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The role of B B Kadomtsev's ideas in shaping the current understanding of turbulent transport

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Contents

Abstract. **B B Kadomtsev's turbulent diffusion models are reviewed. Some of the current approaches to describing 'long-range correlation' effects are presented that are directly based on B B Kadomtsev's ideas (diffusion renormalization of quasi-linear equations, the percolation approach to strong turbulence, stochastic instability and the transverse diffusion of plasma particles as factors affecting transport in a 'braided' magnetic field). It is shown that B B Kadomtsev's analytical methods have great heuristic power and will undoubtedly influence the further development of turbulent transport theory.**

Science thrills us only when we, taking an interest in the life of great researchers, begin to follow the history of their discoveries.
James Clerk Maxwell

1. Introduction

The name B B Kadomtsev is well known to physicists working in various branches of science. This notwithstanding, the 'tokamak' specifics of many fundamental papers by Boris Borisovich impede the understanding of their significance by

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Boris Borisovich Kadomtsev (1928–1998) at the Kurchatov Institute of Atomic Energy (1992).

a scientist not directly involved in research on magnetic plasma confinement. This comes as no surprise, because a high level of theoretical research in high-temperature plasma physics demands invoking refined mathematical tools which allow one to account for the peculiarities of the toroidal geometry of the tokamak plasma column.

Rather unexpectedly, for a reader willing to learn the particulars of B B Kadomtsev's original papers, it turns out that he often used surprisingly clear physical models and that his mathematical apparatus is entirely accessible if aided by intuition common to physicists. This paper presents some of Kadomtsev's ideas in a language comprehensible to a broad audience of researchers. It is certainly impossible to address the whole spectrum of works by Boris Borisovich [1, 2] in a short paper, and we limit ourselves to an analysis of new models for turbulent transport proposed by him, and their current developments.

One of the main problems faced by B B Kadomtsev in the analysis of anomalous transport under the conditions of the strong turbulence in a high-temperature magnetized plasma was the need to account for the contributions of large-scale vortex structures. Similar problems are not uncommon in hydrodynamical turbulence, where the notion of inverse cascade was coined, which implies that the direction of energy transfer is paving towards larger scales. And yet, in tasks of classical hydrodynamics the transport of particles is disassociated from questions of confinement of the 'working material' with temperatures measuring a million degrees. Serious difficulties are also brought about by the need to account for interactions between plasma and a magnetic field, and even the geometry of toroidal traps on its own creates issues reaching beyond technical ones.

Under these conditions, proposing a theoretical model that adequately simulates the main properties of the physical system being explored is a true art. Indeed, in analyzing B B Kadomtsev's work, we are amazed to see how broad his views were and how far ahead of time his ideas advanced. For instance, the study of the impact of convective cells on transport processes, which he initiated in 1965, led to the construction of adequate models only in the mid 1980s. A similar situation occurred with Boris Borisovich's proposal in 1978 to consider the nontrivial (fractal) topology of equipotential lines in two-dimensional turbulence. The first scalings for electron transport coefficients relying on this concept appeared only in 1991.

Many of B B Kadomtsev's ideas are interdisciplinary in character and may, therefore, be expressed in a language sufficiently universal for physicists. This task, however, cannot be solved in one or two papers. Moreover, with time, ever new parallels between Boris Borisovich's work in plasma physics and other research avenues have become apparent. Nevertheless, the range of problems centering on a description of turbulent transport has been worked out sufficiently well to date. In this paper, we briefly review the impact of B B Kadomtsev's ideas on the current views on turbulent transport.

2. Kolmogorov spectrum and discussion via correspondence with Kraichnan

In this section, we will briefly touch on the important comments made by B B Kadomtsev in one of the first reviews covering the theory of plasma turbulence [3]. Interestingly, these comments, frequently cited in the current literature, were concerned not with the aspects of plasma theory but with the general approach to the description of well-developed turbulence put forward in the works by A N Kolmogorov and A M Obukhov at the beginning of the 1940s [4–7]. Review [3] was largely devoted to questions emerging in studies of numerous plasma instabilities that develop in magnetized plasmas, but the author had apparently realized the inevitability of switching from a description of particular instabilities to a scaling description of a turbulent state as a whole.

At this early stage of his research, Kadomtsev was already well familiar with Kolmogorov's ideas on turbulence, which underlie one of the most general approaches to the problem. Indeed, judging by the recollections of the colleagues he worked with in Obninsk on one of the variants of the thermonuclear charge, he carried out a detailed theoretical analysis of the ignition of a large spherical mass of deuterium [8]. An explosion of that power would unavoidably lead to a turbulent mixing of matter.

On the other hand, the qualitative description of the turbulence energy cascade given in the works by Kolmogorov and Obukhov prompted theorists to formulate the theory of turbulence based on the 'first principles'. We note right away that even now, 70 years later, such a theory has not materialized. However, in the 1960s, many physicists considered developing a rigorous theory of turbulence as a promising task and tried to exploit all available means to achieve the goal. Here, we begin very briefly with the Kolmogorov approach and then show how Kadomtsev was able to point out the principal errors made by one leading American theorist in the field of the theory of turbulence.

According to modern views, first accurately formulated by Kolmogorov, pulsating motion in a turbulent flow may be considered to be the result of the simultaneous existence of 'vortices' of various sizes, which are responsible for velocity fluctuations on different scales. In the framework of this approach, only the largest vortices appear as a result of mean flow instability. Indeed, let a fluctuation of order V_l in velocity evolve for some reason in a domain of a laminar flow with the size l. The energy associated with this pulsation is proportional by order of magnitude to V_l^2 or V_k, if we consider scales in terms of the Fourier components, where $k \propto 1/l$, and the time needed for it to evolve is estimated as

$$\tau_{\mathrm{K}} \approx \frac{l}{V_l} \approx \frac{1}{k V_k}\,. \tag{1}$$

Notice that the energy equal to an order of magnitude to

$$\frac{V_l^2}{\tau_{\mathrm{K}}(l)} \approx \frac{V_l^3}{l} \approx V_k^3 k\,, \tag{2}$$

$$l < l_\nu \approx \frac{1}{k_\nu} \approx \left(\frac{\nu_{\mathrm{F}}^3}{\varepsilon_{\mathrm{K}}}\right)^{1/4}$$

is transferred from the mean flow to pulsations per unit time. It is assumed that the turbulent flow maintains a continuous energy flux from large to small vortices. For large Reynolds numbers, $\mathrm{Re} = V_0 L_0/\nu_{\mathrm{F}} \gg 1$, because of the negligible influence of drag on vortices at all scales except the smallest, there is almost no dissipation in a turbulent flow (it will become essential only for $l < l_\nu \approx 1/k_\nu \approx (\nu_{\mathrm{F}}^3/\varepsilon_{\mathrm{K}})^{1/4}$). Here, V_0 is the characteristic velocity scale, L_0 is the characteristic spatial (external) scale of flow, ν_{F} is the viscosity, and $\varepsilon_{\mathrm{K}} = \mathrm{const}$ is the Kolmogorov spectral energy flux. As a consequence, one has for $l \gg l_\nu \approx 1/k_\nu$:

$$V_k^3 k \approx \frac{V_l^3}{l} \approx \varepsilon_{\mathrm{K}} = \mathrm{const}\,. \tag{3}$$

Then, $V_k \approx (\varepsilon_K k)^{1/3}$, i.e. the velocity of pulsations in vortices of size l is proportional to $l^{1/3}$ and depends otherwise on the single parameter ε_K:

$$V_l(l) \approx (\varepsilon_K l)^{1/3} . \tag{4}$$

The preservation of the flux ε_K hinges on energy conservation in nonlinear interactions. In the general case of homogeneous isotropic turbulence considered here, it is convenient to introduce the notion of spectral energy density $E(k)$, putting [4–7]

$$\frac{V_0^2}{2} = \int E(k)\, dk , \tag{5}$$

where $V_k^2/2 \propto kE(k)$. It is then straightforward to derive the Kolmogorov–Obukhov scaling for the spectrum of turbulence [4, 5]:

$$E(k) \propto \frac{V_k^2}{k} \propto \frac{(\varepsilon_K k)^{2/3}}{k} \propto C_K \frac{\varepsilon_K^{2/3}}{k^{5/3}} . \tag{6}$$

Here, $C_K \approx 1.6-1.7$ stands for the Kolmogorov constant. In spite of its phenomenological character, the scaling for the turbulence spectrum $E(k) \propto 1/k^{5/3}$ is in excellent agreement with experimental data and presents a milestone in the theory of turbulence [6, 7].

At the beginning of the 1960s, when the subject of plasma turbulence surfaced as the most pressing in the research on hot plasma confinement, the problem of interaction between waves and particles was one of most important [9, 10]. Despite the successes of quasilinear theory [11], the description of nonlinear effects had called for attracting the wave kinetic equation [3, 9, 10]. A similar technique [the direct interaction approximation (DIA)] was adapted by R H Kraichnan to describe interaction between vortices as well [12]. Schematically, this method can be viewed as adding random links (correlations) between numerous 'copies' of the Navier–Stokes equations with a random Gaussian force:

$$V_\alpha(1) + \frac{1}{N}\,\Gamma(1,2,3)\, V_\beta(2)\, V_\delta(3) = S_\alpha(1) . \tag{7}$$

Here, N is the number of copies, Γ is a nonlinear operator, V_α, V_β, and V_δ are the velocities, and S_α are random forces. This representation reflected the formal Hopf approach to writing the Navier–Stokes equations,

$$\frac{\partial}{\partial t}\, V + \hat{\Gamma}(V, V) = \hat{L} V , \tag{8}$$

where $\hat{\Gamma}$ is the bilinear operator describing nonlinear effects, and $\hat{L}$ is the linear operator describing viscous effects. We will refrain from describing the rather intricate diagram technique used in this branch of research, since it is the subject of extensive literature [6, 7]. We present below only qualitative estimations which help to understand the arguments of B B Kadomtsev, who proved the incorrectness of DIA, even though the equations it leads to share a number of properties with the Navier–Stokes equations (conservation laws, scaling transformations, invariance with respect to shifts in time and space). We also note that at the time Ref. [3] was published, its author Robert Kraichnan enjoyed a reputation as a leading American researcher in the field of turbulent transport processes among specialists developing the theory of turbulence [13].

Working with an integral equation describing nonlocal interaction between vortices, Kraichnan devised a spectrum for the inertial range which was very distinct from the Kolmogorov scaling:

$$E(k) \propto C_K V_0^{1/2}\, \frac{\varepsilon_K^{1/2}}{k^{3/2}} . \tag{9}$$

Kadomtsev realized that, despite the underlying powerful formalism, the derivation of the spectrum suffers from a fundamental flaw. Indeed, Kraichnan overestimated the role of large-scale structures in the description of the evolution of the small-scale component. In fact, he expressed the spectral energy flux as the product of stress σ_T by the strain rate $\omega_l \propto 1/\tau_0$. Since the stress can be written as the product of turbulent viscosity

$$\nu_T \propto V_l^2 \tau_0 \approx V_l^2\, \frac{l}{V_0} \tag{10}$$

and the strain rate ω_l, we get

$$\varepsilon_K \propto V_l^2 \tau_0 \left(\frac{V_l}{l}\right)^2 \approx kE(k)\, \frac{1}{V_0 k}\, k^3 E(k) . \tag{11}$$

Simple manipulations give the Kraichnan spectrum in the form

$$E(k) \propto V_0^{1/2}\, \frac{\varepsilon_K^{1/2}}{k^{3/2}} . \tag{12}$$

It should be noticed now that it is just the use of the external scale V_0 incorrectly accounting for the effect of large vortices, which, in fact, only convey small vortices, thus slightly deforming them (adiabatic interaction of distant harmonics). Kadomtsev in Ref. [3] managed to easily handle the intricate technique of Kraichnan's computations, and his arguments were immediately accepted by the majority of specialists. Possibly, namely this error prompted Kraichnan to formulate almost philosophically the thesis "With scaling we can explain everything without understanding anything" — well known not only to hydrodynamicists. In the Russian translation, it sounds even harder, yet revealing.

Despite the 'precautions' stressed by Kraichnan, Boris Borisovich effectively tapped the concept of scaling in many papers exploring questions of anomalous transport. Moreover, Kraichnan's arguments for the existence, at large Reynolds numbers, of very distinct, stretched, and strongly entangled vortex filaments with a spatial scale on the order of the external flow size were later used by Kadomtsev in formulating his original approach to the description of anomalous transport in the presence of large-scale vortex structures.

