important fundamental physical constants, such as α and h, so we, therefore, have preferred to treat a_μ as an independent constant.

At the same time, it is important to understand that the significance of studies of the anomalous magnetic moment of the muon for the determination of fundamental constants is not exhausted by their participation in correcting muon parameters. It is more essential that many of the computational technologies utilized are also applied in calculations of the anomalous magnetic moment of the electron a_e. We have already noted that additional tests of the theory of a_e are necessary, and calculations of the muon anomaly provide such a test, albeit quite limited. Therefore, an indirect relationship exists between this constant and the data group related to the fine-structure constant α.

The agreement between theory and experiment in the case of the anomalous magnetic moment of the muon has never been perfect, starting from the first publications of results obtained at BNL.[20] Since then, both theory and experiment have improved. The state of the art, in accordance with paper [122], may be inferred from Fig. 10.

Let us note at once that, taking into account the contradiction between theory and experiment, the experimental value of Ref. [123] was chosen as the 2010 CODATA recommended value [7].

We recall that the theoretical expression is composed both of quantum electrodynamic terms and of contributions from hadrons residing in intermediate states. In QED theory, significant success has been achieved (see, for instance, Ref. [124]); however, the hadron contributions still remain a stumbling group. A detailed analysis of the general situation can be found in monographs [125, 126].

The hadron contributions consist of two parts. First, there is a contribution from hadron polarization of vacuum, and, second, there is also a significantly smaller contribution of the hadron light-by-light scattering block. Their uncertainties, nevertheless, are comparable, thus determining the final precision of the theory.

Hadron polarization of vacuum is described by 'direct' experimental data on electron–positron annihilation into hadrons and by hadron decays of the τ-lepton. The integral

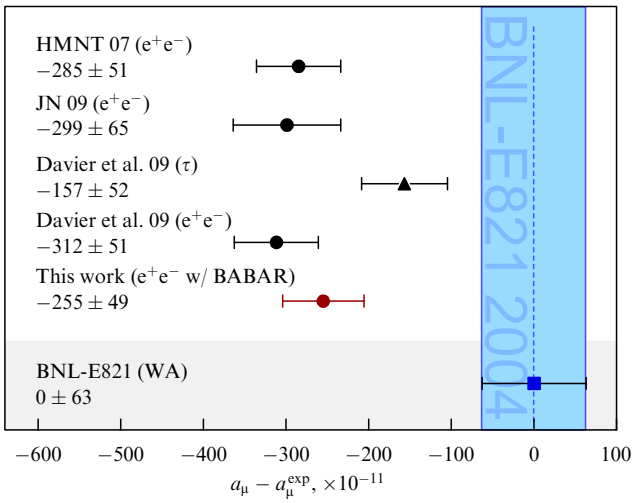

Figure 10. Theoretical and experimental results for the anomalous magnetic moment of the muon α_μ. The picture is reproduced from Ref. [122] with permission of the authors. *This work* relates to the study done in Ref. [122].

is calculated over the measured cross sections. The word 'direct' is in quotation marks, because calculations of the contribution have to be done with an uncertainty not larger than 0.5%, and in this case data are needed that have the same high precision. For this, it is necessary to take into account different nontrivial corrections during interpretation of experimental data on the annihilation and decay of particles. The results obtained from annihilation and decays are far from being in excellent agreement, as is seen from Fig. 10. Different data processing procedures can lead to results that differ in magnitude. For instance, the account of corrections for isotopic symmetry violation may be diverse. Thus, the assertion was made in Ref. [127] that, after such corrections are taken into account in a certain manner, the data on annihilation and decays turn out to be consistent with each other.

The contribution from hadronic light-by-light scattering is small; however, it cannot be calculated either from the first principles or from results of direct experiments. Numerical simulation leads to a scatter of results and to large uncertainties; typical uncertainty amounts to 20–30%.

In particular, the following results are presented in Ref. [122] for the aforementioned key hadron contributions: for the leading contribution from hadronic vacuum polarization we have

$$a_\mu(\mathrm{LO-hVP}) = 695.5(4.1) \times 10^{-10}\,,$$

while for the contribution from the hadron light-by-light scattering block the value utilized was [128]

$$a_\mu(\mathrm{hLbL}) = 10.5(2.6) \times 10^{-10}\,.$$

These values and their uncertainties must be compared with the uncertainty in the experimental determination of a_μ [63], which amounts to 6.3×10^{-10}. The discrepancy between theoretical and experimental results is several times larger than the declared uncertainty (see Fig. 10).

