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Synchronization of delay-coupled oscillator networks

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Contents

Abstract. **Research on the synchronization of delay-coupled oscillator networks is reviewed. A number of key research approaches using different models and methods are described, and major results obtained through their use are presented and generalized. The most characteristic properties of time-delay coupled systems are discussed.**

1. Introduction

The synchronization of self-sustaining oscillations constitutes a phenomenon of fundamental importance encountered in systems of diverse natures in physics, technology, biology, or other disciplines. The essence of this phenomenon consists in the fact that coupling between subsystems often leads to a qualitative change in the dynamics of the system as a whole, namely, it tunes and co-ordinates the rhythms of the interacting parts. The history of the study of synchronization, which already amounts to more than three centuries, begins with the classic work by Huygens [1] on the pendulum clock, and Rayleigh [2] on organ pipes and tuning forks. In the first half of the 20th century, a new impetus to study synchronization was given by the inception of vacuum electronics: van der Pol and Appleton experimentally examined synchronization in electric generators [3, 4], whilst Andronov and Witt coined the theory of this phenomenon [5, 6]. Since then and up to the present time, the body of research on synchronization has been continuously expanding in two directions. On the one hand, synchronization is being discovered in ever new branches of science and engineering. As a far from complete list we mention here studies of mechanical vibrators [7, 8], turbulent flows [9, 10], optical laser systems [11, 12], systems of communication and control [13–15], chemical reactions [16], and living systems [17–21]. On the other hand, theoretical methods of exploring synchronization, which have become an important part of the general theory of nonlinear oscillations, evolve and are perfected further. The synchronization theory, which dates back to the work of van der Pol [4], Andronov and Witt [22], and Krylov and Bogoliubov [23, 24], has seen further development in numerous contributions (see, e.g., Refs [16, 25–41]). The creation of a general synchronization theory has become possible owing to the remarkable fact that the synchronization of self-sustained oscillators of very different natures often relies on qualitatively similar dynamical mechanisms. Synchronization as a universal dynamical phenomenon is discussed in monograph [42]; some mathematical aspects of synchronization can be cleared up from Ref. [43].

Despite such a long and rich history, there are still plenty of unexplored areas in the field of synchronization, which invite further research. One important task consists in studying the time-delay effects in coupling between the interacting subsystems. Its scientific significance and urgency hinge on the following factors.

First, the presence of a time-varying delay in the coupling between the elements is characteristic of many systems of diverse natures, being caused by the finite speed of signal propagation or other factors. Systems with time-delay coupling are typical in electronics and radiophysics [44–54], nonlinear optics [55–66], neural dynamics [67–73], biology and physiology [18, 74–76], ecology, economics, and the social sciences [51, 77, 78], etc. The brain in mammals may serve as a notable example of a system with significant intrinsic time delays in coupling. The propagation of neural pulses between neurons located in its various regions takes time comparable with, or even exceeding, the typical oscillations timescale. It is fascinating that sometimes one observes the full synchronization of distant brain compartments [79–86], which is important for the cognitive processes the brain maintains. One more example is furnished by modern wireless

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communication systems [87, 88]. The time delays in this case are due to the finite speed of radio signal propagation as well as the finite duration of the message, which has to be fully received in order to be correctly interpreted. In this case, the synchronization of various components of a wireless communication system is needed for controling their access to the medium [89, 90]. As yet another very fresh example of a system where time delays play an important role, we may mention the implementation of so-called reservoir computing, which was realized recently on the basis of a laser with a long feedback loop [91, 92].

Second, the occurrence of time delays in coupling may lead to an essential modification and increased complexity of collective system dynamics. One intuitively understands that the delay in coupling would interfere with the establishment of synchronization, which is indeed observed in a number of cases. In other cases, however, an opposite effect shows itself, and adding time delays into the interelement interactions leads, on the contrary, to the establishment of synchronization in the system. Often, delays entail the occurrence of multistable regimes in the system, when different regimes of collective dynamics are realized, for example, synchronous and asynchronous, for the same parameters as a function of the initial conditions. Even more intricate dynamic effects can be evoked by delays; they will be discussed at length further.

It should also be mentioned that studying time-delay systems frequently presents a real mathematical difficulty for researchers, for systems of that type are described by differential-difference equations and possess, in general, a phase space of infinite dimension. A substantial body of literature (see, e.g., Refs [93–96]) is devoted to investigating systems with time delay coupling and to their mathematical theory.

We present in this review the most significant results related to the synchronization of networks of self-oscillating elements interacting with time delay. Given a topic so immense, discussing all pertinent research in a single review seems impossible, and we leave aside such lines of inquiry as synchronization of chaotic self-sustaining oscillations and systems with discrete time (references to some work contributing to these areas are provided in the Conclusions). The main focus of this review is on the synchronization of regular self-sustaining oscillations. Among the approaches applied in numerous studies exploring this topic, we can single out several basic ones, differing in the models used and the research methods. We describe these approaches below in due course, presenting for each of them the main results found with its assistance. Section 2 deals with the approach based on the phase description of self-sustaining oscillations. Section 3 describes the approach also accounting for the amplitude dynamics. Section 4 concerns the approach based on pulse coupling. In the Conclusions (Section 5), we briefly summarize the results of our description and formulate the general properties most pertinent to systems with time delay coupling.

In this review, when referring to a system composed of identical or similar self-oscillatory subsystems interacting with each other, we interchangeably use the terms 'network' or 'ensemble'. Referring to separate subsystems, we call them an 'element' or 'self-sustained oscillator'. Turning to the phase description of self-oscillations, we often call self-sustained oscillators 'phase oscillators'. In the case of pulse-coupled networks, we sometimes call the self-sustained oscillators 'pulsators'.

2. Phase description

The most widely disseminated approach to modeling the dynamics of self-sustained time-delay coupled oscillator networks relies on so-called phase approximation. If the coupling between self-oscillating subsystems is weak, it mainly influences the dynamics of the phases of oscillations, leaving their amplitudes unchanged. In this case, following the approach developed by Kuramoto [16], it is possible to derive closed equations for the phases of self-sustained oscillators in the form

$$\frac{d\varphi_j}{dt} = \omega_j + \sum_{k \neq j} H_{jk}(\varphi_k - \varphi_j)\,. \tag{1}$$

Here, φ_j are the phases of the oscillators, ω_j are the frequencies of their autonomous oscillations, and H_{jk} are the functions describing interelement couplings, which depend only on the phase difference between the interacting elements. Frequently, coupling is given in the form of a harmonic function: $H(\varphi) \sim \sin\varphi$.

Models like (1) are long and widely used to explore phase effects (see, e.g., Refs [97–101]). For two symmetrically coupled self-sustained oscillators, system (1) is reduced to a simple one-dimensional system on a circle for the phase difference $\varphi_1 - \varphi_2$. In this case, the system exhibits frequency entrainment if coupling between the elements is sufficiently strong to compensate for detuning in their natural frequencies. The system then has a synchronous solution at a unique common frequency.

