

METHODOLOGICAL NOTES

PACS numbers: 41.20.Cv, 41.20.Gz

Some useful correspondences in classical magnetostatics and multipole representations of the magnetic potential of an ellipsoid

R Z Muratov

DOI: 10.3367/UFNe.0182.201209d.0987

Contents

Abstract. **It is shown that for a given geometric body, the Ferrers theorem not only relates the potentials of volume- and surface-distributed scalar (charge or mass) sources (which it is known to do) but also relates the vector (scalar) magnetic field potentials produced by the volume- and surface-distributed densities of a stationary current (i.e., vector sources). For a body with a given magnetization, the magnetic multipole moments calculated from expressions for polarization magnetic charges are shown to be equal to those of Ampère currents. Using these results and noting the universality of the multipole expressions, multipole representations of the scalar magnetic potential of an ellipsoid can be (and, indeed, have been) obtained rather straightforwardly.**

1. Introduction

The problems of the theory of potentials discussed in this paper, which are of interest in and of themselves, appeared in our case in connection with the derivation of the multipole representation of the scalar magnetic potential of an ellipsoid. Unlike the theoretical analysis of electrostatic and gravitational potentials of scalar sources (charge or mass density), the theoretical analysis of the magnetic potential involves many calculations due to the vector nature of its sources [the volume $\mathbf{j}(\mathbf{r})$ and (or) surface $\mathbf{i}(\mathbf{r})$ current densities]. Another complication — in contrast to the case of scalar sources — is that the density vector of a stationary current must satisfy the solenoidal condition

$$\operatorname{div} \mathbf{j} = 0 \tag{1}$$

and the boundary condition

$$j_n\big|_S = 0 \tag{2}$$

on the surface S of the current region. Finally, the most radical difference between the scalar potential of charges (masses) and the scalar magnetic potential of currents is that the scalar potential does not exist in the spatial region where currents are present. Therefore, as is well known, the Poisson equation for the scalar potential does not exist, and hence its formal solution representing the potential in the form of an integral of the sources and the static Green function is absent. The standard approach to overcome this difficulty involves the use of the vector potential of a magnetic field.

It is natural, however, that when the external field is the main interest (as, for example, in the derivation of the multipole representation of the magnetic potential[1] of an ellipsoid), the use of the scalar potential becomes possible and efficient, especially when the spatial region with currents is simply connected. Relations discussed in this paper are of a quite general type (with respect to the geometrical shape of the region with currents and the form of functions describing the currents) and are useful for solving particular problems.

2. Ferrers rule for the vector and scalar magnetic potentials

In 1877, Ferrers [3] obtained a relation between the gravitational potentials of volume and surface distributions of sources for the same geometrical body. This result (the Ferrers rule) applied to Coulomb fields consists in the following. If we know an analytic expression for the potential

R Z Muratov Moscow State Mining University, Leninskii prosp. 6, 119991 Moscow, Russian Federation
E-mail: rodeszmu@yandex.ru

Received 21 March 2011
Uspekhi Fizicheskikh Nauk **182** (9) 987 – 997 (2012)
DOI: 10.3367/UFNr.0182.201209d.0987
Translated by M Sapozhnikov; edited by A M Semikhatov

[1] Multipole representations are useful, in particular, because (for sources distributed in an ellipsoidal region of space) they considerably simplify the solution of the problem with equivalent sources, i.e., those producing identical external fields, which was formulated by Frenkel [1, p. 103], [2, p. 524].

Φ^{T} of a charge distributed in a volume V with a density $\rho(x, y, z)$, where ρ is a homogeneous (of degree k) function of coordinates, then the expression

$$\Phi = (k+2)\Phi^{\mathrm{T}} - \mathbf{r}\,\frac{\partial \Phi^{\mathrm{T}}}{\partial \mathbf{r}} \tag{3}$$

allows finding the potential Φ of the surface charge distributed with the density $\sigma = \rho\, p$ on the boundary S of the same volume. Hereafter in this section, for an arbitrary point K on the boundary S, p is the length of a perpendicular lowered from the origin (located inside V) to a plane tangent to the surface S, which passes through K. It is important that the observation point in Ferrers rule (3) can be chosen both inside and outside the region V, while the shape of this region can be arbitrary.

It is interesting to see whether the Ferrers rule can be applied to stationary magnetic fields. We discuss this question following considerations similar to those presented in [4] in the derivation of expression (3).