3. 'Anomalous' plasma diffusion in a magnetic field

The term 'anomalous diffusion' appeared for the first time in a joint paper published in 1960 by B B Kadomtsev and A V Nedospasov both as a preprint of the Kurchatov Institute of Atomic Energy [14] and a journal paper [15]. This work evolved as a result of discussions with V D Shafranov on questions concerning the instability of a positive discharge column in a magnetic field. The authors showed that applying the longitudinal magnetic field leads to stability loss in the current-carrying plasma column and that the oscillations appearing as a consequence give rise to an azimuthal electric field. This inevitably triggers drift motions of electrons in the radial direction. Such drift motion in the direction transverse to the magnetic field were viewed by the authors as the cause of the anomalous diffusion observed in experiments.

The authors had clearly realized that the development of oscillations would end up in plasma turbulence. In this regard, they pointed out at the end of the paper the now classical Bohm idea on the nature of the turbulent diffusion coefficient [16]. Bohm's arguments can be schematically explained by resorting to the equation for the drift velocity of charged plasma particles in the crossed electric **E** and magnetic **B** fields:

$$\mathbf{V}_E = c\,\frac{\mathbf{B}\times\mathbf{E}}{B^2} \propto \frac{\nabla\varphi}{B_0}\,. \tag{13}$$

Here, $\mathbf{V}_E$ is the drift velocity, and φ is the electric potential. For fluctuations of an electric field which are slower than the ion cyclotron frequency (the low-frequency limit), the motion of particles in plasma can be represented as the superposition of rotational motion around the magnetic field line and the drift motion of a leading center with the velocity $\mathbf{V}_E$.

Under the conditions of well-developed plasma turbulence, one may anticipate the appearance of vortex structures similar to those generated in hydrodynamical turbulence, which would lead to chaotic fluctuations of the electric field. Let L_B be the characteristic scale of emerging structures. By applying traditional approach, one can easily derive the estimate of the particle diffusion coefficient:

$$D_B(L_B) \propto \frac{L_B^2}{\tau_{\rm cor}}\,, \tag{14}$$

where L_B plays the role of the characteristic correlation scale, and $\tau_{\rm cor}$ is the characteristic correlation time. Based on dimensionality arguments, one can estimate the characteristic correlation time in terms of the particle drift velocity V_E in the crossed electric and magnetic fields:

$$\tau_{\rm cor} \approx \frac{L_B}{\delta V_E} \approx \frac{L_B^2}{c\delta\varphi}\,B_0\,, \tag{15}$$

where the drift velocity is estimated as

$$\delta V_E \approx \frac{c}{B_0}\,\delta E_{\rm p} \approx \frac{c}{B_0}\,\frac{\delta\varphi}{L_B}\,. \tag{16}$$

Here, $\delta\varphi \approx \delta E_{\rm p} L_B$ is the potential perturbation on the vortex scale, and $\delta E_{\rm p}$ is the related perturbation in the electric field strength. This simplified estimate of the correlation time can be considered as justified at this stage, because even the more elaborate Kolmogorov–Obukhov approach resorts to a purely dimensional estimate for the interaction time between vortices of size l as $\tau_{\rm K}(l) \propto l/V(l)$, where $V(l)$ is the characteristic velocity of turbulent pulsations on the scale l. The fluctuations of the electric field can be easily related to the plasma temperature:

$$e\delta\varphi \approx e\delta E_{\rm p} L_B \approx T_{\rm p}\,. \tag{17}$$

On substitution, we arrive at the Bohm scaling for the anomalous particle diffusion in a turbulent plasma:

$$D_{\rm B} \approx L_B \delta V_E \approx \frac{c}{B_0}\,\delta\varphi \approx \frac{cT_{\rm p}}{eB_0}\left\langle\left(\frac{e\delta\varphi}{T_{\rm p}}\right)^2\right\rangle^{1/2}. \tag{18}$$

The expression obtained lacks the characteristic spatial scale L_B of vortex structures, which we introduced at the first stage of our analysis. In a certain sense, this can be regarded as the universality of the estimate proposed by Bohm. Indeed, the Kolmogorov theory of turbulence hinges on the existence of a hierarchic structure of vortices and the uniformity of the spectral energy flux. In this sense, the presence of a fixed vortex structure scale would be too simplistic, and the fact that this parameter dropped out the final expression is undoubtedly a huge success of this approach.

Beginning from Ref. [14], the questions of plasma turbulence and anomalous transport became major topics of B B Kadomtsev's work. This becomes apparent if one looks through Boris Borisovich's publications in the collection of his papers [1, 2]. The key words 'turbulence', 'turbulent transport', and 'anomalous diffusion' are encountered in most articles over almost 40 years. B B Kadomtsev himself paid special attention to the anomalous (turbulent) transport in relation to the research of plasma confinement in tokamaks. So, in a paper entitled "My view on controlled fusion" [17], published in 1995, there are the following words: "In 1967, L A Artsimovich and I published (separately from each other) short papers in *Uspekhi Fizicheskikh Nauk* (*Physics–Uspekhi*) where the prospects of the development of tokamak type machines were discussed... . In my paper, the problem of anomalous transport was discussed. I always believed that a complete suppression of plasma instabilities is impossible — so that a residue of weak drift turbulence should always exist in a tokamak plasma. But this turbulence, I thought, is not an absolute obstacle which cannot be overcome. Utilizing theoretical estimates for obtaining drift turbulence scaling (at present it is called 'gyro-Bohm' scaling), I have estimated the quantity

$$r_0 B_0 \approx 10\ [m \times T] \tag{19}$$

as a criterion for plasma ignition (here, r_0 is a minor radius of the tokamak, and B_0 is the toroidal magnetic field). All the experimentalists considered this estimate to be unreachably large. However, this quantity is currently corrected to be $r_0 B_0 \approx 15\ [m \times T]$, thus defining the necessary condition for the performance of a tokamak reactor with a noncircular plasma column cross section. My arguments of a theorist convinced L A Artsimovich, and he issued orders to the industry to construct the T-10 facility (T-10 was put into operation in 1975 after L A Artsimovich had already gone)."

Indeed, Kadomtsev clearly understood the importance of the scaling concept for the description of the turbulent plasma state. So, for example, considering high-frequency plasma oscillations, one commonly employs the estimate of correlation time in the form

$$\tau_{\rm cor}(\omega) \approx \frac{1}{\omega}\,. \tag{20}$$

Here, ω is the characteristic frequency of plasma oscillations. Then, the standard expression for the turbulent diffusion coefficient acquires a quasilinear form [18] which is based on defining the velocity autocorrelation function

$\langle V_E(0)\, V_E(t)\rangle$:

$$D_{\rm T} \approx \int \left\langle V_E(0)\, V_E(t)\right\rangle {\rm d}t \approx \delta V_E^2\, \tau_{\rm cor} \approx \frac{\delta V_E^2}{\omega} \approx \left(\frac{c\delta E_{\rm p}}{B_0}\right)^2 \frac{1}{\omega}\,. \tag{21}$$

Understandably, by relying on this expression one cannot describe low-frequency regimes of plasma turbulence, since the decrease in characteristic frequency cannot lead to an infinite increase ($D_{\rm T}|_{\omega\to 0} \to \infty$) in the diffusion coefficient, as follows from the scaling above.

To describe low-frequency regimes, it is essential to take into consideration the reorganization of a flow topology. In this case, the characteristic correlation times determining the transport turn out to be only indirectly connected with the external frequency. Then, if we introduce a parameter which formally characterizes the path travelled by a particle for the time $1/\omega$, it is not difficult to find that in low-frequency regimes it will essentially exceed the characteristic scale L_B of structures contributing most to the transport:

$$l_\omega \approx \frac{\delta V_E}{\omega} \propto c\, \frac{\delta E_{\rm p}}{B_0}\, \frac{1}{\omega}\bigg|_{\omega\to 0} \gg L_B\,. \tag{22}$$

In the presence of reconnection processes, turbulent mixing will inevitably emerge as a key factor. We need, therefore, to relate the characteristic correlation time to both the scale of the structures and the effective transport coefficient. A simple dimensional estimate at high values of the turbulent diffusion coefficient $D_{\rm T}$ is given by the expression tapped by Kadomtsev on repeated occasions:

$$\tau_{\rm cor} \approx \frac{L_B^2}{D_{\rm T}} \approx \frac{1}{k_\perp^2 D_{\rm T}} < \frac{1}{\omega}\,. \tag{23}$$

Notice that this approach later received a more rigorous mathematical framing in the well-known paper by Kadomtsev and Pogutse [19] devoted to the anomalous transport of electrons in a stochastic magnetic field. We will consider it below. Additionally, this illustrates in an obvious way the importance of accounting for the phenomenological principle of choosing the fastest mode, which proved its efficiency not only for the description of transport processes in plasmas, but also for the analysis of turbulent diffusion of scalar particles and processes of thermal convection [20].

Inserting the expression obtained for the correlation time into the original formula, we find

$$D_{\rm T} \propto \left(\frac{\delta E_{\rm p}}{B_0}\right)^2 \frac{L_B^2}{D_{\rm T}}\, c^2\,. \tag{24}$$

Taking into account the estimate $\delta E_{\rm p} L_B \approx \delta\varphi$, one readily arrives at the final expression for the turbulent diffusion coefficient in low-frequency regimes, which coincides in form with the Bohm scaling:

$$D_{\rm T} \propto c\, \frac{\delta E_{\rm p} L_B}{B_0} \approx c\, \frac{\delta\varphi}{B_0} \propto {\rm Ku}\,. \tag{25}$$

The transition from quasilinear regimes with a quadratic dependence $D_{\rm T} \propto {\rm Ku}^2$ to linear (Bohm) regimes with $D_{\rm T} \propto {\rm Ku}$ [21] happens at the Kubo numbers ${\rm Ku} \approx 1$, which characterize the transition from weak turbulence regimes to the well-developed structural turbulence (Fig. 1). These

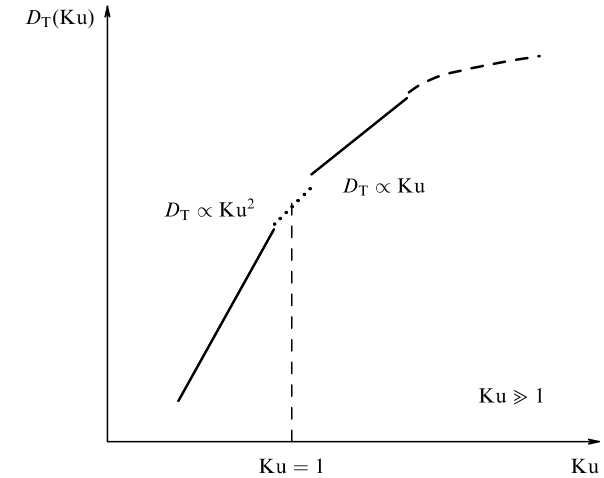


Figure 1. The dependence of the turbulent diffusion coefficient on the Kubo number. The interval ${\rm Ku} < 1$ corresponds to a quasilinear transport regime, where $D_{\rm T} \propto {\rm Ku}^2$. In the region ${\rm Ku} \geqslant 1$, the transport regime is described by the Bohm linear dependence $D_{\rm T} \propto {\rm Ku}$. In the regimes with strong turbulence, the dependence becomes more quiet and is determined by the specific character of vortex structures evolving in a plasma.

strong turbulent regimes demand a more thorough analysis of the coherent structures evolving in them, and, as we shall see further, Kadomtsev proposed an effective method to analyze transport processes in such flows.

4. ‘Braided’ magnetic field and the quasilinear approximation

The specifics of the magnetic configuration of toroidal plasma traps lead to the appearance of resonant magnetic surfaces and the formation of island structures (Fig. 2), with magnetic field stochastization in the vicinity of their separatrices. B B Kadomtsev in his work repeatedly turned to this problem, beginning with paper [21] in 1970. Already in this work attempts were made to explain the anomalous character of electron transport in a tokamak plasma based on a model of a stochastic magnetic field.

However, in a joint paper with O P Pogutse he proposed a fundamentally new approach [19]. Unfortunately, this work

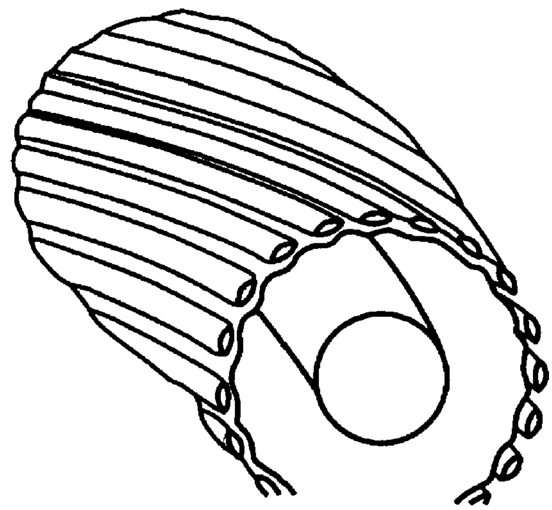


Figure 2. Island structures in a tokamak, evolving as the axial symmetry is broken.

was published only in conference proceedings and was, therefore, known to many physicists by indirect references. In Sections 5–10, we will consider in detail several ideas that are important for the theory of turbulent transport and which have been proposed in this report.