Resolution of the existing contradiction may be achieved by an adequate description of the experiments on e^+e^- annihilation and the τ-lepton decay, as well as by the choice of models of hadronic light-by-light scattering. Since the

[19] LNE — Laboratoire National de Metrologie et d'Essais; INRIM — Istituto Nazionale di Ricerca Metrologica.
[20] BNL — Brookhaven National Laboratory.

results obtained from annihilation and decays differ noticeably from each other, it is to be expected that at least part of the divergences will be discarded by analysis of the experimental data involved in describing the hadron contributions. From this point of view, the choice of the experimental value of a_μ as the recommended value in the adjustment [7] seems reasonable. However, strictly speaking, there are no solid reasons for considering experiment [63] to be absolutely correct and *all* the divergence of theoretical and experimental values to be exclusively due to systematic errors in the theory.

10. Progress in determination of the values of fundamental physical constants

To conclude the short review of new original data and results involved into the 2010 adjustment [7], it is interesting to compare the new results with earlier ones. The dynamics in the improvement of the (declared) precision are presented in Fig. 11 for the entire period of work of the CODATA Task Group on Fundamental Constants.

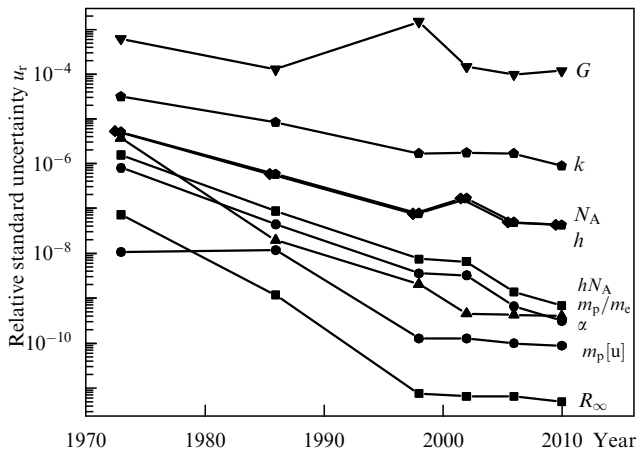

Figure 11. Accuracy in determining the values of fundamental physical constants in CODATA adjustments [1–7].

From the figure, it is seen that in certain cases the precision falls instead of rising. This takes place either on appearing new results contradicting to earlier ones, or when systematic effects, not taken into account previously, are revealed. The shift of certain constants from one adjustment to another may also go beyond the limits of errors, and sometimes even significantly. This, for instance, happened recently with the values of the fine-structure constant and of the Planck constant. For the most important constants, Table 8 shows the changes in their determination accuracy and the shifts of their values based on the results of the 2010 adjustment.

Concerning the quantities presented in the table, the situation has unambiguously improved only in the case of two of them: the electron–proton mass ratio m_e/m_p, and the Boltzmann constant k. The new values are consistent with the old ones, and their precision and reliability have increased. The greater reliability is a consequence of the appearance of new and independent results.

In speaking about the other constants, one must admit that the changes that have taken place are not so favorable. In some cases, the precision somewhat deteriorated owing to the increase in the scatter of the data, as happened with the Newtonian constant of gravitation G. In others, the value went beyond the limits of the existing uncertainty, as in the cases of the fine-structure constant α and the Planck constant h. A scatter of the data, although not reflected in an extension of the uncertainty, also took place in the case of the Rydberg constant R_∞.

Is this good or bad? The answer depends on the goals that were to be achieved by the adjustment. If the issue only concerned a practical problem, in which, say, the (numerical) result itself were important, then one should say that the situation has worsened. From this standpoint, yes, the tables of recommended values have become somewhat worse.

However, the very tables, although they are the most well-known results of the adjustment, do not actually represent its main goal. Above, we explained why the procedure is called adjustment. We shall now emphasize the specific character of many measurements and calculations applied for obtaining data. These are either essentially new approaches or measurements and calculations that just recently were beyond the limits of accessibility.

