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The new life of complete integrability

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Abstract. **We briefly review new trends in how the notion of complete integrability of dynamical systems and their quantum versions have developed over the last four decades. We describe a new technique for working with integrable models and outline its main applications.**

This review is based on my talks at the Symposium for foreign members of the Royal Swedish Academy of Sciences (November 2011), the winter school "Nonlinear Waves" (Nizhnii Novgorod, March 2012), and the Ginzburg Conference on Physics (Lebedev Physical Institute, Russian Academy of Sciences, May 2012). My aim was to introduce a broad audience of theoretical physicists to the rapidly developing domain of mathematical physics concentrating on the idea of complete integrability. Of course, the presentation is largely based on my personal experience and does not cover all the aspects of this development.

The notion of complete integrability in Hamiltonian mechanics was created and developed in the 19th century by famous mathematicians and mechanicists Jacobi, Poisson, Liouville, Hamilton, and others. In modern terms (see, e.g., Arnold's book [1]), this notion can be formulated as follows. An antisymmetric matrix $\Omega^{mn}(\xi)$ defined on a phase space Γ with coordinates $(\xi) = (\xi^1, \ldots, \xi^N)$ determines the Poisson brackets of the coordinates,

$$\{\xi^m, \xi^n\} = \Omega^{mn}(\xi)\,,$$

and of functions on the phase space,

$$\{f, g\} = \Omega^{mn} \partial_m f \, \partial_n g\,.$$

The Poisson bracket satisfies the Jacobi identity

$$\big\{\{f, g\}, h\big\} + \big\{\{h, f\},\, g\big\} + \big\{\{g, h\},\, f\big\} = 0\,,$$

which is ensured by the relation

$$\partial_k \Omega^{mn} \Omega^{kl} + \partial_k \Omega^{lm} \Omega^{kn} + \partial_k \Omega^{nl} \Omega^{km} = 0\,.$$

Evolution is governed by the Hamilton equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \xi^n = [H, \xi^n]\,,$$

with a selected function $H(\xi)$ on the phase space, called the energy.

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The triplet $\{\Gamma, \Omega, H\}$ is called a Hamiltonian structure. In textbooks on mechanics, the phase space is even-dimensional, $N = 2L$, and the ξ^n are taken as the canonical coordinates q_i and momenta p^i, $i = 1, \ldots, L$, with the brackets

$$\{p^i, q_k\} = \delta^i_k\,.$$

The celebrated Darboux theorem states that whenever the matrix Ω^{mn} is nondegenerate, it can be brought (at least locally) to the form

$$\Omega = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

by a coordinate transformation, and hence, the corresponding coordinates ξ are canonical. In general, the coordinates ξ can be chosen as $\{\xi\} = \{\eta, \lambda\}$, where the λ have zero brackets with all the coordinates, and the bracket for the η coordinates is nondegenerate. Functions of λ give rise to trivial integrals of motion, and the entire dynamics occurs in the η variables.

A nondegenerate Hamiltonian structure $\{\Gamma_{2L}, \Omega, H\}$ is called completely integrable [1] if there exist $L - 1$ functions $Q_i(\xi)$, $i = 1, \ldots, L - 1$, such that they are functionally independent of H and functionally independent among themselves, and

$$\{H, Q_i\} = 0\,, \qquad \{Q_i, Q_k\} = 0\,.$$

The functions Q_i are called commuting integrals of motion. For an integrable system, a change of variables

$$(\xi^m) \to (I^i, \alpha_k)$$

exists such that the I and α coordinates are canonical,

$$\{I^i, I^k\} = 0\,, \quad \{\alpha_i, \alpha_k\} = 0\,, \quad \{I^i, \alpha_k\} = \delta^i_k\,,$$

and the energy depends only on the I variables:

$$H = H(I)\,.$$

The equations of motion then have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\, I = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}\, \alpha_k = \frac{\partial H}{\partial I^k}\,,$$

and hence, $I^i(t) = I^i$, $\alpha_k(t) = \alpha_k(0) + \omega_k t$, and $\omega_k = \partial H / \partial I^k$. In typical examples, the variables α_k take values in a torus, and are therefore called angles. The total set of variables (I, α) is called the action–angle-type variables.

In the late 19th–early 20th centuries, the search for action–angle-type variables for specific dynamical systems (typically, exotic pendulums) was a fascinating occupation for experts in classical mechanics. It suffices to mention the Kovalevskaya top or the Chaplygin pendulum. But the interest in this subject phased out gradually.

A new development came from an unexpected angle. In 1967, a group of American experts — Gardner, Greene,

[1] With the "completely" part to be omitted in what follows.