The first theoretical works (see papers [22] and the references cited therein) devoted to a description of the stochastic magnetic field have already suggested exploiting the analogy with the behavior of scalar particles in the field of hydrodynamic turbulence. This approach is built around the stochastic equation for the field lines:

$$\frac{\mathrm{d}\mathbf{r}_\perp}{\mathrm{d}z} = \mathbf{b}(z, \mathbf{r}_\perp)\,, \quad \mathbf{b} = \frac{\mathbf{B}'}{B_0} \approx \mathbf{b}_0\,. \tag{26}$$

Here, a weak random field $\mathbf{B}'(B_x, B_y, 0)$ is superimposed on a strong stationary field $\mathbf{B}(0, 0, B_0)$ directed along the z-axis, and b_0 is the characteristic relative scale of perturbations. In tasks concerning magnetic field line diffusion in a high-temperature plasma, by order of magnitude the perturbations b_0 are estimated as $b_0 \approx 10^{-3}-10^{-4}$ [23]. Then, the classical Taylor expression [19] for the coefficient of transverse diffusion of magnetic field lines takes the form

$$D_\mathrm{m} = \frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}z \big\langle \mathbf{b}(z,0)\,\mathbf{b}(0,0)\big\rangle \propto b_0^2 \lambda_z\,. \tag{27}$$

Here, $\langle\ \rangle$ stands for the commonly accepted notation for averaging, and λ_z is the longitudinal correlation scale of the stochastic magnetic field:

$$\lambda_z = \frac{1}{b_0^2}\int_{-\infty}^{\infty} \mathrm{d}z \big\langle \mathbf{b}(z,0)\,\mathbf{b}(0,0)\big\rangle\,. \tag{28}$$

This makes it possible to consider the diffusion coefficient D_m of magnetic field lines from the correlation viewpoint. In our anisotropic case, the longitudinal and transverse correlation effects need to be thoroughly analyzed. For this reason, the omission of the transverse displacement $\lambda_\perp$ in the Taylor expression

$$\mathbf{b}(z, \lambda_\perp) \approx \mathbf{b}(z, 0) \tag{29}$$

is a serious drawback and will be valid only when the diffusive displacement in the transverse direction is much less than the transverse correlation scale: $b_0\lambda_z \ll \lambda_\perp$. However, in tasks of strong turbulence of the most interest is the case where the transverse correlation effects play an essential role: $b_0\lambda_z \geqslant \lambda_\perp$. Kadomtsev and Pogutse [19] proposed that a new approach be used and formulated its applicability criterion in terms of a dimensionless parameter — the magnetic Kubo number characterizing the ratio of longitudinal and transverse correlation effects:

$$\mathrm{Ku}_\mathrm{m} = \frac{b_0 \lambda_z}{\lambda_\perp} > 1\,. \tag{30}$$

Kadomtsev and Pogutse related this regime to the percolation character [24] of the behavior of streamlines, which allows one to explore the effects of 'long-range correlations'. In fact, it is supposed here that the actually appearing kinematic decorrelation $b_0\lambda_z$ is larger than the formally introduced transverse correlation scale $\lambda_\perp$.

In order to compute the magnetic diffusion coefficient in this limit, it has been proposed to modify the quasilinear equations for the description of scalar particles [25] with due regard for 'turbulent mixing'. Kadomtsev and Pogutse considered a three-dimensional problem, based on the continuity equation for the magnetic field line density n_b:

$$\frac{\partial n_b}{\partial z} + \mathbf{b}\nabla_\perp n_b(\mathbf{r}_\perp, z) = 0\,. \tag{31}$$

The quantity n_b has been represented as the sum of mean density $n_0 = \langle n_b \rangle$ and the fluctuating part n_1:

$$n_b(z, \mathbf{r}) = n_0 + n_1\,. \tag{32}$$

Upon averaging, we then obtain

$$\frac{\partial n_0}{\partial z} + \nabla_\perp \langle \mathbf{b} n_1 \rangle = 0\,, \tag{33}$$

$$\frac{\partial n_1}{\partial z} + \mathbf{b}\nabla_\perp n_0 = v_1 \frac{\partial n_1}{\partial x} - \left\langle v_1 \frac{\partial n_1}{\partial x} \right\rangle. \tag{34}$$

In the quasilinear approximation, the nonlinear term in the equation for the mean density is preserved, but in the equation for perturbation the terms of the second order, $v_1\,\partial n_1/\partial x - \langle v_1\,\partial n_1/\partial x\rangle$, are routinely dropped. This allows one to readily solve the equation for perturbations and go over to averaging in the equation for the mean density.

An important step taken by Kadomtsev and Pogutse was the 'renormalization' of the equation for perturbations, based on the idea that turbulent mixing exhibits a diffusive behavior because of the effects of long-range correlations. They replaced the terms of the second order in the equation for n_1 by the diffusion term $D_\mathrm{m}\nabla_\perp^2 n_1$:

$$\frac{\partial n_1}{\partial z} + \mathbf{b}\nabla_\perp n_0 = D_\mathrm{m}\nabla_\perp^2 n_1\,. \tag{35}$$

Notice that, in contrast to the diffusion coefficient in models of scalar transport constructed by Corrsin [26] and Dreizin and Dykhne [27], they tapped an effective diffusion coefficient D_m of magnetic field lines, yet to be found by solving the renormalized equations. On a qualitative level, we have applied such an approach in Section 3, when analyzing the Bohm scaling in the low-frequency limit.

After these manipulations, the system of renormalized equations is still in a form convenient for solution. The linear character of the equation governing perturbations has not been compromised, but the equation has become a parabolic (diffusion) one instead of hyperbolic, which it was in the standard quasilinear approximation for a passive scalar. Applying the mathematical apparatus of Green's functions for the parabolic equation describing density perturbations, we obtain

$$\frac{\partial G}{\partial z} - D_\mathrm{m}\nabla_\perp^2 G = \delta(\mathbf{r} - \mathbf{r}')\,. \tag{36}$$

The final expression for the mean density n_b takes the form of a diffusion equation

$$\frac{\partial n_b(z, \mathbf{r})}{\partial z} = D_\mathrm{m}\nabla_\perp^2 n_b\,, \tag{37}$$

where the magnetic diffusion coefficient and the Fourier spectrum of perturbed amplitudes are expressed as

$$D_\mathrm{m} = \frac{1}{2}\int \frac{b^2(\mathbf{k})}{\mathrm{i}k_z + k_\perp^2 D_\mathrm{m}}\,\mathrm{d}\mathbf{k}\,, \tag{38}$$

$$b^2(\mathbf{k}) = \frac{1}{(2\pi)^2}\int \big\langle b(0)\,b(\mathbf{r})\big\rangle \exp(-\mathrm{i}\mathbf{k}\mathbf{r})\,\mathrm{d}\mathbf{r}\,. \tag{39}$$

For $\Delta k_z > k_\perp^2 D_{\rm m}$, one gets the classical quasilinear expression

$$D_{\rm m} = \frac{\pi}{2} \int {\rm d}\mathbf{k}\, b^2(\mathbf{k})\, \delta(k_z) \propto b_0^2 \lambda_z \propto {\rm Ku}_{\rm m}^2 \,, \qquad (40)$$

where λ_z is the transverse correlation scale. In the case of strong transverse correlations, $\Delta k_z < k_\perp^2 D_{\rm m}$, we arrive at a Bohm expression:

$$D_{\rm m}^2 = \frac{1}{2} \int \frac{b^2(\mathbf{k})}{k_\perp^2}\, {\rm d}\mathbf{k} \propto (b_0 \lambda_\perp)^2 \propto {\rm Ku}_{\rm m}^2 \,. \qquad (41)$$

Despite the simple form of the estimate obtained, $D_{\rm m} \approx b_0 \lambda_\perp$, the linear character of the dependence of the effective diffusion coefficient $D_{\rm eff}$ on the 'stochastic layer' width $\lambda_\perp$ is widely used to describe turbulent transport in models with convective cells and percolation streamlines, which helped to derive numerous scalings and carry out a comparison between theoretical estimates and data of laboratory and numerical experiments.

5. Kadomtsev–Pogutse scaling

A broad range of problems devoted to turbulent transport involves anisotropic media, which necessitates describing the interaction between longitudinal and transverse correlation mechanisms. This problem is closely connected with the research of processes of turbulent plasma particle diffusion in a stochastic magnetic field. Kadomtsev and Pogutse [19] considered several decorrelation mechanisms responsible for the effective electron transport in a stochastic magnetic field. Their analysis hinges on the idea of magnetic field lines randomly walking in the transverse direction (Fig. 3).

In order to move to the description of particle transport, it is convenient to introduce the diffusion coefficient of magnetic field lines in the classical form [21]:

$$D_{\rm m} \propto \frac{r_\perp^2}{L_{||}} \,, \qquad (42)$$

where $L_{||} > \lambda_z$. Here, $r_\perp$ is the displacement of a perturbed field line in the transverse direction upon displacement over the distance $L_{||}$ along the field line. Finding the relationship between the magnetic diffusion coefficient $D_{\rm m}$ and the effective coefficient $D_\perp$ of transverse particle diffusion in the braided magnetic field presents a difficult task, because the charged particles may 'leave' the field line.

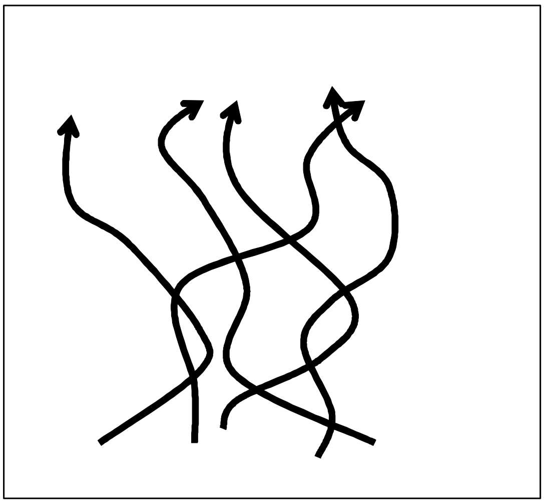


Figure 3. Random wandering of magnetic field lines, which can be described in the framework of the diffusive approximation.

If we assume that a particle in its motion strictly follows the displacement direction of the initially selected magnetic field line (as if the streamline pierces the particles as beads), we readily get the expression for the particle transverse diffusion coefficient:

$$D_\perp \propto \frac{r_\perp^2}{L_{||}} \frac{L_{||}}{t} \approx D_{\rm m} \frac{L_{||}}{t} \,. \qquad (43)$$

When the particles move ballistically along streamlines, the estimate of the transverse diffusion coefficient becomes

$$D_\perp \approx D_{\rm m} V_{||} \,. \qquad (44)$$

Here, $V_{||}$ is the speed of a particle as it moves along the field line.

A nonstandard situation arises when collisions between particles in a braided magnetic field are considered. In this case, it is natural to assume that the particle motion in the longitudinal direction bears a diffusive character (random walks along the field line without the possibility of quitting the initially 'selected' field line):

$$\chi_{||} \approx \frac{L_{\rm cor}^2}{2\tau} \approx \frac{L_{||}^2}{2t} \,. \qquad (45)$$

Here, $L_{\rm cor}$ is the longitudinal correlation length, and τ is the correlation time. The estimate of a longitudinal displacement then assumes the form

$$L_{||}(t) \approx \sqrt{2\chi_{||} t} \,. \qquad (46)$$

Inserting this diffusion-assisted estimate into the formula for the transverse diffusion coefficient, we get the Getmantsev scaling [28]:

$$D_\perp(t) \approx D_{\rm m} \frac{\sqrt{2\chi_{||} t}}{t} \approx \sqrt{2\chi_{||}} \frac{D_{\rm m}}{\sqrt{t}} \,. \qquad (47)$$

The result demonstrates an essential difference between the transverse (compound-diffusion) transport and the classical diffusive transport, since

$$\Delta_\perp^2 \approx D_{\rm m} \sqrt{2\chi_{||}}\, \sqrt{t} \propto t^{1/2} \ll t \quad \text{for} \quad t \gg \tau \,. \qquad (48)$$

This corresponds to subdiffusive transport [29] with the Hurst exponent $H = 1/4$. The result obtained points at the nontrivial character of the relationship between longitudinal and transverse correlation effects occurring in the description of a particle transport in a stochastic magnetic field.

