

REVIEWS OF TOPICAL PROBLEMS

PACS numbers: **05.40.–a**, 47.27.T–, **47.53.+n**

Reconstruction of streamline topology, and percolation models of turbulent transport

O G Bakunin

DOI: 10.3367/UFNe.0183.201303b.0257

Contents

Abstract. **This paper discusses in detail the percolation models of turbulent diffusion that help establish nontrivial relations among theoretical concepts used in the theories of turbulence, dynamical systems, transport, etc. This approach is particularly important due to the need to describe turbulence in the presence of coherent structures, flow reconstructions, and drift and dissipation effects. In such regimes, the conventional quasilinear description is inconsistent with experimental results, necessitating the search for fundamentally new models and approaches. Most attention is given to the scaling concept, an important and widely used tool among theoreticians and experimentalists.**

1. Introduction

Turbulent transport is a fundamental physical phenomenon of immense practical significance. Nevertheless, years of active research have not yet proposed a rigorous physical–mathematical picture of turbulent transport. On the one hand, this opens up broad new avenues for researchers; on the other hand, it creates serious difficulties in resolving particular issues. Indeed, transport phenomena in turbulent flows can very seldom be successfully described with classical diffusion models. The main reason is the complexity of disordered motions intrinsic to turbulent flows. The lack of order in the flow field, manifested in the random character of fluid particle velocities (not defined by the macroscopic flow description), necessitates the wide use of correlation models and the concept of scaling [1–3]. A developed turbulent flow is spawned by a hierarchical set of eddies in which the largest eddy formations reach a size comparable to that of the domain under consideration, while small eddies reside at the 'viscous' scale. Under such circumstances, choosing the characteristic correlation length and time that define the transport of particles in turbulent flows is highly nontrivial. Here, the correlation characteristics of the flow velocity field must be taken into account together with its 'topological' features, which are not always directly related to small-scale turbulent motions [3–5]. Additionally, the description of turbulent transport requires attention to 'competing' factors such as seed (molecular) diffusion, reconnection of stream lines, and stochastic instability [6–9].

Similar problems arise in plasma physics in relation to the motion of charged particles in a stochastic magnetic field, condensed-matter physics in relation to transport in amorphous semiconductors, and in numerous systems where the law describing diffusion essentially deviates from the classical behavior [10–12]. In spite of substantial progress in explaining anomalous transport, many aspects of papers already deemed

O G Bakunin Institute of Tokamak Physics,
National Research Center 'Kurchatov Institute',
pl. Kurchatova 1, 123182 Moscow, Russian Federation
Tel. +7 (499) 492 42 65
E-mail: oleg_bakunin@yahoo.com

Received 2 February 2012, revised 27 May 2012
Uspekhi Fizicheskikh Nauk **183** (3) 257 – 276 (2013)
DOI: 10.3367/UFNr.0183.201303b.0257
Translated by S D Danilov; edited by A M Semikhatov

classical in this area are still relevant. For instance, already in the early phase of exploring turbulent diffusion processes, it was proposed to use the correlation functions, a modification of the classical diffusion equation, renormalization methods, etc. [13–15]. Presenting the development of all these research concepts in this review does not seem possible. Our attention is confined to the ideas of scaling, which are an important and rather universal instrument used by both theoreticians and experimentalists [2–4]. The approach based on scaling ideas allows straightforwardly handling the problem statements and tasks in various branches of modern physics related to turbulence.

Seventy years have already passed since the publication of the fundamental work by Kolmogorov and Obukhov proposing a scaling description of well-developed turbulence [16, 17]. And yet, the fundamental question of the character of eddy interactions in turbulent flows is still open [3, 18]. In fact, scaling remains, as previously, the main instrument of analysis. Quite naturally, we face the same issues concerning turbulent transport under the conditions of strong turbulence. Coherent structures emerging at large Reynolds numbers substantially complicate the description of effective transport. The methods developed for weak turbulence lead to results that contradict those obtained both experimentally and numerically [19–21].

In the absence of a universal analysis method for transport effects in structured turbulence, it is natural to concentrate on particular but sufficiently general approaches, choosing a specific shape of eddy structures forming the relevant class of turbulent flows as a basis. In this review, we concentrate on percolation models of random two-dimensional flows. The idea of using the percolation theory to describe anomalous transport in two-dimensional turbulent flows was proposed by Kadomtsev and Pogutse [22] in 1978. On the one hand, this approach allowed applying the phase transition theory to describe self-organization in turbulent media. On the other hand, it prompted the use of fractal concepts (scaling) to describe the geometric characteristics of the objects being studied. The unconventional idea by Kadomtsev and Pogutse attracted the attention of Zel'dovich, who noted that "the percolation formulation of the problem supplements the problem of linkage of field lines developed in detail by Moffat and others" [24]. Later, the percolation method was repeatedly used in analyzing transport phenomena in both hydrodynamical and plasma turbulence [24–27].

2. Turbulent diffusion and scaling

In this section, we briefly recapitulate the already classical results pertaining to turbulent transport, whose research history has already surpassed the 100-year mark. The reader can consult numerous monographs and reviews for a detailed account [1, 3, 4, 8, 9, 12]. Taylor was the first to recognize the need for a statistical approach to the analysis of particle transport in a turbulent flow [28]. The formula for the turbulent diffusion coefficient that he proposed,

$$D_{\mathrm{T}} = \int_0^t C(t)\,\mathrm{d}t \tag{1}$$

(where $C(t) = \langle V(0)\,V(t)\rangle$ is the velocity autocorrelation function), was based on the Lagrangian representation of particle motion in the field of stationary and isotropic turbulence,

$$\mathbf{x}(t) = \int_0^t \mathbf{V}(t)\,\mathrm{d}t\,. \tag{2}$$

Using the exponential representation for the autocorrelation function, Taylor described the rms displacement of a scalar particle at large times by the traditional diffusion scaling $R^2(t) \propto D_{\mathrm{T}}t$. Later, the approach to the description of transport processes in terms of correlation functions was advanced both in the framework of the quasilinear weak turbulence theory [29] and by considering the general principles of statistical physics in studies by Kubo and Green [1, 13, 21]. This simplified model of turbulent transport already contained not only important information on the dependence of the turbulent diffusion coefficient on the amplitude of turbulent pulsations V_0, $D_{\mathrm{T}} \propto V_0^2\tau_{\mathrm{L}}$, where τ_{L} is the Lagrangian correlation time, but also the possibility of exploring nondiffusive regimes (anomalous diffusion) owing to the use of model power-law correlation functions.

Another important result was the discovery of the anomalous dispersion of two particles in a turbulent flow (relative diffusion) by Richardson in 1926 [30]. The scaling proposed by Richardson, which describes an anomalously fast growth of the mean separation between two particles with time,

$$l_{\mathrm{R}}^2(t) \propto t^3\,, \tag{3}$$

remains a subject of scientific discussions even now [1, 3]. Considering diffusion from a resting source, it is easy to see that the dispersion cannot grow faster than the ballistic motion, $R^2(t) \propto t^2$. Obukhov succeeded in proposing the first theoretical explanation of this anomalous regime based on the idea developed by him and Kolmogorov that the spectral energy flux ε_{K} is preserved in the developed turbulent flow [16, 17]. In the framework of the phenomenological one-parametric approach, ε_{K} [m^2 s^{-3}] is the proportionality factor connecting the relative distance and time:

$$l_{\mathrm{R}}^2(t) \propto C_{\mathrm{R}}\varepsilon_{\mathrm{K}}t^3\,. \tag{4}$$

Here, $C_{\mathrm{R}} \approx 0.2$ is the empirical Richarson constant. Subsequent studies by Batchelor have shown that the relative diffusion evolves through four different stages [1, 31]. At the first stage, the main role is played by the stochastic instability leading to the exponential dependence $l_{\mathrm{R}}^2(t) \propto \exp t$. It is then taken over by a transient ballistic regime, $l_{\mathrm{R}}^2(t) \propto \varepsilon_{\mathrm{K}}^{2/3}t^2$. The next evolution stage is related to cascade processes inside the inertial range and obeys the Richarson scaling. As the distance between particles reaches the characteristic size of energy-containing eddies by order of magnitude, this scaling is replaced by a quasidiffusive regime, $l_{\mathrm{R}}^2(t) \propto 2D_{\mathrm{T}}t$. The reader can find a detailed discussion of questions pertaining to relative diffusion in the vast literature [1, 3, 9, 13].

We emphasize that it is important to clearly distinguish between the diffusion of particles from a fixed source (for example, the Taylor model) and the relative diffusion of two marked particles in a cloud moving in space (the Richarson scaling). Here, we consider only the semiclassical diffusion from a fixed (resting) source. However, for review purposes, it is important to stress that the concept of scaling has been widely used beginning from the first papers on the turbulent diffusion, the diversity of transport regimes in which is an unavoidable consequence of the hierarchical structure of developed (strong) turbulence. Moreover, Richardson already clearly understood the nontrivial influence of turbulent mixing on the intensification of transport processes.

3. Dissipation and advection

The way the turbulent diffusion coefficient depends on the amplitude of turbulent pulsations is an important characteristic to explore, because in the regime with strong structured turbulence this dependence is much smoother than the Taylor prediction, $D_T \propto V_0^2$. Here, we consider the method proposed by Zel'dovich [32] to analyze the equation of passive scalar transport, which is valid for resting and moving media as well. The admixture particles in a flow exerting no effect on the flow dynamics are called a passive scalar. For example, in the case where the temperature of every fluid particle is preserved (the temperature is 'frozen' in the medium), even temperature can be treated as a passive scalar. A similar situation occurs in models dealing with magnetic field transport [8, 11], where the magnetic field, being 'frozen in', is carried by plasma, forming intricate configurations. In the simplest case where the fluid flow is incompressible, $\operatorname{div} \mathbf{V} = 0$, we can rewrite the classical diffusion equation by replacing the partial derivative $\partial n/\partial t$ of a scalar density n with the full (Lagrangian) derivative. The scalar transport equation then becomes

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \mathbf{V}\nabla n = D_0 \nabla^2 n \,. \qquad (5)$$

We note that in the absence of diffusion (the diffusion coefficient $D_0 = 0$), we recover the freezing-in condition $n(\mathbf{r}, t) = n(\mathbf{r}_0, t)$, where $\mathbf{r}_0$ is the initial particle coordinate, $\mathbf{r}_0 = \mathbf{r}(t = 0)$. From a formal standpoint, the Lagrangian behavior of a scalar can result in arbitrarily large density gradients, but in the presence of diffusion (or heat conduction if the problem is reformulated in terms of temperature), the inhomogeneities in density must smooth out. To qualitatively assess the competition between these two factors, we consider a bounded domain with insulated walls in which the mean particle density is preserved,

$$\langle n \rangle = \frac{1}{W} \int n(\mathbf{r})\, d^3\mathbf{r} \,, \qquad (6)$$

where W is the domain volume. Additionally, we let the density perturbations satisfy $n_1 = n - \langle n \rangle$ and $\langle n \rangle = 0$ and

$$\int n_1 \, d^3\mathbf{r} = 0 \,. \qquad (7)$$

The smoothing of inhomogeneities in a scalar field (density, temperature, etc.) with time is a fundamental effect that can be explored with the help of the variational approach in a rather general form, unrelated to the specific form of the velocity field. As a quantity to be varied, we conveniently take the integral

$$I_Z(t) = \int n^2(\mathbf{r}, t)\, d^3\mathbf{r} \,. \qquad (8)$$

Indeed, we consider the identity

$$I_Z = \int n^2 \, d^3\mathbf{r} = \int \left(n - \langle n \rangle\right)^2 d^3\mathbf{r} + \langle n \rangle^2 W \,. \qquad (9)$$

The value of I_Z reaches a minimum when the scalar density n takes the value coinciding with the mean: $n(\mathbf{r}, t) = \langle n \rangle$. The character of the evolution of I_Z can be addressed by multiplying both sides of the transport equation by n. Then, by virtue of the Ostrogradski–Gauss theorem, we arrive at the equation

$$\frac{1}{2}\frac{\partial}{\partial t} \int n^2 \, d^3\mathbf{r} = \int n D_0 (\nabla n)_N \, dS - \int D_0 (\nabla n)^2 \, d^3\mathbf{r} \,. \qquad (10)$$

If boundary fluxes are equal to zero (the density gradient vanishes at the boundary, $(\nabla n)_N = 0$) and if the medium is infinite, we obtain Zel'dovich's dissipation theorem:

$$\frac{\partial I_Z}{\partial t} = \frac{\partial}{\partial t} \int n^2 \, d^3\mathbf{r} = -2D_0 \int (\nabla n)^2 \, d^3\mathbf{r} < 0 \,. \qquad (11)$$

An important fact is the lack of a direct effect from fluid motion. Indeed, the fluid velocity does not enter the last expression. However, the fluid motion contributes indirectly to the evolution of I_Z by dictating the spatial distribution of the scalar density.

In the framework of quasistationary turbulence, it is natural to neglect the term with the time derivative in the left-hand side of Eqn (10). The expression $D_0(\nabla n)_N$ characterizes the contribution from external sources, whereas the term $D_0(\nabla n)^2$ is associated with the scalar redistribution within the volume δW under consideration. Here, it is convenient to introduce the effective diffusion coefficient in the form

$$D_{\text{eff}} = \frac{1}{n^2 L_0} \int_W D_0 (\nabla n)^2 \, dW \,, \qquad (12)$$

where L_0 is the characteristic size of the system. The condition that D_{eff} be minimum (the minimum of the functional) then reduces to the purely diffusive equation $\nabla(D_0 \nabla n) = 0$, where $\min D_{\text{eff}} = D_0$. The Zel'dovich theorem demonstrates that the effective flux can increase owing to the convective mixing in the case where the flow field is incompressible and a seed (molecular) diffusion exists.