Kruskal, and Miura (GGKM) — invented a remarkable construction for the solution of the Korteweg–de Vries (KdV) equation [2]

$$v_t + 6vv_x + v_{xxx} = 0\,.$$

In the original work [3], this equation occurred in the hydrodynamical problem of a shallow water flow, but GGKM found its applications to the theory of plasma. The KdV equation has the remarkable solution

$$v(x,t) = \frac{A}{\cosh^2\big(a(x-vt)\big)}\,,$$

describing a solitary wave of the type that can be observed on a gently sloping beach (see Photo 1).

Photo 1. Solitons on the water surface in the Gulf of Finland, Komarovo.

Kruskal and Zabusky dubbed this solution a "soliton," following the current fashion in the theory of elementary particles. Their intuitive idea was justified, when the theory of solitons established itself within quantum field theory.

The GGKM method consists in a change of variables that involves spectral characteristics of the Schrödinger equation

$$L\psi = k^2\psi\,, \quad L = -\frac{d^2}{dx^2} + v(x)\,, \qquad (1)$$

where the potential $v(x)$ is given by the initial conditions for the KdV equation. In the simplest case where the spatial variable ranges the entire axis $-\infty < x < \infty$ and the potential $v(x)$ is assumed to vanish at infinity,

$$v(x) \to 0\,, \quad |x| \to \infty\,,$$

Eqn (1) with positive k has a solution $\psi(x,k)$ with the asymptotic forms

$$\psi(x,k) \to \exp(ikx) + r(k)\exp(-ikx)\,, \quad x \to -\infty\,,$$

$$\psi(x,k) \to t(k)\exp(ikx)\,, \quad x \to \infty\,,$$

where the transmission and reflection coefficients $t(k)$ and $r(k)$ satisfy the unitarity condition

$$\big|t(k)\big|^2 + \big|r(k)\big|^2 = 1\,.$$

If $v(x)$ takes negative values in some interval, then there exists a discrete spectrum $k^2 = -\kappa_l^2$, $l = 1, \ldots, n$, with exponentially decreasing wave functions

$$\psi_l(x) \to \exp(\kappa_l x)\,, \quad x \to -\infty\,;$$

$$\psi_l(x) \to c_l\exp(-\kappa_l x)\,, \quad x \to \infty\,.$$

The coefficient $t(k)$ is uniquely determined by $r(k)$ and κ_l based on the analyticity property. The independent scattering data $\{r(k), \kappa_l, c_l\}$ uniquely define the potential $v(x)$. The subject of reconstructing the potential from spectral data was vigorously developed in the 1950s by Gelfand, Levitan, Krein, Marchenko, Jost, Kohn, Moses, and others (see the references in [4]). Its version applicable to the Schrödinger operator on the entire real axis, which was required for the GGKM method, was discussed in my PhD thesis in 1959 [5].

It was shown by GGKM that the transformation from a potential $v(x)$ to the scattering data linearizes the KdV equation:

$$\begin{array}{ccc} v(x) & \longrightarrow & v(x,t) \\ \big\downarrow & & \big\downarrow \\ \big(r(k), \kappa_l, c_l\big) & \longrightarrow & \big(r(k,t), \kappa_l, c_l(t)\big)\,, \end{array}$$

where

$$r(k,t) = \exp(-ik^3t)\,r(k,0)\,, \quad c_l(t) = \exp(\kappa_l^3 t)\,c_l\,.$$

Lax [6] gave an important interpretation of the role played by linear problem (1) in describing the dynamics governed by the KdV equation; the $L(t)$ operator with the potential $v(x,t)$ satisfies the equation

$$\frac{d}{dt}L(t) = \big[L(t), A(t)\big]\,,$$

where $A(t)$ is a third-order linear differential operator that can also be constructed in terms of $v(x,t)$. Hence, the KdV dynamics is an isospectral deformation of the operator L.

For some time, the GGKM trick was regarded as a remarkable but isolated stroke of luck, not suggestive of any generalization. But Zakharov and Shabat showed in 1979 [7] that the nonlinear Schrödinger equation (NLSE)

$$i\psi_t = -\psi_{xx} + g|\psi|^2\psi$$

is also solvable by a similar method. The role of the spectral problem is, in this case, taken over by the Dirac equation

$$\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} 0 & g\psi \\ g\bar{\psi} & 0 \end{pmatrix}\right)\phi(x,\lambda) = \lambda\phi(x,\lambda)\,.$$

It then became clear that the method of the inverse scattering problem has a wider applicability.