It should be noted that, in this model, particles never leave their magnetic field lines, which is a significant limitation that was overcome by Kadomtsev and Pogutse through considering the decorrelation mechanism based on the particle change in the field line by virtue of transverse diffusion $\chi_\perp$. In the case of high-temperature plasma confinement in tokamak facilities, the longitudinal diffusion coefficient is much larger than the transverse one: $\chi_{||} \gg \chi_\perp$. This condition can easily be rewritten in plasma physics terms: $\chi_{||} \approx V_{\rm T}^2 \tau_{\rm ei}$, and

$\chi_\perp \approx r_e^2/\tau_{ei} \approx 1/\tau_{ei}(V_T/\Omega_{He})^2$. Here, V_T is the thermal electron velocity, $\nu_{ei} \approx 1/\tau_{ei}$ is the electron–ion collision frequency, r_e is the Larmor radius of electrons, and $\Omega_{He} \gg \nu_{ei}$ is the electron gyrofrequency. We then get the condition $\chi_{||}/\chi_\perp \approx (\Omega_{He}\tau_{ei})^2 \gg 1$. To account for transverse decorrelation effects, it is necessary to replace the time parameter t in the Getmantsev formula for the effective turbulent diffusion coefficient by the characteristic correlation time:

$$D_\perp(\tau) \propto D_m \sqrt{\frac{\chi_{||}}{\tau}} . \tag{49}$$

Substituting this latter time, we obtain the Kadomtsev–Pogutse formula for the effective transverse diffusion coefficient of particles in a stochastic magnetic field:

$$D_{KP} \approx D_m \frac{\sqrt{\chi_{||}\chi_\perp}}{r_0} , \tag{50}$$

where $\chi_\perp \propto r_0^2/\tau$, r_0 is the characteristic transverse spatial scale which equals the Larmor radius of electrons in the case of anomalous electron diffusion in a braided magnetic field.

The consideration is frequently carried out in terms of thermal conductivity coefficients in order not to complicate the analysis with issues of plasma ambipolarity. We will, however, keep the diffusion notation for the uniformity of presentation.

It should be kept in mind that the magnetic diffusion coefficient D_m and the longitudinal and transverse diffusion coefficients $\chi_{||}$ and $\chi_\perp$ are assumed to be known, and $D_{KP}(\chi_{||}, \chi_\perp) > \chi_\perp$. Under the conditions of strong turbulence $D_m(b_0) \propto b_0\lambda_\perp$, and, consequently, we obtain the expression

$$D_{KP} \propto b_0\sqrt{\chi_{||}\chi_\perp} > \chi_\perp , \tag{51}$$

which transforms into the condition $b_0\Omega_{He}\tau_{ei} > 1$. Additionally, the Kadomtsev–Pogutse regime assumes the smallness of the transverse decorrelation time, as opposed to the longitudinal time scale (the principle of fast mode dominance):

$$\tau(\chi_\perp) \propto \frac{r_0^2}{\chi_\perp} \approx \frac{\lambda_\perp^2}{\chi_\perp} < \tau_{||} \approx \frac{\lambda_z^2}{\chi_{||}} . \tag{52}$$

Then, the applicability condition for the transverse decorrelation takes the form

$$b_0\Omega_{He}\tau_{ei} < \mathrm{Ku}_m \approx \frac{b_0\lambda_z}{\lambda_\perp} , \tag{53}$$

where $\mathrm{Ku}_m > 1$ (Fig. 4).

To conclude this section, it is worth noting that the Kadomtsev–Pogutse result can be cast in a different form by making use of a collisionless diffusion coefficient with an additional correcting factor. The longitudinal particle velocity enters the expression for the longitudinal diffusion coefficient:

$$\chi_{||} \approx V_{||}^2\tau_{coll} . \tag{54}$$

On the other hand, the transverse diffusion coefficient takes the form $\chi_\perp \approx \lambda_\perp^2/\tau_{coll}$, where $\lambda_\perp$ is the transverse correlation scale. After simple calculations, we arrive at the formula for the effective diffusion coefficient relevant to the Kadomtsev–

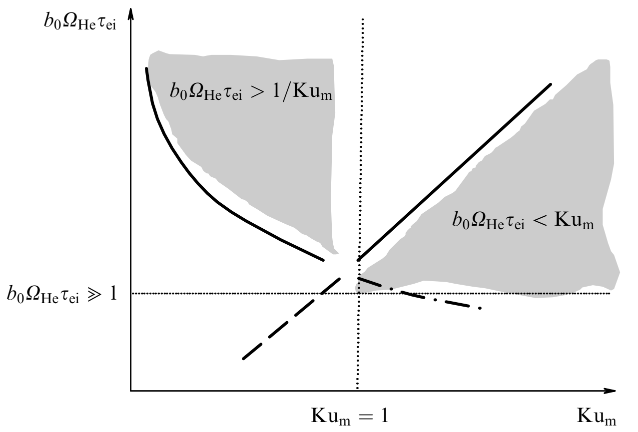


Figure 4. Diagram showing the applicability regions for anomalous regimes of electron transport in a steady 'braided' (stochastic) magnetic field, which were set up by Kadomtsev and Pogutse.

Pogutse regime:

$$D_{eff} \approx D_m V_{||} \frac{\lambda_\perp}{r_0} . \tag{55}$$

We treated the Kadomtsev–Pogutse model in terms of the correlation scale $\lambda_\perp$ and the characteristic time τ_{coll}. To describe plasma physics problems [19], these quantities can readily be related to the electron Larmor radius ρ_e and the electron–ion collision frequency ν_{ei}.

6. Stochastic instability and transport

Kadomtsev and Pogutse also examined a decorrelation mechanism which fundamentally differs from the transverse diffusion, is linked to the dispersion of initially close field lines, and owes its existence to the effect of stochastic instability [19, 30], which plays an important role in problems occurring in plasma physics and astrophysics (Fig. 5). It is assumed that on average two initially close

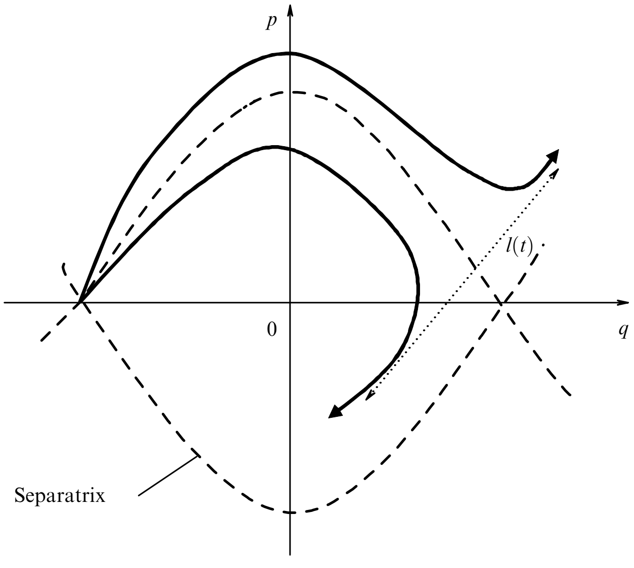


Figure 5. Stochastic instability of phase trajectories in the vicinity of a separatrix. Two initially close trajectories disperse exponentially with time over a distance on the order of $l(t)$.

field lines deflect from each other according to the law

$$l(z) = l_0 \exp\left(\frac{z}{\lambda_{\rm K}}\right). \tag{56}$$

Here, l_0 is the initial distance between the field lines, and z is the distance travelled along the field line. The quantity $h_{\rm K} = 1/\lambda_{\rm K}$ is called the Kolmogorov entropy:

$$\frac{1}{L_{\rm K}} \approx h_{\rm K} = \lim_{l_0 \to 0,\, z \to \infty} \left(\frac{1}{z} \ln \frac{\varDelta(z)}{l_0}\right). \tag{57}$$

Based on the equation of motion in the Lagrange form, already used earlier for the analysis of the field line diffusion, namely

$$\frac{{\rm d}\mathbf{r}_\perp}{{\rm d}t} = \mathbf{b}(z, \mathbf{r}_\perp), \quad \mathbf{b} = \frac{\mathbf{B}'}{B_0} \approx \mathbf{b}_0, \tag{58}$$

Kadomtsev and Pogutse derived the expression

$$\frac{\partial}{\partial z}(r_2 - r_1) = b(z, r_2) - b(z, r_1) \approx \frac{\partial b}{\partial r}(r_2 - r_1) \tag{59}$$

describing the dispersion of field lines of a stochastic field for a small departure $r_2 - r_1$ [19]. Formal calculations result in the exponential dependence

$$r_2(z) - r_1(z) \approx \Delta r(z = 0) \exp\left(\int_0^z \frac{\partial b}{\partial r}\, {\rm d}z\right). \tag{60}$$

The increment of stochastic instability can be found by averaging this expression with the assumption that the random quantity b is Gaussian in character, which allows the mean value to be evaluated by the standard formula

$$\langle \exp A \rangle = \exp \frac{\langle A^2 \rangle}{2}. \tag{61}$$

Whence we obtain

$$\varDelta_\perp(z) = \langle r_2(z) - r_1(z) \rangle = \varDelta_0 \exp\left(\frac{1}{2}\int_0^z\!\!\int_0^z {\rm d}z'\,{\rm d}z'' \left\langle \frac{\partial b(z', r)}{\partial r}\frac{\partial b(z'', r)}{\partial r}\right\rangle\right). \tag{62}$$

The integral expression in formula (62) is analogous to the expression for the quasilinear diffusion coefficient and, after simple manipulations, allows obtaining the increment γ_z of stochastic instability in the form

$$\gamma_z = \frac{1}{2}\int_{-\infty}^{\infty} \left\langle \frac{\partial b(0,0)}{\partial r}\frac{\partial b(z,0)}{\partial r}\right\rangle {\rm d}z. \tag{63}$$

In terms of the dimensionless parameter — the magnetic Kubo number $\rm Ku_m$ — this result takes the form

$$\gamma_z \approx \frac{b_0^2 \lambda_z}{\lambda_\perp^2} \approx \frac{D_{\rm m}}{\lambda_\perp^2} \approx \frac{1}{\lambda_z}\,{\rm Ku}_{\rm m}^2. \tag{64}$$

Naturally, the applicability limits of this estimate coincide with the applicability limits of quasilinear approximation ${\rm Ku_m} \approx b_0 \lambda_z/\lambda_\perp < 1$.

Kadomtsev and Pogutse utilized this scaling to estimate the transport effects in which the main decorrelation mechanism is provided by stochastic instability (Fig. 6). Writing out the correlation time, based on dimensional considerations, in the form

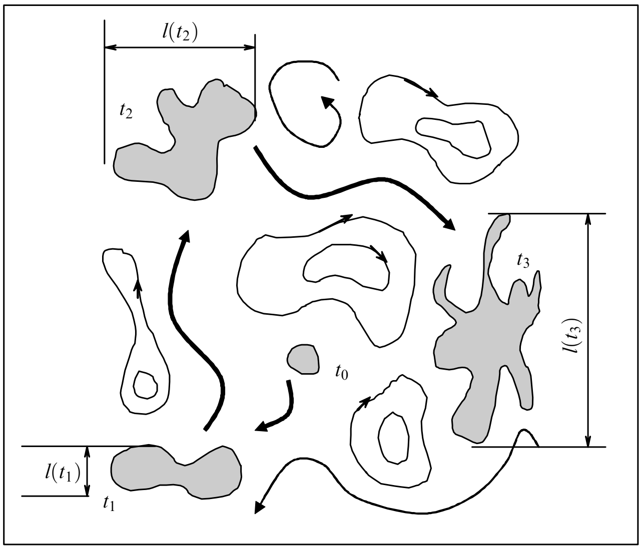


Figure 6. The evolution of a cell as a result of emerging stochastic instability. The characteristic cell size grows with time under the action of a random (turbulent) field: $l_0 \ll l(t_1) \ll l(t_2) \ll l(t_3)$.

$$\tau \approx \frac{1}{\gamma_z^2 \chi_{||}} \approx \frac{\lambda_z^2}{\chi_{||}}\,{\rm Ku}_{\rm m}^{-4}, \tag{65}$$

they modified the expression for the transverse diffusion coefficient on scales $z > \lambda_z$:

$$D_\perp(\tau) \propto D_{\rm m}\sqrt{\frac{\chi_{||}}{\tau}}, \tag{66}$$

having obtained the formula

$$D_\perp \propto D_{\rm m}\,\chi_{||}\gamma_z \approx D_{\rm m}\,\chi_{||}\,\frac{{\rm Ku}_{\rm m}^2}{\lambda_z}. \tag{67}$$

We emphasize that, because the quasilinear approximation has been used for the increment of stochastic instability, the same quasilinear approximation should also be tapped for the field line diffusion coefficient: $D_{\rm m} \propto b_0^2 \lambda_z$. In this case, we arrive at a fundamentally new scaling for the effective coefficient of transverse particle diffusion in a stochastic magnetic field:

$$D_\perp \propto \chi_{||} b_0^2\,{\rm Ku}_{\rm m}^2 \approx \chi_{||} b_0^4 \left(\frac{\lambda_z}{\lambda_\perp}\right)^2. \tag{68}$$

Such a regime of electron transport was later obtained in the well-known study [31]. The applicability condition for this regime can also be found from the principle of fast mode selection, $1/\gamma_\perp \approx \lambda_\perp^2/[D_\perp(\chi_{||})] < \lambda_\perp^2/\chi_\perp \approx \tau_\perp$, which corresponds to the condition $D_\perp \propto \chi_{||} b_0^4 (\lambda_z/\lambda_\perp)^2 > \chi_\perp$. Performing calculations for the applicability region of the regime alluded to, we obtain the inequality $b_0 \Omega_{\rm He} \tau_{\rm ei} > 1/({\rm Ku_m}) \approx \lambda_\perp/(b_0\lambda_z)$ (Fig. 4), where ${\rm Ku_m} \ll 1$. Let us mention that this condition can be interpreted in terms of characteristic spatial scales as well: $l_{\rm cor}(\chi_{||}) \propto (\chi_{||}/\gamma_\perp)^{1/2} \approx \lambda_z/{\rm Ku_m} \gg \lambda_z$.

