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Nonlinear dynamics of quadratically cubic systems

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Contents

Abstract. We propose a modified form of the well-known nonlinear dynamic equations with quadratic relations used to model a cubic nonlinearity. We show that such quadratically cubic equations sometimes allow exact solutions and sometimes make the original problem easier to analyze qualitatively. Occasionally, exact solutions provide a useful tool for studying new phenomena. Examples considered include nonlinear ordinary differential equations and Hopf, Burgers, Korteweg–de Vries, and nonlinear Schrödinger partial differential equations. Some problems are solved exactly in the space–time and spectral representations. Unsolved problems potentially solvable by the proposed approach are listed.

1. Introduction

The processes considered in this article can be called strongly nonlinear. The differences between the effects of strong nonlinearity and strongly pronounced manifestations of weak nonlinearity are discussed at length in review [1], which also offers examples of strongly nonlinear systems and proposes their classification.

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An important example of a strongly nonlinear system of the first kind (see Ref. [1]) is one with a quadratically cubic (QC) nonlinearity. The attribute of quadratically cubic is sometimes applied to systems that simultaneously contain quadratic and cubic nonlinearities. Here, the term 'quadratically cubic system' is used in a rather different sense. We consider a model of cubically nonlinear systems built on the basis of quadratic relations. Such systems do actually exist. No less important is the fact that the relevant nonlinear equations can sometimes be solved with much less effort than the equations for truly cubic systems. The possibility of obtaining an exact analytic description of nonlinear processes in a model problem framework frequently allows a better understanding of their real physical features.

In the simplest case of an oscillatory system with one degree of freedom $x(t)$, the QC nonlinearity is given by the function $x|x|$. The corresponding anharmonic oscillator is described by the equation

$$\frac{d^2x}{dt^2} + \frac{3}{2}\,x|x| = 0\,. \tag{1}$$

In contrast to a 'well' described by the quadratic parabola $U = x^2/2$ (the potential function of a harmonic oscillator), the well $U = x^2|x|/2$ is here formed by two branches of the cubic parabola $U = x^3/2$ continuously sewn at the point $x = 0$. As in the well-known case [2, 3] where, instead of $x|x|$, Eqn (1) contains a standard cubic term x^3, there is no asymptotic transition to a linear system as $x \to 0$. The energy of oscillations

$$W = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\,x^2|x| \equiv \frac{1}{2}\,A^3\,, \tag{2}$$

is defined by the cube of their amplitude ($A > 0$). The period of oscillations

$$T = \frac{4}{\sqrt{A}}\int_0^1 \frac{dy}{\sqrt{1-y^3}}\,,$$

which decreases with an increase in the amplitude as $1/\sqrt{A}$, is given by

$$T\sqrt{A} = \frac{4}{3^{1/4}}\, F\left(\arccos\frac{\sqrt{3}-1}{\sqrt{3}+1}\,,\, \frac{1}{2}\sqrt{2+\sqrt{3}}\right)$$
$$= \frac{4\sqrt{\pi}}{3}\,\frac{\Gamma(1/3)}{\Gamma(5/6)} \approx 5.61\,. \qquad (3)$$

Here, $F(\psi, k)$ is the elliptic integral of the first kind and $\Gamma(z)$ is the gamma function. The shape of oscillations is described by the elliptic Jacobi functions [4] and contains only odd harmonics.

We note that oscillations of the oscillator with a similar nonlinear decay $\sim \dot{x}|\dot{x}|$ are well studied (see, e.g., Ref. [5]).

In recent years, the model of a damped 'bistable' oscillator

$$\dot{x} + x^3 - x = F(t) \qquad (4)$$

has become popular in studies of the so-called stochastic resonance [6]. For example, Eqn (4) describes the motion of a Brownian particle in a strongly viscous medium in the field of the double-well anharmonic potential $U = x^4/4 - x^2/2$. The particle is subject to an external force $F(t)$ given by the sum of noise $\sqrt{2D}\,\xi(t)$ of intensity D and the signal $A\cos(\Omega t)$, periodic in time. Equation (4) is analyzed numerically or analytically under some approximations, for example, with the help of the Fokker–Planck equation. At the same time, Eqn (4) can be brought into correspondence with the QC nonlinear model:

$$\dot{x} + x|x| - x = F(t)\,. \qquad (5)$$

The advantage of model (5) is that it is amenable to linearization. Setting $|x| = \dot{u}/u$, we obtain

$$\ddot{u} - \dot{u} = \pm F(t)\,u\,,$$

where the plus sign corresponds to $x > 0$ and the minus sign corresponds to $x < 0$.