In early 1971, I met with V Zakharov at a conference on inverse problems in Novosibirsk and only then learned about the GGKM method. Discussing explicit formulas, we noticed that in the system of spectral variables, half of them evolve linearly with time,

$$\arg r(k) \to \arg r(k) + k^3 t\,, \quad \ln c_l \to \ln c_l + \kappa_l^3 t\,,$$

and the other half ($|r(k)|$ and κ_l) are independent of time. Similarity with the action–angle variables was evident. Based on that idea and some known results on the spectral characteristics of the Schrödinger operator, we showed that passing to the spectral data is indeed a canonical transformation to action–angle-type variables. Our paper [8], with the title "Korteweg–De Vries equation as a completely integrable Hamiltonian system," open a new development in the theory of integrable models.

Briefly, our results are as follows: the KdV equation is a Hamiltonian system on an infinite-dimensional phase space whose coordinates are given by functions $v(x)$, where x can be regarded as the "coordinate number" of $v(x)$. The Poisson bracket of the coordinates is given by

$$\{v(x), v(y)\} = \delta'(x-y)\,.$$

The right-hand side is here antisymmetric and is independent of the coordinates, and therefore the Jacobi identity is satisfied trivially. The functional

$$N = \int v(x)\,\mathrm{d}x$$

commutes with all $v(x)$ and is, therefore, a central element. In the subspace $N = \text{const}$, the bracket is invertible and canonical coordinates can be chosen as the even and odd components of the Fourier transform

$$v_{\mathrm{e}}(k) = \int_{-\infty}^{\infty} v(x)\cos 2\pi xk\,\mathrm{d}x\,,$$

$$v_{\mathrm{o}} = \int_{-\infty}^{\infty} v(x)\,\frac{\sin 2\pi xk}{2\pi k}\,\mathrm{d}x\,.$$

For $k > 0$, we then have

$$\{v_{\mathrm{e}}(k), v_{\mathrm{e}}(k')\} = 0\,,\quad \{v_{\mathrm{o}}(k), v_{\mathrm{o}}(k')\} = 0,$$

$$\{v_{\mathrm{e}}(k), v_{\mathrm{o}}(k')\} = \delta(k-k')\,.$$

The Hamilton function is

$$H = \int_{-\infty}^{\infty}\left(\frac{1}{2}\,v_x^2 + v^3(x)\right)\mathrm{d}x\,,$$

and is easily seen to give rise to the KdV equation. The quantity

$$P = \int \frac{1}{2}\,v^2(x)\,\mathrm{d}x$$

generates the equation

$$v_t + v_x = 0$$

and plays the role of momentum.

We evaluated the Poisson brackets of scattering data and showed that the role of "action"-type variables was played by the function

$$\rho(k) = \frac{k}{2}\ln\left(1 - |r(k)|^2\right)$$

and the eigenvalues κ_l. The argument $r(k)$ and the constants c_l are "angle"-type variables. The Hamiltonian H and the momentum P can be explicitly expressed in terms of the actions:

$$P = \sum_l \kappa_l + \int_0^{\infty} k\rho(k)\,\mathrm{d}k\,, \qquad (2)$$

$$H = -\sum_l \kappa_l^3 + \int_0^{\infty} k^3\rho(k)\,\mathrm{d}k\,. \qquad (3)$$

The higher odd-degree moments of the density $\rho(k)$ are also local functionals:

$$Q_n = \int_0^{\infty} k^{2n+1}\rho(k)\,\mathrm{d}k = \int_{-\infty}^{\infty}\Phi_n(v, v_x, \ldots)\,\mathrm{d}x\,,$$

whose densities Φ_n depend on v and its first derivatives.

Formulas (2) and (3) are reminiscent of the formulas in many-body quantum theory in a mixed representation of fields and particles. The first terms produce the contribution of particles (solitons) with the dispersion

$$\epsilon(p) = -p^3\,,$$

and the second terms produce the contribution of a secondary quantized field with the dispersion

$$\omega(k) = k^3\,.$$

In this way, Kruskal and Zabusky's intuition was satisfactorily substantiated.

As an expert in quantum field theory, I found this especially appealing. It opened a new possibility for the mechanism of interpreting particles beyond the perturbation theory paradigm "one field, one particle." But the nonrelativistic nature of the KdV theory and the unusual dispersion law $\omega(k) = k^3$ did not invite quantization.