It should be mentioned that a broad diversity of regimes are encountered in particle diffusion in a stochastic magnetic field. A discussion of all the existing variants is beyond the scope of this article, and we utilized the simplest model of randomly walking field lines in order to demonstrate, based on simple dimensional estimates, the importance of Kadomtsev's ideas on the crucial connection between longitudinal and transverse correlations in the analysis of anisotropic transport problems.

7. Vortex structures and anomalous transport

The development of views on strong plasma turbulence has led to the recognition of the importance of convective cells generated in the course of thermal convection in a plasma column with current. This question was addressed in detail in the review by Kadomtsev and Pogutse [32] in 1966 and in *Review of Plasma Physics* [33], where the first estimates were obtained for the transport coefficients in a tokamak plasma with due regard for the presence of large-sized convective cells. It is noteworthy that, in the discussion devoted to Kadomtsev's talk at the conference of the International Atomic Energy Agency (IAEA), these results were called outstanding.

Further research in the field of drift-convective turbulence confirmed the need to account for the impact of vortex structures on turbulent transport processes. Simple estimates can be obtained by considering regular vortex structures (convective cells) (Fig. 7), as has been done in Ref. [34].

In the case of two-dimensional plasma flow, the system of streamlines in a model of convective cells can be described in terms of steady streamfunctions $\Psi(x,y)=\Psi_0\sin(k_x x)\times\sin(k_y y)$, with the component of velocity expressed as

$$V_x=\frac{\partial\Psi}{\partial y}\,,\quad V_y=-\frac{\partial\Psi}{\partial x}\,. \tag{69}$$

Let us select the cell size λ and the characteristic velocity V_0 of convective flow as the parameters. The effective transport in the system of vortices can be described by relying on the decorrelation mechanism involving the diffusive escape of particles from a convective layer of width Δ. It is natural to use the characteristic time it takes a particle to leave the boundary (convective) layer as the correlation time $\tau\approx\Delta^2/D_0$. Here, D_0 is the coefficient of the 'seed diffusion' or the diffusion which shows its worth because of stochastization of plasma particle trajectories in the vicinity of separatrices between the vortices. We write down the effective diffusion coefficient in the convective form

$$D_{\text{eff}}\approx\lambda V_0 P_\infty\,, \tag{70}$$

where P_∞ is the portion of space responsible for convection. The portion P_∞ can readily be estimated for convective cells:

$$P_\infty\approx\frac{\lambda\Delta}{\lambda^2}\approx\frac{\Delta}{\lambda}\,. \tag{71}$$

The expression for the turbulent diffusion coefficient now coincides with the renormalized quasilinear Kadomtsev–Pogutse estimate considered in connection with the field line diffusion [19]:

$$D_{\text{eff}}\propto\lambda V_0\,\frac{\Delta}{\lambda}=V_0\Delta\,. \tag{72}$$

The estimate of stochastic layer width Δ follows from the balance of particles in the layer. This balance suggests that $N_D\propto D_0(n/\Delta)\,\lambda$ particles come from the convective cell per unit time, while the convection along the boundary layer removes $N_{\text{conv}}\propto nV_0\Delta$ particles. As a result, we obtain

$$\Delta(V_0)=\sqrt{\frac{D_0\lambda}{V_0}}\approx\sqrt{D_0\tau}\,. \tag{73}$$

Finally, the formula for the coefficient of effective diffusion in a system of convective cells assumes the form

$$D_{\text{eff}}=\text{const}\sqrt{D_0V_0\lambda}\propto V_0^{1/2}\,. \tag{74}$$

As is seen, this result formally differs from both the quasilinear estimate $D_{\text{eff}}\propto V_0^2$ and the linear dependence $D_{\text{eff}}\propto V_0$.

Nondiffusive transport regimes can also be obtained for chaotic structures [35]. Consider random velocity fluctuations in the form of narrow convective flows of width l_0 and velocity V_0, which, as a whole, compose a system of randomly directed plane-parallel flows (Fig. 8). These flows act in the transverse direction on a particle diffusing with the molecular diffusion coefficient D_0. To calculate the transverse diffusion coefficient $D_\perp$, we employ the estimate

$$D_\perp(t)\approx\frac{\lambda_\perp^2}{t}\,, \tag{75}$$

where the transverse displacement is furnished by a quasiballistic estimate $\lambda_\perp(t)\approx V_0tP_\infty(t)$. Here, $\lambda_\perp$ is the transverse displacement over time t, and P_∞ is the relative fraction, $P_\infty(t)=\delta N(t)/N(t)$, of velocity pulsations δN noncompensated on the mean [27]. The quantity

$$N(t)\approx\frac{\sqrt{2D_0t}}{l_0} \tag{76}$$

represents the number of shear flows a particle crosses while executing longitudinal (diffusive) motion. We can discard the transverse diffusive displacement of a particle, since it is small

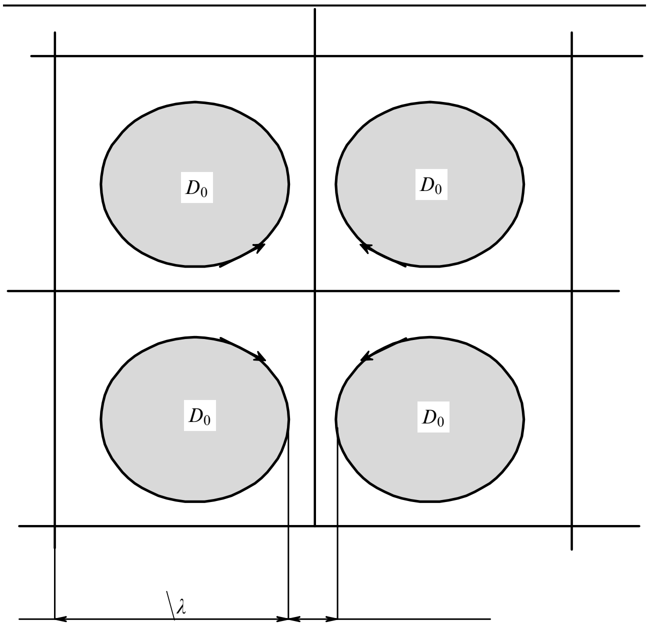


Figure 7. Convective cells: λ is the characteristic cell size, Δ is the width of the stochastic (diffusive) layer, D_0 is the coefficient of molecular (seed) diffusion which provides the particle decorrelation in a steady vortex flow.

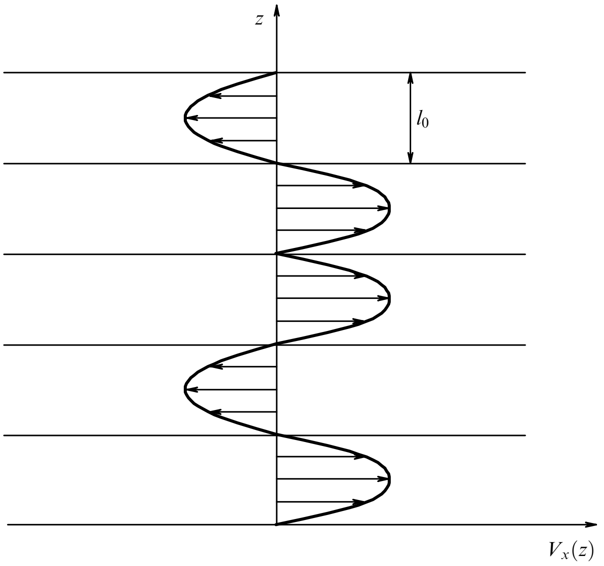


Figure 8. A Dreizin–Dykhne shear flow formed by an ensemble of randomly placed plane-parallel flows; l_0 is the flow characteristic width.

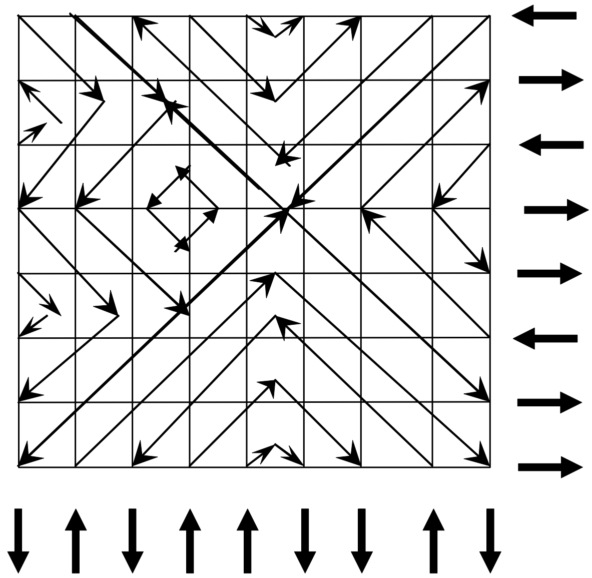


Figure 9. A two-dimensional random flow (Manhattan grid flow) formed as superposition of two mutually perpendicular random Dreizin–Dykhne shear flows.

compared to the convective transport. Then, estimating $\delta N(t) \approx \sqrt{N(t)}$ with the help of 'Gaussian statistics', it is a simple matter to make up the formula for the effective diffusion coefficient:

$$D_\perp(t) \propto V_0^2 l_0 \sqrt{\frac{t}{D_0}}, \tag{77}$$

or for the root-mean-square particle displacement:

$$\lambda_\perp^2(t) \propto D_\perp(t)\, t \propto t^{3/2}. \tag{78}$$

In the superdiffusion case under consideration, we obtain the Hurst exponent $H = 3/4 > 1/2$.

The model we consider can be generalized by superimposing two mutually perpendicular shear flows: $\Psi(x, z) = \Psi^x(z) + \Psi^z(x)$. In this case, we create a random steady system of vortices (Fig. 9), in which the Kadomtsev–Pogutse renormalization method based on the 'isotropization' related to long-range correlations is applicable to the description of particle transport. Thus, replacing the seed molecular diffusion coefficient in the Dreizin–Dykhne formula (for the diffusion coefficient) by the effective diffusion coefficient

$$D_{\text{eff}}(t) \propto V_0^2 \frac{l_0}{\sqrt{D_{\text{eff}}}}\, t^{1/2}, \tag{79}$$

we get the expression which coincides with the exact problem solution obtained using renormalization group methods [35–37]:

$$D_{\text{eff}}(t) \propto V_0 l_0 \left(\frac{V_0}{l_0}\, t\right)^{1/3}, \quad R(t) \propto D_{\text{eff}}(t)\, t \propto l_0 \left(\frac{V_0}{l_0}\, t\right)^{2/3}. \tag{80}$$

This example of superdiffusive transport in a 'mathematically' organized random steady vortex flow demonstrates how nontrivial the transport regimes can emerge in the presence of vortex structures.

B B Kadomtsev undoubtedly realized that under conditions of strong turbulence the appearance of large-scale vortex structures would inevitably lead to turbulent mixing and an enhancement of the effective transport. He attributed the anomalous character of diffusion in such regimes to the percolation character of streamline behavior [19] in two-dimensional random flows (Fig. 10). Such streamlines embrace nearly the entire flow because of their convoluted character (fractality). The criterion of strong turbulence formulated by Kadomtsev and Pogutse in terms of the magnetic Kubo number can be readily interpreted for a system of streamlines of a two-dimensional turbulent flow, $\text{Ku} = V_0/(\lambda\omega) \gg 1$. Here, V_0 is the characteristic amplitude of turbulent pulsations, λ is the characteristic size of vortices, and ω is the characteristic frequency of perturbations. In this framework, one not only succeeds in considering random steady flows, but can also take into account the mechanisms of flow topology reorganization important for low-frequency drift turbulence. We will briefly consider how the percolation approach has been adapted to two-dimensional and quasi-two-dimensional flows in Sections 8–10.