We consider a bounded simply connected spatial region (hereafter, called a body) of volume V in which currents with a density $\mathbf{j}(\mathbf{r})$ circulate. Assuming that a closed surface S enclosing V is smooth and convex, we choose the origin of coordinates inside V. In the region containing currents, the magnetic induction vector $\mathbf{B}$ is not a potential vector. Therefore, as noted above, the scalar potential of the magnetic field cannot be directly expressed in terms of the current density $\mathbf{j}(\mathbf{r})$, similarly to the expression of the electrostatic potential in terms of the corresponding charge density. In this case, we should use the vector potential[2]

$$\mathbf{A}^{\mathrm{T}}(\mathbf{r}) = \frac{1}{\mathrm{c}}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V'.$$

Each Cartesian component of the volume current density is assumed to be a homogeneous function (of degree k) of coordinates:

$$\mathbf{j}(\lambda\mathbf{r}) = \lambda^k\,\mathbf{j}(\mathbf{r}), \tag{4}$$

where λ is an arbitrary constant. The vector potential of the same body at the point $\lambda\,\mathbf{r}$ is

$$\mathbf{A}^{\mathrm{T}}(\lambda\mathbf{r}) = \frac{\lambda^{k+2}}{\mathrm{c}}\int_{\overline{V}} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' = \lambda^{k+2}\,\overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{r}), \tag{5}$$

where $\overline{V}$ is the volume of a body whose boundary is similar to that of the initial body and is oriented similarly, but is located λ times closer to the origin,[3] and $\overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{r})$ differs from $\mathbf{A}^{\mathrm{T}}(\mathbf{r})$ only in that the integration region of $\overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{r})$ is $\overline{V}$ rather than V. Expression (5) was derived using the change of integration variables $\mathbf{r}' \to \lambda\mathbf{r}'$ and property (4).

We now assume that $\lambda = 1 + \varepsilon$, where ε is an arbitrarily small positive number. Then the boundaries of volumes V and $\overline{V}$ form a homothetic shell whose thickness at an arbitrary point is $\mathrm{d}p = p\varepsilon$. According to the superposition principle, the vector potential of the shell is the difference between the vector potentials of the bodies bounding the shell under the condition that the current densities in the bodies coincide in the volume shared by these two bodies. This is also valid, of course, for any Cartesian component of the vector potential. In particular, the component $\mathrm{d}A_x$ of the vector potential of an infinitely thin shell with an infinitesimal surface current density $\mathrm{d}\mathbf{i} = \mathbf{j}\,\mathrm{d}p = \mathbf{j}\,p\varepsilon$ is expressed as

$$\begin{aligned}\mathrm{d}A_x &= A_x^{\mathrm{T}}(\mathbf{r}) - \overline{A}_x^{\mathrm{T}}(\mathbf{r}) \\ &= A_x^{\mathrm{T}}(\mathbf{r}) - (1+\varepsilon)^{-k-2} A_x^{\mathrm{T}}(\mathbf{r}+\varepsilon\mathbf{r}) \\ &= \left[(k+2)A_x^{\mathrm{T}} - \mathbf{r}\nabla A_x^{\mathrm{T}}\right]\varepsilon,\end{aligned} \tag{6}$$

or

$$\mathrm{d}\mathbf{A} = \left\{(k+2)\,\mathbf{A}^{\mathrm{T}}(\mathbf{r}) - (\mathbf{r}\nabla)\,\mathbf{A}^{\mathrm{T}}(\mathbf{r})\right\}\varepsilon.$$

Taking into account that ε is a constant (although arbitrarily small) number, we conclude that the surface current with the *finite* density

$$\mathbf{i}(\mathbf{r}') = \mathbf{j}(\mathbf{r}')\,p(\mathbf{r}'), \qquad \mathbf{r}' \in S, \tag{7}$$

circulating in a zero-thickness shell corresponds to the vector potential

$$\mathbf{A}(\mathbf{r}) = (k+2)\,\mathbf{A}^{\mathrm{T}}(\mathbf{r}) - (\mathbf{r}\nabla)\,\mathbf{A}^{\mathrm{T}}(\mathbf{r}). \tag{8}$$

The observation point in expression (8) can be chosen both inside and outside the volume V.

We next consider the magnetic induction $\mathbf{B} = \operatorname{rot}\mathbf{A}$, in particular, some of the Cartesian components, for example, B_z. We successively obtain

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = (k+1)\,B_z^{\mathrm{T}} - x\,\frac{\partial B_z^{\mathrm{T}}}{\partial x} - y\,\frac{\partial B_z^{\mathrm{T}}}{\partial y} - z\,\frac{\partial B_z^{\mathrm{T}}}{\partial z}, \tag{9}$$

where A_x and A_y are determined by (8) and $\mathbf{B}^{\mathrm{T}} = \operatorname{rot}\mathbf{A}^{\mathrm{T}}$.