If we consider the action of a sequence of short (delta-function) pulses on QC system (5),

$$F(t) = \sum_{n=1}^{N} A_n \delta(t - t_n)\,, \qquad (6)$$

with the 'amplitudes' A_n applied at time instants t_n, Eqn (5) can also be linearized. Setting $x = 1/u$, for the intervals between pulses (6), we obtain

$$\dot{u} + u = \pm 1\,, \quad u_n(t) = \pm 1 + \left[u_n(t_n) \mp 1\right]\exp\left(-(t - t_n)\right)\,, \qquad (7)$$

where $t_{n-1} < t < t_n$. The upper signs correspond to positive values of u_n and the lower ones to negative values. It follows from solution (7) that there are two 'attracting' points: as $t \to \infty$, the positive initial values of the function tend to 1, and the negative ones tend to -1. Returning to the original variable, we obtain

$$x_n(t) = \frac{x_n(t_{n-1})}{\exp\left(-(t - t_{n-1})\right) + \left[1 - \exp\left(-(t - t_{n-1})\right)\right]\left|x_n(t_{n-1})\right|}\,. \qquad (8)$$

Formula (8) describes the evolution of $x_n(t)$ in the time interval $t_{n-1} < t < t_n$. At the instant t_n, $x_n(t)$ 'jumps' to the new value

$$x_{n+1}(t_n) = x_n(t_n) + A_n\,. \qquad (9)$$

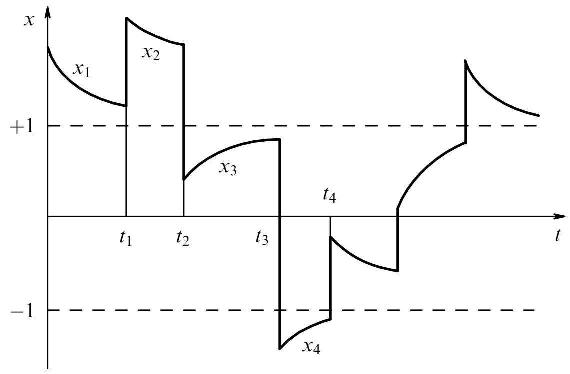

Figure 1. Response of QC system (5) to the sequence of short iulses described by formula (6).

Formula (9) is obtained by integrating Eqn (5) with right-hand side (6) in a small vicinity of t_n. If, for a random sequence of pulses, we know the distribution of the 'amplitudes' and appearance times, then we can use Eqns (8) and (9) to compute the statistical characteristics of the response shown in Fig. 1.

We note that the change of the variable $x = 1/u$ also allows linearizing a more general QC model,

$$\dot{x} + f_1(t)\,x|x| - f_2(t)\,x = 0\,. \qquad (10)$$

In particular, if the functions $f_{1,2}(t)$ are periodic, they can describe the oscillations of wells of a bistable potential, which are used for explaining the phenomenon of stochastic resonance [6]. The solution of dynamical equation (10), for example, for $f_1(t) \equiv 1$ and $f_2(t) = \dot{f}(t)$ on the interval $t_{n-1} < t < t_n$, takes the form

$$x_n(t) = x_n(t_{n-1})\Big[\exp\left(-(f(t) - f(t_{n-1})\right)$$
$$+ \left|x_n(t_{n-1})\right| \int_{t_{n-1}}^{t} \exp\left(-(f(t) - f(t')\right)\,\mathrm{d}t'\Big]^{-1}\,. \qquad (11)$$

The jump of $x_n(t)$ at the instant t_n is, as previously, defined by formula (9).

2. Example of a system with a quadratically cubic nonlinearity

Cubic nonlinearity, discussed here in the context of its modeling with piecewise continuous quadratic relations, is, as is well known, used rather widely in nonlinear dynamics.

For example, cubic nonlinearity is dominant for shear waves in homogeneous isotropic solids. Quadratic effects are prohibited here by symmetry considerations. Shear waves are an efficient tool in diagnosing soft biological tissues, because the shear moduli in normal and pathologically modified tissues differ by several orders of magnitude [7]. At the same time, the speed of sound and density vary only by a few percent. Studying the nonlinearities of shear moduli provides new information on the mechanical parameters of solids, which is important for medical and industrial analyses.

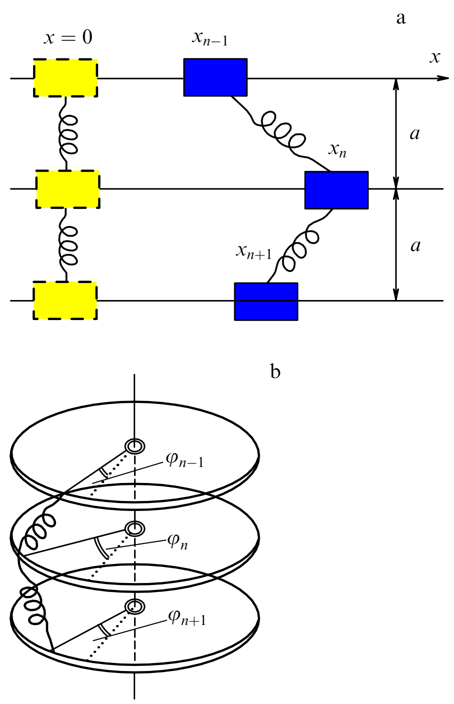

Figure 2. (a) A chain of masses connected by linear springs and moving along rods. (b) A chain of linearly connected disks performing torsional oscillations.