8. Coherent structures and percolation transport

In Section 2, we briefly mentioned Kraichnan's idea about the importance of large-scale vortex formations for the description of hydrodynamical turbulence. B B Kadomtsev pointed out the errors of that approach in relation to the description of the Kolmogorov energy cascade. Nevertheless, the idea itself that coherent (vortex) structures influence the processes related to turbulence did not escape Boris Borisovich's attention. In 1978, he succeeded in proposing an approach [19] in which largely stretched vortex lines contribute substantially to particle transport, even though they occupy only a small volume portion (see Fig. 10).

Until the publication of the work by Kadomtsev and Pogutse [19], which proposed the idea of percolation equipotentials, a rigorous mathematical expression for their description had not yet been available. Only in 1987 was it rigorously demonstrated that the 'shell' of a percolation cluster, which is a prototype for the percolation streamline in the framework of the topographic model, is described by

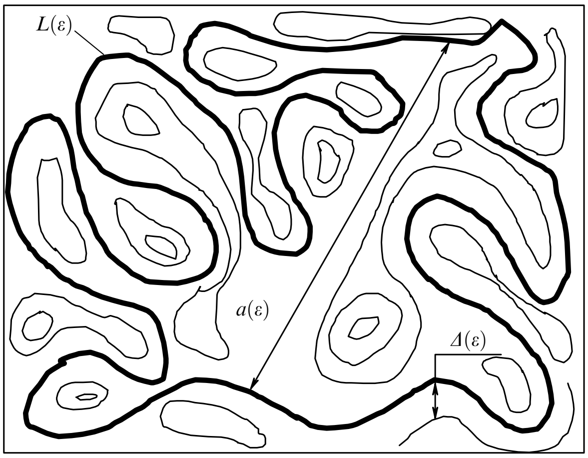


Figure 10. A percolation streamline in a two-dimensional random flow: λ is the characteristic cell size, $\Delta(\varepsilon)$ is the stochastic (diffusive) layer width, $a(\varepsilon)$ is the correlation scale governing the transport, and $L(\varepsilon)$ is the length of the percolation streamline.

the scaling [38]

$$L(\varepsilon) \propto \lambda \left(\frac{1}{\varepsilon}\right)^{\nu D_{\mathrm{H}}}, \quad \nu = \frac{4}{3}, \quad D_{\mathrm{H}} = 1 + \frac{1}{\nu}. \tag{81}$$

Here, λ is the characteristic size, D_{H} is the Hurst exponent, ν is the correlation exponent, and ε is a small parameter indicating how close the system is to the percolation threshold [24].

The streamlines $\Psi = \Psi(x, y)$ in this approach are treated as coastlines appearing because of flooding of a hilly landscape by water (Fig. 11). It is expected that there is a sharp transition from 'separate lakes on an infinite stretch of land to separate islands in an infinite ocean.' The percolation theory relies on the existence of at least one coastline of infinite length. The related streamfunctions can be modelled by 'perturbing' the relief corresponding to the system of convective cells [32, 36, 37, 39–44]. In fact, $\varepsilon \approx \delta\Psi/\lambda V_0$, where $\delta\Psi$ is the magnitude of the streamfunction in the vicinity of the percolation threshold, and V_0 is the characteristic flow velocity. In the theory of continual percolation, the correlation length $a(\varepsilon)$ (the transverse size of the percolation cluster) in the vicinity of the percolation threshold, $\varepsilon \to 0$, is given by the scaling

$$a(\varepsilon) \approx \lambda \left(\frac{1}{\varepsilon}\right)^{\nu} \approx \lambda \left(\frac{L(\varepsilon)}{\lambda}\right)^{1/D_{\mathrm{H}}}. \tag{82}$$

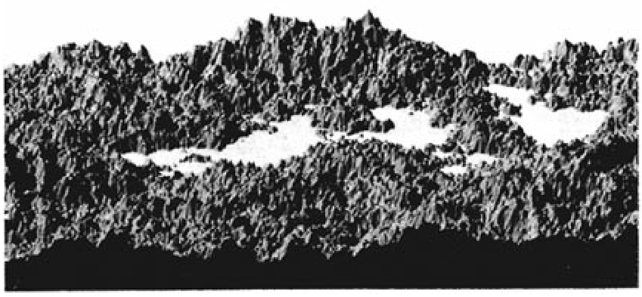


Figure 11. Flooding of a hilly landscape enabling one to model the phase transition in the framework of the continual percolation model.

It is assumed that the particles move along streamlines and, consequently, in the percolation limit $\Delta_{\mathrm{cor}} \approx a(\varepsilon)_{\varepsilon \to 0} \to \infty$. This difficulty can be overcome through renormalization of the small percolation parameter ε. Whereas detailed coverage of the renormalization methods can be found in reviews [25, 36, 45], we discuss some important findings in the theory of turbulent transport, which were derived following the method proposed by B B Kadomtsev.

Reference [40] explores a one-scale unsteady random flow with the streamfunction $\Psi(x, y, t)$, such that

$$\Psi_0 \approx \lambda V_0, \quad \lambda \approx \left|\frac{\Psi}{\nabla\Psi}\right|, \tag{83}$$

where the unsteadiness furnishes the reorganization of the streamline topology. In conditions of low-frequency drift turbulence, when

$$\mathrm{Ku} = \frac{V_0}{\lambda\omega} \gg 1, \quad \text{or} \quad \omega \ll \frac{V_0}{\lambda}, \tag{84}$$

namely the reorganization of the flow is a dominant decorrelation mechanism. Here, ω is the characteristic oscillation frequency. The formal expression for the diffusion coefficient in the percolation limit is written out as

$$D_{\mathrm{eff}} = \int_0^{\infty} \frac{\mathrm{d}\Psi_1}{\Psi_1} P_{\infty}(\Psi_1) \frac{a^2(\Psi_1)}{\tau(\Psi_1)}, \tag{85}$$

where the perturbation of the Hamiltonian in the framework of the mean field theory is given by the expression $\Psi_1 \approx \varepsilon_* \lambda V_0$. Calculations lead to the scaling

$$D_{\mathrm{eff}}(\varepsilon) \approx \frac{a^2}{\tau} P_{\infty} \approx \frac{a^2}{\tau} \frac{L(\varepsilon)\,\Delta(\varepsilon)}{a^2} \approx V_0 \Delta(\varepsilon). \tag{86}$$

Here, the correlation time τ is estimated on the basis of the conception that particles move ballistically along streamlines, $\tau \approx \tau_B \approx L(\varepsilon)/V_0$, $P_{\infty} = L(\varepsilon)\,\Delta(\varepsilon)/a^2(\varepsilon)$ is the portion of the volume occupied by percolation streamlines, Δ is the width of the percolation layer, and a is the correlation scale. In fact, the problem of obtaining the turbulent diffusion coefficient has been reduced in the approximation employed here to computing the width of the stochastic (percolation) layer, whereas the estimate $D_{\mathrm{eff}}(\varepsilon) \approx V_0\Delta(\varepsilon)$ is equivalent to the expression proposed by Kadomtsev and Pogutse for strong turbulence regimes.

Having estimated the time over which the flow pattern changes considerably as $T_0 \approx 1/\omega$, the authors of Ref. [40] supposed that the main parameter in the case of low-frequency perturbations is the lifetime τ of a single percolation line, which also represents the correlation time. In this case, the equation for the small parameter ε can be written down as

$$\tau(\varepsilon_*) = \varepsilon_* \frac{1}{\omega}, \quad \text{or} \quad \frac{L(\varepsilon_*)}{V_0} = \frac{\varepsilon_*}{\omega}. \tag{87}$$

Having used the scaling from the percolation theory, $L(\varepsilon) = \lambda(a/\lambda)^{D_{\mathrm{H}}}$, and taken the stochastic layer width for the 'physical analog' of the small parameter, $\Delta \approx \lambda\varepsilon$, we easily find ε_* as a function of flow parameters ω, V_0, and λ:

$$\varepsilon_* = \left(\frac{\lambda\omega}{V_0}\right)^{1/(2+\nu)} = \left(\frac{1}{\mathrm{Ku}}\right)^{3/10} \propto \omega^{3/10}. \tag{88}$$

Further manipulations now lead to the final expression for D_{eff} (see Fig. 1):

$$D_{\text{eff}}(\varepsilon_*) \approx \frac{a^2(\varepsilon_*)}{\tau(\varepsilon_*)}\, P_\infty(\varepsilon_*) \approx \lambda^2 \omega\, \text{Ku}^{7/10} \propto V_0^{7/10} \omega^{3/10} \,. \tag{89}$$

The dependence $D_{\text{eff}}(\omega)$ here differs fundamentally from the quasilinear scaling $D_{\text{eff}}(\omega) \propto V_0^2/\omega$. The length of the percolation streamline and the correlation scale, defined as

$$L(\varepsilon_*) \approx \lambda\, \frac{1}{\varepsilon_*^{\nu+1}} \approx \lambda\, \text{Ku}^{1+\nu/(2+\nu)} \,, \tag{90}$$

$$a(\varepsilon_*) = \lambda\, \frac{1}{\varepsilon_*^{\nu}} \approx \lambda\, \text{Ku}^{\nu/(2+\nu)} \,, \tag{91}$$

are not infinitely large in this approach, because the small parameter ε_* does not tend to zero, but takes a specific value of ε_* for all flow types with the characteristic D_0, V_0, and λ.

It is readily seen that by considering low-frequency regimes of drift plasma turbulence we may interpret the magnetic Kubo number in terms of drift velocity:

$$\text{Ku} \approx \frac{V_0}{\lambda\omega} \approx \frac{k^2 \varphi}{\omega B_0} \,. \tag{92}$$

Here, k is the wavenumber, φ is the electric potential perturbation, and B_0 is the amplitude of the magnetic field.

Notice that the problem of particle transport in a tokamak, which is in its essence three-dimensional, has been reduced to a two-dimensional one. Indeed, the equation of motion for leading centers has the form

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = V_{\|}\, \mathbf{e}_z + \mathbf{V}_\perp = V_{\|}\, \mathbf{e}_z + c\, \frac{\mathbf{B} \times \nabla\varphi(\mathbf{r}, z, t)}{B^2} \,. \tag{93}$$

In the limit when the collision frequency is smaller than the characteristic oscillation frequency, we can assume that the longitudinal velocities are constant and represent the electric potential in a simplified form

$$\varphi(\mathbf{r}) = \varphi(x, y, z_0 + V_{\|} t, t) \,, \tag{94}$$

where z_0 is the particle's initial coordinate. The respective Hamiltonian (the two-dimensional streamfunction) is given by the expression

$$\Psi(x, y, t) = -\frac{c}{B}\, \varphi(x, y, z_0 + V_{\|} t, t) \,. \tag{95}$$

In fact, we are dealing with a Hamiltonian system with 1.5 degrees of freedom [46, 47]. Namely for this reason we have been considering the reorganization of equipotentials in low-frequency drift turbulence on the basis of ideas associated with the Hamiltonian diffusion:

$$V_x(x, y, t) = -\frac{\partial \Psi(x, y, t)}{\partial y} \,, \tag{96}$$

$$V_y(x, y, t) = \frac{\partial \Psi(x, y, t)}{\partial x} \,. \tag{97}$$

Thus, the scaling obtained for the effective coefficient of turbulent diffusion can be written down, with account for fluctuations of the electric potential, as

$$D_{\text{T}} \approx \frac{\omega}{k^2} \left(\frac{k^2 \varphi}{\omega B} \right)^{7/10} \propto \omega^{3/10} \left(\frac{\varphi}{B} \right)^{7/10} \,. \tag{98}$$

The applicability of this formula was repeatedly tested in numerical experiments, which showed that scalar particle transport in drift turbulence is well described by the percolation model [48–52].

