

METHODOLOGICAL NOTES

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The neutron Berry phase

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Contents

Abstract. **The neutron Berry phase is found from an exact analytic solution of the Schrödinger equation in a constant magnetic field B_0 and a perpendicular radiofrequency field b rotating with an angular frequency ω. The solution is found for arbitrary values of B_0, b, and ω. The Berry phase is shown to be a linear approximation of the exact value in the parameter ω/B_0 when this parameter is small.**

1. Introduction

We consider a neutron in an infinite space filled with a magnetic field $\mathbf{B}_0$. The neutron spinor wave function satisfies the equation

$$\mathrm{i}\,\frac{\mathrm{d}\left|\boldsymbol{\Psi}(t)\right\rangle}{\mathrm{d}t} = \boldsymbol{\sigma}\mathbf{B}_0\left|\boldsymbol{\Psi}(t)\right\rangle. \tag{1}$$

Here, the standard gyromagnetic ratio $\gamma_{\mathrm{n}} = |\mu_{\mathrm{n}}|/\hbar$ $(\mu_{\mathrm{n}} < 0)$ is included in the field definition. The solution of Eqn (1) is

$$\left|\boldsymbol{\Psi}(t)\right\rangle = \exp\left(-\mathrm{i}\boldsymbol{\sigma}\mathbf{B}_0 t\right)\left|\Psi(0)\right\rangle. \tag{2}$$

If $|\Psi(0)\rangle = |\mathbf{B}_0\rangle$, i.e., if the initial state corresponds to the neutron polarization aligned along the field $\mathbf{B}_0$, then (2) reduces to

$$\left|\boldsymbol{\Psi}(t)\right\rangle = \exp\left(-\mathrm{i}B_0 t\right)\left|\Psi(0)\right\rangle. \tag{3}$$

It follows that the initial polarization remains unchanged and the wave function acquires only the so-called dynamical phase $\varphi_{\mathrm{d}}(t) = B_0 t$.

We now imagine that besides the permanent field $\mathbf{B}_0$, which we direct along the z axis, there is also an RF field

$$\mathbf{b}(t) = b\big(\cos(2\omega t), \sin(2\omega t), 0\big), \tag{4}$$

where the factor 2 is separated for convenience to avoid fractional values in what follows. The total field $\mathbf{B}(t) = \mathbf{B}_0 + \mathbf{b}(t)$ is a vector of the length $B = (B_0^2 + b^2)^{1/2}$, whose end runs along a circle with the period $T = 2\pi/2\omega$, as is shown in Fig. 1. Therefore, the vector $\mathbf{B}(t) = \mathbf{B}_0 + \mathbf{b}(t)$ becomes a generatrix of the cone. We assume that $\omega \ll B_0$, i.e., the angular speed of the vector $\mathbf{B}$ rotation is much smaller than the spin precession frequency B around the field $\mathbf{B}$. In this case, the spin adiabatically follows the moving vector $\mathbf{B}(t)$ of the field. However a question arises: when the vector $\mathbf{B}(T)$ after the period $t = T$ returns to its initial position $\mathbf{B}(0)$, does the phase of the spinor wave function have the same dynamical value $\varphi_{\mathrm{d}}(T) = BT$ as for a fixed $\mathbf{B}$? The answer to this question is no. The phase of the spinor wave function after the time period T is $\varphi(T) = \varphi_{\mathrm{d}}(T) + \phi_{\mathrm{B}}$, i.e., it acquires an additional term ϕ_{B}, called the Berry phase [1], which is equal to

$$\phi_{\mathrm{B}} = \frac{1}{2}\,\Omega, \tag{5}$$

i.e., to half the solid angle that from the origin of $\mathbf{B}$ subtends the area $S = \pi b^2$ encircled by the end of $\mathbf{B}$. The factor $1/2$ is characteristic of spin-1/2 particles. At small b, expression (5)