Outside the volume V, the vector $\mathbf{B}^{\mathrm{T}}$ is potential: $\mathbf{B}^{\mathrm{T}} = -\nabla\tilde{\Phi}^{\mathrm{T}}$. This allows rewriting (9) for observation points outside a body in the form

$$B_z = -(k+1)\,\frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial z} + x\,\frac{\partial^2\tilde{\Phi}^{\mathrm{T}}}{\partial x\,\partial z} + y\,\frac{\partial^2\tilde{\Phi}^{\mathrm{T}}}{\partial y\,\partial z} + z\,\frac{\partial^2\tilde{\Phi}^{\mathrm{T}}}{\partial z^2},$$

where $\tilde{\Phi}^{\mathrm{T}}$ is the external scalar potential of the magnetic field of volume currents with the density $\mathbf{j}(\mathbf{r})$ circulating in the body. Taking into account that

$$\frac{\partial}{\partial z}\left(z\,\frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial z}\right) = \frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial z} + z\,\frac{\partial^2\tilde{\Phi}^{\mathrm{T}}}{\partial z^2},$$

we finally obtain

$$B_z = -(k+2)\,\frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial z} + \frac{\partial}{\partial z}\left(x\,\frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial x} + y\,\frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial y} + z\,\frac{\partial\tilde{\Phi}^{\mathrm{T}}}{\partial z}\right). \tag{10}$$

Cyclic permutations in (10) give similar expressions for B_x and B_y, whence the result is written in the vector form as

$$\mathbf{B} = -\nabla\tilde{\Phi}. \tag{11}$$

[2] We distinguish the vector potential $\mathbf{A}^{\mathrm{T}}(\mathbf{r})$ of volume currents from the vector potential $\mathbf{A}(\mathbf{r})$ of surface currents of the same body. The notations for the speed of light c and the semiaxis c of an ellipsoid should also be distinguished.

[3] A layer whose surfaces are similar and are similarly oriented is called a homothetic layer.

Here, the scalar magnetic potential of a shell of the same body whose surface currents have density (7) is given by[4]

$$\tilde{\Phi} = (k+2)\tilde{\Phi}^{\mathrm{T}} - \mathbf{r}\,\frac{\partial \tilde{\Phi}^{\mathrm{T}}}{\partial \mathbf{r}}\,. \tag{12}$$

Expressions (8) and (12) are the statement of the Ferrers rule for constant magnetic fields; the former is valid in all space, whereas the latter is valid only outside the region with currents.

3. Mutual integral correspondence between Poisson magnetic charges and Ampère molecular currents

We now consider magnetic multipole moments. It is known that the general feature of multipole moments of any physical nature (gravitational, electric, or magnetic) is that, being integral characteristics of the corresponding source system, they contain concrete information on the material structure, shape, and size of the system producing the corresponding external field. Therefore, relations expressed in terms of multipole moments always have universal properties. An example is given by multipole expansions of the potentials of electrostatic fields for arbitrary three-dimensional charge distributions bounded in space [5]. These expressions, describing fields outside the source system, are valid for both discrete and continuous distributions (over the volume, surface, and line) of free or bound charges. Similar expressions obtained by replacing electric moments with magnetic moments and equating the monopole moment to zero are also valid for magnetic fields produced by electric currents. In this case, it makes no difference whether these currents are conduction currents or belong to convection or molecular currents.

Another example is the multipole representations of the external electrostatic (gravitational) potential of charges (masses) continuously distributed (with a polynomial dependence on coordinates) over the volume or surface of an ellipsoid [6–8]. These expressions are less universal than the multipole expansion and are valid only for ellipsoidal configurations. In Section 4, we use this universality to obtain multipole representations for the scalar magnetic potential of stationary currents in an ellipsoid.

But first, we consider some mathematical corollaries concerning the two different treatments of the origin of magnetic fields of (nonferromagnetic) magnets in the history of magnetism. We assume that a magnet of this type is a simply connected body located in a nonmagnetic medium placed in an external magnetic field. The magnetic properties of such a magnet are characterized by the magnetization vector **I**. We are interested in the magnetic multipole moments of the body and consider them using a semimicroscopic description of sources.

Poisson [9–11] explained the origin of a magnetic field by the presence of bound magnetic charges with the volume ($\tilde{\rho}$) and surface ($\tilde{\sigma}$) densities given by

$$\tilde{\rho} = -\operatorname{div}\mathbf{I}\,, \qquad \tilde{\sigma} = I_n\,. \tag{13}$$

These expressions are similar to that for the density of bound electric charges in terms of the polarization **P**. Maxwell, in his *Treatise on Electricity and Magnetism* [12, S. 430], rejected the magnetization mechanism proposed by Poisson, because it contradicted experiments. Remarkable, however, are the final words of Maxwell's critical statement: "*Of course the value of Poisson's mathematical investigations remains unimpaired, as they do not rest on his hypothesis, but on the experimental fact of induced magnetization*."