9. Long-range correlations and the Bohm scaling

The percolation approach to the description of transport in two-dimensional drift turbulence, presented in Section 8, neglects numerous aspects having a bearing on the configuration of actual plasma traps. Indeed, the description of particle transport in a tokamak plasma requires refinements in accounting for the toroidal geometry of the facility. Additionally, new specific effects linked to the toroidal drift arise here [23, 51]. Let us use the classical expression for the magnetic field in a tokamak with concentric magnetic surfaces [1, 2, 23, 51]:

$$\mathbf{B} = \left(B_\varphi \hat{\boldsymbol{\varphi}} + B_\theta(r)\, \hat{\boldsymbol{\theta}} \right) (1 - \varepsilon_{\text{T}} \cos\vartheta) \,, \tag{99}$$

and utilize the traditional notation for the parameters of tokamak plasma: $\varepsilon_{\text{T}} = r/R$, and $B_\theta = \varepsilon_{\text{T}} B_0/q$ (Fig. 12). The equations of ion motion in the tokamak magnetic field, with account for turbulent fluctuations, take the form [53, 54]

$$\frac{\mathrm{d}\mathbf{r}_\perp}{\mathrm{d}t} = \frac{c}{B_0}\, \mathbf{E} \times \mathbf{b} + V_{\|} \mathbf{b} + U_{\text{d}} \left(\hat{\boldsymbol{\theta}} \cos\vartheta(t) + \hat{\mathbf{r}} \sin\vartheta(t) \right). \tag{100}$$

For longitudinal motion, we obtain the equation

$$\frac{\mathrm{d}V_{\|}}{\mathrm{d}t} = \frac{e}{m}\, E_{\|} - \frac{\mu B_0 \varepsilon_{\text{T}}}{qR} \sin\theta \,, \quad V_{\|} = \frac{\mathrm{d}z}{\mathrm{d}t} \,, \tag{101}$$

$$\theta(t) = \frac{z(t)}{qR} \,, \quad z(t) = \int^t V_{\|}(t')\, \mathrm{d}t' \,. \tag{102}$$

The magnetic moment is given by the expression $\mu = V_\perp^2/B$, and the drift velocity by the formula [23, 51]

$$U_{\text{d}} \approx \frac{V_\perp^2 + V_{\|}^2/2}{\omega_{B_{\text{i}}} R} \approx \frac{V_{\text{T}_{\text{i}}}^2}{\omega_{B_{\text{i}}} R} \,. \tag{103}$$

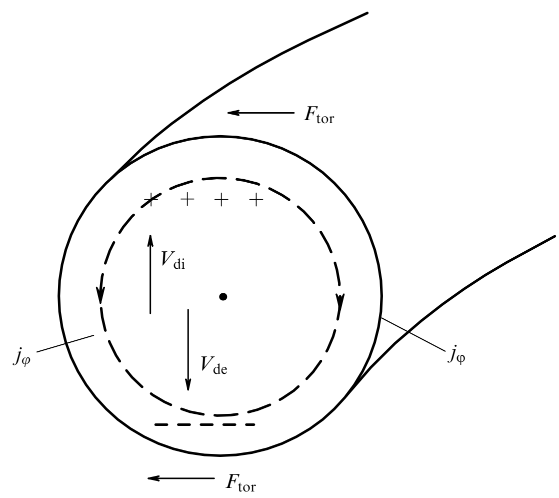


Figure 12. The onset of toroidal drift in a tokamak. Here, V_{de} is the electron drift velocity, V_{di} is the ion drift velocity, F_{tor} is the force creating the toroidal drift, and j_φ is the current in the tokamak plasma column.

For the problem in question, it is assumed [23] that $\omega \approx k_{\perp} cT/(eB_0 L_n) \approx (10^3 - 10^5)\ \mathrm{s}^{-1} \ll \omega_{B_i} \approx 10^8\ \mathrm{s}^{-1}$ and that the amplitude of turbulent fluctuations lies within the limits $U_d \leqslant V_0 \leqslant 10 U_d$. In accordance with the approximations made, we get the Hamiltonian describing the motion of plasma particles:

$$H(x, y, t) = \frac{c\widetilde{\Phi}(r, t) + V_{||}(t)\,\widetilde{A}(r, t)}{B} + U_d \cos\left[\theta(t) + \frac{y}{r_0}\right]. \qquad (104)$$

Here, the notations $\mathbf{B} = \mathrm{rot}\,\widetilde{\mathbf{A}}$, $x = r - r_0$, and $y = r_0\theta(t)$ are used. As in Section 8, we are dealing with a Hamiltonian describing a system with 1.5 degrees of freedom. This is a classical problem in the theory of dynamical systems [55], because chaos does not emerge in Hamiltonian systems with one degree of freedom. Getting a description of dynamical chaos requires the methods of statistical analysis, which substantially complicates the application of traditional methods of Hamiltonian dynamics. Splitting of separatrices arising in such problems, which leads, in turn, to the emergence of stochastic layers, results in the need to consider specific features in the behavior of equipotentials and new decorrelation mechanisms [56–59]. In terms of the streamfunction, we obtain

$$\Psi(x, y, t) = \Psi_0(x, y, t) + \Psi_1(x, y, t), \quad \langle \Psi_0 \rangle = 0. \qquad (105)$$

In our case, the impact of turbulent fluctuations is described by the function

$$\Psi_0(x, y, t) = \frac{c\widetilde{\Phi}(r, t)}{B} \approx \frac{V_0}{k_{\perp}}. \qquad (106)$$

The contribution of drift effects, which are related to the toroidal drift velocity, is defined by the expression

$$\Psi_1(x, y, t) = U_d\, r \cos\left(\theta(t) + \frac{y}{r_0}\right). \qquad (107)$$

In the framework of the scaling analysis, we have the estimate for the ratio between characteristic velocities of the chosen model:

$$\frac{V_0}{U_d} \approx \frac{\rho_i V_{T_i}}{L_n} \frac{\omega_{B_i} R}{V_{T_i}} \approx \frac{R}{L_n} \approx 10 \gg 1, \qquad (108)$$

$$\omega_{B_i} = \frac{eB}{m_i c}, \quad L_n = \frac{n}{\nabla n} \approx 15\ \mathrm{cm}, \qquad (109)$$

where we have exploited the relationship for the potential perturbation (mixing rule):

$$T\nabla n \approx en\nabla\widetilde{\Phi}, \qquad (110)$$

first proposed by Kadomtsev [21], which assumes that electrons rapidly reach equilibrium and that the problem of transport description reduces to the analysis of ion diffusion in plasma. It is then not difficult to give the estimate on the amplitude of potential fluctuations:

$$\frac{e\widetilde{\Phi}}{T} \approx \frac{\delta n}{n} \approx \frac{\rho_i}{L_n} \approx \frac{1}{k_{\perp} L_n}. \qquad (111)$$

Note that we assumed earlier that the characteristic frequencies (stochastic resonance) ω pertaining to the decorrelation time of drift turbulence and the frequency characterizing the toroidal motion of particles, $\omega_z = V_{||}/(qR)$, are of the same order of magnitude.

In the problem posed here, the decorrelation mechanism is directly linked both to the presence of drift causing a reconnection of equipotentials and to topology reorganization under the conditions of low-frequency turbulence. We introduce the Hamiltonian diffusion coefficient D_{Ψ}, which makes allowance for both these factors [54]:

$$D_{\Psi} \approx (\delta\Psi)^2 \omega \approx U_d^2 a^2(\varepsilon)\omega. \qquad (112)$$

Here, $a(\varepsilon) \approx \lambda|\varepsilon|^{-\nu}$ is the correlation scale in the percolation approximation. It is convenient to write out the renormalization condition for the small parameter in the form $\tau_{\Psi}(\varepsilon_*) = \tau_B(\varepsilon_*)$ [54]:

$$\frac{(\varepsilon_* \lambda V_0)^2}{U_d^2 a^2(\varepsilon_*)\omega} = \frac{L(\varepsilon_*)}{V_0}, \qquad (113)$$

where we introduced the notation

$$\tau_{\Psi}(\varepsilon) \approx \frac{\Delta^2}{D_{\Psi}(\varepsilon)}, \quad \tau_B(\varepsilon) \approx \frac{L(\varepsilon)}{V_0}. \qquad (114)$$

Here, Δ is the percolation layer width. The solution of the equation yields the scaling

$$\varepsilon_* \approx \left(\frac{U_d}{V_0}\right)^{2/[3(1+\nu)]} \left(\frac{1}{\mathrm{Ku}}\right)^{1/[3(\nu+1)]} \propto U_d^{2/7} V_0^{-3/7} \omega^{1/7}, \quad \nu = \frac{4}{3}. \qquad (115)$$

In the end, we arrive at the formula for the effective transport coefficient which accounts for drift motions in the low-frequency percolation regime:

$$D_{\mathrm{eff}} \propto D_{\mathrm{plato}} \left(\frac{V_0}{U_d}\right)^{22/21} \left(\frac{1}{\mathrm{Ku}}\right)^{10/21}, \quad D_{\mathrm{plato}} \propto U_d^2 \tau_B. \qquad (116)$$

In the conditions of tokamak plasma, one has $\mathrm{Ku} \approx V_0/(\lambda\omega) \approx 5$, $V_0/U_d \approx 10$, and, consequently, the transport indeed exceeds traditional neoclassical values in the regimes when the collision frequency is unessential: $D_{\mathrm{eff}}(\mathrm{Ku}, V_0, U_d) \approx 5 D_{\mathrm{plato}}$.

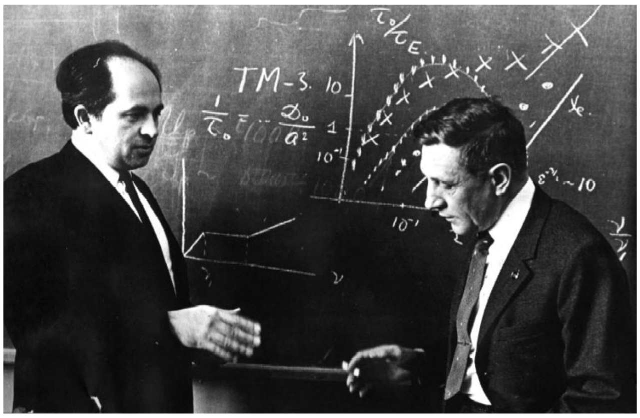
It should be note that the question concerning the transport of particles and heat in tokamaks is one of most important, for the characteristic time τ_E of plasma confinement entering the Lawson fusion ignition criterion

$$n\tau_E > 3 \times 10^{20}\ \mathrm{s\,m^{-3}} \qquad (117)$$

can be estimated through the effective transport coefficient: $\tau_E \propto r_0^2/(D_{\mathrm{eff}}(B_0, T_p))$. Here, n is the plasma density, r_0 is the minor tokamak radius, B_0 is the magnetic field induction, and T_p is the plasma temperature.

From the viewpoint of small-scale turbulence, the classical estimate of the turbulent diffusion coefficient is furnished by the gyro-Bohm scaling:

$$D_{\mathrm{GB}} \approx \frac{\rho_i}{L_n} D_B \approx \frac{cT_p}{eB_0} \frac{\rho_i}{L_n} \propto \frac{1}{B_0^2}, \quad D_B \approx \frac{cT_p}{eB_0} \propto \frac{1}{B_0}, \qquad (118)$$



B B Kadomtsev (left) and M K Romanovskii in 1975 at a scientific seminar devoted to the discussion of turbulent diffusion effects in the tokamak T-3 in comparison with neoclassical estimates.

where $D_{\rm B}$ is the Bohm diffusion coefficient. As an example, at $T_{\rm p} = 1$ keV and $B_0 = 1$ T, the Bohm scaling provides magnitudes which are one order of magnitude higher than the observed magnitudes of $D_{\rm T} \approx 1-5\ {\rm m^2\,s^{-1}}$. Namely for this reason, the gyroscaling derived by Kadomtsev and containing the factor $\rho_{\rm i}/L_n \ll 1$, was considered as a more appropriate candidate for describing plasma confinement. However, the dependence on the magnetic field proves to be too sharp:

$$\tau_E(B_0) \propto \frac{r_0^2}{D_{\rm GB}(B_0)} \propto B_0^2\,, \qquad (119)$$

because experiments point to a much smoother dependence.

The percolation scaling presented above, which takes into account both the drift effects and low-frequency turbulence, allows one to obtain a smoother dependence on the magnetic field intensity, and at the same time contains a reducing factor:

$$D_{\rm eff} \approx V_0 \varDelta \approx \lambda V_0\, \frac{\varDelta}{\lambda} \approx D_{\rm B}\, \frac{\varDelta}{\lambda}\,, \quad \frac{\varDelta(\varepsilon)}{\lambda} \approx \varepsilon \ll 1\,. \qquad (120)$$

Typical parameters of emerging structures can be estimated to an order of magnitude ($U_{\rm d} \approx 10^3\ {\rm m\,s^{-1}}$, $\lambda \approx 10^{-2}$ m, $V_0 \approx 10^{-4}\ {\rm m\,s^{-1}}$). Calculations, in fact, yield

$$D_{\rm eff} \approx \lambda V_0 \left(\frac{\lambda\omega}{V_0}\right)^{1/7} \left(\frac{U_{\rm d}}{V_0}\right)^{2/7} \propto \frac{1}{B_0^{6/7}}\,, \qquad (121)$$

where λV_0 corresponds to the Bohm estimate by order of magnitude. As a result, we arrive at

$$\tau_E(B_0) \propto \frac{r_0^2}{D_{\rm eff}(B_0)} \propto B_0^{6/7}\,. \qquad (122)$$

It is of interest to compare these theoretical estimates with the data from experiments conducted on modern tokamaks. Thus, in the tokamak JET (joint European torus), the scaling is $\tau_{\rm JET} \propto B_0^{0.26}$ [60, 61]. In the framework of 'single-facility scalings' [60], one needs to allow for the invariance of parameters $\beta_{\rm ex} = nT_{\rm p}/B_0^2 = {\rm const}$ and $\nu_{\rm ex} = (n/T_{\rm p}^2)\, L_{\rm ex} = {\rm const}$, which leads to the dependences for temperature and density of plasma as $T_{\rm p} \propto B_0^{2/3}$ and $n \propto B_0^{4/3}$. Then, for the gyro-Bohm scaling, we get $\tau_E \propto B_0$, for the Bohm formula $\tau_E \propto B_0^{0.333}$, and for the percolation model $\tau_E \propto B_0^{0.285}$. It is apparent that the percolation approach provides the best approximation. It is also worth noting that even insubstantial (15–20%) deterioration in plasma confinement may hinder reaching the thermonuclear ignition in the ITER (International thermonuclear experimental reactor), now under construction [61].