More complex cases of large phase oscillator ensembles have also been studied. The classical work [16] considered an ensemble with 'each-to-each' global connections. This model, initially proposed by Kuramoto to describe chemical and biological oscillators, was later recognized as a universal one suitable for the description of synchronization in ensembles of self-oscillatory elements of an arbitrary nature [102]. The model is defined as

$$\frac{d\varphi_j}{dt} = \omega_j + \frac{K}{N}\sum_{k \neq j} \sin(\varphi_k - \varphi_j)\,, \tag{2}$$

where K is the coupling coefficient. The natural frequencies ω_j of self-sustaining oscillations are distributed in some interval around the central frequency $\bar{\omega}$ with the probability density $g(\omega)$. The dynamics of system (2) for large N are studied in Ref. [16], which introduces the concept of the mean field to characterize the degree of ensemble synchronization:

$$Z = R\exp(i\theta) = \frac{1}{N}\sum_{j=1}^{N}\exp(i\varphi_j)\,. \tag{3}$$

The mean field amplitude R is the system's order parameter characterizing the degree of the coherence of ensemble elements, caused by their mutual synchronization. If synchronization is absent altogether, i.e., if all elements oscillate at different frequencies, the parameter R is close to zero. Upon synchronization of some part of the elements in the ensemble at a certain frequency, their oscillations add coherently, and a nonzero mean field arises. Kuramoto showed that, on increasing the coupling strength K, a transition takes place from an asynchronous regime to a synchronous one, in analogy with the second-order phase transition. If the coupling strength exceeds a certain critical value K_c depending on the frequency distribution $g(\omega)$, a nonzero mean field is generated in the system, with the amplitude increasing proportionally to the square root of supercriticality: $R \sim \sqrt{K - K_c}$. Such a transition corresponds to the Andronov–Hopf supercritical bifurcation. The strengthening of the mean field with increasing supercriticality stems from the frequency entrainment of an ever-increasing number of oscillators.

The incorporation of time-delay coupling in systems of coupled phase oscillators endows them with new dynamical properties not observed in systems with instantaneous coupling. As shown in Ref. [103], for small coupling coefficients, the account for delay only modifies the shape of the coupling function if the delay is not too large. However, for stronger coupling and longer time delays, fundamentally new effects emerge in the system.

Research on the effect of the time-delay coupling on the dynamics of systems of coupled phase oscillators was initiated by the pioneering work of Schuster and Wagner [104] who considered an ensemble of two phase oscillators interacting with a symmetric delay:

$$\frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t} = \omega_1 - K\sin\left(\varphi_1(t) - \varphi_2(t-\tau)\right), \tag{4}$$

$$\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t} = \omega_2 - K\sin\left(\varphi_2(t) - \varphi_1(t-\tau)\right), \tag{5}$$

where $\omega_{1,2}$ are the natural frequencies of self-sustaining oscillations. The authors of Ref. [104] explored synchronous solutions to the set of equations (4), (5) in the form $\varphi_{1,2} = \Omega t \pm \alpha/2$, where Ω is the common frequency, and α is the phase shift. Such solutions have been found analytically and analyzed for stability. The most interesting effect arising in the time-delay coupled system is the coexistence of stable synchronous solutions at different common frequencies. This effect is illustrated in Fig. 1a, which displays stable system solutions as a function of coupling strength (the solid line corresponds to the most stable solution[1]). When the coupling coefficient exceeds a certain threshold K_c, the system acquires a synchronous solution at a frequency in the vicinity of the mean frequency of self-sustained oscillators, as is the case for instantaneous coupling. However, the further increase of the coupling strength is accompanied by the appearance of newer and newer solutions on passing the bifurcation points $K = K_{ci}$. The newly appearing solutions are characterized by higher frequencies and stronger local stability, yet the solutions at lower frequencies do not lose their stability. The higher the system multistability, the longer the delay time τ and the stronger the coupling coefficient K. The number of differing solutions that simultaneously exist in the system for given parameters can be estimated as $N \sim (K - K_c)\tau$. The bifurcation lines corresponding to the appearance of new solutions are plotted in Fig. 1b.

Later on, models comprising delay-coupled phase oscillators were tapped to study the dynamics of more complex ensembles. For example, Yeung and Strogatz [105] studied the influence of delayed coupling on the dynamics of the Kuramoto model by modifying it in the following way:

$$\frac{\mathrm{d}\varphi_j(t)}{\mathrm{d}t} = \omega_j + \frac{K}{N}\sum_{k\neq j}\sin\left(\varphi_k(t-\tau) - \varphi_j(t)\right). \tag{6}$$

[1] A nonzero negative Lyapunov exponent with the smallest absolute value is considered by the authors of Ref. [104] as a measure of local stability.

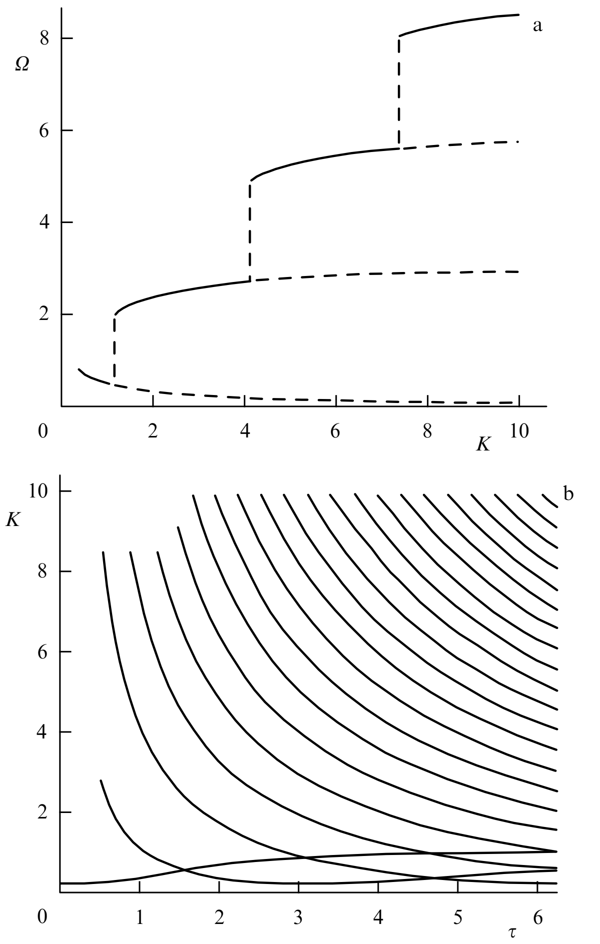

Figure 1. Synchronous solutions at $\tau = 1$ (a) and the bifurcation lines corresponding to the appearance of new periodic solutions (b) in the system of two coupled phase oscillators (4), (5). The parameters of the system are $\omega_1 = 0.6$, and $\omega_2 = 1.4$. (Taken from Ref. [104].)

The authors of Ref. [105] first studied a system of identical phase oscillators $(g(\omega) = \delta(\omega - \omega_0), \omega_0 = 2/\pi)$. Considering the Fokker–Planck equation for the density $\rho(\varphi, \omega, t)$, they devised analytical conditions for the stability of the asynchronous oscillation regime, turning then to global synchronous solutions in the form $\varphi_j(t) = \varphi(t) = \Omega t + \beta$. From the condition of self-consistency, they derived the expression for the common oscillation frequency:

$$\Omega = \omega_0 - K\sin\Omega\tau, \tag{7}$$

and the stability condition for the synchronous solution:

$$\cos\Omega\tau > 0. \tag{8}$$

The diagram of dynamical regimes for system (6) is given in Fig. 2a. The black color marks the areas where the only stable solution is the asynchronous regime. Interestingly, these areas form a periodic structure along the τ-axis, with a characteristic period corresponding to that of autonomous oscillations of isolated oscillators. The area where only synchronous oscillations are stable was painted white. On increasing the coupling strength, the transition from asynchronous to synchronous regimes passes through the domain of bistability, shown in gray. The existence of a bistability domain between the areas of asynchronous and synchronous