Examples of mechanical systems with a strong cubic nonlinearity are given in Fig. 2. Figure 2a depicts a chain of masses whose motion is constrained by parallel guiding rods lying in a plane and separated by a distance a. The spring elongation satisfies the linear Hooke law $F = -k\Delta l$. If all masses except one with a number n are fixed, its small oscillations, $|x_n| \ll a$, for small static elongation $\Delta l_0 \ll a$ are governed by the nonlinear equation

$$m\,\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} + k\,\frac{\Delta l_0}{a}\,x_n + \frac{k}{2a^2}\,x_n^3 = 0\,. \tag{12}$$

Interestingly, for $\Delta l_0 = 0$, i.e., when the spring is not stretched initially, the linear motion regime does not occur. The nonlinearity fully governs the system motion for oscillations of an arbitrarily small amplitude:

$$\frac{\mathrm{d}^2 X}{\mathrm{d}\tau^2} + X^3 = 0\,, \qquad X = \frac{x_n}{a\sqrt{2}}\,, \qquad \tau = t\sqrt{\frac{k}{m}}\,. \tag{13}$$

This nonlinearity can be called 'geometrical' because it stems from the presence of constraints (restricting the motion), and not from the 'physical' nonlinearity of the spring deformation.

In the continuum limit, waves in the discrete chain obey the partial differential equation

$$\frac{\partial^2\zeta}{\partial t^2} - c^2\,\frac{\partial^2\zeta}{\partial z^2} = \frac{\beta}{3}\,\frac{\partial^2\zeta^3}{\partial z^2}\,, \qquad \zeta = \frac{\partial x}{\partial z}\,. \tag{14}$$

Here, ζ is the deformation and $z = na$ is the coordinate along which the oscillations propagate. The nonlinear coefficient and the speed of sound are given by $\beta = (3k/2m)a^2$ and $c^2 = (k/m)\Delta l_0 a$.

Equation (14) can be referred to as a nonlinear wave equation if its right-hand side is small compared to any term in the left-hand side. However, in the absence of spring stretching, or if it is too weak, when $c \to 0$, the nonlinear evolution of the wave profile 'dominates' the process of its propagation. Thus, speaking about the distortion of a traveling wave profile does not seem plausible in this case.

Equation (14) is related to a first-order equation, which is a model of a simple distributed QC nonlinear system:

$$\frac{\partial\zeta}{\partial t} = \pm\sqrt{\beta}\,|\zeta|\,\frac{\partial\zeta}{\partial z}\,. \tag{15}$$

Indeed, differentiating Eqn (15) over t and replacing the derivative over t by the derivative over z in the right-hand side of the resulting relation, we arrive at second-order equation (14). The properties of strongly nonlinear systems like (14) are explored in Ref. [2]. In particular, it is shown that such systems maintain periodic localized oscillations, but traveling wave solutions may be absent in them.

We note that for experimental modeling of strongly nonlinear dynamics [2], a chain of disks is more convenient (Fig. 2b). The disks perform torsional oscillations, for which friction in the axes is much lower than that for masses sliding along guiding rods (Fig. 2a). The mathematical models of both chains coincide.

3. Riemann waves

If dispersion and dissipation are absent, plane waves in a QC system are described by the equation for simple (Riemann) waves:

$$\frac{\partial u}{\partial z} = \frac{\varepsilon}{c^2}\,|u|\,\frac{\partial u}{\partial\tau}\,. \tag{16}$$

Mathematicians sometimes refer to equations like Eqn (16), but with a quadratic nonlinearity, as the Hopf equation, and physicists call it the Riemann equation. Here, we use the same notation as for ordinary media with quadratic nonlinearity [8]. Namely, u is the oscillatory velocity, ε is the nonlinear parameter, and $\tau = t - z/c$ is the time in the reference frame moving along the z axis at the speed of sound c. For small Mach numbers $|u|/c$, Eqn (16) can describe weakly nonlinear waves; in this case, z is a 'slow' coordinate. A similar equation (15), as shown in Section 2, is also valid for strongly nonlinear shear and torsional oscillations in distributed systems [2]. In this case, Eqn (15) cannot be considered an evolutionary one.

The solution of Eqn (16) corresponding to a wave with the time dependence $u = \Phi(\tau)$ at the medium boundary $z = 0$ is provided by the implicit function

$$u = \Phi\left(\tau + \frac{\varepsilon}{c^2}\,|u|z\right). \tag{17}$$

If the original wave is harmonic, $u = u_0 \sin(\omega\tau)$, then it is of interest to follow the modification of its spectral composition. We use the standard technique for expansion of implicit functions in Fourier series, described in book [8],

$$u = u_0 \sum_{n=1}^{\infty} C_n(Z)\sin\big(n\omega\tau + \varphi_n(Z)\big)\,.$$

Here, Z is the distance normalized with the characteristic nonlinear length z_{SH} of discontinuity formation in the wave profile,

$$Z = \frac{z}{z_{\mathrm{SH}}} = \frac{\varepsilon}{c^2}\,\omega u_0 z\,.$$

After simple but cumbersome manipulations, we find

$$C_n = \left[1 - (-1)^n\right] \frac{2}{nZ} \left[\left(\frac{2}{\pi n} - E_n(nZ)\right)^2 + J_n^2(nZ)\right]^{1/2},$$
$$\tan\varphi_n = \frac{(2/\pi n) - E_n(nZ)}{J_n(nZ)}, \tag{18}$$

where E_n is the Weber function and J_n is the Bessel function. Result (18) represents an analog of the Bessel–Fubini solution known for media with quadratic nonlinearity [8]. The derivation of an analogous expansion for a truly cubic system is discussed in Ref. [9].