A remarkable relativistic example was produced by another equation solvable by the inverse scattering method:

$$\phi_{tt} - \phi_{xx} + \frac{m^2}{\gamma}\sin\gamma\phi = 0\,,$$

bearing the jargony name sine-Gordon (SG).[2] The action–angle variables for this, evidently Hamiltonian, system were obtained by Takhtajan and myself [12]. In addition to solitons of two types, differing by the sign of their topological charge, that equation also had a periodic solution: breathers. The phase space of a soliton is two-dimensional, as in the KdV case, and the space of a breather has dimension 4, which comprises one degree of freedom for the center-of-mass motion and another degree of freedom for internal oscillations. The part of the phase space corresponding to the second degree of freedom is compact, and under semiclassical quantization gives finitely many states, which can be interpreted as soliton–antisoliton bound states. As a result, in addition to the contribution of a scalar particle with mass m, the semiclassical spectrum consists of solitons and antisolitons with the mass $8m/\gamma$ and their bound states with the masses

$$M_n = \frac{16m}{\gamma}\sin\frac{n\gamma}{16}\,.$$

Independently, this spectrum was also obtained by Dashen, Hasslacher, and Neveu [13]; our paper was sent for publication to *Physics Letters*, but the correspondence was lost in the mail to Italy and was published only later.

The SG model has demonstrated a number of remarkable properties.

[2] The rhyme sine-Klein is of somewhat doubtful taste, but is addictive. What we call the Klein–Gordon equation, however, should instead be called the Klein–Fock equation (see [9–11]).

1. Solitons have the topological charge

$$Q = \int_{-\infty}^{\infty} J_0 \, \mathrm{d}x \,, \quad J_0 = \partial\phi \,,$$

which takes integer values.

2. The soliton mass and scattering phases are inversely proportional to the coupling constant γ. Hence, the original theory with weakly interacting particles acquires new, strongly interacting particles. I was vigorously advertising this standpoint in lectures given at Harvard and Princeton in 1975. It is also reflected in my erroneous paper [14], with the ambitious title "Hadrons from leptons?"; the legacy of that paper, however, is the model of a three-dimensional soliton with the Hopf invariant playing the role of topological charge.

In the first half of the 1970s, the list of integrable models was expanding rapidly; it included systems such as the Heisenberg magnet (HM) and the nonlinear chiral field; in these models, the respective dynamical variables take values in the nonlinear manifold $\mathbb{S}^2$ and in a compact group G. The list also embraced some discrete-space problems, for example, the Toda chain. The number of people investigating the theory was growing, and groups soon appeared in the USA, Japan, and France. In Leningrad (Saint Petersburg), Takhtajan and I were joined by Kulish, Korepin, Sklyanin, Semenov-Tyan-Shansky, Reiman, Matveev, and Its. At the Landau Institute, together with Zakharov and Shabat, S P Novikov's group formed, including young researchers Manakov, Krichever, and others. In the book "Hamiltonian methods in the theory of solitons," written by Takhtajan and myself [15], a detailed account of the events and an exhaustive list of references are given.

But the principal aim of our group was to construct the quantum theory of solitons. It became clear by the late 1970s that quantizing them is possible by starting with the auxiliary linear problem with noncommutative dynamical variables and developing an analog of spectral theory for it. As a result, the quantum inverse scattering method (QISM) was developed, which naturally involved the Bethe-ansatz technique developed by Bethe in [16], with the example of a spin chain of spins 1/2. This led to the formulation of the algebraic Bethe ansatz (ABA) in [17,18]. The team mentioned above was gradually joined by others, including Izergin, Smirnov, Tarasov, and Bogoliubov.

I describe the key points of the ABA with the simplest example of the Heisenberg spin chain of spins 1/2, following [19].

The dynamical variables are the spin operators s_n^a, $n = 1, \ldots, N$, $a = 1, 2, 3$, where N is the length of the chain, with the periodicity condition $N + 1 = 1$. The operators s_n^a satisfy the commutation relations

$$[s_m^a, s_n^b] = \mathrm{i}\delta_{mn}\, \epsilon^{abc} s_n^c \,,$$

where ϵ^{abc} is the unit antisymmetric tensor. The total Hilbert space $\mathcal{H}$ of the system is the tensor product of $\mathbb{C}^2$ spaces for each spin:

$$\mathcal{H} = \prod_{n=1}^{N} \otimes \mathbb{C}^2 .$$

Spin operators with a subscript n act nontrivially only in the space at the nth position in this product and are expressed in terms of the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as

$$s^i = \frac{1}{2}\,\sigma^i .$$

The Hamiltonian has the form

$$H = J \sum_{n=1}^{N} s_n^a s_{n+1}^a \,,$$

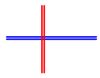
with $s_{N+1}^a = s_1^a$. The ABA is based on the use of the auxiliary linear problem

$$\Phi_{n+1} = L_n(\lambda)\, \Phi_n \,, \qquad (4)$$

with the matrix $L_n(\lambda)$ of the form

$$L_n(\lambda) = \begin{pmatrix} \lambda + s_n^3 & \mathrm{i}s_n^- \\ \mathrm{i}s_n^+ & \lambda - s_n^3 \end{pmatrix}, \quad s_n^{\pm} = s_n^1 \pm \mathrm{i}s_n^2 \,.$$