Ampère [12–14] assumed that the magnetic field of magnets is produced by molecular currents, which are related to the magnetization by

$$\mathbf{j}_{\mathrm{A}} = c\operatorname{rot}\mathbf{I}\,, \qquad \mathbf{i}_{\mathrm{A}} = c\operatorname{Rot}\mathbf{I}\,. \tag{14}$$

Here, $\mathbf{j}_{\mathrm{A}}$ and $\mathbf{i}_{\mathrm{A}}$ are the respective volume and surface densities of the Ampère molecular currents, and $\operatorname{Rot}\mathbf{I}$ is the so-called *surface rotor* of the vector **I** (see, e.g., [15]),

$$\operatorname{Rot}\mathbf{I} = \left[\mathbf{n}(\mathbf{I}_2 - \mathbf{I}_1)\right] = -\left[\mathbf{n}\mathbf{I}\right]. \tag{15}$$

The unit vector **n** of the external normal to the body surface S is directed from medium 1, in which $\mathbf{I}_1 = \mathbf{I}$, to medium 2, where $\mathbf{I}_2 = 0$ in our case. We note that Ampère currents satisfy conditions (1) and (2) automatically.

Obviously, multipole systems can be produced by both charges and currents. In this connection, the question arises: What is the relation between the magnetic multipole moments of Poisson magnetic charges (13) and the magnetic multipole moments of Ampère molecular currents (14)? In this case, of course, we consider the same body with a magnetization **I**. We recall that the vector **I** is the mean density on the magnetic (dipole) moment in a physically infinitely small volume, i.e., the magnetic moment of the body

$$\mathfrak{M} = \int \mathbf{I}\,\mathrm{d}V\,. \tag{16}$$

The magnetic multipole moments of magnetic charges are defined by the expressions

$$\mathfrak{M}_{i_1\ldots i_l} = \begin{cases} \displaystyle\int \tilde{\rho}(\mathbf{r})\,\theta_{i_1\ldots i_l}(\mathbf{r})\,\mathrm{d}V\,, \\ \displaystyle\oint \tilde{\sigma}(\mathbf{r})\,\theta_{i_1\ldots i_l}(\mathbf{r})\,\mathrm{d}S\,, \end{cases} \qquad l = 1, 2, \ldots, \tag{17}$$

which are absolutely similar to the definition of electrostatic multipole moments (in which the value $l = 0$ is also allowed, however) (see, e.g., [16]). The tensors $\theta_{i_1\ldots i_l} = \hat{D}_{i_1}\ldots\hat{D}_{i_l}\times 1$ introduced in [17] are the result of the action of the product of components of the vector operator $\hat{\mathbf{D}} = 2\mathbf{r}(\mathbf{r}\nabla) - r^2\nabla + \mathbf{r}$ on unity. In particular,

$$\theta^{(0)} = 1\,, \quad \theta_i = x_i\,, \quad \theta_{ij} = 3x_ix_j - r^2\delta_{ij}\,,$$
$$\theta_{ijk} = 3\left(5x_ix_jx_k - r^2\langle\langle\delta_{ij}x_k\rangle\rangle\right),$$
$$\theta_{ijkl} = 3\left(35x_ix_jx_kx_l - 5r^2\langle\langle\delta_{ij}x_kx_l\rangle\rangle + r^4\langle\langle\delta_{ij}\,\delta_{kl}\rangle\rangle\right),$$
$$\theta_{ijklm} = 15\left(63x_ix_jx_kx_lx_m - 7r^2\langle\langle\delta_{ij}x_kx_lx_m\rangle\rangle + r^4\langle\langle\delta_{ij}\,\delta_{kl}x_m\rangle\rangle\right).$$

Here, δ_{kn} is the Kronecker delta and the double angular brackets denote the special symmetrization operation in which all noncoincident tensors obtained from the tensor in angular brackets are added to this tensor with all possible permutations of the indices. For example,

$$\langle\langle\delta_{ij}x_kx_l\rangle\rangle \equiv \delta_{ij}x_kx_l + \delta_{ik}x_jx_l + \delta_{jk}x_ix_l + \delta_{il}x_jx_k + \delta_{jl}x_ix_k + \delta_{kl}x_ix_j\,.$$

[4] An arbitrary constant appearing in our derivation of (12) is set equal to zero, such that both potentials $\tilde{\Phi}^{\mathrm{T}}(\mathbf{r})$ and $\tilde{\Phi}(\mathbf{r})$ tend to zero as $\mathbf{r}\to\infty$.

The magnetic multipole moments of currents (in our case, Ampère currents) are defined by the expressions [5, 18]

$$\mathfrak{M}_{i_1\ldots i_l} = \begin{cases} \dfrac{1}{(l+1)\mathrm{c}} \displaystyle\int_V [\mathbf{r}\,\mathbf{j}_\mathrm{A}]_k \frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}(\mathbf{r})\,\mathrm{d}V\,, \\ \dfrac{1}{(l+1)\mathrm{c}} \displaystyle\oint_S [\mathbf{r}\,\mathbf{i}_\mathrm{A}]_k \frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}(\mathbf{r})\,\mathrm{d}S\,. \end{cases} \quad l = 1, 2, \ldots\,. \tag{18}$$

For both points of view (Poisson's and Ampère's) to coexist simultaneously and self-consistently, expressions (17) and (18) must, obviously, lead to coincident results. This section is devoted to the proof of this statement.