It can be seen that only amplitudes of odd harmonics differ from zero in solution (18). The amplitude of a wave at the frequency of its first harmonic decays with the distance traveled by the wave (the energy is pumped into higher harmonics). The amplitudes of higher harmonics increase, and the third harmonic increases at $Z \ll 1$ in accordance with the linear law $C_3 \approx 4Z/(5\pi)$.

Exact formulas (18) are valid for the distance $0 < Z < 1$. At $Z = 1$, a shock front is formed, and its behavior requires a separate analysis.

4. Weak shock waves of compression and rarefaction

We use the well-known approach and analyze the Riemann waves graphically [8]. For this, we write solution (17) as an explicit function of time:

$$\tau = \Phi^{-1}(u) - \frac{\varepsilon}{c^2}\,|u|z\,. \tag{19}$$

Formula (19) implies that in order to construct the wave profile at a certain distance z, the original function $\Phi^{-1}(u)$ has to be augmented by the function $-\varepsilon|u|z/c^2$ proportional to $|u|$. Such a construction is drawn in Fig. 3 for a compression shock wave in the form of a symmetric jump between the values $u = -u_0$ and $u = +u_0$. In agreement with Eqn (19), point A at the front is displaced to the position A′, and point B is displaced to the position B′. The new positions of all other points form a broken line (shown as the dashed line) connecting points A′ and B′ with the origin.

The front displacement $\tau_{\mathrm{SH}}(z)$ is governed by the momentum conservation law [8] and corresponds to the equality of the areas of the two triangles hatched in Fig. 3,

$$\tau_{\mathrm{SH}}(z) = -(\sqrt{2} - 1)\,\frac{\varepsilon}{c^2}\,u_0 z\,. \tag{20}$$

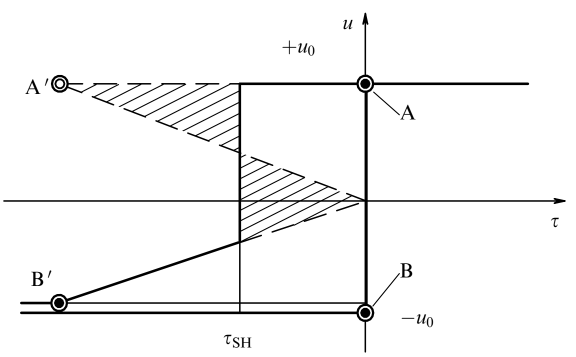

Figure 3. Transformation of a symmetric compression shock front.

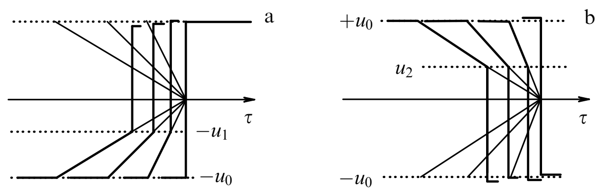

Figure 4. Formation and propagation of (a) stable compression and (b) rarefaction shock fronts.

It follows from the construction that the symmetric jump, in contrast to that in a quadratically nonlinear medium, is unstable in general. A sloping region evolves there. A 'forerunner' precedes the front. The height of the jump decreases from the initial value $2u_0$ to the 'stable' value $\sqrt{2}u_0$, which does not depend on the distance z. In a medium characterized by 'rigidity' increasing with pressure ($\varepsilon > 0$), the jump moves according to Eqn (20) with the supersonic speed

$$v_{\mathrm{SH}} \approx c_0\left[1 + \left(\sqrt{2} - 1\right)\varepsilon\,\frac{u_0}{c}\right].$$

The process of compression front transformation is shown in Fig. 4a. It can be seen that the profile becomes broken at small distances; then a sloping region forms, and it expands with time, becoming increasingly gently sloped. Within the sloping region, the velocity of oscillations increases linearly with time from $-u_0$ to $-u_1 = -(\sqrt{2} - 1)u_0$. At the front, the velocity jumps from $-u_1$ to u_0.

Hence, a stable compression shock in the medium under consideration should be characterized by the ratio of perturbations (immediately before and after the shock front) $|u_1|/u_0 = \sqrt{2} - 1$.

A construction analogous to the one shown in Fig. 3 for a symmetric rarefaction jump leads to the picture in Fig. 4b. It is well known that in a quadratically nonlinear medium, any rarefaction jump is unstable. Here, the jump between the oscillation velocity values $-u_0$ and $u_2 = (\sqrt{2} - 1)u_0$ turns out to be stable. Interestingly, the propagation speed for this jump also proves to be supersonic; moreover, it coincides with the propagation speed of the compression shock shown in Fig. 4a. As a consequence, alternating stable compression and rarefaction shock waves can be connected by smooth profile intervals. They are separated by equal-time intervals that do not vary with the distance traveled. This picture corresponds to a periodic traveling signal.