The precision of most of the input data is determined not by statistical but systematic uncertainties. Meanwhile, estimation of systematic effects is often the most nontrivial part of an experiment or a calculation (but not the most technically difficult part).

And we arrive at a paradoxical situation. Although there are highly qualified specialists and use is made of the most advanced technologies, a lack is revealed of accumulated experience, which, actually, could not even exist in pioneering studies.

Experience can only be accumulated by performing experiments and calculations and by comparing the results obtained with available others. The main goal of the adjustment resides in checking the consistency of advanced methods with each other and with traditional methods.

From this point of view, the more inconsistencies there are, the better. Thus, the vulnerable points in new measurement and calculation technologies are revealed.

Let us now formulate the main results of the 2010 adjustment from the standpoint of fundamental physics.

Table 8. Recent progress in determination of the values of fundamental physical constants in the 2006 [5, 6] and 2010 [7] adjustments. Here, Δ denotes the relative change of the value A: $\Delta(A) = (A(2010) - A(2006))/A(2006)$; u_r is the relative standard uncertainty.

Quantity	u_r (2006)	Δ	Δ/u_r (2006)	u_r (2010)	u_r (2010)/u_r (2006)
R_∞	6.6×10^{-12}	1.1×10^{-12}	0.17	5.0×10^{-12}	0.76
m_e/m_p	4.3×10^{-10}	0.1×10^{-10}	0.03	4.1×10^{-10}	0.95
α	6.8×10^{-10}	44.1×10^{-10}	6.50	3.2×10^{-10}	0.47
h	5.0×10^{-8}	9.2×10^{-8}	1.84	4.4×10^{-8}	0.88
k	1.7×10^{-6}	-1.2×10^{-6}	−0.68	9.1×10^{-7}	0.53
G	1.0×10^{-4}	-0.7×10^{-4}	−0.66	1.2×10^{-4}	1.2

• A drastic improvement has taken place in the precision of tests of quantum electrodynamic calculations of the anomalous magnetic moment of the electron. This has become possible both owing to the development of theory and experiment relevant to the anomalous magnetic moment of the electron and due to the significant success achieved in determining the fine-structure constant by methods of Raman spectroscopy on rubidium atoms. A comparison between theory and experiment in the case of the anomalous magnetic moment has become sensitive to five-loop contributions.

• For the first time after a long break, a successful result was achieved in determining the energy levels of the muonic atom [18]. This is the first successful measurement on muonic atoms by laser spectroscopy methods.

• A noticeable contradiction was revealed in determining the proton charge radius by different methods, which, in particular, should stimulate the analysis of data on electron–proton scattering and on the atomic spectroscopy of hydrogen and deuterium.

• Successful application of laser spectroscopy is under way for precision measurements in another unstable atom, namely, in antiprotonic helium. The high experimental measurement accuracies permit raising the issue of the extent to which 'exotic' atoms should be considered exotic.

• Significant success has been achieved in theoretical [129–131] and experimental [132, 133] studies of the fine structure of the helium atom. We believe that the importance of the value of the fine-structure constant, obtained from studies of the fine structure of the helium spectrum, was underestimated, and that in the future it will be included in the main data processing procedure. The role of quantum electrodynamics in determining the values of fundamental physical constants is undeniable. An important role in determining the fine-structure constant (from the anomalous magnetic moment of the electron) is played by the QED of free particles; the quantum electrodynamics of two-particle bound systems are important, in particular, for determining the Rydberg constant, the electron–proton mass ratio, and the electron–muon mass ratio. Checking the QED theory of hydrogenlike atoms [15] has something closely in common with the determination of the fundamental constants. At present, three-particle atoms and molecules, such as antiprotonic helium, ordinary helium, and hydrogen molecular ions [134–136], are becoming important from the standpoint of metrology. Bearing this circumstance in mind, it is expedient to fully involve the spectral data on helium in the data processing.

• The existence of precise results permits imposing a number of constraints on the new physics. While constraints can be imposed by standard methods of elementary particle physics on the effects related to heavy particles with normal coupling constants, the precision physics of simple atoms, represented to a significant extent in the adjustment, permits us to impose constraints on ultraweak interactions with particles of ordinary mass (see, for instance, Ref. [137]).