But first, we verify how the dipole moments of Ampère volume currents

$$\mathfrak{M}_V = \frac{1}{2\mathrm{c}}\int [\mathbf{r}\,\mathbf{j}_\mathrm{A}]\,\mathrm{d}V = \frac{1}{2}\int [\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]\,\mathrm{d}V \tag{19}$$

and the dipole moments of Ampère surface currents

$$\mathfrak{M}_S = \frac{1}{2\mathrm{c}}\oint [\mathbf{r}\,\mathbf{i}_\mathrm{A}]\,\mathrm{d}S = \frac{1}{2}\oint [\mathbf{r}\,\mathrm{Rot}\,\mathbf{I}]\,\mathrm{d}S \tag{20}$$

agree with expression (16) for the magnetization. We consider the double sum in (19) and (20),

$$2\mathfrak{M} = \int [\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]\,\mathrm{d}V + \oint [\mathbf{r}\,\mathrm{Rot}\,\mathbf{I}]\,\mathrm{d}S = \int [\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]\,\mathrm{d}V - \oint \big[\mathbf{r}[\mathbf{nI}]\big]\,\mathrm{d}S\,, \tag{21}$$

where (15) is taken into account. For the x-component of the integrand in the volume integral, we obtain

$$[\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_x = y\,\mathrm{rot}_z\,\mathbf{I} - z\,\mathrm{rot}_y\,\mathbf{I}$$
$$= -\left(y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) I_x + \frac{\partial}{\partial x}(yI_y + zI_z)$$
$$= -\frac{\partial}{\partial y}(yI_x) - \frac{\partial}{\partial z}(zI_x) + \frac{\partial}{\partial x}(yI_y + zI_z) + 2I_x\,,$$

or

$$[\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_x - 2I_x = \frac{\partial}{\partial x}(\mathbf{r}\,\mathbf{I}) - \mathrm{div}\,(I_x\,\mathbf{r})\,. \tag{22}$$

Integrating (22) over the body volume and using the Gauss–Ostrogradskii theorem, we find

$$\int \big([\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_x - 2I_x\big)\,\mathrm{d}V = \oint n_x(\mathbf{r}\,\mathbf{I})\,\mathrm{d}S - \oint I_x(\mathbf{rn})\,\mathrm{d}S\,. \tag{23}$$

Taking into account that $\big[\mathbf{r}\,[\mathbf{nI}]\big] = \mathbf{n}(\mathbf{r}\,\mathbf{I}) - \mathbf{I}(\mathbf{rn})$, we rewrite (23) in the vector form

$$\int [\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]\,\mathrm{d}V = 2\int \mathbf{I}\,\mathrm{d}V + \oint \big[\mathbf{r}\,[\mathbf{nI}]\big]\,\mathrm{d}S\,. \tag{24}$$

Substituting (24) into (21), we see that relation (16) is satisfied and that additive integral characteristics related to currents (in particular, molecular currents) should be calculated taking the contribution of the surface currents into account whenever they are present These characteristics include, of course, not only the magnetic dipole moment but also the moments of higher-order magnetic multipoles.

We do not verify that the dipole moment of Poisson magnetic charges (13) also corresponds to expression (16). We prove a more general statement about the coincidence of magnetic multipole moments of an arbitrary rank for the systems of molecular currents (14) and magnetic charges (13) for the same body with the magnetization **I**.

We use expression (18) for the rank-l magnetic multipole moment tensor and first consider the contribution

$$\mathfrak{M}^{(V)}_{i_1\ldots i_l} = \frac{1}{(l+1)\mathrm{c}}\int_V [\mathbf{r}\,\mathbf{j}_\mathrm{A}]_k \frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}V = \frac{1}{l+1}\int_V [\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_k \frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}V \tag{25}$$

to this moment from volume currents with the density $\mathbf{j}_\mathrm{A} = \mathrm{c}\,\mathrm{rot}\,\mathbf{I}$ and the contribution

$$\mathfrak{M}^{(S)}_{i_1\ldots i_l} = \frac{1}{(l+1)\mathrm{c}}\oint_S [\mathbf{r}\mathbf{i}_\mathrm{A}]_k \frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S = -\frac{1}{l+1}\oint_S \big[\mathbf{r}[\mathbf{nI}]\big]_k \frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S \tag{26}$$

from surface currents with the density $\mathbf{i}_\mathrm{A} = \mathrm{c}\,\mathrm{Rot}\,\mathbf{I} = -\mathrm{c}\,[\mathbf{nI}]$ [see expressions (14) and (15)].