To conclude, we write the general formula for the position of the shock wave front

$$\tau_{\mathrm{SH}}(z) = -\frac{\varepsilon z}{2c^2}\,\frac{u_2|u_2| - u_1|u_1|}{u_2 - u_1}\,. \tag{21}$$

Here, the values u_1 and u_2 can have any sign. The compression front is stable in general ($u_1 < 0, u_2 > 0$) if the perturbation u_1 lies simultaneously on the discontinuity and on the part of the simple wave adjacent to it. This implies that $\tau_{\mathrm{SH}} \leqslant \tau_1$, where, according to Eqn (19),

$$\tau_1 = -\frac{\varepsilon z}{c^2}\,|u_1|\,. \tag{22}$$

Formulas (21) and (22) lead to the stability condition for the compression front:

$$|u_1| \leqslant u_{\text{CR}} = \left(\sqrt{2}-1\right)u_2\,. \tag{23}$$

For a compression jump propagating in an unperturbed medium ($u_1 = 0$) and also for a jump both of whose perturbation values are positive ($u_1 > 0,\ u_2 > u_1$), the specifics of QC nonlinearity are not manifested. In this case, formula (21) leads to the well-known result from the theory of quadratically nonlinear Riemann waves [8, 10–12].

Similarly, using formula (21) allows showing that the rarefaction jump ($u_1 > 0,\ u_2 < 0$) is generally stable if $u_1 \leqslant u_{\text{CR}} = (\sqrt{2}-1)|u_2|$. A rarefaction jump propagating in an unperturbed medium ($u_1 = 0$) or a rarefaction jump with negative perturbations before and after the front ($u_1 < 0$, $u_2 < u_1$) is also stable. In these cases, manifestations of the cubic nonlinearity also disappear. We note that there is no contradiction with the known conjecture on the instability of a rarefaction wave in a medium with quadratic nonlinearity: the medium 'rigidity' for such a rarefaction front does not increase with pressure, but decreases.

5. Periodic trapezoidal saw-tooth wave

It is known that quadratically nonlinear media can support quasistable periodic structures in the form of saw-tooth waves composed of a periodic sequence of linearly sloping parts connected via shock compression fronts. The shape of these waves is preserved as they propagate. What varies is the amplitude of the jump in each period. The stable character of these structures allows considering processes of their interaction and self-interaction [11, 13] in the same way as this is done for quasiharmonic signals in nonlinear dispersive media.

We find a quasistable wave profile in a QC medium. We seek a solution of Eqn (16) that preserves its form $\Phi(\tau)$ as it propagates:

$$u = A(z)\,\Phi\left(\tau - \tau_{\text{SH}}(z)\right).$$

The solution of the resulting ordinary differential equations with the boundary conditions $A(0) = u_0$ and $\tau_{\text{SH}}(0) = 0$ is

$$A(z) = \frac{u_0}{1 + z/z_0}\,, \qquad \omega\left(\tau - \tau_{\text{SH}}(z)\right) + C = \ln|\Phi| + Z_0|\Phi|\,,$$
$$\omega\tau_{\text{SH}}(z) = -\ln\left(1 + \frac{z}{z_0}\right), \qquad Z_0 = \frac{z_0}{z_{\text{SH}}} = \frac{\varepsilon}{c^2}\,\omega u_0 z_0\,, \tag{24}$$

where z_0, ω, and C are some constants. The function Φ in solution (24) is multi-valued and unbounded; therefore, at first glance, it lacks physical meaning. However, the arbitrary constant C allows translating the branches of the function $\Phi(\tau - C)$ along the time axis τ, forming their periodic sequence (Fig. 5). By joining these branches with jumps, it becomes possible to construct a saw-tooth profile that is periodic in time. In contrast to the case of a quadratically nonlinear medium, each half-period of the wave is not triangular but trapezoidal. The period contains both compression and rarefaction shock waves, which are the respective jumps between the values $-u_1 = -(\sqrt{2}-1)A(z)$ and $u_2 = A(z)$ and between $u_1 = +(\sqrt{2}-1)A(z)$ and $-u_2 = -A(z)$. Both jumps are stable shock waves traveling at the same speed as the smooth parts Φ of the profile.

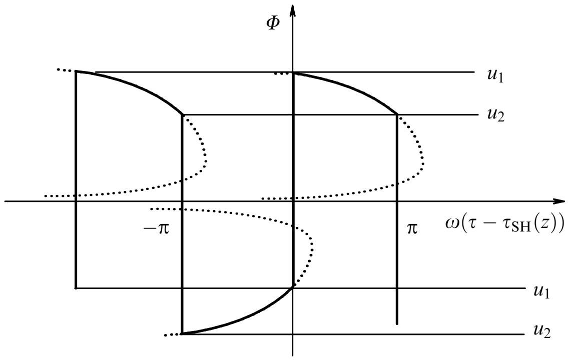

Figure 5. Trapezoidal saw-tooth wave in a quadratically cubic medium without dispersion.