We see that the effects of long-range correlations — which allow us to consider the impact of large-scale structures forming in plasma — on transport can be analyzed with the help of the percolation concept framed by Kadomtsev and Pogutse.

10. Evolution of the stochastic layer and scalings

A salient feature of the results obtained by B B Kadomtsev is their universality, owing to which their range of applicability stretches far beyond the bounds one may anticipate based on the assumptions made by the author. For example, work by Isichenko [62, 63], who developed ideas of long-range correlations, offers calculations of electron transport in a stochastic magnetic field in the percolation approximation. A new regime proposed by the author of these studies takes into account both the evolution of the fractal structure of a magnetic field tube (Fig. 13) and the particle collisional decorrelation through the modification of the balance between characteristic scales. Nevertheless, the final expression for the diffusion coefficient turned out to be coincident with the scaling proposed by Kadomtsev and Pogutse [19] and derived qualitatively.

We dwell on the main elements of the analysis made in Refs [62, 63] in order to show how Kadomtsev's ideas are enriched if applied to studies of ever more complex transport mechanisms (Fig. 14). Let us consider the stage of percolation structure formation based on the formula for the perimeter $L(t)$ of a percolation cluster (shell). Then, the characteristic correlation size will be expressed by the formula

$$a(t) \propto \lambda_\perp \left(\frac{L(t)}{\lambda_\perp}\right)^{1/D_{\rm H}}. \qquad (123)$$

We can now express the evolving width of the stochastic layer based on the following scaling (Fig. 15):

$$\varDelta_\perp(t) \propto \lambda_\perp \left(\frac{\lambda_\perp}{a(t)}\right)^{1/\nu} \propto \lambda_\perp \left(\frac{\lambda_\perp}{L(t)}\right)^{1/(\nu D_{\rm H})}. \qquad (124)$$

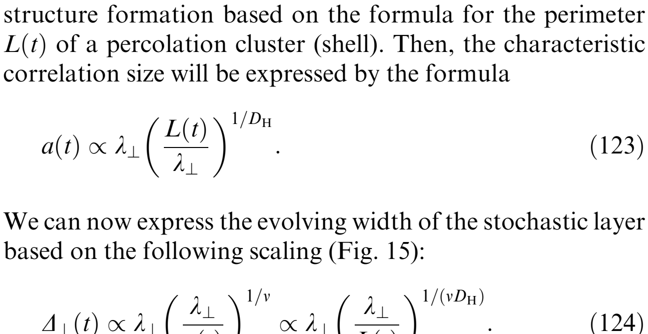


Figure 13. The evolution of a tube of magnetic field lines as a result of the action of stochastic instability. Here, $\varDelta_\perp$ is the stochastic layer width in the percolation limit.

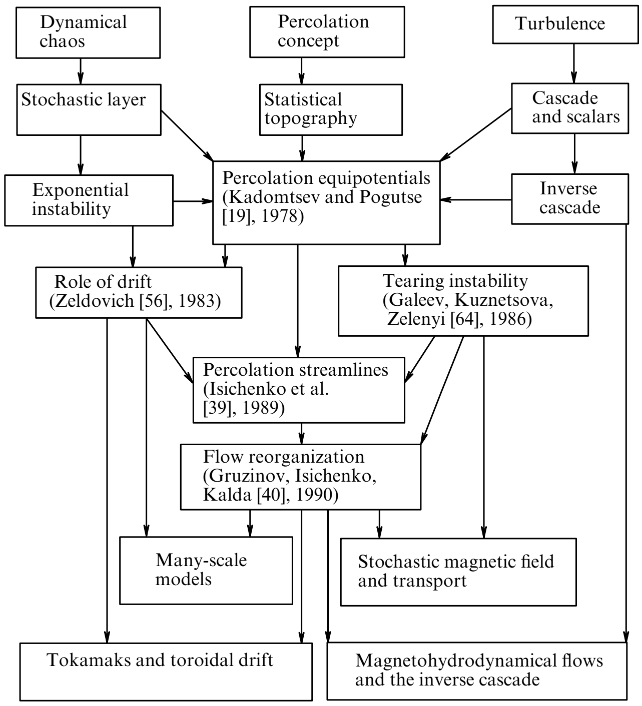


Figure 14. Schematic illustrating the inception and development of the percolation approach in application to the description of anomalous transport under conditions of strong turbulence in the presence of large-scale vortex structures.

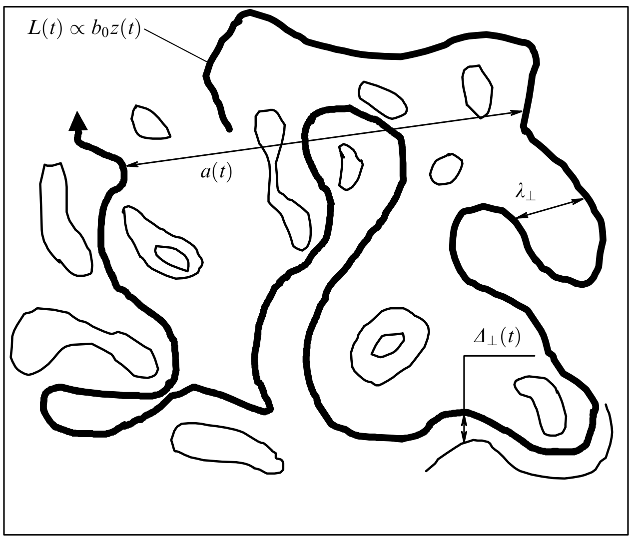


Figure 15. Evolutionary stage of percolation structure growth. Here λ is the characteristic cell size, $\Delta(t)$ is the width of the stochastic (diffusive) layer, $a(t)$ is the correlation scale governing the transport, and $L(t) \propto b_0 z(t)$ is the length of the percolation stream line.

Substituting the ballistic estimate for the projection of the path travelled by a magnetic field line onto the transverse section of the plasma ring, $L(z(t)) \approx b_0 z(t)$, in this relationship, we get the scaling describing the reduction in the stochastic layer width:

$$\Delta_\perp(t) \approx \lambda_\perp \left(\frac{\lambda_\perp}{b_0 z(t)} \right)^{1/(\nu D_{\mathrm{H}})} \propto \frac{1}{(b_0 z)^{3/7}} . \qquad (125)$$

In agreement with the ideas of Batchelor, Rechester, and Rosenbluth [65, 66], decorrelation will ensue as a result of particle transitions from one field line to another when the characteristic scale of the stochastic layer would compare with the characteristic diffusive scale: $\Delta_\perp(\tau) \approx \Delta_{\mathrm{dis}}$. In our case, it is the transverse diffusion coefficient $\chi_\perp$ of particles:

$$\lambda_\perp \left(\frac{\lambda_\perp}{b_0 z(\tau)} \right)^{1/(\nu D_{\mathrm{H}})} \approx \sqrt{4 \chi_\perp \tau} . \qquad (126)$$

Assuming, as in the model constructed by Kadomtsev and Pogutse, that longitudinal motions of particles bear a diffusive character, $z^2(\tau) \approx 2\chi_{||}\tau$, we can resolve Eqn (126) and determine the characteristic correlation time

$$\tau \approx \frac{\lambda_\perp^2}{\chi_\perp} \left(\frac{\chi_\perp}{b_0^2 \chi_{||}} \right)^{1/(\nu+2)} \approx \tau_\perp \left(\frac{\chi_\perp}{b_0^2 \chi_{||}} \right)^{1/(\nu+2)} . \qquad (127)$$

This result is valid provided $\tau < \tau_\perp \approx \lambda_\perp^2 / \chi_\perp$, which is correct if the inequality

$$\frac{\chi_\perp}{b_0^2 \chi_{||}} < 1 \qquad (128)$$

holds true, where $b_0 \ll 1$ and $\chi_\perp / \chi_{||} \ll 1$. As a result, we get the expression in terms of the small percolation parameter and characteristic model parameters:

$$\tau \approx \tau_\perp \varepsilon_*^2 \left[\frac{\chi_\perp}{\chi_{||}} \left(\frac{\lambda_{||}}{\lambda_\perp} \right)^2 \right]^{1/(\nu+2)} \approx \tau_\perp \varepsilon_*^2 \left(\frac{\tau_{||}}{\tau_\perp} \right)^{1/(\nu+2)} . \qquad (129)$$

Using the expression for the effective diffusion coefficient in the percolation limit, already mentioned several times in the preceding sections, we get the scaling

$$D \propto \frac{r_\perp^2(\tau)}{\tau} \approx \frac{a^2(\tau)\, P_\infty(\tau)}{\tau} \approx \frac{\lambda_\perp^2}{\tau} \left(\frac{L(z)}{\lambda_\perp} \right)^{1/D_{\mathrm{H}}} , \qquad (130)$$

where we took into account that the Kolmogorov spatial scale λ_{K} for the development of stochastic instability is approximately equal to the mixing scale λ_{m}:

$$\lambda_z < z < \lambda_{\mathrm{m}} \approx \varepsilon_* \lambda_z \approx \lambda_z \left(\frac{1}{\mathrm{Ku_m}} \right)^{1/(\nu+2)} . \qquad (131)$$

As before, $P_\infty \approx \lambda_\perp / a$ is the effective portion of space contributing to the percolation transport: $L(z) \approx b_0 z \approx b_0 \sqrt{2\chi_{||}\tau}$. On substituting, we arrive at the scaling which describes the evolution of the stochastic layer width:

$$\Delta_\perp(t) \approx \lambda_\perp \left(\frac{\lambda_\perp}{b_0 z(t)} \right)^{1/(\nu D_{\mathrm{H}})} \propto \frac{1}{t^{3/14}} , \qquad (132)$$

and the expression for the effective transverse diffusion coefficient, which coincides with the formula devised by Kadomtsev and Pogutse:

$$D_\perp(\tau) \approx \lambda_\perp^2 \, \frac{b_0 \sqrt{\chi_{||}}}{\lambda_\perp} \, \tau^{(\nu+2)/[2(\nu+1)]} \approx b_0 \sqrt{\chi_{||} \chi_\perp} , \quad \nu = \frac{4}{3} , \qquad (133)$$

which is valid under the conditions

$$1 > \frac{\chi_\perp}{b_0^2 \chi_{||}} > \frac{1}{\mathrm{Ku_m^2}} . \qquad (134)$$

In fact, these are the conditions of strong turbulence, for which the percolation method has, in reality, been proposed: $Ku_m \gg 1$ [19].

These results point to the feasibility of the evolution model being proposed. Additionally, with the help of a similar method, one succeeds in obtaining principally new regimes of electron transport in a stochastic magnetic field in the presence of effects of unsteadiness [55, 62–64, 67], which proves the efficiency of models that can be treated analytically. The percolation approach to the description of anomalous transport in plasma under conditions of strong turbulence was only B B Kadomtsev's first step in the search for mechanisms of self-organization liable for plasma confinement in tokamaks [59, 68–71]. It is actively being developed at present as well [69, 70]. One may with certainty argue that this and other ideas of Boris Borisovich will invariably attract newer and newer researchers.

11. Conclusion

This paper considers models proposed by B B Kadomtsev to describe turbulent diffusion. We discuss some of the current approaches to the description of the effects of 'long-range correlations', which are directly based on the ideas of B B Kadomtsev concerning the diffusive renormalization of quasilinear equations, the percolation approach to the description of strong turbulence, and the impact of stochastic instability and transverse diffusion of plasma particles on transport in a 'braided' magnetic field. It is shown that the methods of analysis developed by B B Kadomtsev carry an immense 'heuristic potential' and will undoubtedly influence further advancement of the turbulent transport theory.

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