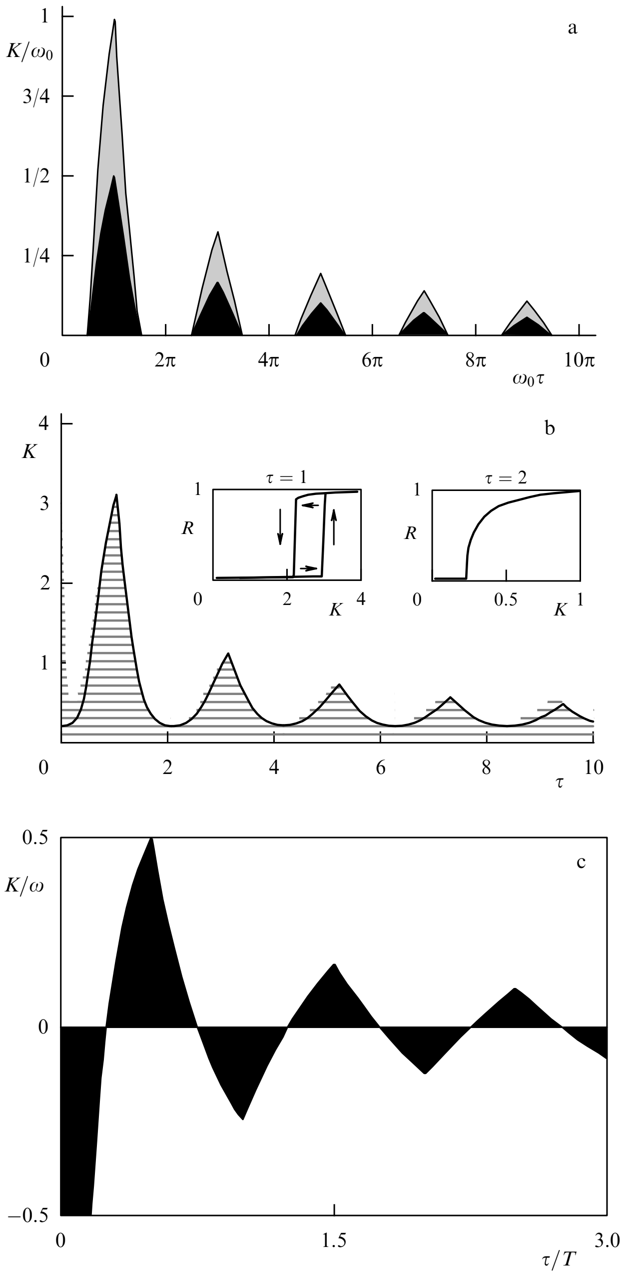

Figure 2. (a) Diagram of dynamical regimes of system (6) for identical oscillators. The stability areas for asynchronous regimes are shaded black, the stability area of synchronization is white, and the domains of bistability are shaded gray (taken from Ref. [105]). (b) The same as in figure (a), but in the presence of frequency spread. The curve denotes the boundary of the synchronization area. Insets show transitions to synchronization in the case of subcritical Hopf bifurcation at $\tau = 1$ (left inset) and $\tau = 2$ (right inset) (taken from Ref. [105]). (c) Diagram of dynamical regimes of ensemble (9) for the coupling function $f(\varphi) = \sin^m \varphi$ and odd m. The stability areas for the asynchronous regimes are shaded black (taken from Ref. [106]).

regimes is a completely new dynamical feature lacking in the Kuramoto system with instantaneous coupling.

On allowing spread in the frequencies of self-sustained oscillators, the picture in the parameter space undergoes some modifications (Fig. 2b). The areas of asynchronous motions 'rise' along the K-axis, and now the asynchronous regime becomes possible for arbitrary delay times. However, the periodic character of the boundary of the synchronization area is preserved, and the critical value K_c of the coupling coefficient, for which the transition from the asynchronous to synchronous regime takes place, depends on the time delay in a periodic way. We also note that, depending on the values of τ, this transition may occur through both the supercritical and subcritical Andronov–Hopf bifurcations. In the latter case, the transition is realized through the bistability domain and is characterized by a hysteresis (see inserts to Fig. 2b).

Earl and Strogatz [106] have generalized the results above to coupling functions of an arbitrary form and to a more general class of coupling topologies. The model proposed by them takes the form

$$\frac{\mathrm{d}\varphi_j(t)}{\mathrm{d}t} = \omega + \frac{K}{M}\sum_{k=1}^{N} a_{jk}\, f\big(\varphi_k(t-\tau) - \varphi_j(t)\big)\,. \tag{9}$$

Here, $f(\varphi)$ is an arbitrary 2π-periodic function, and the matrix a_{jk} reflects the topology of coupling: $a_{jk} = 1$ if oscillator j affects oscillator k, and $a_{jk} = 0$ otherwise. The coupling matrix is subject to the constraint that each oscillator be coupled to the fixed number M of other phase oscillators. This constraint is satisfied, in particular, by the topologies of a ring or an ensemble with global couplings.

Exploring synchronous solutions of the set of equations (9) in the form $\varphi_j(t) = \Omega t$, the authors of Ref. [106] obtained the expression for the common frequency of synchronous oscillations:

$$\Omega = \omega + K f(-\Omega\tau)\,, \tag{10}$$

and proved that the synchronous solution is stable if and only if the inequality

$$K f'(-\Omega\tau) > 0 \tag{11}$$

is satisfied. It is worth mentioning that stability criterion (11) coincides with condition (8) if the coupling function is chosen as $f(\varphi) = -\sin\varphi$.

It is notably that the stability criterion appears to be remarkably simple: the stability or instability of the synchronous solution depends only on the sign of the coupling function derivative. Because the coupling function is periodic, the boundaries of the synchronization area also show a well pronounced periodic structure. This is seen from Fig. 2c, which displays a diagram of dynamical regimes of system (9) for the coupling function in the form of $f(\varphi) = \sin^m \varphi$ for odd m.

Networks of phase oscillators with more complex types and topologies of coupling have also been addressed in the literature. Reference [107] reports that, for an ensemble with a global feedback (of the so-called comparator type), the incorporation of delay causes system desynchronization. Reference [108] studied a two-dimensional lattice with local time-delay coupling, for which it showed the existence of a large set of synchronous regimes with differing frequencies. However, only the regime with minimum frequency finds itself stable among them, the rest being metastable. Even for a short time delay, the frequency of the stable synchronization regime proves to be substantially smaller than the natural frequencies of individual elements. This effect was called 'frequency suppression' of a system.

The formation mechanisms of frequency clusters were studied for a chain of oscillators in Ref. [109]. Reference [110] demonstrated that the delay in coupling causes different effects in different topologies. For example, the delay enhances the system multistability in a ring of unidirectionally coupled oscillators, whilst in contrast it expands the attraction basin of one of the solutions, which is characterized by the highest symmetry, in a ring of oscillators coupled bidirectionally.

Interesting results have been obtained for ensembles of phase oscillators in which the interelement signal delay depends on their mutual location. The model assumes that the elements reside in a medium characterized by a constant speed c of signal propagation. The interaction between elements separated by a distance r_{jk} is then characterized by the time delay $\tau_{jk} = r_{jk}/c$. References [111, 112] consider ensembles of globally coupled phase oscillators arranged as a one-dimensional chain with periodically posed boundary conditions (ring). They show that the presence of space-dependent delays in coupling destroys the global synchronization regime and leads to the generation of some structures — the phase waves — propagating in the ring. A similar result is observed in the two-dimensional plane topology [113, 114]. In this case, global synchronization is destroyed for sufficiently long time delays, and various phase structures, such as rolls, rectangular and diamond-shaped lattices, and ring and spiral waves, emerge in the system.