We transform the integrand in (25). We obtain successively

$$[\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_k \frac{\partial\theta}{\partial x_k} = \varepsilon_{klm}x_l\,\varepsilon_{mnr}\frac{\partial I_r}{\partial x_n}\frac{\partial\theta}{\partial x_k} = \varepsilon_{klm}\varepsilon_{nrm}x_l\frac{\partial I_r}{\partial x_n}\frac{\partial\theta}{\partial x_k}$$
$$= (\delta_{kn}\delta_{lr} - \delta_{kr}\delta_{ln})\,x_l\frac{\partial I_r}{\partial x_n}\frac{\partial\theta}{\partial x_k} = x_l\frac{\partial I_l}{\partial x_k}\frac{\partial\theta}{\partial x_k} - x_l\frac{\partial I_k}{\partial x_l}\frac{\partial\theta}{\partial x_k}\,, \tag{27}$$

where ε_{klm} is the totally antisymmetric unit pseudotensor. In addition, to shorten the representation of the tensor $\theta_{i_1\ldots i_l}$, we temporarily omit all the tensor indices as long as these are not involved in transformations being performed. It is convenient to reduce each of the terms in the rightmost expression in (27) to the forms

$$x_l\frac{\partial I_l}{\partial x_k}\frac{\partial\theta}{\partial x_k} = \frac{\partial}{\partial x_k}\left(x_l I_l\frac{\partial\theta}{\partial x_k}\right) - I_l\frac{\partial\theta}{\partial x_l} - x_l I_l\,\Delta\theta\,, \tag{28}$$

$$x_l\frac{\partial I_k}{\partial x_l}\frac{\partial\theta}{\partial x_k} = \frac{\partial}{\partial x_l}\left(x_l I_k\frac{\partial\theta}{\partial x_k}\right) - 3I_l\frac{\partial\theta}{\partial x_l} - I_k x_l\frac{\partial}{\partial x_l}\frac{\partial\theta}{\partial x_k}\,. \tag{29}$$

We next use the fact that each component of the tensor $\theta_{i_1\ldots i_l}(\mathbf{r})$ is a spherical function, i.e., a homogeneous harmonic polynomial of degree l (see [17]). This means that, first, the last term in (28) vanishes because $\Delta\theta = 0$ and, second, the last term in (29) allows applying the Euler theorem for homogeneous functions to the function $\partial\theta/\partial x_k$, whence [5]

$$x_l\frac{\partial}{\partial x_l}\frac{\partial\theta}{\partial x_k} = (l-1)\frac{\partial\theta}{\partial x_k}\,.$$

[5] It is in view of the anticipated use of the Euler theorem that we used the old-fashioned notation $(\partial/\partial x_l)(\partial\theta/\partial x_k)$ for the second derivative $\partial^2\theta/\partial x_l\,\partial x_k$.

As a result, Eqn (27) takes the form

$$[\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_k\,\frac{\partial\theta}{\partial x_k}=\frac{\partial}{\partial x_k}\left(x_l I_l\,\frac{\partial\theta}{\partial x_k}\right)-I_k\,\frac{\partial\theta}{\partial x_k}-\frac{\partial}{\partial x_l}\left(x_l I_k\,\frac{\partial\theta}{\partial x_k}\right)+3I_k\,\frac{\partial\theta}{\partial x_k}+(l-1)I_k\,\frac{\partial\theta}{\partial x_k}\,,$$

or

$$\begin{aligned}[\mathbf{r}\,\mathrm{rot}\,\mathbf{I}]_k\,\frac{\partial\theta}{\partial x_k}&=\frac{\partial}{\partial x_k}\left(x_l I_l\,\frac{\partial\theta}{\partial x_k}\right)-\frac{\partial}{\partial x_l}\left(x_l I_k\,\frac{\partial\theta}{\partial x_k}\right)+(l+1)I_k\,\frac{\partial\theta}{\partial x_k}\\&=\frac{\partial}{\partial x_k}\left(\mathbf{rI}\,\frac{\partial\theta}{\partial x_k}\right)-\frac{\partial}{\partial x_l}\left(x_l I_k\,\frac{\partial\theta}{\partial x_k}\right)\\&\quad+(l+1)\,\frac{\partial}{\partial x_k}(I_k\theta)-(l+1)\theta\,\mathrm{div}\,\mathbf{I}\,.\end{aligned}\qquad(30)$$

Substituting (30) in volume integral (25) and using the Gauss–Ostrogradskii formula three times, we obtain the final expression

$$\begin{aligned}\mathfrak{M}^{(V)}_{i_1\ldots i_l}&=\frac{1}{l+1}\oint_S\mathbf{rI}\,\frac{\partial}{\partial n}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S-\frac{1}{l+1}\oint_S(\mathbf{rn})\,I_k\,\frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S\\&\quad+\oint_S(\mathbf{In})\,\theta_{i_1\ldots i_l}\,\mathrm{d}S-\int_V\theta_{i_1\ldots i_l}\,\mathrm{div}\,\mathbf{I}\,\mathrm{d}V\,.\end{aligned}\qquad(31)$$