On the other hand, the upper bound on the effective diffusion coefficient in a quasistationary turbulent flow can easily be obtained by considering, for simplicity, the one-dimensional transport equation

$$D_0 \Delta n - \mathbf{V}\nabla n = 0 \qquad (13)$$

with the help of the perturbation method,

$$n = \langle n \rangle + n_1 = n_0 + n_1 \,, \quad V = \langle V \rangle + v_1 = v_1 \,, \qquad (14)$$

where $\langle V \rangle = 0$, $n_1 \ll n_0$ and $D_0 \Delta n_0 = 0$. Computations result in

$$D_0 \frac{\partial^2 n_1}{\partial x^2} = v_1 \frac{\partial n_0}{\partial x} \,. \qquad (15)$$

Limiting our analysis to a dimensional estimate only, we find $n_1 \approx v_1 L_0 n_0 / D_0$. The last result can be conveniently rewritten in terms of the Peclet number $\text{Pe} = \lambda V / D_0$, which is responsible for the comparison of convective effects with the action of molecular diffusion. In this case, $n_1 \approx n_0 \text{Pe}$, the condition $\text{Pe} \ll 1$ characterizes weakly turbulent flows. We note that we relied on the smallness of the term $v_1 \nabla n_1$

compared to $v_1 \nabla n_0$ in deriving the estimate. Resorting to the definition of the effective diffusion coefficient in terms of $(\nabla n)^2$, we find

$$D_{\text{eff}} \approx \frac{1}{n_0^2 L_0} \int_W D_0 (\nabla n_0)^2 (1 + A\,\text{Pe}^2)\,\mathrm{d}W \approx D_0 (1 + A\,\text{Pe}^2)\,, \tag{16}$$

where A is a dimensionless constant. The term $\nabla n_0 \nabla n_1$ is eliminated by virtue of conditions imposed on n_1 and n_0. Simple estimates indicate that the diffusion caused by turbulence frequently far exceeds the molecular one. For example, in the surface layer in the atmosphere, $V_0 \approx 10^4$ cm s^{-1}, $l \approx 10^{-5}$ cm, and $D_0 \approx 0.1$ cm^2 s^{-1} for molecular motions. Turbulent motions involve quantities of a substantially higher order of magnitude, $V_0 \approx 10$ cm s^{-1}, $l \approx (10^{-2} - 10^{-3})$ cm, and $D_0 \approx (10^3 - 10^4)$ cm^2 s^{-1}. For this reason, the particular form of the correction describing the contribution from turbulent motions is extremely important.

Applying Zel'dovich's variational ideas to the exploration of turbulent mixing of a scalar, we can derive the fluctuation–dissipation relation (FDR) that allows estimating the scale of scalar density fluctuations $\langle (\nabla n)^2 \rangle$ in developed turbulent flows [8]. For quasistationary turbulence, the Zel'dovich relation can be written in the form

$$0 = Q(n_1 - n_2) - D \int_W (\nabla n)^2\,\mathrm{d}W\,, \tag{17}$$

where the contribution from the flux through the boundary Q is estimated in terms of the mixing length L_0 and the velocity fluctuation scale V_0:

$$Q \approx S \langle V_0 L_0 \rangle \nabla n \approx S D_{\text{T}} \left(\frac{\Delta n}{L_0} \right)_{\text{macro}}. \tag{18}$$

With Eqn (18) substituted in relation (17), straightforward manipulations lead to the Zel'dovich (FDR) scaling

$$\langle (\nabla n)^2 \rangle = \left(\frac{V_0 L_0}{D_0} \right) \frac{(\Delta n)^2}{L_0^2}\,, \quad \text{Pe} = \frac{V_0 L_0}{D_0} \gg 1\,, \tag{19}$$

or, in the terms of the Peclet number,

$$\nabla n \big|_{\text{local}} \approx \left(\frac{D_{\text{T}}}{D_0} \right)^{1/2} \frac{(\Delta n)^2}{L_0^2} \approx \text{Pe}^{1/2}\, \nabla n \big|_{\text{macro}}\,. \tag{20}$$

The last relation implies that for $\text{Pe} \gg 1$, the character of turbulent motion ensures that two fluid elements with strongly dissimilar scalar densities (or temperatures) turn out to be close to each other. Experiments and numerical simulations confirm this conclusion. The patterns of scalar density distributions evolving through turbulent mixing are often no less beautiful than the canvases of abstractionists.

We can also readily find the scaling for density perturbations $\delta n|_{\text{turb}}$, similar to that derived when estimating D_{eff} for $\text{Pe} \ll 1$. We introduce the local Peclet number $\text{Pe} = \lambda V_0 / D_0$ based on the local scale λ. It can then be assumed that

$$\delta n \big|_{\text{turb}} \propto \nabla n \big|_{\text{local}} \lambda \approx \text{Pe}^{1/2}\, \nabla n \big|_{\text{macro}} \lambda\,. \tag{21}$$

Considering the Peclet and Reynolds number to be directly proportional to each other, we estimate the scale λ as

$$\lambda \approx l_v \propto \frac{L_0}{\text{Re}_\lambda^{3/4}} \propto \frac{L_0}{\text{Pe}_\lambda^{3/4}}\,, \quad \text{where} \quad \text{Pe} \propto \text{Re} \gg 1\,. \tag{22}$$

In this case, the expression for the amplitude of scalar density perturbations on scales l_v takes the form of the scaling relation

$$\delta n \big|_{\text{turb}} \approx \text{Pe}^{-1/4}\, \delta n \big|_{\text{macro}} \approx \text{Re}^{-1/4}\, \delta n \big|_{\text{macro}}\,. \tag{23}$$

Accordingly, the scaling for n_λ in the case $\text{Pe} \gg 1$ is essentially different from its counterpart for $\text{Pe} \ll 1$, $\delta n(\text{Pe}) \propto n_0\,\text{Pe}$.

We use this result of Zel'dovich for a preliminary estimate of transport in a developed turbulent flow. Under the conditions of strong turbulence, a substantial contribution to the turbulent transport comes from convective motions, which allows estimating the particle flux as

$$q \propto \delta n V_0\,, \quad \text{Pe} \propto \text{Re} \gg 1\,. \tag{24}$$

Using the above expression for density perturbations, we derive the scaling relation for the turbulent diffusion coefficient

$$D_{\text{eff}}\,(\text{Pe}) \propto \frac{q\,(\text{Pe})}{n_0}\, L_0 \propto D_0\, \text{Pe}^{3/4}\,. \tag{25}$$

Expression (25) is notably different from the Zel'dovich scaling $D_{\text{eff}}(\text{Pe}) \propto D_0\,\text{Pe}^2$ and, as we see in what follows, just the 'flat' scaling emerges in a rigorous theoretical analysis of turbulent transport in regimes of strong (structured) turbulence.

4. Fractal behavior and percolation stream lines

By turning the consideration from general three-dimensional flows to incompressible two-dimensional flows, we not only benefit from the advantages of the Hamiltonian representation of the equations of motion for a fluid element, but also are in a position to propose a more rigorous classification of the transport regimes. With the incompressibility condition $\operatorname{div} \mathbf{V} = 0$, the equations of motion can be written as

$$\dot{x} = V_x = \frac{\partial \Psi(x,y)}{\partial y}\,, \tag{26}$$

$$\dot{y} = V_y = -\frac{\partial \Psi(x,y)}{\partial x}\,, \tag{27}$$

where $\Psi(x,y,t)$ is the stream function, which is a random field in models of two-dimensional turbulent flows [13, 33]. Scalar particles move along the stream lines, but because of molecular diffusion or reconnection of stream lines, they can leave the initially selected ones. The analysis of different decorrelation mechanisms responsible for the existence of different regimes of turbulent diffusion is one of the goals pursued by this review.

It is well known that processes of self-organization that lead to the formation of large-scale eddy structures play an important role in two-dimensional turbulence [1, 3]. In the framework of fluid dynamics, this process is frequently associated with the inverse spectral energy cascade [16, 17], and in the physics of strongly magnetized plasma, with the onset of drift-convective instabilities [21, 22, 25]. Using the stream function formalism for random two-dimensional flows allows considering eddies of various scales in the framework of statistical topography. Moreover, the coherent structures evolving as a result of self-organization in

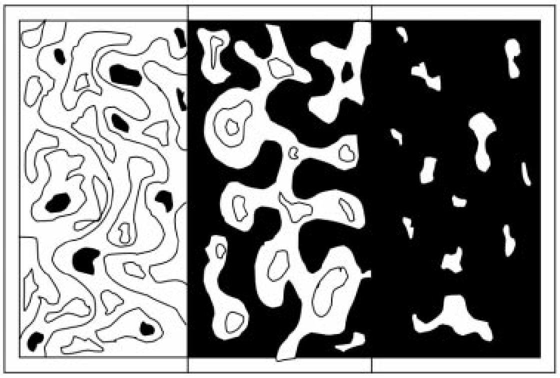

Figure 1. Percolation transition in the two-dimensional case. Dark and light areas respectively correspond to 'water' and 'land'.

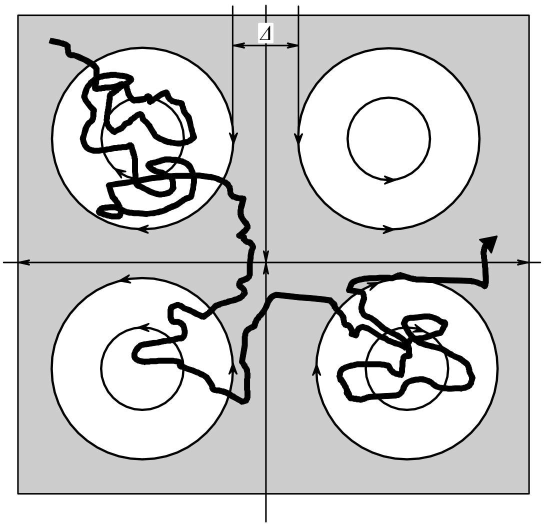

Figure 2. Convective cells. Δ is the thickness of the stochastic (diffusive) layer. The bold line shows an example of a particle trajectory in the system of convective cells.

turbulent flows may noticeably differ from regular or small-scale eddy features and occupy a substantial part of the flows, thus creating the conditions for the emergence of anomalous transport regimes.

In this review, we focus on the percolation model of self-organization of two-dimensional random flows [22, 25]. In the percolation approach, the stream lines $\Psi = \Psi(x, y)$ are regarded as coastlines appearing as a result of the inundation of a hilly landscape (Fig. 1). It is expected that there is a sharp transition from the region with bounded lakes on the otherwise endless land to the region with islands in an endless ocean. The percolation theory assumes the existence of at least a single coastline of infinite length. The corresponding stream functions can be modeled by 'perturbing' the landscape described by the system of convective cells (Fig. 2)

$$\Psi(x, y) = \Psi_0 \sin(k_x x) \sin(k_y y) \,. \tag{28}$$

A small initial perturbation $\delta\Psi$ displaces 'saddle points' from the zero level $\Psi = 0$. In fact, we are dealing here with random splitting of saddles and the build-up of long winding stream lines.

Kadomtsev and Pogutse [22] linked the anomalous character of diffusion in regimes with strong turbulence to the fractal character of stream lines of a two-dimensional flow in the vicinity of the flooding level (Fig. 3). It was proposed to use the following scaling, found in numerical experiments on statistical topography, for the length of the percolation stream line [22, 34]:

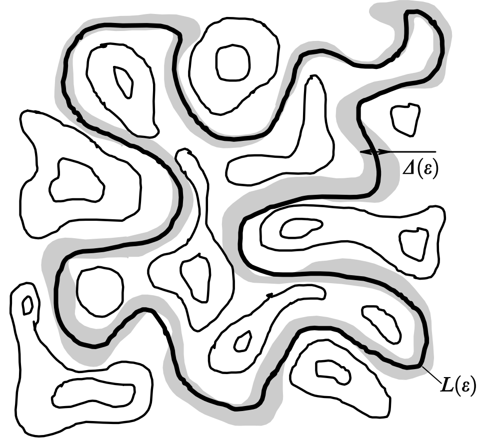

Figure 3. Percolation stream line (bold curve) in a two-dimensional random flow. The stochastic layer is shaded. $\Delta(\varepsilon)$ is the thickness of the stochastic (diffusive) layer and $L(\varepsilon)$ is the length of the percolation stream line.

$$L(\varepsilon) \propto \frac{1}{\varepsilon^{2.4}} \,. \tag{29}$$

Here, ε is a small dimensionless quantity characterizing the degree of the system deviation from the critical state (the percolation threshold), $\varepsilon \approx \delta\Psi/(\lambda V_0)$, where $\delta\Psi$ is the magnitude of the stream function in the vicinity of the percolation transition, λ is the characteristic size, and V_0 is the characteristic flow velocity. Expression (29) for $L(\varepsilon)$ corresponds to a fractal representation of the curve length [35]. Percolation (fractal) stream lines in fact embrace the entire flow in the vicinity of the percolation threshold, furnishing conditions needed for the onset of anomalous transport.

But the fractal representation of stream lines alone is insufficient for the description of the effects related to the substantial increase in transport coefficients. Moreover, the fractal character of lines sometimes leads to a slower diffusion (subdiffusion). In the percolation limit, the main characteristics of turbulent diffusion are the typical correlation scales in space and time. In Section 5, we present basic scaling relations that enable us to characterize the hierarchy of space scales in a simple percolation model.

5. Percolation hierarchy of spatial scales

The percolation approach looks very promising because it offers a simple, yet universal, model of behavior based on the effects of strong correlations, which, in the case of turbulent diffusion, are related to convective transport along the branching network of stochastic layers formed as a result of separatrix splitting. On the other hand, numerical simulations for two-dimensional percolation lattices can be used to verify the theoretical analysis.

To illustrate the main notions of percolation theory, we consider a simple model on a square lattice. Let the cells of this lattice be filled with the probability p or be empty with the probability $1 - p$. The filled neighboring cells sharing a common edge form a cluster. If $p \ll 1$, the clusters are small and isolated. As p increases from 0 to 1, the number of the largest clusters also increases. There is a critical value $0 < p_c < 1$ at which a single cluster is formed that connects the opposite lattice sides. If the lattice size tends to infinity, $L_0 \to \infty$, the size of this percolation cluster also becomes infinite. The value p_c at which an infinite cluster forms for the first time is called the percolation threshold or the critical probability. Numerical simulations for finite lattices show that $p_c \approx 0.59275$ for clusters formed by neighboring vertices. Such simulations also prove that clusters represent fractal distributions of filled cells.