The 'trapezoidal saw-tooth' plotted in Fig. 5 represents the asymptotic form (at distances $z \gg z_{\text{SH}}$) of a wave that was harmonic when it entered the medium ($z = 0$). We note that the procedure of constructing this wave is analogous to that described in Refs [10, 13] for a usual cubically nonlinear medium.

6. Quadratically cubic Burgers equation

To account for linear damping, Eqn (16) has to be generalized by the addition of a dissipative term with the second derivative:

$$\frac{\partial u}{\partial z} = \frac{\varepsilon}{c^2}\,|u|\,\frac{\partial u}{\partial \tau} + \frac{b}{2c^3\rho_0}\,\frac{\partial^2 u}{\partial \tau^2}\,. \tag{25}$$

Here, b is the dissipative coefficient expressible in terms of shear and bulk viscosities and the thermal conductivity of the medium [8]. For brevity, we write Eqn (25) in dimensionless variables:

$$\frac{\partial V}{\partial Z} = \frac{1}{2}\,\frac{\partial}{\partial \theta}\left(|V|V\right) + \Gamma\,\frac{\partial^2 V}{\partial \theta^2}\,, \tag{26}$$

where

$$Z = \frac{z}{z_{\text{SH}}}\,, \qquad \theta = \omega\tau\,, \qquad V = \frac{u}{u_0}\,, \qquad \Gamma = \frac{z_{\text{SH}}}{z_{\text{DISS}}} = \frac{b\omega}{2\varepsilon c\rho_0 u_0}\,.$$

The quantity Γ, equal to the ratio of nonlinear and dissipative lengths, is referred to as the inverse acoustic Reynolds number (or the Goldberg number). For $\Gamma \gg 1$, dissipative effects prevail over nonlinear ones. For $\Gamma \ll 1$, by contrast, the effect of nonlinearity is more pronounced.

Equation (26) can be linearized using the following generalization to the Hopf–Cole–Florin transformation [8]:

$$|V| = 2\Gamma\,\frac{\partial}{\partial \theta}\ln U\,. \tag{27}$$

But in contrast to the linearization of the standard Burgers equation in a quadratically nonlinear medium, the formal linearization in this case is less productive. The break in the derivative arising in the vicinity of the point $V = 0$ leads to a front displacement in the comoving reference frame (see Fig. 3), which calls for nontrivial generalizations of known results.

We find one of the stationary solutions of Eqn (26) describing a dissipative structure of the shock front. For definiteness, we consider a stable shock wave with a compression corresponding to the transition from one constant value $V_1 = -(\sqrt{2} - 1)$ to another one, $V_2 = 1$. In accordance with formula (20), we seek the solution in the form

$$V(Z, \theta) = V\big[\theta_* = \theta - \theta_{\mathrm{SH}}(Z)\big]\,, \qquad \theta_{\mathrm{SH}}(Z) = -\big(\sqrt{2} - 1\big)Z\,. \tag{28}$$

Upon integration, substitution of Eqn (28) in Eqn (26) leads to the ordinary differential equation

$$\Gamma\,\frac{\mathrm{d}V}{\mathrm{d}\theta_*} + \frac{1}{2}\,|V|V - \alpha V = \frac{1}{2}\,\alpha^2\,, \qquad \alpha \equiv \sqrt{2} - 1\,. \tag{29}$$

The integration constant [the right-hand side of Eqn (29)] is fixed by the condition $V(\theta_* \to \infty) \to 1$. From Eqn (29), for negative values of V, it follows that

$$\Gamma\,\frac{\mathrm{d}V}{\mathrm{d}\theta_*} = \frac{1}{2}\,(V + \alpha)^2\,. \tag{30}$$

Hence, the second boundary condition $V(\theta_* \to -\infty) \to -\alpha$ is satisfied automatically. The solution of Eqn (30) that satisfies the condition $V(\theta_* = \theta_0) = 0$ has the form

$$V = \alpha^2\left(\frac{\theta_* - \theta_0}{2\Gamma}\right)\left[1 - \alpha\left(\frac{\theta_* - \theta_0}{2\Gamma}\right)\right]^{-1}, \quad -\infty < \theta_* < \theta_0\,, \tag{31}$$

where θ_0 is the integration constant to be determined. The 'negative' part of the front described by solution (31) should continuously match the 'positive' branch, which is found from Eqn (29):

$$\Gamma\,\frac{\mathrm{d}V}{\mathrm{d}\theta_*} + \frac{1}{2}\,(V - \alpha)^2 = \alpha^2\,. \tag{32}$$

The solution of Eqn (32) that satisfies the condition at infinity $V(\theta_* \to \infty) \to 1$ is

$$V = \alpha\left[1 + \sqrt{2}\,\tanh\left(\alpha\sqrt{2}\,\frac{\theta_*}{2\Gamma}\right)\right], \qquad \theta_0 < \theta_* < \infty\,. \tag{33}$$

Requiring that the solution be continuous and matching formulas (31) and (32) at $V = 0$ or $\theta_* = \theta_0$, we find

$$\frac{\theta_0}{2\Gamma} = -\frac{1}{\alpha\sqrt{2}}\,\mathrm{artanh}\,\frac{\sqrt{2}}{2} \approx -1.52\,. \tag{34}$$

Interestingly, dissipation smooths the jump in the derivative at $V = 0$. At the point $\theta_* = \theta_0$, not only the function but also its derivative is continuous. This is directly seen from a comparison of Eqns (30) and (32). The structure of the compression shock front is given by formulas (31) and (33) and is depicted in Fig. 6. The characteristic front length $\Delta\theta \sim \Gamma$ decreases as the nonlinearity increases, but increases with dissipation. The perturbation V monotonically increases across the front.