It should be noted that many constraints on the new physics should not be addressed too seriously. Thus, it is possible to draw a conclusion concerning the antiproton mass from the comparison of optical transition frequencies in antiprotonic helium [58] with transitions in hydrogen. Whereas a comparison with the standard measurement of the proton mass (see, for example, Ref. [54]) permits reaching a conclusion on the proton–antiproton mass ratio [58].

It may be appropriate here to recall that the very existence and the main properties of antifermions follow from the Dirac equation. If it is considered to hold true, nothing but the equality between the proton and antiproton masses is to be expected. But if it is considered to be incorrect, the reference point for calculations of energy levels vanishes, for instance, in the hydrogen atom (the spectrum of which is compared to the spectrum of antiprotonic helium).

Realistic violation of CPT invariance should, first of all, alter the Dirac equation. Then, further possible constraints on the parameters of CPT violation should already be based on an analysis of the 'violated' Dirac equation, in which a certain deviation will take place with respect to the hydrogen and deuterium spectra, the g-factor of a bound electron in a hydrogenlike ion, and the antiprotonic helium spectrum.

Although individual particular assertions, such as the requirement that the particle and antiparticle masses be equal, did become distinctive symbols of one symmetry or another, it would be unreasonable to subject such individual facts to tests on their own. A standard physical theory comprises a construction, within the framework of which separate assertions are interrelated. Here, some of the assertions, which it seems desirable to test, turn out to be related to assertions that have already been tested or that actually serve as the foundation of the very test. In this sense, when the results of precision studies are applied for imposing restrictions on the new physics, it is necessary to clearly comprehend the framework construction of the new physics and also how the very input data can undergo modifications within this framework. Such tests, as a rule, are model-dependent.

The recommended values collected in the tables [7] are, first of all, needed for convenience of use and for uniformity. If high-precision values are indeed required, the course of action here should be different. Thus, if the value of the fine-structure constant is required for comparison of theory and experiment, it is important not to substitute its recommended value into the formulae, but to obtain α from the comparison of theory and experiment, and then to compare it with other values (for details, see paper [93]).

For example, the present recommended value of the Planck constant is a certain average of two inconsistent values (with a somewhat extended uncertainty owing to this inconsistency). This is seen from Fig. 5. Recently, a new result appeared [91] with a relative uncertainty of 6.5×10^{-8}. Comparison of the result reached in Ref. [91] with the recommended value is not so informative, while agreement of the value [91] with the result for $N_{\mathrm{A}}(^{28}\mathrm{Si})$ (project for measurement of the Avogadro constant) and its inconsistence with the value of $K_{\mathrm{J}}^2 R_{\mathrm{K}}$ (NIST-07) (the NIST watt balance) sends us quite a definite signal. This example shows that a comparison with original results yields information which is absent in a comparison with the recommended value.

From the point of view of determining the reliability of recommended values, it would be interesting to consider the results which are the second and the third in accuracy. This, however, is not always possible to do owing to the existence of numerous correlations. Thus, the results of measurements of several transitions may be presented in one of the publications (and in several atoms, for instance, in different isotopes, as in the spectroscopy of atomic hydrogen and deuterium and antiprotonic helium-3 and 4). Several separate experiments may also be carried out with the same setup, as it occurs in the spectroscopy of hydrogen and deuterium. Clearly, these results are correlated, while independent confirmations are required, first and foremost, to provide reliability in deter-

mining the values of fundamental physical constants. The reliability issue is a key one here, since the determination of the values of fundamental constants by pioneering methods is extremely vulnerable.

A certain change is also to be noted in the character of input data and in the methods of working with them. This field of studies has traditionally developed on the basis of laboratory measurements. Somewhat overestimating, it can be said that the adjustment is an adjustment of everything that is relevant to the fine-structure constant α and to the Planck constant h. Refining these two constants involves in the consideration standards of the units of the base electrical quantities, which is of great importance from both metrological and practical points of view.

The other quantities appear, mainly, when they are required. Thus, the value of the Rydberg constant is needed with account for the relationship to α, while for its determination it is necessary to know the charge radii of the proton and of the deuteron. The proton magnetic radius, which is no more and no less fundamental than its charge radius, is not needed for these purposes. As a result, the proton charge radius is included in the tables of recommended values [7], while its magnetic counterpart is not.