As p is driven closer to p_c, the finite-size clusters grow in size; their correlation length a, being the size of clusters contributing most to this growth, tends to infinity at p_c. As $a(p)$ tends to infinity, the characteristic length that would enter scaling dependences of the system physical properties ceases to exist. Similarly to fractal structures, the system looks the same at different magnifications. The system properties become insensitive to many local details, including even the lattice structure, which ensures the universality of the critical indices describing the divergence of parameters as p approaches p_c. These universal indices depend on the model considered and the system dimensionality, but not on the local structure. Importantly, in the vicinity of $p = p_c$, the geometric percolation can be expressed in the same language as a second-order phase transition, e.g., from a paramagnetic high-temperature state to a ferromagnetic low-temperature one. For example, the probability p of a vertex being occupied can serve as an analog of the temperature T.

The correlation length a also tends to infinity as p approaches p_c (from below and from above) with some new critical index ν:

$$a \propto |p - p_c|^{-\nu} = |\varepsilon|^{-\nu}, \qquad (30)$$

where ε is a small percolation parameter characterizing the proximity of the system state to the percolation threshold. This behavior is reminiscent of the divergence of the correlation length in the vicinity of critical points of temperature phase transitions. The percolation critical indices are universal because they depend only on the system dimensionality d, i.e., $\nu = \nu(d)$. The goal of percolation theory is to compute critical indices and establish relations between them. The reader can learn about methods of computing the indices from numerous sources (see, e.g., Refs [9, 25, 27, 35]). For two-dimensional lattices, the index $\nu = 4/3$ can be found exactly. The situation is more intricate in three dimensions, and percolation indices have been obtained thus far only with the help of numerical simulations [25].

In the theory of two-dimensional continual percolation, it is rigorously proven that the correlation scale follows the same scaling as in the lattice model [9, 25, 27, 35]. This allows considering the correlation scale $a(\varepsilon)$ as the transverse size of the percolation stream line. In the theory of continual percolation, the correlation length $a(\varepsilon)$ is also the parameter characterizing the length of percolation stream lines in the vicinity of percolation transition, $\varepsilon \to 0$:

$$L(\varepsilon) \propto \frac{a(\varepsilon)}{\varepsilon} \propto a(\varepsilon)^{D_H}, \quad D_H = 1 + \frac{1}{\nu}, \quad \nu = \frac{4}{3}. \qquad (31)$$

Here, D_H is the Hurst exponent [9, 25, 27, 35]. Thus, the percolation approach allows realizing the idea of 'long-range correlations'.

However, a serious problem has emerged here, because the diffusion coefficient is directly connected with the expression for the correlation length Δ_{cor}: $D \approx \Delta_{cor}^2/\tau$, where τ is the correlation time. In the case that we consider, it is plausible to assume that particles of the scalar move along stream lines; hence,

$$\Delta_{cor} \approx a(\varepsilon)_{\varepsilon\to 0} \to \infty \qquad (32)$$

in the percolation limit. This difficulty can be removed by renormalizing the small percolation parameter ε. This review is precisely devoted to the methods of renormalization of the parameter ε in problems of turbulent transport.

Chaotic behavior of stream lines gives rise to a complex topological pattern. In this nontrivial situation, the ideas of fractality and percolation are fruitful, because the notion of the correlation scale and characteristic spatial scale of the explored structures can be linked to the fractal topological characteristics of the flows. On the other hand, in contrast to conventional diffusion models, the theory of continual percolation includes not only the hierarchy of spatial scales but also the hierarchy of time scales, which allows considering time-dependent models for which processes of flow topology reorganization and effects driven by stochastic instability gain in importance.

6. Renormalization and percolation

The classical percolation representation $\Delta_{cor} \approx a(\varepsilon)_{\varepsilon\to 0} \to \infty$ is too abstract for actual tasks of practical significance. We consider a simple approach that allows recovering the scaling behavior for the characteristic correlation scale at finite values of the percolation parameter. For systems of a finite size L_0, the condition $a(\varepsilon) \leqslant L_0$ must be satisfied. It is therefore reasonable to introduce a new, 'renormalized' small parameter ε_* [36] such that

$$a(\varepsilon_*) \approx \lambda \frac{1}{|\varepsilon_*|^{\nu}} \leqslant L_0 . \qquad (33)$$

Elementary manipulations give

$$\varepsilon_* \approx \left(\frac{\lambda}{L_0}\right)^{1/\nu}. \qquad (34)$$

The result in (34) can be interpreted from the standpoint of 'percolation experiments with samples' of finite size. In such 'samples', the percolation transition occurs for ε_* lying in some range $\Delta\varepsilon$ in close proximity to zero. The value for ε_* above can be treated as a characteristic estimate for this range (Fig. 4), $\Delta\varepsilon \approx \varepsilon_*$. We note that we are in fact dealing with a physically motivated small parameter

$$\varepsilon_0 \approx \frac{\lambda}{L_0} \ll 1, \qquad (35)$$

related to a system with characteristic scales L_0 and λ. As a result of 'renormalization', we obtain the new (small, but finite) percolation parameter

$$\varepsilon_*(\varepsilon_0) \approx \Delta\varepsilon \approx \varepsilon_0^{1/\nu}. \qquad (36)$$

As could be anticipated, ε_* decreases with the increase in the size of system L_0. In fact, we 'hide' the singularity in the model

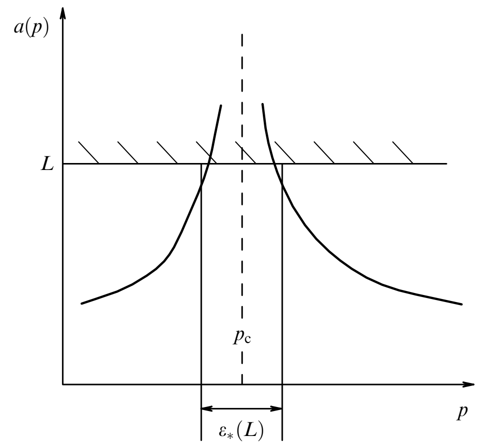

Figure 4. Renormalization of the small percolation parameter.

phenomenological parameter characterizing the finiteness of the sample size.

In the problem of constructing the percolation flow in a two-dimensional random flow, an analog of the quantity $\Delta\varepsilon \approx \varepsilon_*$ is furnished by the width of the stochastic layer. In the model of convective cells, the width of the stochastic (diffusive) layer was explored in the limit of large Peclet numbers $\mathrm{Pe} = \lambda V_0/D_0 \gg 1$ in Ref. [37]. We suppose that scalar particles move along the stream lines, but can leave them because of the action of seed (molecular) diffusion D_0. The particles leave the stochastic layer diffusively:

$$\frac{\partial n}{\partial t} \propto D_0 \, \frac{n}{\Delta^2} \, . \tag{37}$$

Convection along the boundary layer contributes as

$$\frac{\partial n}{\partial t} \propto V_0 \, \frac{n}{\lambda} \, . \tag{38}$$

Comparing estimates (37) and (38), we find the stochastic layer width

$$\Delta(V_0) = \sqrt{\frac{D_0 \lambda}{V_0}} \propto \frac{1}{\sqrt{V_0}} \, . \tag{39}$$

This is an important result: under conditions of strong turbulence ($\mathrm{Re} \gg 1$), the stochastic layer turns out to be very narrow.

We consider the effective transport of scalar particles in the system of convective cells based on the reduced convective estimate in the limit $\mathrm{Pe} = \lambda V_0/D_0 \gg 1$:

$$D_{\mathrm{eff}} \approx \lambda V_0 P_\infty \approx \lambda V_0 \, \frac{\Delta}{\lambda} = V_0 \Delta(V_0) \, , \quad P_\infty \approx \frac{\lambda \Delta}{\lambda^2} = \frac{\Delta}{\lambda} \, . \tag{40}$$

Here, P_∞ is the fraction of space related to convection. We finally arrive at the following estimate for the turbulent diffusion coefficient:

$$D_{\mathrm{eff}} = \mathrm{const} \sqrt{D_0 V_0 \lambda} \approx D_0 \, \mathrm{Pe}^{1/2} \propto V_0^{1/2} \, . \tag{41}$$

The scaling derived in Ref. [37] provided a theoretical interpretation of the results of numerical experiments on transport in the system of drift convective cells [38], which have been met with considerable excitement. The result in (41) in terms of the Peclet number ($\mathrm{Pe} \gg 1$) differs notably from the quasilinear estimate $D_{\mathrm{eff}} \propto \mathrm{Pe}^2 \propto V_0^2$.

In percolation models of turbulent diffusion, the key components are the selection of the small parameter ε_0 for the problem under study and finding an appropriate 'renormalization' condition ε_* that would enable obtaining physically relevant results in terms of the parameters of velocity, characteristic scale, seed diffusion, etc. The system of convective cells is not a percolation one, but it is not difficult to realize that a small perturbation of its stream function would spawn a percolation structure without significantly modifying the character of scalar particle motion. We can expect the Peclet number to become the small parameter in steady percolation flows with seed diffusion, and the renormalization condition to include the diffusive (stochastic) layer width.

7. Percolation transport in a steady flow

We consider a two-dimensional steady flow with zero mean velocity, which is given by a 'generic' bounded stream function $\Psi(x, y)$. It is understood to be on average an oscillating and isotropic function, behaving quasirandomly with respect to the distribution of saddle points over their heights. Symmetry considerations lead to the conclusion that for any function $\Psi(x, y)$ belonging to the generic type, only one closed zero stream line of infinite length exists. For instance, a monoscale random flow

$$\Psi_0 \approx \lambda V_0 \, , \quad \lambda \approx \left| \frac{\Psi}{\nabla \Psi} \right| \tag{42}$$

was explored in [33]. In general, a monoscale flow is formed by a superposition of a large number of harmonics with the same wavelength λ but with different amplitudes, phases, and the direction of wave vector $\mathbf{k}$:

$$\Psi(x, y) = \sum_j^N \psi_j \cos(\mathbf{k}_j \mathbf{r} + \varphi_j) \, , \quad N \gg 1 \, . \tag{43}$$

Computation of the turbulent diffusion coefficient relies on the idea of balance between the ballistic motion of scalar particles along the percolation stream lines and diffusive drift out of the stochastic layer. The influence of 'long correlations' is taken into account in the expression for the diffusion coefficient through the correlation scale $a(\varepsilon)$. The formal expression for the diffusivity in the percolation limit is written as

$$D_{\mathrm{eff}} = \int_0^\infty \frac{\mathrm{d}\Psi_1}{\Psi_1} \, P_\infty(\Psi_1) \, \frac{a^2(\Psi_1)}{\tau(\Psi_1)} \, , \tag{44}$$

where the perturbation of the Hamiltonian is given by $\Psi_1 \approx \varepsilon_* \lambda V_0$ in the mean field theory framework. Computations lead to the scaling

$$D_{\mathrm{eff}}(\varepsilon) \approx \frac{a^2}{\tau} \, P_\infty \approx \frac{a^2}{\tau} \, \frac{L(\varepsilon) \, \Delta(\varepsilon)}{a^2} \approx V_0 \Delta(\varepsilon) \, , \tag{45}$$

where the correlation time τ is estimated ballistically, $\tau \approx \tau_{\mathrm{b}} \approx L(\varepsilon)/V_0$, $P_\infty = L(\varepsilon) \, \Delta(\varepsilon)/a^2(\varepsilon)$ is the fraction of the volume occupied by percolation stream lines, and Δ is the width of the percolation layer. In fact, the problem of computing the turbulent diffusivity has been reduced in the chosen approximation to finding the width of the stochastic (percolation) layer.

The estimate $D_{\rm eff}(\varepsilon) \approx V_0 \Delta(\varepsilon)$ is equivalent to the expression used in models of convective cells. It is therefore natural to use the balance of characteristic times (diffusive τ_D and ballistic $\tau_{\rm b}$) as the starting point in estimating the percolation layer width Δ:

$$\tau_{\rm b} \approx \frac{L(\varepsilon)}{V_0} = \frac{\Delta^2(\varepsilon)}{D_0} \approx \tau_D\,, \quad \Delta \approx \sqrt{\frac{D_0 L(\varepsilon_*)}{V_0}}\,. \tag{46}$$

Indeed, the time a particle travels along the percolation stream line $\tau_{\rm b} \approx L/V_0$ must have the same order of magnitude as the diffusive time $\tau_D \approx \Delta^2/D_0$ it takes a particle to leave the percolation stochastic layer of width Δ.