3. Account for amplitude dynamics

The phase description of self-sustaining oscillations serves as a plausible approximation if coupling between the oscillatory subsystems is weak. If coupling between the self-sustained oscillators becomes sufficiently strong, it affects not only their phase dynamics but also their amplitudes. For weakly nonlinear self-sustaining oscillations, the analysis of accompanying effects can be carried out with the van der Pol averaging method. Mutual synchronization of two self-sustained oscillators with account for the amplitude dynamics was studied by Aronson et al. [115], who considered an ensemble of two coupled van der Pol systems in the vicinity of the Andronov–Hopf bifurcation. Such an ensemble is described by the following system of equations for the phases and amplitudes:

$$\frac{\mathrm{d}r_1}{\mathrm{d}t} = r_1(1 - \kappa\gamma - r_1^2) + r_2\gamma\cos\phi\,, \tag{12}$$

$$\frac{\mathrm{d}r_2}{\mathrm{d}t} = r_2(1 - \kappa\gamma - r_2^2) + r_1\gamma\cos\phi\,, \tag{13}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \varDelta + q_1 r_1^2 - q_2 r_2^2 - \gamma\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\sin\phi\,. \tag{14}$$

Here, r_1 and r_2 are the oscillation amplitudes of self-sustained oscillators, $\phi = \theta_1 - \theta_2$ is the phase difference between their oscillations, γ is the coupling strength, and the parameter κ describes the type of coupling (for example, $\kappa = 1$ corresponds to the so-called diffusive coupling). The parameter $\varDelta$ stands for the frequency detuning between the oscillators, and q_1 and q_2 define the dependence of the oscillation frequency on the amplitude in uncoupled systems.

The most interesting fact discovered in system (12)–(14) is the so-called quenching of oscillations, also called oscillation death. Its essence is that, upon coupling the self-sustained oscillators, they cease to oscillate and their amplitudes r_j decay to zero values. In the phase space, this effect corresponds to a globally stable equilibrium state at the coordinate origin. The authors of Ref. [115] performed a linear stability analysis of this equilibrium state and showed that it can only be stable for $|\varDelta| > 2/\kappa$. Thus, the death of oscillations in the presence of diffusive coupling is observed only for a sufficiently large detuning, $\varDelta > 2$.

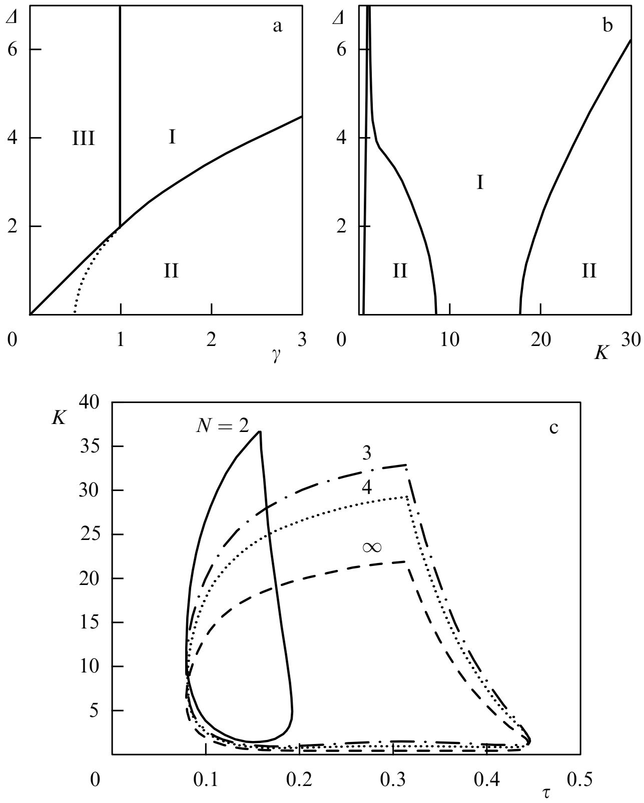

Figure 3. (a) Bifurcation diagram of a system of two coupled weakly nonlinear self-sustained oscillators (12)–(14) in the parameter plane of the 'coupling strength–frequency detuning'. I denotes the domain of oscillation death, II is the synchronization domain, and III is the domain of asynchronous oscillations (taken from Ref. [115]). (b) Bifurcation diagram in the parameter plane $K-\varDelta$ for the system of two self-sustained oscillators with a delay coupling (12)–(18); the parameters $\tau = 0.0817$, $\bar{\omega} = 10$ (taken from Ref. 116). (c) The domain of oscillation death in the system of identical oscillators (19) in the plane 'time delay–coupling strength' for various values of N (taken from Ref. [116]).

Reference [115] also reports on a full bifurcation analysis for system (12)–(14) in the case of isochronous self-sustaining oscillations and diffusive coupling ($q_j = 0$, $\kappa = 1$). The system's parameter space was divided into three domains with qualitatively different behaviors (Fig. 3a). For a small detuning $\varDelta$ and strong coupling strength γ, the system exhibits synchronization, in which case its elements oscillate preserving a constant phase shift. For large detuning and weak coupling, the system dynamics are asynchronous — the phase difference of oscillators grows indefinitely. In the limit of large detunings and strong coupling, the oscillations in the system are quenched. When making oscillations nonisochronous ($q_j \neq 0$), the bifurcation diagram of the system becomes more complicated and includes the appeared zones of multistability.

Inclusion of amplitude dynamics into consideration may qualitatively modify the collective behavior of even more complex ensembles of self-oscillatory elements. In chains and lattices comprising coupled self-sustained oscillators, the change in amplitudes of self-sustaining oscillations may lead to complex dynamics of phase clusters [30]. In a system of globally coupled self-sustained oscillators, accounting for varying amplitudes results in the formation of phase–amplitude clusters [117].

An analysis of the impact of a time delay in coupling on the phase–amplitude dynamics of a self-oscillatory system was first carried out by Reddy et al. [116, 118, 119]. The authors of these studies considered a system of two diffusively coupled self-sustained oscillators, namely

$$\frac{\mathrm{d}r_1}{\mathrm{d}t} = r_1(1 - K - r_1^2) + Kr_2 \cos\left[\theta_2(t-\tau) - \theta_1\right], \qquad (15)$$

$$\frac{\mathrm{d}r_2}{\mathrm{d}t} = r_2(1 - K - r_2^2) + Kr_1 \cos\left[\theta_1(t-\tau) - \theta_2\right], \qquad (16)$$

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + K\,\frac{r_2(t-\tau)}{r_1}\sin\left[\theta_2(t-\tau) - \theta_1\right], \qquad (17)$$

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + K\,\frac{r_1(t-\tau)}{r_2}\sin\left[\theta_1(t-\tau) - \theta_2\right]. \qquad (18)$$

Here, r_j and θ_j are the amplitudes and phases of self-sustained oscillators, ω_j are the frequencies of autonomous self-sustaining oscillations, K is the coupling strength, and τ is the time delay in coupling. An analysis of the linear stability of the equilibrium state at the coordinate origin, which corresponds to oscillation death, was also carried out, with a rather unexpected outcome. It turned out that, for a delayed coupling, the oscillations may die out even for identical self-sustained oscillators, i.e., for the zero frequency detuning $\Delta = \omega_1 - \omega_2$. This property is illustrated in Fig. 3b; the domain of oscillation death is expanding here down to $\Delta = 0$. This effect was dubbed 'death by delay' [120]: in the absence of a delay in the coupling, the oscillators become synchronized, whereas its presence causes the death of oscillations. A qualitative explanation of this effect was proposed by Strogatz [120]. For instantaneous coupling, the points representing the self-oscillators in the phase space 'attract' each other and tend to approach in the limit cycle, whilst in the case of delay coupling each point is 'attracted' to a site the other one occupied some time before. For an appropriately chosen delay, this may lead to 'pulling' points from the cycle to the coordinate origin and the oscillation death.