To determine the value of a_μ, it is necessary to take into account the small effect due to weak interactions, and to do so it is necessary to know the value of the Fermi (weak interaction) constant G_F. It was included in the tables [7]. Certain characteristics of the neutron and of the muon, such as their masses and magnetic moments, were also included, but their lifetimes were not.

The choice of the above-referred examples is not arbitrary, and we shall come back to them somewhat later. However, it seems appropriate to start with mentioning one more circumstance. Only laboratory data were made use of in the initial data processing procedures. The volume of these data was observable. The adjustment implied the input of all directly determined data and their joint processing.

Gradually, the situation changed and at present significant data arrays have originated that are processed outside the framework of the adjustment. Thus, for example, the proton and deuteron charge radii derived from scattering data are taken from ready processed data provided by one or several accelerator experiments. This is done because the charge radius is not a directly measured quantity, but the result of extrapolation of scattering data for different momentum transfers.

In the course of data processing for elastic electron–proton scattering [19, 138], several parameters originate, and, in particular, the proton electric and magnetic radii. (In Ref. [138], the same processing result is presented as in the earlier publication [18] quoted above in Section 3.2. However, unlike Ref. [18], the more recent article [138] reports more details, and it also presents the result for the magnetic radius.) The charge radius included in the list of recommended values and the magnetic radius, which is not included, are correlated. Here, it turns out that the results for the charge radii, given in these two publications, are in excellent agreement, while the results for the magnetic radii are in strong contradiction (see also the discussion in articles [21, 22]), as is seen in Fig. 12.

Strictly speaking, it is also necessary to take into account the correlation in determining the radii from one data processing procedure or another in the case of e–p scattering. Such an account will result in turning the axes of the ellipses, but will not alter the picture qualitatively. The

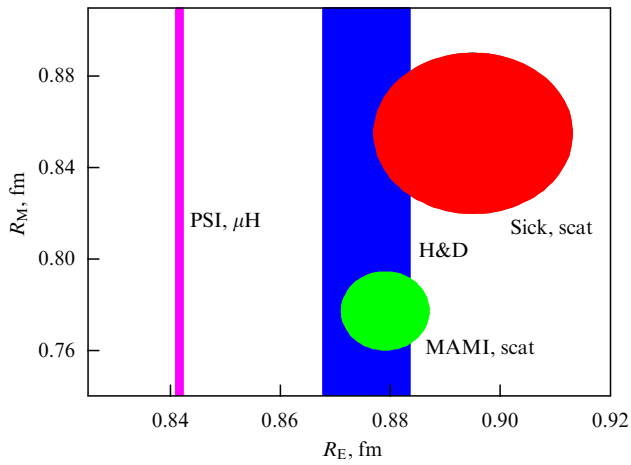

Figure 12. Determination of the proton electric (charge) and magnetic radii R_E and R_M, respectively. In Fig. 1 and in other sections, where only the proton charge radius appears, we follow the notation adopted in Ref. [7] and make use of the symbol R_p. The notation for experiments and experimental data processing procedures follows that in Fig. 1.

indirect value obtained from the deuteron charge radius and from the isotopic shift in hydrogen (indicated in Fig. 1 by an open square) is not shown in the figure, since it would be represented by the related vertical belt fully covering the entire figure and would not provide any useful information.

The Fermi constant, as one more parameter pertaining to high-energy physics, is determined from the lifetimes of the muon and the neutron, as well as from some other experiments. The situation with the lifetimes of the two aforementioned particles has not always been unambiguous (see, for instance, Refs [139] and [140, 141]). The recommended value [7] of the Fermi constant is taken from the data processing procedure implemented by the PDG[21] [142]. Its precision is certainly quite more than sufficient for a_μ.

Questions inevitably arise as to which data must be 'adjusted' and which are to be adopted, as well as which quantities should be included in the tables and which should not. These issues become especially important on the threshold of the adoption of new definitions for SI units, regarding which part of the experimental data, namely, those related to the international prototype of the kilogram and to the temperature of the triple point of water, will turn out to have nothing to do with the fundamental physical constants.

11. Towards a quantum system of units (quantum SI)

In this connection, we shall conclude with a discussion of two issues: of the possibility and expedience of new definitions of SI units, and of what will happen with the values of the fundamental physical constants after such a redefinition.