Following the spirit of work dealing with the phase transition theory, the authors of Ref. [33] proposed 'renormalization', i.e., a way to compute the universal small parameter ε (for a given class of flows), by identifying the small 'width' of the percolation stream line with the small parameter of the percolation theory,

$$\Delta(\varepsilon) = \lambda\varepsilon\,. \tag{47}$$

This renormalization is actually the main result in Ref. [33]; it initiated the active use of similar methods in other problems of turbulent transport. To find the expression for the small percolation parameter ε_* in the model of a steady random flow with the seed diffusion D_0, we write the algebraic equation

$$\sqrt{\frac{D_0 L(\varepsilon_*)}{V_0}} = \lambda\varepsilon_*\,. \tag{48}$$

The computations can be carried out to the end if we use rigorous scaling results of percolation theory [39] for the correlation scale a and the length of fractal stream line L as a function of ε:

$$a(\varepsilon) = \lambda\varepsilon^{-\nu}\,, \quad L(\varepsilon) = \lambda\left(\frac{a}{\lambda}\right)^{D_{\rm H}}, \quad D_{\rm H} = 1 + \frac{1}{\nu}\,, \quad \nu = \frac{4}{3}\,. \tag{49}$$

The renormalized percolation parameter and the effective diffusivity coefficient are expressed via the Peclet number $\mathrm{Pe} = \lambda V_0/D_0 \gg 1$ (Fig. 5) as

$$D_{\rm eff} \approx V_0 \Delta(\varepsilon_*) \approx V_0 \lambda \left(\frac{1}{\mathrm{Pe}}\right)^{1/(3+\nu)} = D_0\,\mathrm{Pe}^{10/13}\,,$$
$$\varepsilon_* \approx \left(\frac{1}{\mathrm{Pe}}\right)^{1/(3+\nu)}\,. \tag{50}$$

Result (50) was subjected to numerous tests in numerical experiments as well as a probabilistic theory analysis [40, 41]. There is every reasons to regard Eqns (48)–(50) as rigorously proved. For the renormalization of the initial small parameter $\varepsilon_0 \approx 1/\mathrm{Pe}$, we obtain the expression

$$\varepsilon_* = (\varepsilon_0)^{1/(3+\nu)} \gg \varepsilon_0\,. \tag{51}$$

Some 'arbitrariness' in choosing $\lambda\varepsilon$ (but not $\lambda\varepsilon^2$ or $\lambda\varepsilon^3$) can be interpreted as an intention to have a universal small parameter, in analogy with the only characteristic scale in the phase transition theory. The length of the percolation stream line and the correlation scale,

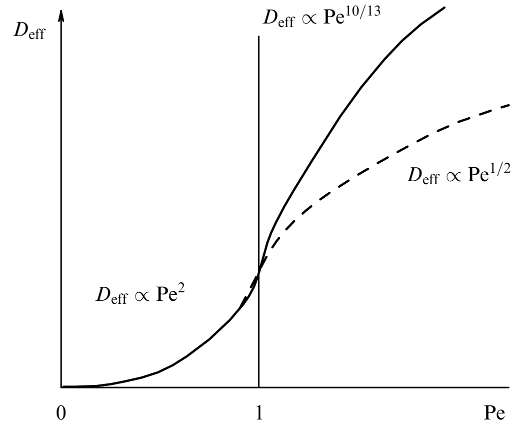

Figure 5. The turbulent diffusion coefficient in a steady random flow in the percolation limit.

$$L(\varepsilon_*) \approx \lambda\,\frac{1}{\varepsilon_*^{\nu+1}} \approx \lambda\,\mathrm{Pe}^{(1+\nu)/(3+\nu)} \propto V_0^{7/13}\,, \tag{52}$$

$$a(\varepsilon_*) = \lambda\,\frac{1}{\varepsilon_*^{\nu}} \approx \lambda\,\mathrm{Pe}^{\nu/(3+\nu)} \propto V_0^{4/13}\,, \tag{53}$$

are never infinite in this approach because the small parameter ε_* does not tend to zero, but attains a particular value ε_* for all types of flow with characteristic D_0, V_0, and λ. This is, in a nutshell, the universality of the scaling $D_{\rm eff} \propto D_0\,\mathrm{Pe}^{10/13}$. Because the condition $a(\varepsilon_*) \leqslant L_0$ must be ensured, the condition on Pe (the amplitude of turbulent pulsations) can be readily obtained as

$$1 \ll \mathrm{Pe} \leqslant \left(\frac{L_0}{\lambda}\right)^{(3+\nu)/\nu}\,, \quad \frac{D_0}{\lambda} \ll V_0 \leqslant \frac{D_0}{\lambda}\left(\frac{L_0}{\lambda}\right)^{13/4}\,. \tag{54}$$

To conclude this analysis, we note that we strived to propose a simple exposition, but the findings in Ref. [33] only seems to be simple! It suffices to recall the full 'hierarchy' of scales used in the analysis:

$$L(\varepsilon) \approx \frac{a(\varepsilon)}{\varepsilon} \gg a(\varepsilon) \gg \lambda \gg \Delta(\varepsilon) \approx \lambda\varepsilon\,. \tag{55}$$

It is just in the context of elaboration on the spatial and temporal scale hierarchy that the percolation theory of turbulent diffusion underwent further development [42–45].

8. Turbulent diffusion and flow topology reconstruction

The unsteadiness of flows is among the most important factors influencing transport processes. For instance, the change in the stream line topology is one of the major decorrelation mechanisms. We need one more dimensionless parameter to describe this situation, which, in contrast to the Reynolds and Peclet numbers, includes the characteristic time scale of the flow topology variation $T_0 \approx 1/\omega$, where ω is the characteristic frequency of external perturbations. Such a parameter is furnished by the Kubo number $\mathrm{Ku} = V_0/(\lambda\omega)$, which can be easily interpreted using ideas on the character of scalar particle motion along stream lines. In the case of high frequencies ω, the path covered by a test particle can be estimated ballistically, $l_\omega(\omega) \approx V_0/\omega$, and, consequently, $\mathrm{Ku} = l_\omega/\lambda$. On the other hand, in the high-frequency limit, l_ω is used as the correlation length,

$$D_{\mathrm{eff}}(V_0, \omega) \approx \frac{l_\omega^2}{\tau} \approx V_0^2 \omega \,, \tag{56}$$

where $\tau \propto 1/\omega$ is the correlation time. Substituting $1/\omega$ for the correlation time leads to the quasilinear scaling for the diffusion coefficient:

$$D_{\mathrm{eff}} = \lambda^2 \omega \, \mathrm{Ku}^2 \propto V_0^2 \,. \tag{57}$$

However, in the low-frequency limit $\omega \ll V_0/\lambda$, the actual correlation scale a is much smaller than l_ω, since it is affected by the modification of stream line topology (for example, the decrease in stream line lengths owing to their reconnections) (Fig. 6). Resorting to ideas on percolation hierarchy of spatial scales, it is possible to consider the percolation limit of turbulent diffusion for a scalar in a time-dependent incompressible plane flow for $\mathrm{Ku} \geqslant 1$. Having estimated the time of the complete flow pattern renewal as $T_0 \approx 1/\omega$, the authors of Ref. [42] proposed that the main parameter in the case of low-frequency perturbations is the lifetime of a single percolation stream line τ. It is natural to assume that just τ is the correlation time, which can be estimated as

$$\tau \approx \varepsilon \, \frac{1}{\omega} \approx \varepsilon T_0 \,. \tag{58}$$

Here, ε is the small percolation parameter associated with the problem. In the time-dependent case considered here, we expect a useful result if we manage to compute the concrete 'universal' value of ε_* by using the simple expression that accounts for the convective motion of scalar particles along the percolation stream line during the lifetime of this stream line,

$$\tau \approx \tau_{\mathrm{b}} \approx \varepsilon_* \, \frac{1}{\omega} \approx \varepsilon_* T_0 \,. \tag{59}$$

The equation for the small percolation parameter $\varepsilon_* = \varepsilon_*(\omega, V_0, \lambda)$ is obtained by the simple substitution

$$\frac{\varepsilon_*}{\omega} = \frac{L(\varepsilon_*)}{V_0} \,. \tag{60}$$

Using the percolation theory scaling $L(\varepsilon) = \lambda (a/\lambda)^{D_{\mathrm{H}}}$, we easily express ε_* as a function of the flow parameters ω, V_0, and λ:

$$\varepsilon_* = \left(\frac{\lambda \omega}{V_0} \right)^{1/(2+\nu)} = \left(\frac{1}{\mathrm{Ku}} \right)^{3/10} \propto \omega^{3/10} \,. \tag{61}$$

From the standpoint of renormalization of the primary small parameter $\varepsilon_0 \approx 1/\mathrm{Ku}$, we obtain the expression

$$1 > \varepsilon_* = (\varepsilon_0)^{1/(2+\nu)} > \varepsilon_0 \,. \tag{62}$$

The expression for the diffusion coefficient in the percolation limit is written as

$$D_{\mathrm{eff}} = \int_0^\infty \frac{\mathrm{d}\Psi_1}{\Psi_1} \, P_\infty(\Psi_1) \, \frac{a^2(\Psi_1)}{\tau(\Psi_1)} \,, \tag{63}$$

where $\Psi_1 \approx \varepsilon_* \lambda V_0$. Computations now lead to the final result for D_{eff} (Fig. 7):

$$D_{\mathrm{eff}}(\varepsilon_*) \approx \frac{a^2(\varepsilon_*)}{\tau(\varepsilon_*)} \, P_\infty(\varepsilon_*) \approx \lambda^2 \omega \, \mathrm{Ku}^{7/10} \propto V_0^{7/10} \omega^{3/10} \,. \tag{64}$$

This dependence differs principally from the quasilinear dependence

$$D_{\mathrm{eff}}(\omega) \propto \frac{V_0^2}{\omega} \,. \tag{65}$$

Indeed, there are no grounds to believe that the reduction in the perturbation frequency would result in infinitely growing transport. We see that the percolation approach gives a realistic dependence $D_{\mathrm{eff}} \propto \omega^{3/10}$. The values of ε_* and D_{eff} depend only on the flow parameters ω, V_0, and λ. Numerous experiments lend support to the percolation scaling [46, 47].

We note that the percolation scale $a(\varepsilon_*)$ in the model considered is indeed much less than l_ω:

$$a(\varepsilon_*) \approx L(\varepsilon_*) \, \varepsilon_* \approx V_0 \tau \varepsilon_* \approx \varepsilon_*^2 \, \frac{V_0}{\omega} \approx \varepsilon_*^2 l_\omega \ll l_\omega \,. \tag{66}$$

On the other hand, with account for the finite system size L_0, the following condition must hold:

$$a(\varepsilon_*) = \frac{\lambda}{\varepsilon_*^\nu} \approx \lambda \, \mathrm{Ku}^{\nu/(2+\nu)} \leqslant L \,, \tag{67}$$

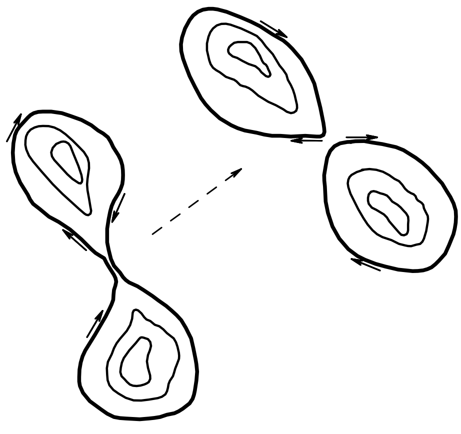

Figure 6. Reconnection of stream lines.

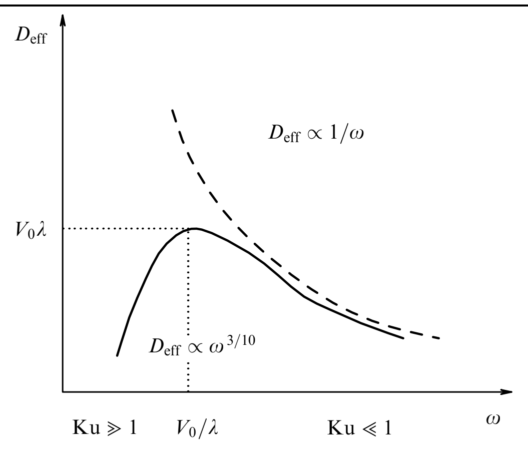

Figure 7. The dependence of the turbulent diffusion coefficient on the frequency in the percolation limit.

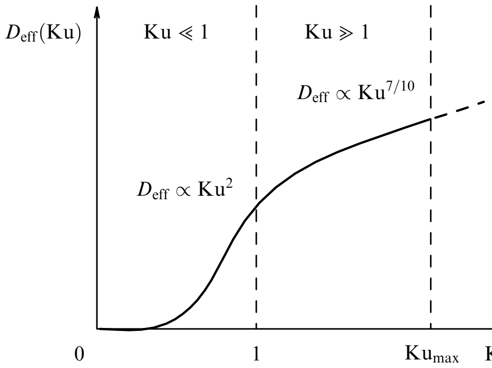

Figure 8. The dependence of the turbulent diffusion coefficient on the Kubo number in the percolation limit.

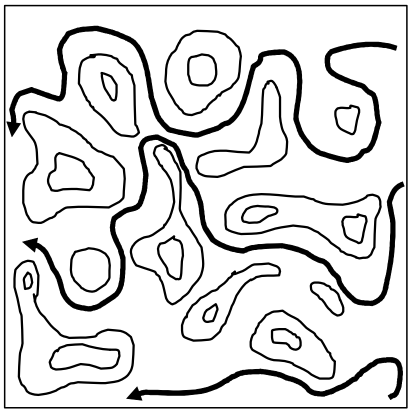

Figure 9. Open percolation stream lines.

which bounds the Kubo number as (Fig. 8)

$$l \ll {\rm Ku} \leqslant \left(\frac{L_0}{\lambda}\right)^{(2+\nu)/\nu} . \tag{68}$$

We need to perform additional estimates of the effect of diffusive particle drift off the stream lines in terms of the characteristic time τ_D. In our estimates, we use the coefficient of 'seed' (molecular) diffusion D_0:

$$\tau \approx \frac{\varepsilon_*}{\omega} < \tau_D , \quad \tau_D \approx \frac{\Delta^2(\varepsilon_*)}{D_0} \approx \frac{\lambda^2 \varepsilon_*^2}{D_0} . \tag{69}$$

In fact, this is the limitation on the magnitude of seed diffusion D_0 in the class of flows considered:

$$D_0 < \lambda^2 \omega \varepsilon_* ({\rm Ku}) \propto \frac{\omega^{13/10}}{V_0^{3/10}} . \tag{70}$$

On the other hand, we obtain a hierarchy of characteristic times related to the problem of percolation in a time-dependent flow:

$$\left(\tau \approx \frac{\varepsilon}{\omega}\right) \ll \left(\frac{\lambda^2 \varepsilon^2}{D_0} \approx \tau_D\right) \ll \left(\frac{1}{\omega} \approx T_0\right) . \tag{71}$$

We note that expression (64) for the coefficient of turbulent diffusion of a scalar in flows with changing topology (in the low-frequency limit) is only valid in the approximation $\tau < \tau_D$.

9. Influence of small drift velocity

The percolation models in Sections 7 and 8 are based on the assumption that a stochastic (percolation) layer exists and that the system is subject to 'seed' classical diffusion or low-frequency fluctuations of a characteristic frequency D_0. Other physical situations in which percolation effects essentially influence the character of transport can be addressed by analyzing the mechanisms responsible for the processes in the stochastic layer. In fact, we need to consider changes in the character of percolation transport under an external influence. An important example is the analysis of effects brought about by the presence of drift or zonal flows (Fig. 9). The idea of such an approach was first suggested by Zel'dovich [23], who predicted the appearance of "percolation along thin bundles" after imposing a weak uniform field on a random two-dimensional flow (magnetic field). Simultaneously, Trugman [48] proposed a percolation model to analyze effects of an external electric field.