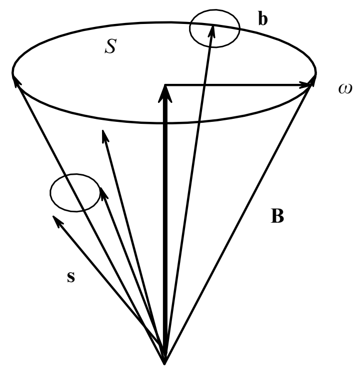

Figure 1. Neutron spin **s** precesses around the magnetic field **B**, which itself slowly rotates about the z axis with a small frequency ω. After the period T, when the magnetic field returns to its original direction, the precession phase of the spin is not the same as the precession phase $\varphi_{\mathrm{d}} = Bt$ around the fixed field vector **B**. There is an additional term, called the Berry phase, which has the value $\phi_{\mathrm{B}} = \Omega/2$, where Ω is the solid angle under which the area circumscribed by the end of the vector **B** is seen from its beginning.

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reduces to the frequently used form

$$\phi_{\mathrm{B}} = \frac{1}{2}\,\frac{\pi b^2}{B^2}\,. \tag{6}$$

It is shown in Section 2 that the Schrödinger equation

$$\mathrm{i}\,\frac{\mathrm{d}\big|\boldsymbol{\Psi}(t)\big\rangle}{\mathrm{d}t} = \boldsymbol{\sigma}\mathbf{B}(t)\big|\boldsymbol{\Psi}(t)\big\rangle \tag{7}$$

in the field $\mathbf{B}(t) = \mathbf{B}_0 + \mathbf{b}(t)$ is easily solved analytically for arbitrary values of the parameters b, B_0, and ω. The solution gives a precise value of the phase $\varphi(T)$, which in the adiabatical case, i.e., at a small parameter $\eta = \omega/B_0$, can be expanded in a power series in this parameter η, and the linear approximation in this parameter is just expression (5) for the Berry phase, where the solid angle Ω for arbitrary values of the field b is equal to

$$\Omega = 2\pi\left(1 - \frac{B_0}{B}\right) \equiv 4\pi\sin^2\frac{\theta}{2} \equiv \frac{\pi b^2}{B^2\cos^2(\theta/2)}\,, \tag{8}$$

where θ is the angle between the axis and the generatrix $\mathbf{B}(t)$ of the cone: $\cos\theta = B_0/B$. The solution method was originated by Rabi [2], and its form shown here was used, for instance, in [3], and also by many other authors.

2. Solution of the Schrödinger equation

To solve Schrödinger equation (7), we use the identity

$$\begin{aligned}\boldsymbol{\sigma}\mathbf{b}(t) &= b\big(\sigma_x\cos(2\omega t) + \sigma_y\sin(2\omega t)\big)\\ &= b\exp(-\mathrm{i}\omega\sigma_z t)\sigma_x\exp(\mathrm{i}\omega\sigma_z t)\,,\end{aligned} \tag{9}$$

and substitute $|\boldsymbol{\Psi}(t)\rangle$ in the form

$$\big|\boldsymbol{\Psi}(t)\big\rangle = \exp(-\mathrm{i}\omega\sigma_z t)\,\big|\chi(t)\big\rangle\,. \tag{10}$$

As a result, the equation becomes

$$\mathrm{i}\,\frac{\mathrm{d}\big|\chi(t)\big\rangle}{\mathrm{d}t} = \boldsymbol{\sigma}\boldsymbol{\mathcal{B}}\,\big|\chi(t)\big\rangle\,, \tag{11}$$

where $\boldsymbol{\mathcal{B}} = (b, 0, B_0 - \omega)$ is a vector independent of the time t. The solution of (11) is similar to solution (2) of Eqn (1):

$$\big|\chi(t)\big\rangle = \exp(-\mathrm{i}\boldsymbol{\sigma}\boldsymbol{\mathcal{B}}t)\big|\chi(0)\big\rangle\,, \tag{12}$$

and with Eqn (10) taken into account, the solution is

$$\big|\boldsymbol{\Psi}(t)\big\rangle = \exp(-\mathrm{i}\omega\sigma_z t)\exp(-\mathrm{i}\boldsymbol{\sigma}\boldsymbol{\mathcal{B}}t)\big|\chi(0)\big\rangle\,. \tag{13}$$