The delay-induced death of oscillations in a system of identical self-sustained oscillators is observed not only for two coupled elements but also for larger ensembles. In Refs [116, 119], Reddy and his coauthors considered this effect for different coupling topologies — global and ring. For 'each-to-each' type coupling, an ensemble of N self-sustained oscillators is described by the following set of equations for complex-valued amplitudes $Z_j = r_j \exp(\mathrm{i}\theta_j)$:

$$\frac{\mathrm{d}Z_j(t)}{\mathrm{d}t} = \left(1 + \mathrm{i}\omega_j - \left|Z_j(t)\right|^2\right)Z_j(t) + \frac{K}{N}\sum_{k\neq j}\left[Z_k(t-\tau) - Z_j(t)\right]. \qquad (19)$$

At $\tau = 0$, such a system was studied by Mirollo and Strogatz [121], who derived analytical conditions for the onset of oscillation death. The result of this work is that, as for two coupled self-sustained oscillators, the oscillations die out only for a substantial spread in frequencies and sufficiently strong coupling. Reddy et al. [122] showed that, in large ensembles with delayed coupling, oscillation death is possible even without frequency detuning. They identified the boundaries of the domain of oscillation death (death island) for an ensemble of identical self-sustained oscillators. The shape of such islands is illustrated in Fig. 3c for various N and in the thermodynamic limit $N \to \infty$. One is led to conclude that the delay-induced death of oscillations represents a rather general effect characteristic of ensembles comprising self-sustained oscillators with various topologies, provided coupling among them is not weak. We add that this effect was discovered experimentally in a system of two nonlinear LC-circuits coupled through a digital delay line [122].

An approach to exploring ensembles of self-sustained oscillators with delayed coupling, which accounts not only for the phase but also for the amplitude dynamics, was elaborated in a number of later studies. Thus, D'Huys and coauthors [123] showed that accounting for amplitude dynamics may lead to the emergence of chaotic regimes. A detailed analysis of a system composed of two [124] and three [125] van der Pol generators with a diffusive delayed coupling was also carried out. Recently, an extended series of studies has been devoted to the dynamics of ensembles of Stewart–Landau oscillators (the normal form of the Andronov–Hopf bifurcation) with so-called time-delay phase-dependent coupling. In this case, a complex-valued coupling coefficient is selected: $\tilde{K} = K\exp(\mathrm{i}\theta)$, where θ is the coupling phase. It was demonstrated that, by choosing the coupling phase, one may effectively control the system dynamics by switching between different regimes of oscillations [126–129].

4. Description based on pulse coupling

Another approach to studying systems with delayed coupling alternative to the phase and phase–amplitude approaches described in Sections 2 and 3, respectively, resorts to the idea of so-called pulse coupling. Systems characterized by pulse coupling include neural networks, heart pacemaker cells, populations of fireflies, and some others [130–133]. In such systems, the dynamics of self-oscillatory elements are characterized by the periodic generation of short signals (pulses) against the background of long quiescent intervals. The action of these pulses on other elements abruptly changes their state. Quite often, transient processes decay rather rapidly, and the self-sustained oscillator being affected rapidly returns to the stable oscillatory state. This allows one to assume that a self-sustained oscillator always stays in the self-oscillatory regime and that pulse actions instantaneously change or reset its oscillation phases.

When modeling systems with pulse coupling, phase oscillator type models are routinely used as basis elements. Their state, in this case, is described by a single variable — their phase φ, which monotonously increases with a constant rate $\mathrm{d}\varphi/\mathrm{d}t = \omega$. When the phase attains a maximum value, for example, unity, the element generates a pulse and switches to the state with a zero phase. The dynamics of the basis element in an autonomous case are thus the periodic generation of pulses with a constant period $T = 1/\omega$. Sometimes, such elements are also called pulsators. The interaction of pulsators reduces to an exchange by pulses. The pulse fired by one element and affecting another one causes an instantaneous shift in the phase of the latter by some value, so that $\varphi(t+0) = \varphi(t-0) + \Delta\varphi$.

The phase shift $\Delta\varphi$ depends only on the instantaneous value of $\varphi(t-0)$ of the phase directly before the interaction. The phase shift $\Delta\varphi$ can be either positive, which corresponds to the 'acceleration' of the oscillator, or negative, which testifies to its 'retardation'. The dependence $\Delta\varphi = f(\varphi)$, called the 'phase resetting curve' or the 'phase response curve' (PRC) [134], describes the acceleration/retardation of the rhythm of the oscillator caused by variation of its phase under an external action. The notion of a phase resetting curve was widely used in work dealing with oscillations in biological systems, such as heart muscle cells or neural networks [135–141]. The universality of the approach relying on the phase resetting curve is that other popular models, for example, integrate-and-fire and isochronous clocks, etc. [139] can be reduced to it by an appropriate choice of the curve shape.

In the most general form, a network of N phase oscillators with delayed pulse coupling is described by the set of equations [142]

$$\frac{\mathrm{d}\varphi_j(t)}{\mathrm{d}t} = \omega_j + \sum_{k=1}^{N} f_{jk}\big(\varphi_j(t)\big) \sum_{t_k^p} \delta\big(t - t_k^p - \tau_{jk}\big)\,. \tag{20}$$

Here, φ_j and ω_j are instantaneous phases and natural frequencies of the elements, respectively, where $j = 1, 2, \ldots, N$. The function $f_{jk}(\varphi)$ describes the phase resetting curve of the kth element under the action of pulses arriving from the jth element. It can usually be assumed that the form of the curve $f_{jk}(\varphi)$ is the same for all elements and that couplings between them differ only in strength, being characterized by different coupling coefficients μ_{jk}, so that $f_{jk}(\varphi) = \mu_{jk} f(\varphi)$. In such cases, we refer to $f(\varphi)$ as the coupling function. The time delay between elements is determined by the quantities τ_{jk}. The second sum on the right-hand side of Eqn (20) is taken over all moments t_k^p of pulse generation by the kth element. This sum takes on nonzero values only at the instants of time $t_k^p + \tau_{jk}$ when these pulses act on the jth element. Namely then the instantaneous phase shifts in the jth element occur.

Strictly speaking, the approach based on pulse coupling is a particular case of the phase description addressing self-oscillatory networks for a specific selection of coupling functions in the form of a delta function [cf. systems (1) and (20)]. Such a choice of the coupling function, however, leads to the jumpwise dynamics of phases of the network elements, making the task of describing and studying these dynamics fundamentally different from the methodical perspective, which provides the rationale for singling out those models with pulse coupling into a separate class. A convenient and natural instrument to explore systems with pulse coupling comprises point maps [143]. In the absence of time delays, at each instant t_1 of exciting one of the elements the pulse generated by it acts immediately on all other ensemble elements and consequently causes an immediate change in their states. Given the values of phases directly before this instant of time, one may compute the phase shifts under the action of the pulse and, knowing the new phases, determine the next instant some other self-oscillator will fire. As a result, one gets a map describing the change in the state of the ensemble between the two subsequent instants of time its elements generate the pulses.

However, the construction of the point map becomes more involved in the presence of time delays in the system. Information on the instantaneous values of element phases is, in this case, insufficient to predict the system dynamics, since it can be affected by signals generated in the system earlier. For this reason, to fully describe the ensemble state one needs information not only on the current state but also on the past activity of the ensemble. Accordingly, the longer the time delays, the longer the time interval in the past which may influence the system's dynamics in the future. Because of the pulse character of coupling in system (20), only the instants of time t_k^p of the pulse generation are essential out of all the information about ensemble activity in the past.