We can say that $L_n(\lambda)$ is a block matrix in the tensor product space $\mathbb{C}^2 \otimes \mathbb{C}^2$; spin operators act in the first $\mathbb{C}^2$, and the second is explicitly given by the matrix $L_n(\lambda)$. The first space can be naturally referred to as quantum and the second as auxiliary. With the matrix $L_n(\lambda)$, it is convenient to associate the graph

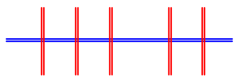
$L_n(\lambda) \sim$  ,

where the vertical line specifies the action in the quantum space and the horizontal line shows the action in the auxiliary space.

The monodromy matrix of system (4), which is expressed as

$$M(\lambda) = \overrightarrow{\prod} L_n(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

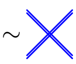
plays the role of "scattering data." Its graph has the form

$M(\lambda) \sim$ 

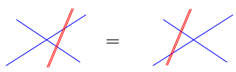
and corresponds to an operator in $\mathbb{C}^2 \otimes \mathcal{H}$. The commutation relations for matrix elements can be derived from the local relations for spins s_n^a as follows. For matrices $L_n^1(\lambda)$ and $L_n^2(\mu)$ associated with the same quantum space n and two auxiliary spaces 1 and 2, we verify the relation

$$R^{12}(\lambda - \mu)\, L_n^1(\lambda)\, L_n^2(\mu) = L_n^2(\mu)\, L_n^1(\lambda)\, R^{12}(\lambda - \mu) \,,$$

where $R^{12}(\lambda)$ is a matrix in the tensor product of auxiliary spaces. Its graph is similar to the graph for L_n:

$R(\lambda) \sim$  ,

where both lines correspond to auxiliary spaces. Commutation relations for the L_n can be depicted as

 .

The local relation immediately implies a similar relation for the monodromy:

$$R^{12}(\lambda-\mu)\,M^1(\lambda)\,M^2(\mu)=M^2(\mu)\,M^1(\lambda)\,R^{12}(\lambda-\mu)\,. \qquad (5)$$

The laborious calculations of Poisson brackets in papers of the 1970s have been wonderfully replaced with elementary algebra.

It follows from (5) that the family of operators

$$T(\lambda) = \mathrm{tr}\ M(\lambda) = A(\lambda) + D(\lambda)$$

is commutative:

$$\big[T(\lambda), T(\mu)\big] = 0\,.$$

It can be shown that this family contains the Hamiltonian

$$H = \frac{\mathrm{d}T(\lambda)}{\mathrm{d}\lambda}\, T^{-1}(\lambda)\big|_{\lambda=i/2}\,.$$

Evidently, $T(\lambda)$ is a polynomial in λ of degree N with $N-1$ nontrivial coefficients — functions of the dynamical variables s_n^a:

$$T(\lambda) = 2\lambda^N + \sum_{n=1}^{N-1} Q_n(s)\,\lambda^n\,.$$

The operators Q_n, $n = 1, \ldots, N-1$, together with the third component of the total spin

$$S^3 = \sum_n \sigma_n^3\,,$$

make up a family of N commuting integrals of motion. It is natural to assume that the spin chain defines a system with N degrees of freedom (in the semiclassical case, the phase space of a single spin is the two-dimensional sphere $\mathbb{S}^2$, and its quantized counterpart is the finite-dimensional Hilbert space $\mathbb{C}^2$). Hence, the system under consideration is completely integrable and the role of action variables is played by operators from the $T(\lambda)$ family. The role of angle-type variables is played by off-diagonal elements of the monodromy matrix.

In the space $\mathcal{H}$, we consider a highest-spin vector Ω:

$$S_n^+\Omega = 0\,.$$

It is annihilated by the operator $C(\lambda)$:

$$C(\lambda)\,\Omega = 0\,.$$

The state

$$\Omega(\lambda_1, \ldots, \lambda_l) = \prod_{i=1}^{l} B(\lambda_i)\,\Omega$$

is an eigenvector of H,

$$H\Omega\big(\{\lambda\}\big) = J\sum_{i=1}^{l} \epsilon(\lambda_i)\,\Omega\big(\{\lambda\}\big)\,,$$

if the λ_i satisfy a system of algebraic equations that first appeared in Bethe's work. I do not give it here and refer the reader to [19], only noting that the dispersion $\epsilon(\lambda)$ is negative for all λ.

The discovery of a relation between natural quantization of the inverse scattering method and Bethe-ansatz formulas originating in entirely different ideas is a remarkable example of the development of modern mathematical physics. This discovery has become the starting point for generalizations going far beyond the initial trick by Bethe.