We next consider integral (26). Identifying the double vector product $\left[\mathbf{r}\,[\mathbf{nI}]\right]$ in it, we obtain the expression

$$\mathfrak{M}^{(S)}_{i_1\ldots i_l}=-\frac{1}{l+1}\oint_S n_k(\mathbf{rI})\,\frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S+\frac{1}{l+1}\oint_S I_k(\mathbf{rn})\,\frac{\partial}{\partial x_k}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S\,,$$

which, combined with (31), gives the expression

$$\mathfrak{M}_{i_1\ldots i_l}=\int\tilde{\rho}\,\theta_{i_1\ldots i_l}\,\mathrm{d}V+\oint\tilde{\sigma}\,\theta_{i_1\ldots i_l}\,\mathrm{d}S$$

for the total multipole moment only in terms of densities (13) of the magnetic bound charge.

Hence, the concept of bound magnetic charges used in calculations of total multipole moments taking the total contribution from volume and point sources into account gives results that quantitatively agree with those obtained using Ampère currents. This conclusion, which, following Maxwell, we regard as purely mathematical, can be used as a link between multipole expressions in electrostatics and magnetostatics (see Section 4).

In this paper, we are interested in magnetic fields mainly at observation points located outside the current region. The separation of currents into volume and surface currents at such points is conventional to some extent because volume currents can always be replaced by equivalent surface currents. This is confirmed by the following consideration. Let all the space be occupied with a superconductor in which a cavity exists with a stationary electric current circulating in it. Because the magnetic field is always absent inside a superconductor, this means, from the standpoint of the superposition principle, that the magnetic field produced by the volume currents in the cavity is 'quenched' by the magnetic field of surface currents induced on the cavity surface. It then follows that currents on the cavity surface taken with the opposite sign are equivalent to volume currents.

4. Multiple representations of the scalar magnetic potential of an ellipsoid

In this section, we consider a three-axial ellipsoid as a geometrical object in which constant magnetic field sources are distributed.

It is known that for a ball or a sphere with a polynomial (of degree k) source density (charges or currents), all the multiple moments of rank greater than k are zero, and therefore the multipole expansion is represented by a finite series. In contrast to the sphere, no moment for an ellipsoid can exist without the simultaneous existence of all higher multipole moments of the same parity. Therefore, the multipole expansion of the potential of an ellipsoid is always an infinite series. At the same time, the volume and surface external potentials of an ellipsoid generated by *polynomial* densities can be described [6–8] in the so-called *multipole representation*,[6] which, as the multipole expansion of the potential of a sphere mentioned above, has the form of a finite sum of components of multiple moments times some standard tensor functions.

We recall the form of multipole representations of the electrostatic potential of an ellipsoid. If the ellipsoid surface is described by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,,\qquad(32)$$

then one of the multipole representations has the same form both for volume charges with the polynomial density

$$\rho(\mathbf{r})=\rho_L\equiv P_L(x,y,z)\,,\qquad L=0,1,\ldots,\qquad(33)$$

and for charges distributed over the ellipsoid boundary with the surface density $\sigma(\mathbf{r})=\sigma_{L+2}$, where

$$\sigma_L\equiv\rho_L\,p\,,\qquad L=0,1,\ldots;\qquad(34)$$

this multipole representation is given by

$$\left.\begin{aligned}\Phi_{L-2}(\mathbf{r})&\equiv\int\frac{\rho_{L-2}}{R}\,\mathrm{d}V\\\Psi_L(\mathbf{r})&\equiv\oint\frac{\sigma_L}{R}\,\mathrm{d}S\end{aligned}\right\}=\sum_{\varkappa=0}^{L}q_{i_1\ldots i_\varkappa}\psi_{i_1\ldots i_\varkappa}(\mathbf{r})\,,\quad\begin{cases}L=2,3,\ldots,\\L=0,1,\ldots,\end{cases}\qquad(35)$$

while another representation is valid only for a volume charge of the density

$$\rho(\mathbf{r})=\rho_L^{\Gamma}\equiv\Gamma_L\left(\frac{x}{a}\,,\frac{y}{b}\,,\frac{z}{c}\right),\qquad L=0,1,\ldots,\qquad(36)$$

[6] We note that all analytic results in [6–8] obtained for gravitational fields are directly transformed to their electric analogs written in the Gaussian system of units if the gravitational constant is replaced by unity and the density and multiple moments of masses are respectively treated as the charge density and electrostatic multiple moments.