The structure of the rarefaction front is recovered analogously or can be found from solutions (31) and (33) with the help of obvious symmetry transformations.

Knowing the solution for the smooth-profile parts of a QC nonlinear wave (see Fig. 5) and the solution describing the structure of the fronts (see Fig. 6), we can derive 'boundary-layer asymptotic forms' by resorting to the method of matching asymptotic expansions, and thus fully describe the wave profile, proceeding further to the analysis of spectral composition, nonlinear losses, and other important characteristics.

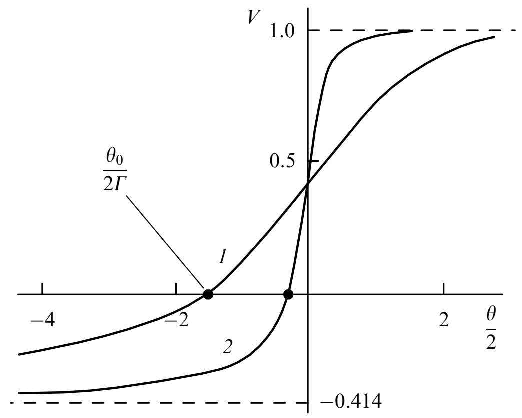

Figure 6. The shape of the compression shock front in a QC medium for $\Gamma = 1.0$ (curve *1*) and $\Gamma = 0.2$ (curve *2*). The locations of matching are marked with filled circles.

To conclude this section, we mention that the replacement of a cubic nonlinearity by the QC model was used in a number of analyses known to us. For instance, with its assistance, the problem [14]

$$\Gamma\,\frac{\mathrm{d}V}{\mathrm{d}\theta} + V^3 + 3\left(\frac{1}{2\pi}\int_0^{2\pi} V^2(\theta)\,\mathrm{d}\theta - \delta\right)V = \sin\theta\,, \quad \langle V\rangle = 0\,, \tag{35}$$

was solved exactly and the frequency characteristics of an acoustic resonator with shock waves 'running' inside it were computed. In Eqn (35), δ is the dimensionless shift between the frequencies of wall oscillations and of resonator eigenmodes.

7. Quadratically cubic nonlinear Schrödinger equation

The replacement of a cubic nonlinearity by the QC nonlinearity has been used above for equations associated with mechanical systems. It is of interest to consider a wider set of problems by analyzing QC generalizations of the Korteweg–de Vries, nonlinear Schrödinger, and other 'reference' equations of nonlinear wave theory.

We consider the QC Schrödinger equation

$$2\mathrm{i}k\,\frac{\partial A}{\partial z} = \Delta_\perp A + \gamma k^2 |A| A\,. \tag{36}$$

When model (36) is used to describe self-action effects in light beams, the variable A is a complex-valued wave amplitude and $\Delta_\perp$ is the Laplace operator in the beam cross section [15]. Like the cubic Schrödinger equation, Eqn (36) has certain conservation laws. In this case, the conservation laws are expressed as

$$I_1 = \iint |A|^2\,\mathrm{d}^2\mathbf{r}_\perp = \mathrm{const}\,,$$
$$I_2 = \iint \left\{|\nabla_\perp A|^2 - \frac{2}{3}\,\gamma k^2 |A|^3\right\}\mathrm{d}^2\mathbf{r}_\perp = \mathrm{const}\,. \tag{37}$$

The first conserved quantity has the meaning of total energy and the second takes nonlinear and diffractional effects into account.

We pass from Eqn (36) to the equations for the real-valued amplitude B and phase Ψ by setting $B = A\exp(-\mathrm{i}k\Psi)$. Separating the real and imaginary parts, we obtain the coupled eikonal and transfer equations [15]:

$$2\,\frac{\partial\Psi}{\partial z} + (\nabla_\perp\Psi)^2 = \gamma B\,,$$
$$2\,\frac{\partial B}{\partial z} + 2\nabla_\perp B\nabla_\perp\Psi + B\Delta_\perp\Psi = 0\,. \qquad (38)$$

Dropping the term $\Delta_\perp B$ in the eikonal equation of system (38) corresponds to passing to the nonlinear geometric optics approximation. For plane (collimated) beams, system (38) takes the form

$$\frac{\partial\alpha}{\partial z} + \alpha\,\frac{\partial\alpha}{\partial y} - \frac{\gamma}{2}\,\frac{\partial B}{\partial y} = 0\,,$$
$$\frac{\partial B}{\partial z} + \alpha\,\frac{\partial B}{\partial y} + \frac{1}{2}\,B\,\frac{\partial\alpha}{\partial y} = 0\,, \qquad \alpha = \frac{\partial\Psi}{\partial y}\,. \qquad (39)$$

Equations (39), whose form is known in fluid dynamics, are linearized via a hodograph transformation. They describe the isoentropic flow of a barotropic fluid. These equations can be solved exactly [15, 16], which corresponds to the aberration self-action in wave beams in nonlinear optics. However, while the intensity is an independent variable for the cubic nonlinearity, Eqns (39) contain the real-valued wave amplitude B in the QC model considered here.