Bethe equations allow passing to the limit as $N\to\infty$. Evidently, this requires strict control over the selection of allowed states, because a naive limit leads to an infinite tensor product with a nondenumerable basis. The relevant selection depends on the sign of the constant J.

For $J < 0$, excitations over the state Ω have positive energy, and we have to consider states for which the operator $Q = S^3 - N/2$ has finite positive values. The operators $S^{\pm}$ are ill-defined in the limit $N \to \infty$. Therefore, the SU(2) symmetry is violated and the states are magnons with the charge $Q = 1$ and their bound states with $Q = 2, 3, \ldots$. We deal here with a ferromagnet.

For $J > 0$, the picture is much more interesting. Constructing the vacuum requires filling the Dirac sea of negative-energy states. This is possible because the spectrum that follows from the Bethe ansatz has a Fermi nature: the roots λ_i cannot coincide. The vacuum was constructed by Hulthén [20] in 1937, but the correct construction of excitations has long been erroneous. In [21], Takhtajan and I showed that one-particle excitations have spin 1/2 (and not 1, as had long been assumed). Therefore, the SU(2) symmetry is not violated for $J > 0$, all three components of the total spin are meaningful, and the excitation is a single particle with spin 1/2. We deal here with an antiferromagnet.

The theory of spin chains can be considered a remarkable example in the theory of many-body systems: it features symmetry breaking, the occurrence of new charges, the construction of a nontrivial vacuum, and so on. I believe that my collaborators mentioned above have gone through very stimulating training with the relevant mathematics.

In the 1990s, the subject of the ABA was developing rapidly and led to major generalizations.

1. Higher-spin models. It was shown that naive generalizations of the Heisenberg Hamiltonian to spins 1 and higher are not integrable. But Kulish, Reshetikhin, and Sklyanin showed that a local energy density exists for which the integrability does hold [22]. For spin 1, this density, which had been found previously by Zamolodchikov and Fateev [23], has the form $\sigma_n^a\sigma_{n+1}^a - (\sigma_n^a\sigma_{n+1}^b)^2$.

2. Anisotropy. A magnet with spin 1/2 and the local density

$$J_1\,\sigma_n^1\,\sigma_{n+1}^1 + J_2\,\sigma_n^2\,\sigma_{n+1}^2 + J_3\,\sigma_n^3\,\sigma_{n+1}^3$$

is called the XYZ model. The partly broken symmetry of the XXZ model with $J_1 = J_2$ is amenable to the ABA formalism. But as Kulish and Reshetikhin showed in [24], the auxiliary linear problem retains its meaning for higher spins only if the dynamical variables satisfy the relation

$$[s_n^+, s_n^-] = \frac{\sin\gamma s_n^3}{\sin\gamma}\,,$$

where γ is the anisotropy parameter. This formula has led to essential progress in mathematics: the creation of the theory of quantum groups (Sklyanin [25], Drinfeld [26], Jimbo [27], Reshetikhin–Takhtajan–Faddeev [28]), which later returned to physics as a symmetry of conformal field theory (Faddeev–Takhtajan [29], Gervais–Neveu [30]). The history of this development can be found in my review [31].

3. Other groups. The BA formalism can be generalized to higher-rank groups, with the BA equations formulated in terms of Dynkin diagrams (Reshetikhin [32]).

4. Continuum limits, i.e., taking the limit as $\Delta \to 0$. For example, the NLSE model can thus be obtained from a spin chain.

5. Inhomogeneous problems. Important examples can be obtained by choosing different values of the parameter λ at different lattice points. In particular, alternating as $\lambda_{2n} = \lambda + \kappa$, $\lambda_{2n+1} = \lambda - \kappa$ allows constructing a natural discrete analog of the quantum SG model.

It has become clear, as a result, that spin chains are a universal class of quantum integrable systems. More details on the subject can be found in my review [33].

It is worth noting that numerous points of the ABA permit an interpretation in the theory of classical statistical physics models on a two-dimensional lattice. This subject, traced back to Onsager [34], was further developed by Lieb [35] and Baxter [36] and has a history of its own. In this country, this direction was pursued by the group in Protvino, organized by Stroganov and Bazhanov [37, 38]. The role of the local $L_n(\lambda)$ operator is played there by the statistical weight. In this case, however, the quantum and auxiliary spaces are identical, and hence, for example, spin chains with higher spins have no statistical-physics interpretation.