The idea of 'gradient percolation' has a broad area of applicability [39]. For example, Isichenko and Kalda [43, 44] and Yushmanov [49] have addressed the influence of a small drift velocity $U_{\rm d}$ on the fractal topology of percolation stream lines,

$$V = V_0 + U_{\rm d} , \tag{72}$$

where the condition $U_{\rm d} \ll V_0$ holds. The simplest choice of the small parameter is here the quantity $\varepsilon_0 = U_{\rm d}/V_0$. Although the renormalization condition in the case of gradient percolation $\varepsilon_* = (\varepsilon_0)^{1/(1+\nu)}$ is obtained from general considerations [48], its interpretation in terms of drift flows proves useful.

We consider a parametric estimate of the drift velocity, natural in the percolation case. For the formulated question about the influence of drift velocity on the behavior of stream lines of a random two-dimensional flow, we obtain

$$V_0 \Delta(\varepsilon) = U_{\rm d}\, a(\varepsilon) , \tag{73}$$

where $a(\varepsilon)$ is the characteristic correlation scale. In fact, condition (73) implies that the mean scalar flux is carried at the speed V_0 in narrow channels of the width $\Delta(\varepsilon)$ oriented on average along the drift velocity direction. We assume that adding the drift velocity creates only a small number of open stream lines that form narrow convective channels. The amplitude of wandering and mean separation between the channels can be estimated with the help of a self-consistent correlation size $a(\varepsilon)$. The condition that the percolation layer is narrow, $\Delta(\varepsilon) \approx \lambda\varepsilon$, has been used many times. It can be readily verified that on substitution of the standard expression for the correlation size $a \approx \lambda |\varepsilon|^{-\nu}$, the parametric dependence for the small parameter ε_* in terms of the drift velocity $U_{\rm d}$ and the amplitude of turbulent pulsations V_0 becomes

$$\varepsilon_* = (\varepsilon_0)^{1/(1+\nu)} = \left(\frac{U_{\rm d}}{V_0}\right)^{1/(1+\nu)} = \left(\frac{U_{\rm d}}{V_0}\right)^{3/7} , \tag{74}$$

where $\nu = 4/3$. First, it is easy to see that expression (74) fully coincides with the result in [48]; second, this expression can be

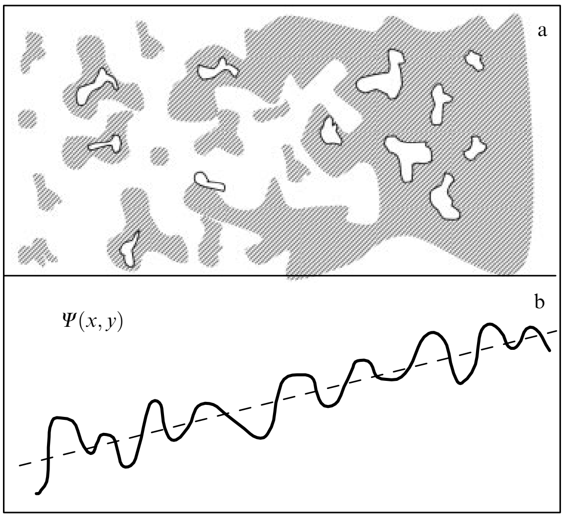

Figure 10. (a) The characteristic shape of isolines in the drift approximation. The hatched areas correspond to 'water'. (b) Percolation relief in the drift approximation.

interpreted in terms of the stream function (Fig. 10):

$$U_{\rm d} \approx \frac{\Psi_1}{a(\varepsilon)} \approx \frac{\varepsilon \Psi_0}{a(\varepsilon)} \approx V_0 \varepsilon^{\nu+1} . \tag{75}$$

Here, the quantity

$$\Psi_1 \equiv \Delta\Psi \approx \varepsilon \lambda V_0 \approx U_{\rm d}\, a(\varepsilon) \tag{76}$$

characterizes perturbations of the stream function in the vicinity of the percolation transition. Moreover, it can be seen that the spatial hierarchy of scales is included in the description of perturbations [50]:

$$U_{\rm d} \approx \varepsilon \frac{\lambda}{a(\varepsilon)} V_0 \approx \frac{\lambda}{L(\varepsilon)} V_0 \approx \frac{\Delta}{a(\varepsilon)} V_0 . \tag{77}$$

However, the parameter λ does not enter the expression for the renormalized quantity ε_*. The hierarchy of velocities used in this substitution is incomplete. Indeed, in the hierarchy of percolation scales, we had the 'lattice scale' λ located between $a(\varepsilon)$ and $\Delta(\varepsilon)$. It is therefore reasonable to introduce the velocity scale $w_{\rm d}$ that corresponds to λ:

$$V_0 \Delta(\varepsilon) \approx w_{\rm d}\, \lambda \approx U_{\rm d}\, a(\varepsilon) . \tag{78}$$

On the one hand, $w_{\rm d}$ can be expressed in terms of $P_\infty(\varepsilon)$,

$$w_{\rm d}(\varepsilon)\, P_\infty(\varepsilon) \approx U_{\rm d} , \tag{79}$$

where the expression $P_\infty(\varepsilon) \approx \lambda / a(\varepsilon)$ proposed in the steady case is used. On the other hand, elementary manipulations show that $w_{\rm d}$ is a convenient parameter for estimating the effects of scalar transport,

$$w_{\rm d} \approx V_0 \frac{\Delta(\varepsilon)}{\lambda} \approx V_0 \varepsilon \approx \frac{a(\varepsilon)}{\tau_{\rm b}} , \quad \tau_{\rm b}(\varepsilon) \approx \frac{L(\varepsilon)}{V_0} . \tag{80}$$

In the framework considered here, the transport of the scalar along percolation channels is convective, $\langle V \rangle \neq 0$:

$$q \approx V_0 \Delta(\varepsilon_*)\, \delta n \approx \lambda V_0\, \delta n(\varepsilon_0)^{1/(1+\nu)} \propto V_0 \left(\frac{U_{\rm d}}{V_0} \right)^{3/7} , \tag{81}$$

where δn is the perturbation of the scalar density.

We have diffusive transport in the transverse direction. For a weak mean flow, the transverse transport is defined by the competition between the seed (molecular) diffusion and 'ballistic' motion of scalar particles along the closed stream lines with a characteristic scale of the order of the correlation scale. This exactly matches the regime $D_{\rm eff} \approx D_0\, {\rm Pe}^{10/13}$ discussed in Section 7.

We note that for the appearance of open stream lines, the imposed drift flow should not be too weak. Indeed, the boundedness requirement for the correlation scale $a(\varepsilon) \leqslant L_0$ results in a constraint on the drift velocity:

$$U_{\rm d} \leqslant V_0 \left(\frac{\lambda}{L_0} \right)^{(1+\nu)/\nu} = V_0 \left(\frac{\lambda}{L_0} \right)^{7/4} . \tag{82}$$

The next logical step in studying drift flows is the inclusion of the effects of time dependence, which play an important role in the analysis of transport processes.

10. Quasilinear approximation and the effects of time dependence

The time dependence of drift flow allows us to inquire about the effective coefficient of turbulent diffusion related to the reorganization of flow topology. We begin learning about the effects induced by the flow unsteadiness from the quasilinear approach. This allows us to keep the expression for the small percolation parameter in the form corresponding to the gradient model, $\varepsilon_* = (U_{\rm d}/V_0)^{1/(1+\nu)}$. In this case, the expression for $D_{\rm eff}$ can be reduced, in a rather straightforward way, to the renormalized quasilinear form

$$D_{\rm eff}(\varepsilon) \approx P_\infty \frac{a^2}{\tau} \approx w_{\rm d}^2(\varepsilon)\, \tau(\varepsilon)\, P_\infty(\varepsilon) , \tag{83}$$

where we use the drift velocity estimate $U_{\rm d} \approx w_{\rm d} P_\infty$ and the estimate $P_\infty \approx \lambda / a(\varepsilon_*)$ for the fraction of space occupied by percolation stream lines. Inserting $\tau \approx 1/\omega$, we arrive at the Yushmanov scaling, which accounts for the effects of time dependence in the quasilinear approximation [49]:

$$D_{\rm eff} \approx \frac{U_{\rm d}^2}{\omega} \left(\frac{1}{\varepsilon_0} \right)^{\nu/(1+\nu)} \approx \frac{U_{\rm d}^2}{\omega} \left(\frac{V_0}{U_{\rm d}} \right)^{4/7} \propto U_{\rm d}^{10/7} V_0^{4/7} \frac{1}{\omega} . \tag{84}$$

In terms of the dependence of $D_{\rm eff}$ on the perturbation amplitude V_0, expression (84) corresponds to the transition from the quasilinear regime with $D_{\rm eff} \propto V_0^2$ to the percolation one with $D_{\rm eff} \propto V_0^{4/7}$. Scaling (84) has been used to interpret the results of numerical simulations dealing with neoclassical transport in tokamaks [49]. The dependence $D_{\rm eff} \propto V_0^{4/7}$ is confirmed by numerical modeling (Fig. 11).

The complexity of accounting for several factors simultaneously has a consequence that time-dependent flows are frequently considered only in the framework of the quasilinear approach. For flows with the characteristic parameter variability scale $\omega \approx 1/T_0$, we can use the quasilinear estimate [51]

$$D_{\rm eff} = \int_0^\infty C(t)\, {\rm d}t \approx \frac{V_0^2}{\omega} , \tag{85}$$

where $C(t)$ is the velocity autocorrelation function. This approach, while working well in the case of very high

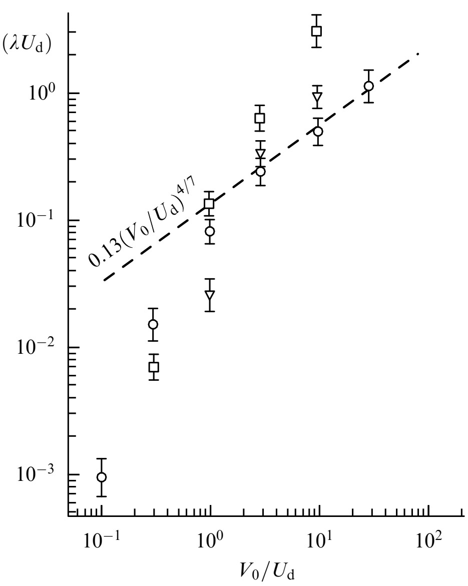

Figure 11. The dependence of the effective diffusion coefficient D on the amplitude of turbulent fluctuations based on numerical simulation data.

frequencies, does not reflect the physical essence of processes at low frequencies (the low-frequency limit),

$$D_{\rm eff} \approx \frac{V_0^2}{\omega}\bigg|_{\omega\to 0} \to \infty\,, \tag{86}$$

when the particle path $l \approx V_0/\omega$ over the time $T_0 \approx 1/\omega$ can be essentially larger than the actual correlation scale. Indeed, in the low-frequency limit, the reconnection processes occur at time intervals much smaller than $1/\omega$, which prevents the scalar particles from completing the path along the stream line they 'ride' initially. In this case, we have to account for the flow topology rearrangement, which affects the mechanisms of decorrelation in a radical way.

11. Low-frequency limit and drift

Obviously enough, the substitution $\tau \approx 1/\omega$ and the use of a small parameter derived from the steady model provide a rather rough approximation, especially if we recall that the characteristic frequencies for which this model was implied (turbulent transport in tokamaks) lie in a very wide range: $10 < \omega_{\rm c_i} < 150$ kHz [52, 53]. According to the percolation approach, we need to find an equation for the small parameter ε_* which would 'seamlessly' include both the perturbation frequency ω and the drift velocity scale $U_{\rm d}$.

The model stream function considered in this section has the form

$$\Psi = \Psi_0(x, y, \omega t) + U_{\rm d}(x\cos\omega t - y\sin\omega t)\,, \tag{87}$$

where Ψ_0 is the main part of the stream function, $V_0 \propto \Psi_0/\lambda$. As a result of stream line reconnections, a random system of drift flows emerges. In the case considered, the key quantity is the parameter D_Ψ describing the reconnection processes in terms of the stream function,

$$D_\Psi(\varepsilon) \propto \frac{\big(\Delta\Psi(\varepsilon)\big)^2}{\tau_{\rm cor}(\varepsilon)}\,, \tag{88}$$

where $\tau_{\rm cor}$ is the characteristic correlation time. Two ways of estimating D_Ψ exist. The first relies on a formal idea about the topology perturbed by the stream function drift:

$$D_\Psi(\varepsilon) \propto \frac{(\varepsilon\Psi_0)^2}{\tau_{\rm b}} \approx \frac{(\varepsilon\Psi_0)^2}{L(\varepsilon)}\,V_0\,. \tag{89}$$

The ballistic time for the motion of scalar particles along the percolation stream line $\tau_{\rm b} \approx L/V_0$ is used as the correlation time. The second way resorts to the use of external factors ($U_{\rm d}$, ω) determining the reconstruction of the flow field:

$$D_\Psi(\varepsilon) \propto \big(U_{\rm d}\,a(\varepsilon)\big)^2\,\omega\,. \tag{90}$$

It should be kept in mind that in the absence of topological modifications, the condition $\Delta\Psi \approx V_0\Delta(\varepsilon) \approx U_{\rm d}\,a(\varepsilon)$ served as the renormalization equation for the small percolation parameter in Ref. [49], where it was proposed to use the renormalization condition in the form of the equation [54, 55]

$$\frac{(\varepsilon\Psi_0)^2}{\tau_{\rm b}(\varepsilon)} = \big(U_{\rm d}\,a(\varepsilon)\big)^2\,\omega\,, \tag{91}$$

which is based on the equivalence of definitions (89) and (90) for the stream line diffusion coefficient. The left-hand side of (91) contains quantities reflecting the geometric features of the random flow. Its right-hand side contains quantities accounting for the external influence. In terms of characteristic times, renormalization condition (91) can be rewritten as

$$\tau_{\rm b}(\varepsilon) = \tau_\Psi(\varepsilon)\,, \quad \text{or} \quad \frac{L(\varepsilon)}{V_0} = \frac{\Delta\Psi(\varepsilon)^2}{D_\Psi(\varepsilon)}\,. \tag{92}$$