Important questions are what precisely is the number of pulses fired earlier that have to be taken into account, and is it finite. We answered them in Ref. [142], where it was shown that, provided the coupling is not excessively strong and certain initial conditions are taken, information on no more than finite number P of last pulses for each element is sufficient to fully describe the system's state.[2] In this case, it is convenient to introduce a finite-dimensional state vector defined as

$$\xi(t) = \big(\varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t), x_1^1, x_1^2, \ldots, x_1^P, x_2^1, x_2^2, \ldots, x_2^P, \ldots, x_N^1, x_N^2, \ldots, x_N^P\big)\,, \tag{21}$$

where $x_j^p = t - t_j^p$ is the time elapsed after the instant when element j generates pulse p, x_j^1 corresponds to the last generated pulse, x_j^2 corresponds to the next to the last, and so on. The vector $\xi(t)$ contains a full description of system (20). Reference [142] constructs a point map defining the system evolution. The system dynamics in this case are regarded as a sequence of some discrete events, so-called H-events, each linked with an instantaneous jumpwise change in the components of the state vector. Such events belong to two types: they are linked to either the pulse generation by one of the elements or the action of a pulse generated earlier on some of the ensemble elements. The point map describes the change in the state vector between subsequent H-events and enables studying the dynamics of ensembles with an arbitrary coupling structure.

4.1 System with two elements

The simplest example of an ensemble with pulse time-delayed couplings is that with just two elements. For short delays, the system of two coupled pulsators was studied by Ernst et al. [144, 145] in the context of the dynamics of neural ensembles. Two cases were considered: that of excitatory coupling, when the outer action favors the generation of the next pulse, and that of inhibitory coupling, when the outer action inhibits pulse generation. It has been shown that the dynamics of the system studied are essentially different in these two cases. In the presence of a delay, the excitatory coupling leads to out-of-phase synchronization in the ensemble: all the elements are excited with the same period, but not simultaneously. This makes the dynamics distinct from the case of instantaneous interaction, in which an ensemble with excitatory couplings is synchronized strictly in phase [130]. A sufficiently strong inhibitory coupling, in contrast, synchronizes the elements of the ensemble in phase, even in the presence of time delay.

The impact of various factors, such as asymmetry in the coupling [146] or the finiteness of a pulse duration [147, 148], on the dynamics of a pair of coupled pulsators has also been explored.

For arbitrary time delays, including long ones, the ensemble of two elements was analyzed in Ref. [149] which considered the case with symmetric couplings characterized by the coupling function $f_{12}(\varphi) = f_{21}(\varphi) = f(\varphi)$ and the time

[2] Reference [142] also offers an example of nontrivial dynamics in a system with strong coupling, where these conditions are not met.

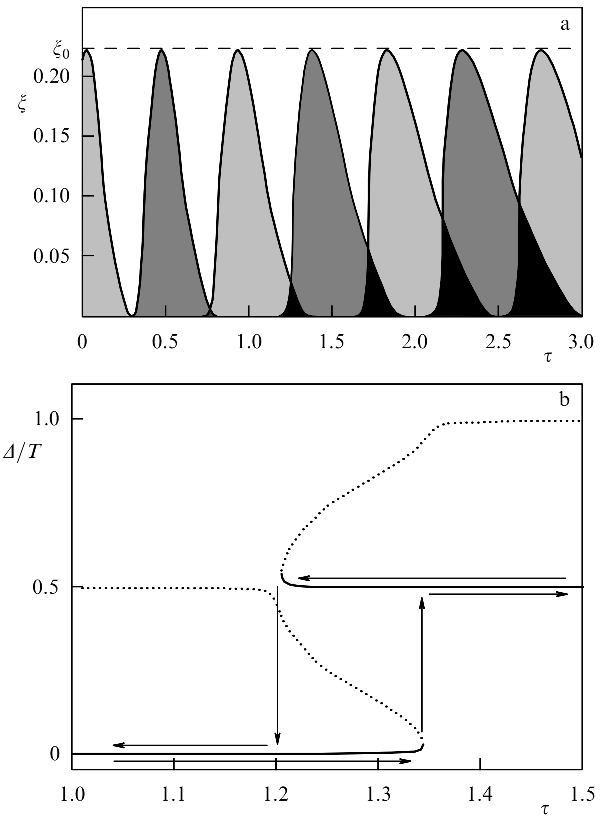

Figure 4. (a) Bifurcation diagram for the ensemble of two coupled pulsators (20) in the delay (τ)–frequency detuning (ξ) plane. The system parameters are $\mu = 0.1$ and $\omega_1 = 1$. The light (dark) gray shading shows zones of in-phase (antiphase) synchronization. The black domains correspond to the overlapping zones and multistability. (b) Phase-flip bifurcation in the ensemble of two coupled pulsators (20). The dependence of the interelement phase difference Δ relevant to system's periodic solutions on the delay time τ. The solid lines mark stable branches of solutions, and dashed lines show the unstable ones. (Taken from Ref. [149].)

delay $\tau_{12} = \tau_{21} = \tau$. Solutions to this system that correspond to the regimes of synchronization and their domains of existence and stability were explored by the Poincaré map method. It was shown that there is a set of domains in the parameter space, so-called synchronization zones, where the synchronization of the small ensemble is observed. In Fig. 4a, these zones are displayed in the plane of the delay time τ and frequency detuning $\xi = \omega_2 - \omega_1$ of elements for a sinusoidal coupling function $f(\varphi) = -\mu \sin 2\pi\varphi$. Synchronization is evident in a bounded band of frequency detuning ξ_0, defined as

$$\xi_0 = \frac{2\mu\omega_1}{1-\mu} \, . \tag{22}$$

The zones of synchronization form a periodic structure along the τ-axis. They extend to infinity along this axis, and synchronous regimes are found in the system for arbitrarily long coupling delays. Noteworthy is the observed alteration of different types of zones which correspond to in-phase or antiphase synchronization of the ensemble elements. The 'width' of the zones increases as the delay time τ is made longer, so that the different types of zones overlap, which entails the onset of bistability between the in-phase and antiphase regimes. We touch here on this phenomenon in more detail.

The overlapping of neighboring synchronization zones happens in the domain where the frequency detuning ξ is small. In this case, the regimes of in-phase and antiphase synchronization coexist in the system within some interval of delays τ. The behavior of the ensemble as the parameter τ slowly changes is characterized by the presence of hysteresis. An example of such behavior is given in Fig. 4b. Let the ensemble initially be in the regime of in-phase synchronization, and the parameter τ be slowly varied from $\tau = 1$ to $\tau = 1.5$. On reaching a critical value $\tau = \tau_1 \approx 1.34$, the ensemble abruptly switches to the regime of antiphase synchronization. This phenomenon got the name of 'phase-flip bifurcation' [150–152]. The dynamic mechanism beyond this effect is due to the disappearance in the phase space of the stable in-phase solution through a saddle–node bifurcation at $\tau = \tau_1$. Moving in the other direction along the parameter τ with it decreasing from $\tau = 1.5$ to $\tau = 1.0$, the ensemble switches in the opposite way from the antiphase to the in-phase regime at $\tau = \tau_2 \approx 1.21$. At this value of τ, the antiphase stable solution also disappears through a saddle–node bifurcation.

The width of intervals of bistability for the in-phase and antiphase regimes, on the boundary of which phase-flip bifurcation takes place, expands with increasing τ. Large τ are characterized by the occurrence of the multistability of another type, associated with the existence of stable synchronous and asynchronous regimes. Yet another specific property of a system with long time delays in the coupling is the substantial increase in the duration of transient processes preceding the establishment of the synchronous regime. The estimate $T \sim \tau^3$ for the duration of the transient stage was obtained in Ref. [149].