Lastly, another source of relations of type (5) is given by scattering theory, where the leading role was played by the work of Yang [39] and Brezin–Zinn-Justin [40]. The work by Yang [39] and Baxter [41] had heuristic significance for our construction of the ABA. That was why Leon Takhtajan and I called relations of type (5) Yang–Baxter equations in [18].

Within the theory of factored scattering, A B Zamolodchikov and Al B Zamolodchikov obtained an exact solution of the nonlinear σ-model [N2].

As the reader would have noticed, I here discuss the methods and results derived mainly in Leningrad. The subject of quantum integrable models gained much popularity in the 1980s–1990s, however. This included establishing remarkable relations to conformal field theory, whose foundations were laid in [43]. In the work of A B Zamolodchikov and his collaborators, integrable models were regarded as deformations of conformal models by special local operators. Interestingly, quantization of the KdV equation [44] turned out to be important for quantization of conformal field theory.

Yet another avenue, originating in the work of C N Yang and C P Yang [45] and developed by Al B Zamolodchikov [46] and Destri and de Vega [47], is associated with the construction of a thermodynamic Bethe ansatz.

The attractiveness of these methods notwithstanding, we must not forget that their physical applications were until recently restricted to problems in a one-dimensional space. The turn of events was Lipatov's discovery in the mid-1990s [48] that high-energy scattering can be described in the Reggeization framework by the formalism of spin chains. The role of a lattice site is then taken by the operator number in a correlation function. Korchemsky and I interpreted Lipatov's observation in terms of the ABA for an SL(2) chain with spin -1 [49].

I want to note that interest in spin chains in relation to high-energy physics was expressed by Feynman toward the end of his career. The World Scientific Publishers once sent me a few lines from the abstract of his talk, which included the phrase, "If someone gives me a Bethe ansatz for the number of polarization greater than two, then I will be able to describe high-energy scattering." Two polarizations, naturally, correspond to spin 1/2, and he needed integrable chains of higher spins — precisely what had been done by our group. Unfortunately, I was unaware of Feynman's words until after his death, and had no chance to communicate our results to him. But, when visiting Pasadena, I was there in time to enter Feynman's office before John Schwarz moved in. A large blackboard showed some fragments of calculations and, among them, a memo (as I remember it):

To learn:
1. Bethe Ansatz.
2. Quantum Hall effect.
3. Turbulence.

Curious as I was, I asked if any materials on the subject remained, and the secretary was so nice as to bring me a pile of paper with Feynman's notes. The handwriting was very accurate and each sheet had a number and a date. Already on the first pages I found an outline of some of our papers, with the names of Reshetikhin and Sklyanin mentioned in particular. But because of a lack of time, I could not study these sheets of paper in detail, and I had to leave. Attempts to recover this material in the Caltech archive have so far been unsuccessful.[3]

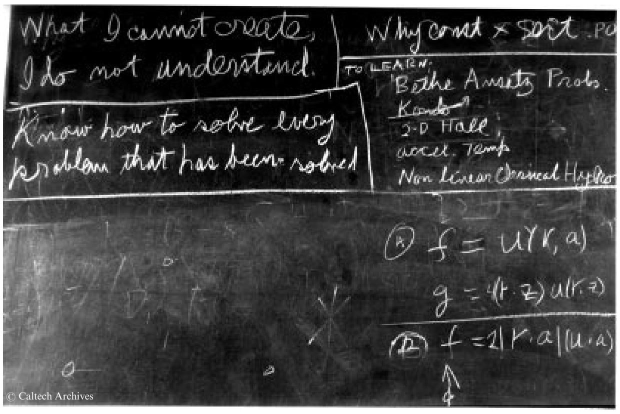


Photo 2. Part of the blackboard in Richard Feynman's office in the California Institute of Technology (Caltech), the last writings of Feynman retained after his death. (Courtesy of the Archives, California Institute of Technology.)

[3] While this paper was being prepared for publication, the managing editor of *Physics–Uspekhi*, M S Aksenteva, had a chance to familiarize herself with some documents from Feynman's archive deposited in Caltech. Here is what she wrote to me when sending the copies of those documents: "Your reminiscences were confirmed. As can be seen in the photo (see Photo 2), Feynman did then indeed define the following problems as the most interesting ones, requiring further study: the Bethe ansatz problem, the two-dimensional Hall effect, and nonlinear classical hydrodynamics (which, obviously, includes the turbulence problem). In addition, I was pleased to see references to works by Faddeev, Takhtajan, Sklyanin, and Zamolodchikov in Feynman's notes to his talk on 22 January 1987 (Lunch Talk on Bethe Ansatz [50])." Now that I can see how Feynman's handwriting looks, I tend to think that the sheets shown to me in Caltech were written by another hand, and that is why they may not be in Feynman's archive.