Equation (92) can be readily solved. The new expression for the small parameter obtained as a result,

$$\varepsilon_* \approx \left(\frac{U_{\rm d}}{V_0}\right)^{2/[3(1+\nu)]}\left(\frac{1}{\rm Ku}\right)^{1/[3(\nu+1)]} \propto U_{\rm d}^{2/7}V_0^{-3/7}\omega^{1/7}, \tag{93}$$

relies simultaneously on two dimensionless quantities: $U_{\rm d}/V_0$ and the Kubo number. The corresponding expression for the effective diffusion coefficient becomes

$$D_{\rm eff} = \int_0^\infty \frac{{\rm d}\Psi_1}{\Psi_1}\,P_\infty(\Psi_1)\,\frac{a^2(\Psi_1)}{\tau(\Psi_1)}$$
$$\approx \lambda V_0\left(\frac{U_{\rm d}}{V_0}\right)^{2/[3(1+\nu)]}\left(\frac{\lambda\omega}{V_0}\right)^{1/[3(\nu+1)]} \propto U_{\rm d}^{2/7}V_0^{4/7}\omega^{1/7}\,. \tag{94}$$

The last result corresponds to the low-frequency limit, which, in contrast to the quasilinear dependence $D_{\rm eff} \propto 1/\omega$, is characterized by an increase in the effective transport for higher frequencies, $D_{\rm eff} \propto \omega^{1/7}$. This form of the dependence offers a proper description of long-range correlation effects and agrees well with numerical simulation results (Fig. 12) presented in Ref. [49]. Consideration of the dependence of $D_{\rm eff}$ on the drift flow velocity amplitude also points to the correct character of the regime change. The quasilinear regime is described by a dependence $D_{\rm eff} \propto U_{\rm d}^{10/7}$ steeper than the dependence $D_{\rm eff} \propto U_{\rm d}^{2/7}$ in the low-frequency regime analyzed by us. For the coefficient of Hamiltonian diffusion, we find

$$D_\Psi \propto V^{24/21}\omega^{13/21}U_{\rm d}^{26/21}\,. \tag{95}$$

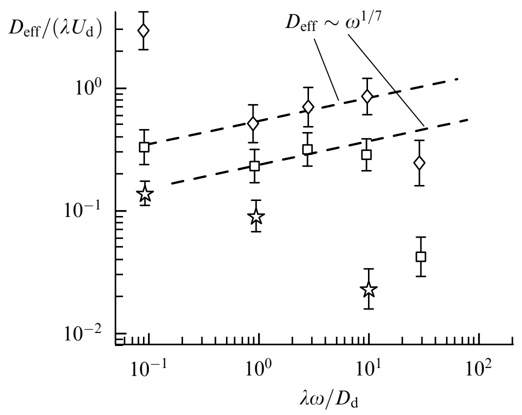

Figure 12. The dependence of the effective diffusion coefficient D on the frequency ω based on numerical simulation data. The rhombi correspond to the Kubo number $\mathrm{Ku} = 5$, squares to $\mathrm{Ku} = 3$, and stars to $\mathrm{Ku} = 0.5$.

We estimate the impact of the diffusive drift of particles off their stream lines using the coefficient of 'seed' (molecular) diffusion D_0:

$$\tau_b(\varepsilon_*) = \tau_\Psi(\varepsilon_*) \ll \tau_D\,, \quad \tau_D \approx \frac{\Delta^2(\varepsilon_*)}{D_0} \approx \frac{\lambda^2 \varepsilon_*^2}{D_0}\,. \tag{96}$$

This allows obtaining the constraint on the magnitude of the seed diffusion D_0 and form the hierarchy of time scales:

$$\frac{\lambda}{V_0} \ll \tau_b(\varepsilon_*) = \tau_\Psi(\varepsilon_*) \ll \tau_D(\varepsilon_*) \ll \frac{1}{\omega}\,, \tag{97}$$

which corresponds to low-frequency drift regimes.

12. Evolution of percolation scales and percolation transport

Initially, the balances of characteristic times or fluxes were used [24, 51] as a normalization condition for the small parameter of the percolation model. Here, we consider an evolutionary approach to the formulation of the renormalization condition [56]. This approach is based on a juxtaposition of two independent expressions used in the percolation theory to describe the correlation scale:

$$L(\varepsilon) \approx \lambda \left(\frac{a(\varepsilon)}{\lambda}\right)^{D_H} \quad \text{and} \quad a(\varepsilon) \approx \frac{\lambda}{|\varepsilon|^\nu}\,. \tag{98}$$

At the initial stage of structure formation, it is natural to expect the increase in the stochastic layer width $\Delta = \Delta(t)$ associated with the magnitude of the small parameter $\varepsilon_* \approx \Delta/\lambda$. Formally, this leads to a decrease in the correlation scale:

$$a_D(t) \approx \frac{\lambda}{|\varepsilon|^\nu} \propto \lambda \left(\frac{\lambda}{\Delta(t)}\right)^\nu. \tag{99}$$

On the other hand, simultaneously with the increase in the length of percolation stream line (or the increase in the path a scalar particle travels along the stream line), $L(t) \propto V_0 t$, the correlation scale

$$a_I(t) \approx \left(\frac{L(t)}{\lambda}\right)^{1/D_H} \approx \left(\frac{V_0 t}{\lambda}\right)^{1/D_H} \tag{100}$$

also increases. In the mean field theory framework, considering the balance between $a_D(t)$ and $a_I(t)$ yields an estimate of the characteristic time t_0 that has to be used as a correlation scale involved in the turbulent diffusion coefficient D_{eff}.

We consider physically motivated approximations to describe the increase in the stochastic layer width. If the increase occurs in a diffusive way, $\Delta^2(t) \approx D_0 t$, we find

$$\lambda \left(\frac{V_0 t_0}{\lambda}\right)^{1/D_H} = \frac{\lambda}{\left(\sqrt{D_0 t_0}/\lambda\right)^\nu}\,. \tag{101}$$

Performing computations, we obtain the estimate

$$t_0 \approx \frac{\lambda^2}{D_0} \left(\frac{1}{\mathrm{Pe}}\right)^{1/(\nu+3)} = \frac{\lambda^2}{D_0} \left(\frac{D_0}{\lambda V_0}\right)^{1/(\nu+3)}, \tag{102}$$

which, with the formula for the stochastic layer width, gives

$$\Delta \approx (D_0 t_0)^{1/2} \approx \lambda \left(\frac{1}{\mathrm{Pe}}\right)^{1/(\nu+3)}. \tag{103}$$

It can be readily found that expression (103) exactly coincides with the respective expression for the steady percolation model, while the estimate for the effective diffusivity is provided by the already classical formula $D_{eff} \approx V_0 \Delta \approx \lambda V_0 [D_0/(\lambda V_0)]^{3/13}$ [33].

Understandably, other expressions for the increase in the stochastic layer width can also be used. In the dynamic system theory, the linear estimate $\Delta(t) \propto t$ is widely used. In the context related to the description of effects due to unsteadiness, the last expression can readily be rewritten as $\Delta(t) = (\lambda\omega)\, t$, where ω is the characteristic frequency of external perturbations. In this case, considering the balance of correlation scales in the form

$$\lambda \left(\frac{V_0 t_0}{\lambda}\right)^{1/D_H} = \frac{\lambda}{(\omega t_0)^\nu} \tag{104}$$

yields yet another estimate for the characteristic time t_0:

$$t_0 \approx \frac{1}{\omega} \left(\frac{\lambda\omega}{V_0}\right)^{1/(\nu+2)} \approx \frac{1}{\omega} \left(\frac{1}{\mathrm{Ku}}\right)^{1/(\nu+2)}. \tag{105}$$

The expression for the stochastic layer thickness then acquires the form that corresponds to a time-dependent percolation model:

$$\Delta = \lambda \varepsilon_* = \lambda \left(\frac{1}{\mathrm{Ku}}\right)^{1/(\nu+2)}, \tag{106}$$

where the estimate for the turbulent diffusivity is given by the expression $D_{eff} \approx V_0 \Delta \approx \lambda V_0 [(\lambda\omega)/V_0]^{3/10}$ derived previously for a time-dependent percolation model [42].

The analysis above proves the efficiency of the alternative renormalization method presented here, which allows using the balance of correlation scales as the basis to build new models of percolation transport by using various approximations for the evolution of the stochastic layer width $\Delta(t)$ [57, 58]. Of course, these approximations must express the physical properties of the flows being studied.

13. Turbulent diffusion in flows with the inverse energy cascade

We emphasize an important feature of the percolation approach to the description of turbulent transport. The renormalization method used for this purpose is built on singling out the main process responsible for stochastic layer formation. An analysis of the physical model must provide a small parameter directly related to the 'key' physical quantity. A very important quantity in turbulence theory is the energy dissipation rate ε_K introduced by Kolmogorov. For one thing, this is a fundamental characteristic of developed turbulent flow, for another, just in two-dimensional flows, ε_K is the key quantity describing the inverse energy cascade. The inverse cascade furnishes the mechanism 'launching' the formation of large-scale structures. Precisely in such cases can percolation stream lines form that permeate almost the entire flow and thus essentially modify the character of scalar transport.

We describe the initial stage of stochastic layer formation with the linear dependence

$$\Delta(t) \propto V_R t \propto \left(\frac{\lambda}{\tau_R}\right) t\,. \tag{107}$$

We note that the linear estimate for the rate of stochastic layer expansion is rather natural for the initial evolution stage and is widely used in dynamic system theory [59]. The characteristic time scale τ_R in our problem statement can depend on only two parameters, ε_K and Δ. Combining these two key quantities, $\varepsilon_K[\mathrm{m^2\,s^{-3}}]$ and $\Delta[\mathrm{m}]$, we obtain the estimate

$$\tau_R(\Delta, \varepsilon_K) \propto \left(\frac{\Delta^2}{\varepsilon_K}\right)^{1/3}. \tag{108}$$

Using the scaling form to evaluate the characteristic time was a common practical choice in the analysis of both hydrodynamic and convective turbulence [60, 61]. For example, in order to describe relative diffusion, Monin used an analogous estimate:

$$\frac{1}{\tau(k, \varepsilon_K)} \approx (k^2 \varepsilon_K)^{1/3}\,, \tag{109}$$

where k is the wave number involved in the Fourier representation of the diffusion equation. Similar arguments have also been utilized by Batchelor when considering the transitional (ballistic) regime of relative diffusion [32].

We obtain the scaling for the characteristic velocity in the form

$$V_R(\Delta, \varepsilon_K) \propto \lambda\left(\frac{\varepsilon_K^2}{\Delta}\right)^{1/3}. \tag{110}$$

A similar estimate was used in models by Parker [62] for the magnetic field line reconnection rate $V_P \propto D_m/\Delta_\varepsilon$. Here, V_P is the reconnection rate of magnetic field lines, D_m is the magnetic diffusion coefficient, and Δ_ε is the thickness of the reconnection layer. It can be seen that our estimate for the stream line reconnection rate qualitatively agrees with the Parker formula. The rate is directly proportional to the parameter describing dissipation and is inversely proportional to the layer thickness. We note that in the framework of the percolation approach, we would like to keep the scale hierarchy $\Delta \approx \lambda\varepsilon \ll \lambda \ll a \ll L$, which has proven its efficiency. The other important argument is that this choice of the characteristic scale allows regarding τ,

$$\tau_R \propto \left(\frac{\Delta^2}{\varepsilon_K}\right)^{1/3} \ll \left(\frac{\lambda^2}{\varepsilon_K}\right)^{1/3}, \tag{111}$$

as an estimate for the time of reconnections occurring inside the stochastic layer [63]. The reconnection of stream lines in the presence of continuous generation of new structures by the inverse cascade is the mechanism of utmost importance, which determines the transport of the scalar in the framework of this model.

The time evolution of the stochastic layer in the model under consideration is given by

$$\Delta \approx \left(\frac{\lambda}{\tau_R(\Delta)}\right) t \approx \lambda\left(\frac{\varepsilon_K}{\Delta^2}\right)^{1/3} t\,. \tag{112}$$

Performing computations, we arrive at the scaling

$$\Delta(t) \approx \lambda\left(\frac{\varepsilon}{\lambda^2}\right)^{1/5} t^{3/5}. \tag{113}$$

Assuming the equality of correlation scales $a_D(t_0) \approx a_I(t_0)$, which in this case is the renormalization condition [56, 63]

$$\lambda\left(\frac{V_0 t_0}{\lambda}\right)^{1/D_H} = \frac{\lambda}{\left(\Delta(t_0)/\lambda\right)^{\nu}}\,, \tag{114}$$

we estimate the characteristic time as

$$t_0 = \left(\frac{\lambda}{V_0}\right)^{5/12}\left(\frac{\lambda^2}{\varepsilon_K}\right)^{7/36}. \tag{115}$$

Here and henceforth, $D_H = 1 + 1/\nu$, $\nu = 4/3$. Performing simple manipulations, we find the small percolation parameter

$$\varepsilon_* \approx \frac{\Delta(t_0)}{\lambda} \approx \left[\frac{(\varepsilon_K\lambda)^{1/3}}{V_0}\right]^{1/4} \tag{116}$$

and the stochastic layer width

$$\Delta(t) \approx \lambda\left(\frac{U_K}{V_0}\right)^{1/4}, \tag{117}$$

which involves the characteristic velocity $U_K \approx (\varepsilon_K\lambda)^{1/3}$. The effective stream line reconnection time τ_R shows the inverse proportionality to velocity, $\tau_R \propto 1/V_0^{1/6}$, characteristic of strong-turbulence regimes. The expression for the effective diffusivity acquires the form [63]

$$D_{\mathrm{eff}}(V_0) \approx V_0\Delta(\varepsilon_*) \approx V_0\lambda\left[\frac{(\varepsilon_K\lambda)^{1/3}}{V_0}\right]^{1/4} \propto V_0^{3/4}\varepsilon_K^{1/12}\,. \tag{118}$$

As could be anticipated, the dependence of the effective diffusivity coefficient on ε_K, in contrast to that for quantities characterizing the reconnection processes, is fairly weak. The dependence of the effective diffusivity on the amplitude of turbulent velocity pulsations fits well with the currently accepted ideas on transport in strong turbulence regimes. Applying the criterion that the system is bounded by a size L_0, $a(\varepsilon) \leqslant L_0$, we obtain

$$(\varepsilon_K\lambda)^{1/3} \ll V_0 \leqslant (\varepsilon_K\lambda)^{1/3}\left(\frac{L_0}{\lambda}\right)^3. \tag{119}$$

Constraint (119) limits the scale of turbulent pulsations in the flows considered. We estimate the impact of the diffusive drift of scalar particles off the stream lines using the molecular diffusivity coefficient D_0:

$$\tau_{\mathrm{R}}(\varepsilon_*) \ll \tau_D(\varepsilon_*)\,, \quad \tau_D \approx \frac{\Delta^2(\varepsilon_*)}{D_0} \approx \frac{\lambda^2 \varepsilon_*^2}{D_0}\,. \tag{120}$$

This provides a constraint on D_0.