4.2 Small networks

The next level of complexity after the two-element systems is encountered in networks comprising a few elements. As an example of such a network, Ref. [153] considers an ensemble of four pulsators with a heterogeneous symmetric coupling (Fig. 5a). The values of delays between the elements of each pair are presented in Fig. 5b, the frequencies of the pulsators are equal, $\omega = 1$, and the coupling is the same for each pair and is described by the function $f(\varphi) = -\mu \sin 2\pi\varphi$. For instantaneous coupling ($\tau = 0$), a regime of global synchronization is observed in the ensemble, in which excitatory phases of all four elements are equal: $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (0, 0, 0, 0)$. If the time delay is short, the global synchronization is preserved, but it breaks provided the delay is sufficiently long. Distinct patterns of rhythmic activity

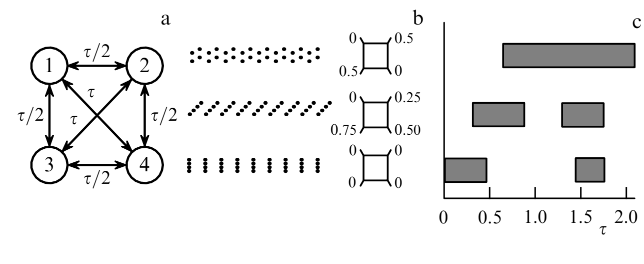

Figure 5. (a) An ensemble of four coupled pulsators. (b) Dynamical regimes of this ensemble. Small black dots mark the excitation instants of time for the elements in the ensemble, and the squares with numbers show the phase relationship of respective patterns. (c) Shaded bands illustrate the intervals of parameter τ where the dynamical regimes of plate b are observed. (Taken from Ref. [153].)

evolve in the ensemble, which are characterized by the periodic excitation of the elements with different phase relations. The patterns fall into two types: (1) sequential excitation, in which all the elements get excited one after another with a quarter period phase shift, $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (0, 0.25, 0.5, 0.75)$, or symmetric to it, and (2) pairwise synchronization, in which the elements are split into two pairs. Each pair is synchronized in phase, but the different pairs are in antiphase: $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (0, 0.5, 0, 0.5)$. Each of these patterns exists in certain intervals of the parameter τ, as illustrated in Fig. 5c.

We see that, as in the case of two elements, the presence of delays in a small ensemble spawns new dynamical regimes of the system. And similar to the case of two elements, transitions between the dynamical regimes in an ensemble accompanying a slow change in the delay times exhibit a hysteresis behavior: the intervals corresponding to different synchronization patterns partly overlap. Additionally, domains with intricate irregular dynamics are observed near the boundaries of pattern existence domains. We note that the ability of the delay to maintain new dynamical regimes in a system is strongly dependent on the coupling structure. So, Ref. [154] shows for an ensemble of four pulsators with excitatory and inhibitory couplings that the presence from couplings of both types enables the ensemble to preserve the regime of global synchronization in a wide range of delay variation.

As the number of elements in an ensemble is increased, its dynamics become more complex. Reference [155] considered an ensemble of five elements with symmetric coupling. Such an ensemble maintains a broad diversity of dynamical regimes with different configurations of element assembling into clusters and phase relationships between the clusters. Under the action of weak external perturbations, the system may undergo transitions between different configurations according to certain rules, which lays the basis for effectively controling the ensemble dynamics.

4.3 Large networks

We turn to the case of networks comprising a large number of pulsators with time-delay coupling. Gerstner [156] studied the dynamics of such networks in an arbitrary topology, but for a particular coupling function $f_{jk}(\varphi) = J_{jk} = \text{const}$. A limitation is also imposed on the net weight of the coupling, which has to be normalized and equal for all elements: $\sum_j J_{ij} = A < 1$ for all i. It was shown that if the maximum delay in the network does not exceed the value of $\Delta = 1 - A$, a periodic regime is established in the network. In this case, the oscillation periods are equal for all elements, but their phase relationships are undefined: the elements may be excited out-of-phase. For discrete homogeneous delays, the system evolves into a synchronous state in a finite time, and it does so asymptotically in general.

Reference [157] considered a network composed of identical elements ($\omega_j = 1$) with each-to-each global coupling, when the coupling function $f_{jk}(\varphi) = f(\varphi)$ and delay time $\tau_{jk} = \tau$ are arbitrary. Such an ensemble is described by the set of equation

$$\frac{\mathrm{d}\varphi_j(t)}{\mathrm{d}t} = \omega + \sum_{k=1}^{N} f\big(\varphi_j(t)\big) \sum_{t_k^p} \delta(t - t_k^p - \tau)\,. \qquad (23)$$

Most attention was focused on the global synchronization regime in the ensemble, in which all the pulsators fire

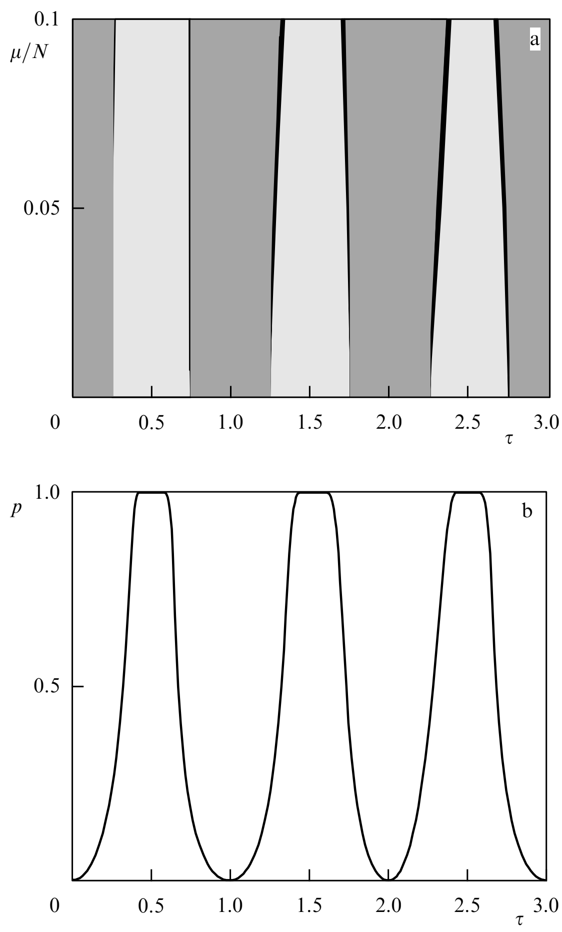

Figure 6. (a) Diagram of dynamical regimes for an ensemble of 10 pulsators with the harmonic coupling $f(\varphi) = -\mu \sin 2\pi\varphi$. Dark gray shading marks domains of stability of the global synchronization regime; the narrow black bands are the domains of bistability of synchronous and asynchronous regimes. (b) Probability of the establishment of the global synchronization regime in an ensemble of 10 pulsators with the coupling function $f(\varphi) = \mu(1 - 2\varphi)$ decreasing everywhere. Initial conditions have been selected at random, $\mu = 0.01$. (Taken from Ref. [157].)

periodically at the same instants of time. The common period in this case is described by the relationship

$$T = 1 - \mu(N-1)\, f(\tau \bmod T)\,. \qquad (24)$$

A stability analysis of the periodic solution led to the formulation of a simple stability criterion for the global synchronization regime. It turned out that it is determined by the sign of the derivative of the coupling function: the synchronous solution is stable for

$$f'(\tau \bmod T) < 0\,, \qquad (25)$$

where $f'(\varphi) = \mathrm{d}f(\varphi)/\mathrm{d}\varphi$. Stability criterion (25) implies a periodic structure of the parameter space for ensemble (23). It can be most readily illustrated for a weak coupling when $\mu N \ll 1$. In this case, the period of joint oscillations is $T \approx 1$, and the stability criterion becomes $f'(\tau \bmod 1) < 0$. The last inequality is periodic in the parameter τ, and the domains of global synchronization in the space of ensemble parameters have the periodic structure, accordingly. These domains are displayed in Fig. 6a for a harmonic coupling function $f(\varphi) = -\mu \sin 2\pi\varphi$. For small coupling coefficients, $\mu \ll 1$, the boundaries of synchronization domains are given by the relationship

$$\tau = \frac{1}{4} + \frac{k}{2}, \qquad (26)$$

where k is an arbitrary nonnegative integer number. The synchronous regime is stable for $\tau \bmod 1 < 1/4$ or $\tau \bmod 1 > 3/4$, and it is unstable for $1/4 < \tau \bmod 1 < 3/4$. The stability domains of the synchronous regime expand for larger coupling coefficients. Outside them, the ensemble elements also fire pulses periodically, but with different phases; in this case, differing phase relationships between pulsators may be realized for the same parameters. Narrow bands of multistability of synchronous and asynchronous regimes are evidenced at the periphery of synchronous regime stability zones.