SI is based on six units of physical quantities: the kilogram, the meter, the second, the ampere, the kelvin, and the mole. (There is also a seventh unit, the candela, which is used for physiological quantities related to illumination, but it will not be dealt with here.)

Defining the six units in terms of natural constants and quantum phenomena implies the adoption, by definition, of values for the six constants. They can be chosen in several ways [88–90]. The most attractive from a practical point of

[21] Particle Data Group.

view is based on taking advantage of the frequency of hyperfine splitting in caesium atom, the speed of light in vacuum, the Planck constant, the electron charge, the Boltzmann constant, and the Avogadro constant (for details, see review [90]).

In moving to the new definitions, it is important that there be no loss of a measurement accuracy (in SI units) and that there be no 'jumps' in the numerical values of ones or other observational quantities. To implement the second condition, it is necessary to know the values of the constants being fixed with a precision comparable to that with which the units are realized in standards. Above, we have already discussed the situation with several of these values.

Success has not been achieved in fully satisfying the first condition. The point is that it is fairly easy to carry out relative measurements of masses, resistances, and voltages. It is possible to achieve uncertainties amounting to several significant figures in the ninth decimal place for masses and to several units in the tenth decimal place in the case of resistances and voltages. It is also possible to build standard measures for these quantities that are reproducible at a level comparable with the accuracy of relative measurements. However, the units for these quantities are not independent. To relate electrical and mechanical units, there is need to perform experiments at a certain stage that are now aimed at measurements of the Planck constant and the Avogadro constant. The precision of these experiments is doubtless lower than the precision of the aforementioned relative measurements (for details, see Ref. [90]), and, what is more, their results are contradictory.

Therefore, in the course of restructuring the SI, the case in point is that the difficulties of measurements in SI units, which are now manifested in electrical measurements, will move to the measurements of masses. The advantage here consists in the fact that, first, the precision of relative electrical measurements and the reproducibility of quantum electrical measures are higher than in the case of weighing, so it is expedient to take advantage of system units rather in this region, and, second, it turns out to be possible to get rid of the only artefact underlying the modern system of units.

The results of comparison of the international prototype of the kilogram with national standards during their verification [143], displayed in Fig. 13, indicate that the mass value of the standard weight averaged over the ensemble of national standards drifts with respect to the international prototype. The value of the long-term drift is already quite comparable to the precision of the best measurements of the Planck constant (see Section 6). In the meanwhile, the international prototype of the kilogram was chosen from a set of identically made weights, in which the other ones are utilized as national or auxiliary standards. There is no reason to consider the international prototype of a kilogram to be conserved significantly better than the national standards, and the observed systematic and stable drift cannot but put one on one's guard. We also note that, contrary to the extremely simple definition of the kilogram, in practice the international prototype has to undergo thorough cleaning before each of the rare times it is made use of [143]. Between verifications, of which since the Metric Convention, i.e., more than a century ago, there have been only three (including the first verification performed immediately after the standards were made), the value of the kilogram is actually determined by extrapolations.

Although advantages exist in the transition to new definitions, the situation is not quite clear in what concerns

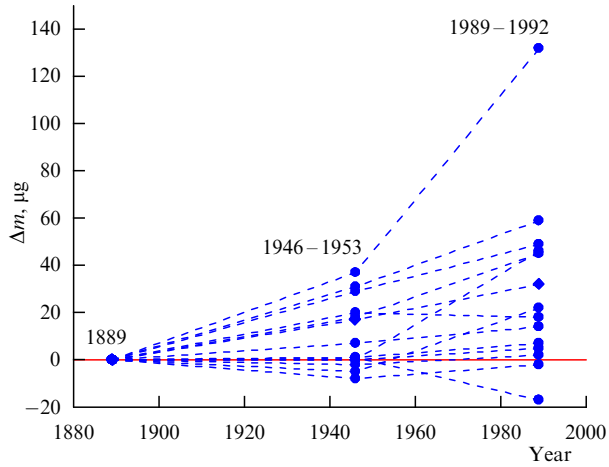

Figure 13. The results of verification of the international prototype of the kilogram with the 14 national standards that all took part in all the three verifications [143]. Following tradition, the changes in the mass of kilogram weights are given in micrograms, i.e., units of 10^{-9} of the mass value of the weight.

the uncertainties in determining the Planck constant. It is clear that new definitions will be adopted, but it is not quite clear precisely when. The appropriate time for the transition to the new definitions will be chosen on the basis of practical reasons and, generally speaking, this may already happen in 2013–2014.