The computed turbulent diffusion coefficient enables estimates of transport effects in two-dimensional and quasilinear turbulent flows in the inverse cascade range. For example, using the measurement data for the radioactive admixture transport in the atmosphere [64], we obtain quantities of the order of $\varepsilon_{\mathrm{K}} \approx 1$ cm^2 s^{-3}, $\lambda_0 \approx 10^4$ cm $\approx$ 100 m, $V_0 \approx 10^3$ cm s^{-1} ≈ 10 m s^{-1}, and $D \approx 5\times 10^6$ cm^2 s^{-1} ≈ 500 m^2 s^{-1}. The result for the turbulent diffusion coefficient agrees well with the experimentally found value $D_{\mathrm{exper}} \approx (10^2 - 10^3)$ m^2 s^{-1} [63].

14. Increment of stochastic instability in the percolation limit

Consideration of time-dependent two-dimensional flows is of special interest because it is well known [65, 66] that in a steady flow, a 'fluid element' in a bounded two-dimensional domain does not experience exponential stretching and, as a consequence, the exponential divergence of initially close stream lines is absent. In fact, in the steady case, we are dealing with a one-dimensional integrable Hamiltonian problem, and only if the flow topology changes does the possibility emerge of exploring the flow stochastic behavior, because the time dependence is equivalent to the appearance of an additional degree of freedom.

The percolation 'hierarchy' of scales that we use starts with the stochastic layer width:

$$L \approx \frac{a}{\varepsilon} \gg a \gg \lambda \gg \frac{h}{V_0} \approx \Delta \approx \lambda\varepsilon\,. \tag{121}$$

The area associated with the stochastic layer Δ can be estimated as

$$L(\varepsilon)\,\Delta(\varepsilon) \approx \frac{a(\varepsilon)\,\lambda\varepsilon}{\varepsilon} \approx a\lambda \gg \lambda^2\,, \tag{122}$$

which is natural because the stochastic layer hosts far more than one stream line (Fig. 13). This also implies that a spatial scale l_{s} must exist that characterizes the size of domains where the adiabatic invariant is no longer conserved (an exponentially narrow layer near the separatrices). Indeed, a good estimate for l_{s} is provided by the expression [67]

$$L(\varepsilon)\,l_{\mathrm{s}}(\varepsilon) \approx \lambda^2\,, \quad \text{or} \quad l_{\mathrm{s}} \approx \frac{\lambda^2}{L(\varepsilon)} \approx \lambda\varepsilon^{\nu+1} \ll \Delta \approx \lambda\varepsilon\,. \tag{123}$$

The extended hierarchy of spatial scales becomes

$$\lambda\varepsilon^{\nu} \approx l_{\mathrm{s}} \ll \Delta \approx \lambda\varepsilon \ll \lambda \ll a \approx \frac{\lambda}{\varepsilon^{\nu}} \ll L \approx \frac{a}{\varepsilon} \approx \frac{\lambda}{\varepsilon^{\nu+1}}\,. \tag{124}$$

The introduction of a new scale $l_{\mathrm{s}} \approx \lambda\varepsilon^{\nu+1}$ is useful in obtaining an estimate of the stochastic instability increment in the low-frequency limit. In the framework of the monoscale approach, the respective scaling was derived in Ref. [42]. The characteristic time of reconnection between two close separatrices can be estimated as

$$\gamma_{\mathrm{s}} \approx \frac{1}{\tau_{\mathrm{s}}} \approx \frac{V_{\mathrm{s}}}{l_{\mathrm{s}}} \approx \frac{\lambda\omega}{l_{\mathrm{s}}} \approx \frac{L(\varepsilon)\,\omega}{\lambda}\,, \tag{125}$$

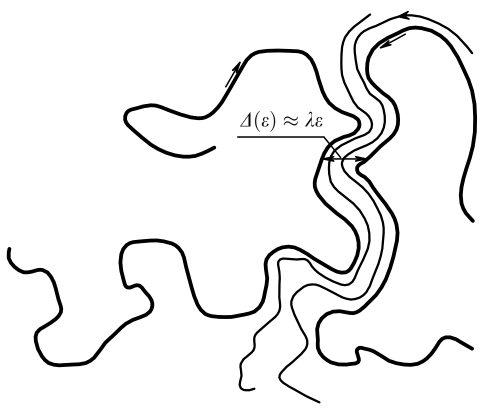

Figure 13. A stochastic layer in a two-dimensional random flow. The solid lines bound the stochastic layer, the thin lines show individual stream lines inside it.

where $V_{\mathrm{s}} = \lambda\omega$ is the estimate for the separatrix motion velocity. We note that the quantity V_{s} has already been implicitly used above in the balance of characteristic times pertaining to a time-dependent percolation flow,

$$\frac{L}{V_0} = \frac{\varepsilon}{\omega}\,, \tag{126}$$

where $\varepsilon/\omega = \Delta/(\lambda\omega) = \Delta/V_{\mathrm{s}}$. Additionally, the condition $\mathrm{Ku} = V_0/(\lambda\omega) > 1$ is equivalent to the conditions

$$\frac{L}{\Delta} = \frac{V_0}{V_{\mathrm{s}}} > 1\,. \tag{127}$$

We write the full hierarchy of time scales in the percolation model considered here [42, 67]:

$$\tau_{\mathrm{s}} \approx \frac{l_{\mathrm{s}}}{V_{\mathrm{s}}} \approx \frac{\lambda}{L\omega} \approx \frac{\lambda}{a}\frac{\varepsilon}{\omega} \ll \frac{\varepsilon}{\omega} \ll T_0 \approx \frac{1}{\omega}\,. \tag{128}$$

On the other hand, the relation between the characteristic times $\tau_{\mathrm{s}} \approx \tau_{\mathrm{b}}$, which can be represented as

$$\frac{\lambda}{L(\varepsilon_*)\,\omega} = \frac{L(\varepsilon_*)}{V_0}\,, \tag{129}$$

allows defining the small percolation parameter

$$\varepsilon_* \approx \left(\frac{\lambda\omega}{V_0}\right)^{1/[2(\nu+1)]} \approx \left(\frac{1}{\mathrm{Ku}}\right)^{3/14} \quad \text{for} \quad \nu = \frac{4}{3}\,. \tag{130}$$

The final expression for the stochastic instability increment γ_{s} then takes the form (Fig. 14)

$$\gamma_{\mathrm{s}} \approx \frac{1}{\tau_{\mathrm{s}}} \approx \omega\,\frac{L(\varepsilon_*)}{\lambda} \approx \omega\sqrt{\mathrm{Ku}}\,. \tag{131}$$

Expression (131) is notably different from the quasilinear result proposed by Kadomtsev and Pogutse ($\gamma_{\mathrm{s}} \approx \omega\,\mathrm{Ku}^2$ [22]) and agrees well with numerical simulations [68].

15. Stochastic instability and the inverse cascade

The method used to derive the scaling $\gamma_{\mathrm{s}} \approx \omega\,\mathrm{Ku}^2$ is tightly connected to the views of Batchelor on the character of evolution of a scalar cloud (Fig. 15) in the dissipative range

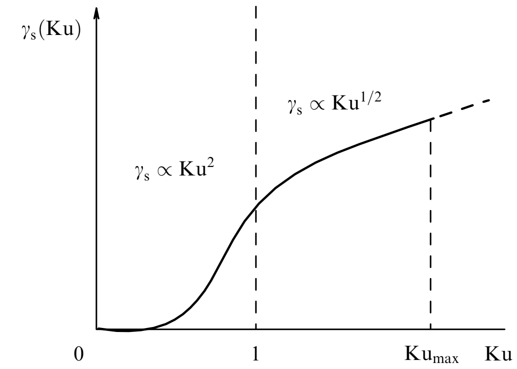

Figure 14. The dependence of the stochastic instability increment on the Kubo number in the percolation limit.

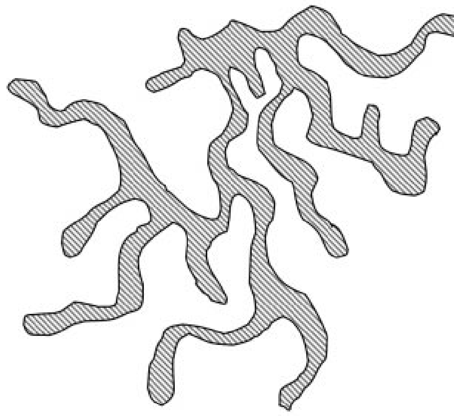

Figure 15. Spreading of a cloud of a scalar owing to stochastic instability.

under the action of stochastic instability [31]. Here, we are still dealing with a cascade character of scalar evolution, despite the lack of an energy cascade:

$$k < k_\nu \approx \left(\frac{\varepsilon_K}{\nu_f^3}\right)^{1/4}, \tag{132}$$

where ν_f is the viscosity coefficient. The lower bound for the cascade of the scalar is set by the scale $k_b \equiv 1/l_b$, which is determined by the balance between the diffusive τ_D and dissipative τ_ν characteristic times,

$$\tau_D \approx \frac{l_b^2}{D_0} = (\nu_f\,\varepsilon_K)^{1/2} \approx \tau_\nu\,. \tag{133}$$

Here, τ_ν also plays the role of the characteristic time related to the stochastic instability, $\tau_s \approx \tau_\nu$, because $V_l \propto l$ in the viscous interval, and therefore the initially neighboring fluid elements diverge exponentially [31, 69]. The authors of Ref. [42] used the potential of the percolation method, which enabled them to connect the characteristic time related to the effects of stochastic instability with the topological features of two-dimensional random flows. In the percolation balance in Ref. [42],

$$\tau_b \approx \frac{L}{V_0} = \frac{\lambda\omega}{l_s} \approx \tau_s\,, \tag{134}$$

the diffusive time τ_D is substituted by τ_b, which is associated with the 'mixing' of the scalar in the given problem statement. However, the relation to the cascade character of processes is lost in this case, because only the frequency ω is used as the model parameter.

We note that in the Kolmogorov turbulence theory, a question regarding the role of stochastic instability arises with respect to both the decorrelation mechanisms (the Batchelor scale l_B) and transitional quasiballistic regimes of relative diffusion,

$$l_R^2 \propto V_R(\varepsilon_K)\,t^2 \tag{135}$$

(the Batchelor scaling). These effects can also be analyzed by considering the balance of characteristic times with the Kolmogorov spectral energy flux ε_K as a parameter [1, 31, 69]. In the framework of the percolation approach, we have a similar possibility. Modifying the estimate for the characteristic stream line reconnection time

$$\tau_R(\Delta, \varepsilon_K) \propto \left(\frac{\Delta^2}{\varepsilon_K}\right)^{1/3} \tag{136}$$

in the stochastic layer by replacing the stochastic layer width $\Delta \approx \varepsilon\lambda$ with the spatial scale $l_s \approx \lambda^2/L$, we obtain

$$\tau_s(\varepsilon_K, l_s) \approx \left(\frac{l_s^2}{\varepsilon_K}\right)^{1/3} \approx \left[\left(\frac{\lambda^2}{L(\varepsilon_*)}\right)^2 \frac{1}{\varepsilon_K}\right]^{1/3}. \tag{137}$$

This scaling allows estimating the characteristic evolution time associated with a separate stream line. In the case considered, we are dealing with several time scales:

$$\tau_s \approx \left(\frac{l_s^2}{\varepsilon_K}\right)^{1/3} \ll \tau_R \approx \left(\frac{\Delta^2}{\varepsilon_K}\right)^{1/3} \ll \left(\frac{\lambda^2}{\varepsilon_K}\right)^{1/3}. \tag{138}$$

It is well known that a cloud of scalar particles evolves in a rather intricate way in a turbulent flow as a consequence of stochastic instability. In our problem statement, the main decorrelation mechanism is characterized by the time scale τ_s. To determine the percolation parameter τ_s, we use the balance of characteristic times $\tau_s(\varepsilon) \approx \tau_b(\varepsilon)$. The resulting equation

$$\left[\left(\frac{\lambda^2}{L(\varepsilon_*)}\right)^2 \frac{1}{\varepsilon_K}\right]^{1/3} = \frac{L(\varepsilon_*)}{V_0}\,, \tag{139}$$

is readily solvable with the conventional dependence for the percolation description of stream lines,

$$L(\varepsilon) \approx \frac{a(\varepsilon)}{\varepsilon} \approx \lambda\,\frac{1}{\varepsilon^{\nu+1}}\,. \tag{140}$$

The small percolation parameter of the model is given by

$$\varepsilon_*^{\nu+1} \approx \left(\frac{\lambda}{V_0}\right)^{3/5} \left(\frac{\varepsilon_K}{\lambda^2}\right)^{1/5}. \tag{141}$$

Accordingly, the stochastic instability increment is

$$\gamma_s \approx \frac{1}{\tau_s} \approx \frac{V_0}{L(\varepsilon_*)} \approx \left(\frac{V_0}{\lambda}\right)^{2/5} \left(\frac{\varepsilon_K}{\lambda^2}\right)^{1/5} \propto V_0^{2/5}\,\varepsilon_K^{1/5}\,. \tag{142}$$

The dependence of the increment on the amplitude of turbulent pulsations V_0 turns out to be generally slow in the percolation limit, $\gamma_s \propto V_0^{2/5}$, whereas the dependence on the spectral energy flux ε_K is much more pronounced than in the formula for the turbulent diffusion coefficient. It is natural to expect this because the intensity of reconnection processes and the evolution of stochastic layers, in contrast to those in the case of transport processes, are directly dependent on the power input in the system. In this model, we have actually succeeded in partly realizing Zel'dovich's thesis that the theory of percolation and cascade phenomenology complement each other. Here, the spectral energy flux ε_K is the key parameter defining the characteristic reconnection time related to a single percolation stream line.