Studies dealing with collective dynamics in neural ensembles frequently mention the fundamental difference between the effects of excitatory and inhibitory coupling on synchronization. Notably, it has been pointed out that in the presence of delays the synchronization of ensembles is more often due to the inhibitory coupling than to the excitatory coupling [156, 158–161]. Interestingly, according to criterion (25), the stability of the global synchronization regime is defined only by the sign of the derivative of coupling function $f(\varphi)$, independent of the sign and magnitude of the function.

Thus, the synchronization of an ensemble of elements is possible for both positive ('excitatory') and negative ('inhibitory') couplings, which is confirmed in Ref. [157]. For modeling an ensemble with excitatory coupling, the coupling function was chosen in the form $f(\varphi) = \mu(1 - \varphi)$, which gives $f(\varphi) \geqslant 0$ for all φ. A coupling of this type is excitatory because an external stimulus always brings nearer the generation of the next pulse by the elements. To model an ensemble with inhibitory coupling, the coupling function had the form $f(\varphi) = -\mu\varphi$. Here, $f(\varphi) \leqslant 0$ for all φ. Therefore, an external stimulus always retards the pulse generation and, hence, the coupling can be called inhibitory. It was shown that global synchronization is observed in the ensembles in both cases.

According to criterion (25), global synchronization in ensembles with a coupling function decreasing everywhere will be stable for any values of parameter τ. This criterion, however, tells us nothing about the uniqueness of this regime. For example, for the coupling function $f(\varphi) = \mu(1 - 2\varphi)$ decreasing everywhere, global synchronization is indeed observed for almost all values of τ, except for integer ones falling on points of discontinuity for f, where a linear analysis is not applicable. The attractor corresponding to global synchronization is, however, not always the only one in the system.

Figure 6b plots the dependence of the probability of establishing the regime of global synchronization, as a function of the value of delay time τ for arbitrary initial conditions. This probability equals one only in narrow vicinities of half-integer values of the parameter τ and drops to zero on approaching integer values of τ. In regions where this probability is less than unity, synchronous and asynchronous regimes coexist. An analogous effect was observed for small time delays in Ref. [162].

5. Conclusions

We described the main approaches to studying networks of self-oscillatory elements which interact in a time-delayed fashion. Historically, the first and most widely disseminated approach is based on the model of Kuramoto phase oscillators. The use of phase models helped to obtain important results and to show that the delay in coupling brings about a substantial modification of ensemble dynamics, making them more complex. In the framework of the phase approach, such dynamical effects as synchronization and the formation of clusters and wave structures have been explored. The natural development of this approach consists in accounting for the amplitude of self-sustaining oscillations. An analysis of phase–amplitude dynamics led to the discovery of a series of new effects, first and foremost, the effect of delay-induced oscillation death.

An alternative approach to modeling ensembles of self-sustained oscillators with time-delayed coupling relies on the concept of pulse-coupled oscillators. This approach offers an important advantage which simplifies analytical and numerical treatment. In general, an ensemble with time-delayed coupling is described by an infinite-dimensional system of differential-difference equations. Making use of pulse-coupled models allows reducing this system to a finite-dimensional point map. The technique of obtaining such a map and the related applicability conditions are presented in Ref. [142]. Similar techniques were tapped earlier, but only for particular cases, for example, for short delays [144–146] or assuming the presence of a particular activity pattern in the ensemble [163–165]. The technique elaborated in Ref. [142] allows one to explore the dynamics in ensembles of an arbitrary configuration for any initial conditions. The reduction of ensemble dynamics to point maps equips one with well-developed techniques of handling them, permitting one to find fixed points and periodic solutions and to analyze their stability and bifurcations. Additionally, the point maps are more convenient for numerical studies than systems with continuous time.

The research pertaining to the dynamics of delay-coupled ensembles of self-sustained oscillators is not limited to the approaches described in this review. Numerous papers are devoted to studies of ensembles of various strongly nonlinear systems involving, for example, relaxation self-oscillations [166, 167], chaotic self-oscillations [168–172], and excitatory [173] and bistable [174] elements. The properties of the dynamics of such ensembles strongly depend on the properties of elements the ensembles are composed of, making the classification of numerous results rather difficult. Nor can we avoid mentioning a fundamentally distinct approach which considers systems with discrete time, i.e., point maps, as elements of an ensemble (see, e.g., Refs [175–179]). All the positive aspects of point maps notwithstanding, this approach has significant limitations: the time delay can only take discrete values and cannot change arbitrarily.

Generalization of the results obtained using various models and approaches allows one to describe characteristic properties intrinsic to delay-coupled ensembles. Apparently, we have to mention the periodicity in the dependence of system's dynamical regimes on the delay time as the most pertinent property of such systems. The characteristic period of this dependence corresponds to the period of autonomous self-sustaining oscillations of the elements composing the ensemble.

At first glance, this property seems obvious: if oscillations have a period T, the delay of a signal over time τ is equivalent to its delay over $\tau + nT$, where $n \in \mathrm{N}$. In this respect, one may get an impression that exploring a system with long time delays is altogether irrelevant and only cases of small delays need to be explored, i.e., $\tau < T$. This conclusion is, however, wrong. The point is that the period of oscillations of ensemble elements is set by the collective dynamics, which in turn depend on the ensemble parameters, including the magnitudes of delay. Despite the presence of characteristic periodicity in the structure of ensemble parameter space, it is, nevertheless, not perfect (see, e.g., Fig. 2 and Fig. 4a).

It is incorrect to assume that a system with a large delay τ is fully equivalent to a system with a short delay $\tau \bmod T$. According to the analysis, the system exhibits new dynamic properties for large delays, such as a significant increase in the duration of the transient processes, the expansion of multistability domains, and the appearance of new types of multistability [149]. Generally speaking, multistability on its own is also one characteristic attribute of delay-coupled ensembles. It is, as a rule, observed for long delay times or strong forces coupling the elements. Two kinds of multistability are possible: one associated with the coexistence of different periodic regimes, and the other associated with the coexistence of periodic and aperiodic regimes.

In discussing the role of time delay in the synchronization of self-oscillatory networks, one cannot unambiguously answer the question of whether the presence of delay in coupling favors or hinders the establishment of synchrony. The answer to this question depends on the concrete coupling function and system parameters. A series of papers shows that the most relevant factor influencing the establishment of synchronization in an ensemble is the sign of the coupling function derivative. Remarkably, this result is valid for both continuous [105, 106] and pulse [157, 165] couplings.

To conclude, we note that synchronization in self-oscillatory delay-coupled networks is an important branch of modern physics, attracting the incessant attention of researchers. In general, such topics as the influence of time delays in the coupling on the dynamics of small ensembles of oscillators and the global synchronization of large networks with a homogeneous structure are explored rather comprehensively; yet many actual avenues still remain unexplored, such as the dynamics of networks with heterogeneous delays, cluster synchronization and activity pattern formation, and the use of delay-coupled systems in applications to information control and processing [180].

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