Photo 3. Richard Feynman in front of his red car (1972). [4]

I had a chance to meet Feynman only once. It was in January 1972, during my first visit to the USA, which was organized by Peter Lax and Luis Nirenberg of the Courant Institute of Mathematical Sciences. Kip Thorn arranged my brief visit to Pasadena to meet Feynman (after memorable skiing with Peter Lax in Aspen, halfway from New York to California). We spent several hours together. He already new about my work with Viktor Popov on the application of Feynman integrals to the construction of a diagram technique in the Yang–Mills field theory. I wanted to learn more about his approach to the construction of the S-matrix in terms of asymptotic fields without invoking Green's functions, which in the Yang–Mills theory did not allow a gauge-invariant treatment. But the discussion gradually (via the asymptotic fields and the problem of mass in the Yang–Mills theory) moved to the approach to infrared divergences in quantum electrodynamics that had just been formulated in my paper with Peter Kulish. To my surprise, Feynman requested a detailed explanation and, insofar as I could see, approved our method. But this discussion took all the time, unfortunately, and we went for a beer in a strip bar. On the way, I was given a chance to take a photo of Feynman in front of his red car. In the bar, we resumed the discussion while a naked girl was walking on the table above us. It seems Feynman decided to lure the young Soviet colleague with the sweets of western life. But the girl was freezing, and even turning livid with cold, and I felt she had to be pitied. And that was the end of my contact with Feynman.

Recently, I was pleased to learn that the reprint of my paper with Popov [51] that I then handed over with the dedication, "To prove that younger generation knows and respects Feynman Integral," did not go into the wastebin and stayed in Feynman's archive at Caltech [52].

To continue with my recollections of my first visit to the USA, I mention two more things. During my stay at the Massachusetts Institute of Technology in February 1972, Victor Weisskopf took me to the famous Oppenheimer Seminar at Princeton, and I could witness how the American scientific community was united in promoting Steven Weinberg's nomination for a Nobel prize. Nothing of the kind has existed in this country, even now. Also, during my second visit to Princeton, on Arthur Wightman's invitation, I gave four talks in two days, and in particular spoke about our paper with Zakharov, the one referred to above, at Kruskal's seminar.

I consider Richard Feynman my spiritual Teacher, along with Hermann Weyl and Paul Dirac.

[4] Two memorable anniversaries associated with Feynman occur in 2013: May 11 is his 95th birthday, and February 15 marks 25 years after his death. (Editor's note.)

Soon after Lipatov's breakthrough, aspects of integrability occurred in the theory of supersymmetric gauge field models. It was shown by Gorsky, Krichever, Marshakov, Mironov, and Morozov (GKM^3) [53] that the algebrogeometric technique developed by Dubrovin, Krichever, and Novikov [54] provides a suitable interpretation for the Seiberg–Witten formula [55] for the superpotential in the $N = 2$ gauge theory. Following the appearance of that work, numerous papers started rapidly appearing with the keyword "integrability" in their titles. Unlike in the original story with the KdV equation, the classical dynamical problem is involved here in solving the quantum problem. But quantization of the integrability technique provided by Nekrasov and Shatashvili turned out to be also applicable to supersymmetric gauge theory and was used in [56,57] in classifying vacuum states. A quantum deformation of the GKM^3 algebraic curve turned out to be related to the Bethe ansatz for XXX, XXZ, and XYZ spin chains in the respective space–time dimensions $D = 2$, 3, and 4.

Spin chains also occurred in the approach to anomalous dimensions in the supersymmetric Yang–Mills field theory. Minahan and Zarembo [58] considered correlation functions of a chain of two local operators $W(x)$ and $Z(x)$, which were set in correspondence with a spin chain, with the spin-up state corresponding to $W(x)$ and the spin-down state corresponding to $Z(x)$. The energy of that chain was interpreted as the anomalous dimension of the product of operators. It was shown that this chain coincides with the spin-1/2 XXX model. Paper [58] gave rise to extensive studies, mainly in Europe (Sweden, France, and Germany). The thermodynamic Bethe-ansatz methods have become especially relevant. It is impossible to fully describe the progress here, and I restrict myself to referring to reviews [59, 60].

New applications to relativistic quantum field theory have shown the power and universality of the notion of integrability and the ABA formalism. We can say that the integrability technique has made a breakthrough to the frontier of modern theoretical high-energy physics (see, e.g., the introduction to [61]), and new remarkable results can be expected to appear. I conclude with this optimistic note.

I am grateful to Leon Takhtajan for the important remarks and M S Aksenteva for the fruitful research at the Caltech Archive, and also to Shelley Erwin and Loma Karklins from the archive for their help in selecting archived documents and offering their copies for study.

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