With this given solution, we can find the direction of the neutron spin at any instant t if it is known at $t = 0$. The initial spin direction, i.e., the spinor $|\chi(0)\rangle$, can be arbitrary. For convenience, we choose it to be $|\chi(0)\rangle = |\boldsymbol{\mathcal{B}}\rangle$, which means that the initial state is normalized, and its spin is directed along the unit vector $\boldsymbol{\mathcal{B}}/|\boldsymbol{\mathcal{B}}|$, where $|\boldsymbol{\mathcal{B}}| = \big(b^2 + (B_0-\omega)^2\big)^{1/2}$. This state is an eigenspinor of the matrix $\boldsymbol{\sigma}\boldsymbol{\mathcal{B}}$ with the eigenvalue $|\boldsymbol{\mathcal{B}}|$. The probability amplitude $A(t)$ of finding the same polarization during time t is

$$\begin{aligned}A(t) &= \langle\boldsymbol{\mathcal{B}}|\exp(-\mathrm{i}\omega\sigma_z t)\exp(-\mathrm{i}\boldsymbol{\sigma}\boldsymbol{\mathcal{B}}t)|\boldsymbol{\mathcal{B}}\rangle\\ &= \exp\big(-\mathrm{i}|\boldsymbol{\mathcal{B}}|t\big)\langle\boldsymbol{\mathcal{B}}|\exp(-\mathrm{i}\omega\sigma_z t)|\boldsymbol{\mathcal{B}}\rangle\\ &= \exp\big(-\mathrm{i}|\boldsymbol{\mathcal{B}}|t\big)\big(\cos(\omega t) - \mathrm{i}\sin(\omega t)\langle\boldsymbol{\mathcal{B}}|\sigma_z|\boldsymbol{\mathcal{B}}\rangle\big)\,.\end{aligned} \tag{14}$$

At $t = T$, we have $\omega t = \pi$, $\sin(\omega T) = 0$, and $\cos(\omega T) = -1 \equiv \exp(-\mathrm{i}\omega T)$. Therefore, (14) reduces to the form

$$A(t) = \exp\big[-\mathrm{i}\big(|\boldsymbol{\mathcal{B}}| + \omega\big)T\big]\,, \tag{15}$$

i.e., at the end of the rotation cycle, the particle state remains to be polarized along the vector $\boldsymbol{\mathcal{B}}$, and the only change in the wave function is the phase factor with the phase

$$\varphi(T) = \big(\omega + |\boldsymbol{\mathcal{B}}|\big)T\,. \tag{16}$$

It differs from the dynamical phase $\phi_{\mathrm{d}}(T) = BT$ by an additional term:

$$\phi_{\mathrm{B}} = \Delta\varphi(T) = \varphi(T) - \varphi_{\mathrm{d}}(T) = \big(\omega + |\boldsymbol{\mathcal{B}}| - B\big)T\,. \tag{17}$$

We note that we have not required ω to be small so far. Therefore, (16) is valid for arbitrary values of the parameters B_0, b, and ω. In the case of a small ratio $\omega \ll B_0$, the additional phase ϕ_{B} can be represented as

$$\phi_{\mathrm{B}} = \left(\omega + \frac{|\boldsymbol{\mathcal{B}}|^2 - B^2}{|\boldsymbol{\mathcal{B}}| + B}\right)T \approx \left(1 - \frac{B_0}{B}\right)\omega T\,, \tag{18}$$

where only the term linear in ω was taken into account. Because $\omega T = \pi$, the additional phase can be represented as

$$\phi_{\mathrm{B}} = \frac{1}{2}\left[2\pi\left(1 - \frac{B_0}{B}\right)\right] \equiv \frac{\Omega}{2} \equiv \frac{\pi b^2}{B(B + B_0)} \equiv \frac{\pi b^2}{B^2(1 + \cos\theta)}\,. \tag{19}$$

At $b \ll B_0$, we have $\theta \ll 1$, and Eqn (19) reduces to (6).

3. Conclusion

The derivation of the Berry phase presented here is similar to the one published in [4] but is much simpler, accessible even to a student audience. It helps understand the physical meaning of the Berry phase for spin-1/2 particles. For a more comprehensive coverage of the Berry phase in the case of arbitrary-spin particles, review [5] is recommended. For those to whom the simple derivation seems unpersuasive, we recommend to look at the theoretical justification of the Berry phase with secondary quantization [6] or the one based on the study of topological properties of space [7].

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