We shall now discuss what will happen with the determination of the values of the fundamental physical constants in the case of the transition to new units. For convenience, we shall consider the consequences stage by stage, assuming fixed values for the Planck constant, the elementary charge, the Avogadro constant, and the Boltzmann constant, one by one.

After the transition to the definition of the kilogram by the Planck constant, the value of the latter, being known exactly, will apparently become an auxiliary constant. The data groups required for determining α and h will unite. Indeed, we recall that a series of combinations of constants from the less precise h-group, involving the Planck constant, are known with the precision of the α-group, for example: hN_{A}, h/m_{e}, e^2/h. As soon as the Planck constant becomes known exactly, the precisions of the values of N_{A}, m_{e}, and e in SI units will immediately improve.

All the old values in the h-group implied, in some way or another, a comparison with the international prototype of the kilogram, and discarding it from the measurements will obviously alter the physical interpretation of the numerical values. The physical meaning of the kilogram itself will also change: actually, all masses will now be measured in units of frequency. This will result in measurements aimed at determining the very Planck constant h (a watt balance), which no longer take part in the adjustment of the values of fundamental constants and become the basis for realization of the new mass unit.

The ampere is determined in the standard SI version by the fixed value of μ_0. The transition to a new definition with a fixed value of the elementary charge (with the value of the Planck constant already fixed) will render the vacuum constants ϵ_0 and μ_0 measurable quantities which are involved in the α-group.

The situation with the Avogadro constant is somewhat more complicated. Actually, there are two different fundamental constants that provide information on how many atoms are to be found in some mass of a substance. First, the value of the atomic mass of carbon-12 in grams unambiguously states the number of atoms of this type in 12 grams of carbon. Second, the Avogadro constant tells us about the number of atoms in a *mole*, and we *call* a mole the amount of carbon atoms contained in 12 grams.

If we start by defining the kilogram, and then change the definition of a mole, adopting, by definition, a fixed value for the Avogadro constant, then its physical meaning will change: in answering the question of how many atoms may be found in a mole, it will no longer answer the question of how many carbon atoms reside in 12 grams of carbon. Nevertheless, the 'old' Avogadro constant, being capable to answer this question, will remain in the equations, and it will be expressed in terms of the mass of a carbon atom in kilograms, which is closely related to the proton mass measured in kilograms.

As a result, a number of constants related to the Avogadro constant will be divided into two parts. On the one hand, such constants as N_A, hN_A, and $F = eN_A$ will maintain their expressions old in form, but with a new, now fixed, Avogadro constant. Their precision will improve, and part of them (this concerns all three aforementioned constants) will be known exactly. On the other hand, a significant role will be played by the value of the atomic mass unit m_u in kilograms. Only it will substitute for the Avogadro constant in some substantial relationships. We recall that in modern definitions the numerical value of the inverse atomic mass unit in grams is equal to the numerical value of the Avogadro constant.

The experiment carried out for determining the Avogadro constant searches for the number of atoms in a macroscopic sample of a substance of known mass. This experiment will not disappear from the adjustment. It will measure the mass of an atom in kilograms. However, it must be understood that the quantity obtained, according to contemporary terminology, instead of corresponding to N_A, will rather correspond to hN_A (since the mass of the substance will be determined in new kilograms related to the Planck constant). In this case, the accuracy of the experiment may turn out to be somewhat lower than the precision of the α-group. Owing to its small statistical weight, the experiment will not play a significant role in the determination of α; however, it will be important for reproducing the kilogram. (At present, determination of the fine-structure constant by electrical methods and, in particular, of $\alpha(R_K)$, namely, the result of comparing the capacity of a calculable capacitor and the Hall resistance, happens to be in quite the same situation. These experiments only slightly affect the recommended value of α, but they are important for reproduction of the farad and ohm.)

Adoption, by definition, of the value of the Boltzmann constant will remove from the adjustment experiments for its determination; now these experiments will measure the temperature of the triple point of water, which does not belong to fundamental physical constants. They will be important for realization of the ITS-90.