Another interesting case corresponds to imaginary values of the coefficient γ in QC Schrödinger equation (36). Substituting $\gamma \to \mathrm{i}\gamma/k$ in Eqn (36) and separating real-valued amplitude and phase, we arrive at the system

$$\frac{\partial\alpha}{\partial z} + \alpha\,\frac{\partial\alpha}{\partial y} = 0\,,$$
$$\frac{\partial B}{\partial z} + \alpha\,\frac{\partial B}{\partial y} + \frac{1}{2}\,B\,\frac{\partial\alpha}{\partial y} = \frac{\gamma}{2}\,B^2\,. \qquad (40)$$

The right-hand side of the transfer equation (the second equation in system (40)), as can be readily seen, is responsible for quadratic (two-photon) wave beam absorption [17, 18]. By the variable change $B = 1/U$, the transfer equation is reduced to the linear form

$$\frac{\partial U}{\partial z} + \alpha\,\frac{\partial U}{\partial y} - \frac{1}{2}\,\frac{\partial\alpha}{\partial y}\,U = -\frac{\gamma}{2}\,, \qquad (41)$$

while the eikonal equation [the first one in Eqns (40)] is independent of it and is readily solvable. Consequently, the entire system (40) can be solved exactly.

8. Quadratically cubic Korteweg–de Vries equation

We write the Korteweg–de Vries (KdV) equation in the same notation as Eqn (16):

$$\frac{\partial V}{\partial Z} = \frac{\alpha}{2}\,\frac{\partial}{\partial\theta}\big(|V|V\big) + \Gamma\,\frac{\partial^3 V}{\partial\theta^3}\,. \qquad (42)$$

The first three conservation laws for Eqn (42) are $I_{1,2,3} = \mathrm{const}$ with

$$I_1 = \int_{-\infty}^{\infty} V(z,\theta)\,\mathrm{d}\theta\,, \qquad I_2 = \int_{-\infty}^{\infty} V^2(z,\theta)\,\mathrm{d}\theta\,,$$
$$I_3 = \int_{-\infty}^{\infty}\left[\alpha V^2|V| - 3\Gamma\left(\frac{\partial V}{\partial\theta}\right)^2\right]\mathrm{d}\theta\,. \qquad (43)$$

The first two expressions (the conservation of momentum and energy) coincide with those for the cubic KdV, and the third one is somewhat different.

The equation for stationary waves $V = V(T = \theta - \beta Z)$ is

$$\Gamma\left(\frac{\mathrm{d}V}{\mathrm{d}T}\right)^2 = E - W(V)\,, \qquad W = \frac{\alpha}{3}\,|V|V^2 + \beta V^2\,. \qquad (44)$$

The phase plane for Eqn (44) does not differ qualitatively from that for the cubic KdV [19]. Depending on the signs of the coefficients α and Γ at the nonlinear and dispersive terms, and also on the stationary wave propagation speed, which depends on the parameter β, the potential $W(V)$ can have one or two wells. As a result, periodic solutions exist, as do solutions in the form of solitary waves — solitons and kinks.

The qualitative considerations presented above stress the importance of formulating a rigorous theory of the QC KdV equation. First of all, it is of interest to find whether it can be solved with the inverse scattering method.

9. Conclusions

The list of QC models mentioned in this work can be augmented by QC analogs of other mathematical models, which include the Klein–Gordon nonlinear equation

$$\Delta u - \frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} = au - bu|u|\,,$$

the Newell–Whitehead nonlinear equation [21]

$$\frac{\partial u}{\partial t} - \kappa\,\frac{\partial^2 u}{\partial x^2} = au - bu|u|\,,$$

the Ginzburg–Landau nonlinear equation [22]

$$\frac{\partial u}{\partial t} = u + (1 + \mathrm{i}b)\Delta u - (1 + \mathrm{i}c)\,u|u|\,,$$

a QC modification of the Khokhlov–Zabolotskaya equation [23]:

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial z} - \frac{\partial}{\partial t}\big(u|u|\big)\right] = N\Delta_\perp u\,,$$

and general nonlinear integro-differential equations [24]

$$\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial z} - \frac{\partial}{\partial t}\big(u|u|\big) - \frac{\partial^2}{\partial t^2}\int_0^{\infty} K(s)\,u(t - s)\,\mathrm{d}s\right] = N\Delta_\perp u$$

with a nondegenerate kernel $K(s)$. This list certainly includes many other known mathematical models. It is possible that some of them can be solved more easily for this (QG) type of nonlinearity. It is also possible that their analysis will help in studying new features of some physical phenomena. Examples of such already realized possibilities are described in this note. Even if the analysis does not bring new knowledge, attempts at doing it may serve as a good practical exercise for the interested reader.

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