16. Conclusions

We have discussed percolation models of turbulent diffusion that are of fundamental importance in describing anomalous transport in the case of strong turbulence. The main focus was on the scaling analysis of various regimes. We have considered the quasilinear approximation, monoscale percolation transport models, and stochastic instability. The choice of renormalization conditions for the small parameter of percolation models has been discussed in detail. It has been shown how the effects of 'long correlation' enter the percolation description of transport.

We considered new approaches to the description of turbulent transport, with due regard to the effects related to the reconnection of stream lines. In particular, it has been shown that the use of the Kolmogorov spectral energy flux ε_K as the basic parameter is helpful in describing the transport in the case where large-scale structures form.

We explored in detail the influence of drift flow and the effects of time dependence on the behavior of a passive scalar in the percolation approach framework. It is shown that the estimates obtained previously are quasilinear and rely on an ungrounded use of the results of the steady case. The novel approach enabled us to take the characteristic velocity of drift flow U_d and the characteristic frequency of perturbations ω into account. The approximation obtained agrees with both the quasilinear dependence and the monoscale percolation model.

The percolation method facilitates the analysis of the effects pertaining to the evolution of stochastic layers in two-dimensional random flows. This is helpful in obtaining important information on stochastic instability increments, which has become possible owing to the percolation hierarchy of spatial and temporal scales. We note that in most cases, the effects of stochastic instability are still estimated only by the order of magnitude, $\gamma_s \propto 1/\tau_s \propto V_0/\lambda$. The models developed in the percolation approach allow exploring more intricate effects, relying on the idea that characteristic scales exist related to individual stream lines, $l_s(\Delta) \propto \lambda^2/L(\Delta)$.

The analysis of numerous turbulent transport models shows that the combination of the balance of correlation scales with the renormalization of the effective correlation time facilitates the description of various flow regimes with both regular and percolation structures. Indeed, the transition to the regime with large fluctuation amplitudes in the system of convective cells, steady percolation flows, or evolving flows with reconnection effects is invariably accompanied by a change in the character of the dependence of the time correlation scale as the fluctuation amplitude increases.

Additionally, just a small modification enables the approaches discussed in the review to be adapted to exploring problems of neoclassical transport [70] and the diffusion of magnetic field lines in plasma [71]. In the latter case, the computations reduce to the replacement of the Kubo number $\mathrm{Ku} = V_0/(\lambda\omega)$ with the magnetic Kubo number $R_m = b_0 L_z/\Delta_\perp$, where b_0 is the perturbation amplitude of the stochastic magnetic field, and L_z and $\Delta_\perp$ are the longitudinal and transverse correlation scales.

The percolation approach to the description of turbulent diffusion has by no means exhausted its potential. For example, we did not touch here on an important aspect of multiscale percolation models, in which the stream function characterizing the flow is given by the scaling

$$\Psi(\lambda) \propto \Psi_0 \left(\frac{\lambda}{\lambda_0}\right)^M ,$$

where λ_0, Ψ_0, and M are model parameters. This framework assisted the appearance of new results, both for the anomalous diffusion coefficients in two-dimensional flows and for stochastic instability increments [25, 51, 56]. Here, a possibility emerges for establishing a link between the multiscale approach and equations in fractional-order derivatives that describe anomalous transport in configurations with random shear flows [72–75].

To conclude, we note that although the applicability domain of the scaling method is limited, the approaches highlighted in the review prove to be among the rather effective tools for analyzing the effect of turbulent transport, and offer a better understanding of the correlation aspects of transport processes. As we have seen, the percolation approach proposed by Kadomtsev and Pogutse more than 30 years ago enables the analysis of new, increasingly complex, problems. It can be argued that, without a doubt, the work in this research area is still far from being completed.

The author is indebted for the valuable comments and discussions to G S Golitsyn, N S Erokhin, G M Zaslavski, V I Kogan, S V Konovalov, E A Kuznetsov, A B Mikhaylovski, V D Pustovoytov, A V Timofeev, V D Shafranov, and E I Yurchenko.

References

1. Monin A S, Yaglom A M *Statistical Fluid Mechanics* Vols 1, 2 (Cambridge, Mass.: MIT Press, 1971, 1975) [Translated from Russian: Monin A S, Yaglom A M *Statisticheskaya Gidromekhanika* Vols 1, 2 (Moscow: Nauka, 1965, 1967)]
2. Barenblatt G I *Scaling, Self-similarity, and Intermediate Asymptotics* (Cambridge: Cambridge Univ. Press, 1996) [Translated from Russian: Barenblatt G I *Podobie, Avtomodel'nost', Promezhutochnaya Asimptotika* (Leningrad: Gidrometeoizdat, 1982)]
3. Frisch U *Turbulence*: *The Legacy of A N Kolmogorov* (Cambridge: Cambridge Univ. Press, 1995) [Translated into Russian (Moscow: FAZIS, 1998)]
4. Klyatskin V I *Phys. Usp.* **51** 395 (2008) [*Usp. Fiz. Nauk* **178** 419 (2008)]
5. Golitsyn G S *Issledovanie Konvektsii s Geofizicheskimi Prilozheniyami i Analogiyami* (Exploring Convection, with Geophysical Applications and Analogies) (Leningrad: Gidrometeoizdat, 1980)
6. Lebedev V V *Fluktuatsionnye Effekty v Makrofizike* (Fluctuation Effects in Macrophysics) (Moscow: MTsNMO, 2004)
7. Childress S, Gilbert A D *Stretch, Twist, Fold*: *The Fast Dynamo* (Berlin: Springer, 1995)
8. Zel'dovich Ya B , Ruzmaikin A A, Sokoloff D D *Magnetic Fields in Astrophysics* (New York: Gordon and Breach, 1983) [Translated into Russian (Moscow – Izhevsk: RKhD, Inst. Komp'yut. Issled., 2006)]
9. Bakunin O G *Turbulence and Diffusion. Scaling Versus Equations* (New York: Springer, 2008)
10. Sornette D *Critical Phenomena in Natural Sciences* (Berlin: Springer, 2006)

11. Zelenyi L M, Veselovskii I S (Eds) *Plazmennaya Geliogeofizika* (Plasma Heliogeophysics) Vols 1, 2 (Moscow: Fizmatlit, 2008)
12. Bakunin O G *Chaotic Flows. Correlation Effects, Transport, and Structures* (Berlin: Springer-Verlag, 2011)
13. Klyatskin V I *Stokhasticheskie Uravneniya Glazami Fizika* (Stochastic Equations in Physicist's Eyes) (Moscow: Fizmatlit, 2001)
14. Uchaikin V V *Phys. Usp.* **46** 821 (2003) [*Usp. Fiz. Nauk* **173** 848 (2003)]
15. Wiggins S *Chaotic Transport in Dynamical Systems* (New York: Springer-Verlag, 1991)
16. Kolmogoroff A N *C.R.* (*Dokl.*) *Acad. Sci. USSR* **30** 301 (1941) [*Dokl. Akad. Nauk SSSR* **30** 299 (1941)]
17. Obukhov A M *C.R.* (*Dokl.*) *Acad. Sci. USSR* **32** 22 (1941) [*Dokl. Akad. Nauk SSSR* **32** 19 (1941)]
18. Kuznetsov E A, Ruban V P *JETP* **91** 775 (2000) [*Zh. Eksp. Teor. Fiz.* **118** 893 (2000)]
19. Budaev V P, Savin S P, Zelenyi L M *Phys. Usp.* **54** 875 (2011) [*Usp. Fiz. Nauk* **181** 905 (2011)]
20. Bakunin O G *Phys. Usp.* **46** 309 (2003) [*Usp. Fiz. Nauk* **173** 317 (2003)]
21. Balescu R *Aspects of Anomalous Transport in Plasmas* (Bristol: IOP Publ., 2005)
22. Kadomtsev B B, Pogutse O P, in *Plasma Physics and Controlled Nuclear Fusion Research 1978. Proc. of the 7th Intern. Conf., IAEA, Innsbruck, Austria, August 23–30, 1978* Vol. 1 (Vienna: Intern. Atomic Energy Agency, 1978) p. 649
23. Zel'dovich Ya B *JETP Lett.* **38** 57 (1983) [*Pis'ma Zh. Eksp. Teor. Fiz.* **38** 51 (1983)]
24. Galeev A A, Kuznetsova M M, Zeleny L M *Space Sci. Rev.* **44** 1 (1986)
25. Isichenko M B *Rev. Mod. Phys.* **64** 961 (1992)
26. Bakunin O G *Phys. Usp.* **46** 733 (2003) [*Usp. Fiz. Nauk* **173** 757 (2003)]
27. Zelenyi L M, Milovanov A V *Phys. Usp.* **47** 749 (2004) [*Usp. Fiz. Nauk* **174** 809 (2004)]
28. Taylor G I *Proc. London Math. Soc. 2* **20** 196 (1921)
29. Vedenov A A, Velikhov E P, Sagdeev R Z *Nucl. Fusion* **1** 82 (1961)
30. Richardson L F *Proc. R. Soc. Lond. A* **110** 709 (1926)
31. Batchelor G K *Quart. J. R. Meteorolog. Soc.* **76** 133 (1950)
32. Zel'dovich Ya B *Zh. Eksp. Teor. Fiz.* **7** 1466 (1937)
33. Isichenko M B et al. *Sov. Phys. JETP* **69** 517 (1989) [*Zh. Eksp. Teor. Fiz.* **96** 913 (1989)]
34. Saleur H, Duplantier B *Phys. Rev. Lett.* **58** 2325 (1987)
35. Sokolov I M *Sov. Phys. Usp.* **29** 924 (1986) [*Usp. Fiz. Nauk* **150** 221 (1986)]
36. Levinshtein M E, Shur M S, Éfros A L *Sov. Phys. JETP* **42** 1120 (1976) [*Zh. Eksp. Teor. Fiz.* **69** 2203 (1975)]
37. Osipenko M V, Pogutse O P, Chudin N V *Sov. J. Plasma Phys.* **13** 550 (1987) [*Fiz. Plazmy* **13** 953 (1987)]
38. Kleva R G, Drake J F *Phys. Fluids* **27** 1686 (1984)
39. Stauffer D *Introduction to Percolation Theory* (London: Taylor and Francis, 1985)
40. Fannjiang A, Papanicolaou G *J. Stat. Phys.* **88** 1033 (1997)
41. Avellaneda M, Elliott F (Jr.), Apelian C *J. Stat. Phys.* **72** 1227 (1993)
42. Gruzinov A V, Isichenko M B, Kalda Ya L *Sov. Phys. JETP* **70** 263 (1990) [*Zh. Eksp. Teor. Fiz.* **97** 476 (1990)]
43. Isichenko M B, Kalda J *J. Nonlin. Sci.* **1** 255 (1991)
44. Isichenko M B, Kalda J *J. Nonlin. Sci.* **1** 375 (1991)
45. Bakunin O G *Chaos Solitons Fractals* **23** 1703 (2005)
46. Misguich J H et al. *Physicalia Mag.* **20** 103 (1998)
47. Zimbardo G, Veltri P, Pommois P *Phys. Rev. E* **61** 1940 (2000)
48. Trugman S A *Phys. Rev. B* **27** 7539 (1983)
49. Yushmanov P N, Oliakov A N *Comm. Plasma Phys. Control. Fusion* **14** 313 (1992)
50. Bakunin O G *Plasma Phys. Control. Fusion* **45** 1909 (2003)
51. Bakunin O G *Rep. Prog. Phys.* **67** 965 (2004)
52. Horton W *Rev. Mod. Phys.* **71** 735 (1999)
53. Terry P W *Rev. Mod. Phys.* **72** 109 (2000)
54. Bakunin O G *J. Plasma Phys.* **71** 435 (2005)
55. Bakunin O G *Physica A* **347** 289 (2005)
56. Bakunin O G, in *Review of Plasma Physics* Vol. 24 (Ed. V D Shafranov) (Berlin: Springer-Verlag, 2008) p. 53
57. Zaslavsky G M *Physics of Chaos in Gamiltonian Systems* (River Edge, N. J.: Imperial College Press, 1998) [Translated into Russian: *Fizika Khaosa v Gamil'tonovykh Sistemakh* (Moscow – Izhevsk: Inst. Komp'yut. Issl.., 2004)
58. Bohr T et al. *Dynamical Systems Approach to Turbulence* (Cambridge: Cambridge Univ. Press, 2005)
59. Zaslavsky G M *Phys. Rep.* **371** 461 (2002)
60. Vainshtein S I, Zel'dovich Ya B, Ruzmaikin A A *Turbulentnoe Dinamo v Astrofizike* (Turbulent Dynamo in Astrophysics) (Moscow: Nauka, 1980)
61. Pope S B *Turbulent Flows* (Cambridge: Cambridge Univ. Press, 2000)
62. Biskamp D *Magnetohydrodynamic Turbulence* (Cambridge: Cambridge Univ. Press, 2003)
63. Bakunin O G *Physica A* **345** 1 (2005)
64. Nieuwstadt F T M, van Dop H (Eds) *Atmospheric Turbulence and Air Pollution Modelling* (Dordrecht: D. Reidel, 1982)
65. Arnold V I, Khesin B A *Topological Methods in Hydrodynamics* (New York: Springer, 1998) [Translated into Russian: *Topologicheskie Metody v Gidrodinamike* (Moscow: Izd. MTsNMO, 2007)]
66. Moffatt H K et al. (Ed.) *Topological Aspects of the Dynamics of Fluids and Plasmas* (NATO ASI Series, E, Vol. 218) (Dordrecht: Kluwer Acad. Publ., 1992)
67. Bakunin O G *Plasma Phys. Control. Fusion* **47** 1857 (2005)
68. Zimbardo G et al. *Phys. Plasmas* **2** 2653 (1995)
69. Batchelor G K *Phys. Fluids* **12** II-233 (1969)
70. Wesson J *Tokamaks* (Oxford: Clarendon Press, 1987)
71. Vainshtein S I *Magnitnye Polya v Kosmose* (Magnetic Fields in Space) (Moscow: Nauka, 1983)
72. Bakunin O G *J. Plasma Phys.* **72** 647 (2006)
73. Bakunin O G, Schep T J *Phys. Lett. A* **322** 105 (2004)
74. West B J, Bologna M, Grigolini P *Physics of Fractal Operators* (New York: Springer, 2003)
75. Bakunin O G *Physica A* **351** 241 (2005)