As a result, a number of macroscopic experiments will either leave the adjustment completely, or lose their past significance.

12. Conclusions

On the whole, one can state that in 2007–2010 significant progress began to show itself in the determination of values of the fundamental physical constants. Several essentially new experiments were realized, such as measurement of the Lamb shift in muonic hydrogen and measurement of the Avogadro constant with an enriched silicon crystal. In a number of other experiments and calculations, significant progress was also achieved.

The values of certain fundamental constants were improved. The precision and reliability were increased in determining the value of the fine-structure constant α, of the electron–proton mass ratio m_e/m_p, and of the Boltzmann constant k. In some values, systematic effects not taken into account previously were revealed, as in the case of α. In part of the data (on the proton charge radius R_p, the Planck constant h, and the Newtonian constant of gravitation G), a significant scatter is observed, which serves as a very important motivation for further studies.

The situation with the fundamental constants necessary for redefinition of the base SI units has, on the whole, improved. This renders more probable a transition, on the basis of natural constants and of quantum phenomena, to new definitions of the base units which will exclude artifacts, such as the international prototype of the kilogram kept at BIPM.

A change in the definitions of SI will remove some of the experiments from the adjustment. They will no longer determine the values of the fundamental physical constants, but will, instead, reproduce novel units. This is quite evident for those experiments where, by means of a chain of comparisons, the international prototype of the kilogram is involved or where the triple point of water is utilized. These values are of no fundamental significance, and they take part in the determination of the values of fundamental constants only to a certain degree, since at present they determine two units (the kilogram and the kelvin) and, consequently, the numerical values of the fundamental quantities in these units.

This will narrow the experimental base for determining the values of the fundamental constants. At the same time, another process is also under way, namely, a broadening of this base. Initially, only the results of laboratory experiments were made to match each other. At present, experiments in high-energy physics are starting to play a role in adjusting the fundamental physical constants. Determination of the values of the Rydberg constant and of the proton charge radius serves as a striking example. The accelerator experiment was actually the referee in the dispute between two spectroscopic experiments. The excellent agreement between the results of electron–proton scattering and atomic spectroscopy of hydrogen and deuterium atoms was a weighty argument in favor of neglecting the result obtained by the spectroscopy of muonic hydrogen, which contradicted them. At the same time, the very scattering data were not processed within the framework of the adjustment; instead, advantage was taken of ready data processing results.

The past two decades have been marked by precision astrophysical and cosmological measurements. While previously it was clear that the heliocentric gravitational constant was a parameter related only to the Solar System, and measurements of the masses of stars in units of the solar mass were no more than the choice of a convenient and

illustrative scale, recent precision tests of the general relativity in the system of double pulsars have changed the situation qualitatively. The solar mass makes up a unit that permits performing the most precise measurements, and in this case the heliocentric gravitational constant also becomes a 'true' gravitational constant for distant star systems, unlike the value of the Newtonian gravitation constant measured in the kilograms.

Analysis of the CMB radiation spectra has resulted in precision measurements in primary thermometry, turning out to be important for fundamental physics. In laboratory studies, the role of primary thermometry was essentially limited. On the whole, the very fact of precision measurements being performed in outer space leads to a certain rethinking of how necessary it is to define units and what quantities are of fundamental significance.

Changing the units of physical quantities will lead to a change in the concept of what should be included in the tables of recommended values of fundamental physical constants, and it may well turn out that the next tables will contain certain quantities from elementary particle physics or astrophysics, which hitherto were never included in such tables.

The author is a member of the [International] CODATA Task Group on Fundamental Constants and chairman of a similar Russian group. This article represents a review of the data and results of the adjustment of the values of the fundamental physical constants [7], carried out recently by the Task Group. While the recommended values are a result of group work, various critical remarks and comments express the opinion of the author only and do not necessarily coincide with the opinions of the Task Group.

Acknowledgments

The author is grateful to his colleagues in the International and Russian groups for the fruitful discussions. The author is also grateful to M Davier for permission to make use of the figure from his article, and to E M Churazov for the useful discussions. The work was partly performed with the support of DFG (grant GZ: HA 1457/7-2) and the RFBR (grant 12-02-91341 NNIO_a).

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