and is given by

$$\Phi_L^\Gamma(\mathbf{r}) \equiv \int \frac{\rho_L^\Gamma}{R}\,\mathrm{d}V = \sum_{\varkappa=0}^{L} \tilde{q}_{i_1\ldots i_\varkappa}\,\varphi_{i_1\ldots i_\varkappa}(\mathbf{r})\,. \tag{37}$$

Here, it is more convenient to let the potentials of volume and surface sources (which were denoted in Section 2 identically) be denoted differently, as Φ and Ψ respectively. In expressions (33) and (36), P_L and Γ_L are polynomials of degree L, Γ_L is a harmonic function of its arguments,

$$p = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-1/2} \tag{38}$$

is the distance from the ellipsoid center to the tangent plane at the point (x, y, z), and R is the distance from the integration element $\mathrm{d}V$ or $\mathrm{d}S$ to the observation point outside the ellipsoid. The irreducible totally symmetric tensors $\varphi_{i_1\ldots i_\varkappa}(\mathbf{r})$ and $\psi_{i_1\ldots i_\varkappa}(\mathbf{r})$ are respectively the so-called *tensor potentials* of an ellipsoid and a homoeoid.[7] These tensors serve as reference special functions generated by the multipole representation theory. Each component of these tensors is a harmonic function of Cartesian coordinates. The explicit form of tensor potentials is known, but is not used in this paper.

The coefficients $q_{i_1\ldots i_\varkappa}$ and $\tilde{q}_{i_1\ldots i_\varkappa}$ in (35) and (37) are the components of tensors of so-called *partial* electric multipole moments. It is important that $q_{i_1\ldots i_\varkappa}$ and $\tilde{q}_{i_1\ldots i_\varkappa}$ can be expressed via linear recurrent relations in terms of the components of tensors of the *total* electric multipole moments $Q_{i_1\ldots i_\varkappa}$ of the ellipsoid. These relations for $q_{i_1\ldots i_\varkappa}$ have the form (which depends on the tensor rank parity) [7]

$$\begin{aligned} q_{j_1\ldots j_{2l+1}} &= Q_{j_1\ldots j_{2l+1}} - (4l+1)!!\sum_{n=0}^{l-1}\frac{4n+3}{(2l+2n+3)!!} \\ &\quad\times \langle\!\langle q_{j_1\ldots j_{2n+1}}\varkappa_{j_{2n+2}j_{2n+3}}\ldots\varkappa_{j_{2l}j_{2l+1}}\rangle\!\rangle\,, \\ q_{j_1\ldots j_{2l}} &= Q_{j_1\ldots j_{2l}} - (4l-1)!!\sum_{n=0}^{l-1}\frac{4n+1}{(2l+2n+1)!!} \\ &\quad\times \langle\!\langle q_{j_1\ldots j_{2n}}\varkappa_{j_{2n+1}j_{2n+2}}\ldots\varkappa_{j_{2l-1}j_{2l}}\rangle\!\rangle\,, \end{aligned} \tag{39}$$

where[8]

$$\varkappa_{ij} \equiv a_{(i)}a_{(j)}\delta_{ij} = \begin{cases} a_{(i)}^2\,, & i = j\,, \\ 0\,, & i \neq j\,. \end{cases}$$

Expressions for $\tilde{q}_{i_1\ldots i_\varkappa}$ differ from expression (39) for $q_{i_1\ldots i_\varkappa}$ only in that $4n+3$ in the numerators of the fractions in the sum should be replaced by $(4n+3)(4n+5)$ and $4n+1$ should be replaced by $(4n+1)(4n+3)$.

Thus, any polynomial volume distribution [including (36)] admits a multipole representation of the potential in form (35), harmonic distribution (36) also having the additional representation (37) of its potential using lower-rank multipoles. But multipole representation (35) has a more general character, being valid not only for volume but also for surface charge distributions for which the ratio σ/p is polynomial. In this case, the two types of surface distributions on an ellipsoid, σ_0 and σ_1, have multipole representations of their potentials that cannot be reproduced by any volume charge. At the same time, an arbitrary polynomial volume distribution can always be replaced by a surface charge distribution with the same multipole representation of the potential.

As shown in [21] with the example of simplest particular cases, multipole representations of the scalar potential of an ellipsoid also exist in magnetic fields produced by direct electric currents. The derivation of the corresponding general expressions presented below also illustrates the use of the results obtained in Sections 2 and 3.

We consider a (nonferromagnetic) magnet of the shape of ellipsoid (32) with an induced magnetization $\mathbf{I}$. If the conduction current is absent, then, as is known [22, 23], the magnetostatic problem considered here (in our case, in the ellipsoid field) is mathematically equivalent to an electrostatic problem (in the absence of free charges), the only difference being the replacements $\mathbf{E} \to \mathbf{H}$ and $\mathbf{D} \to \mathbf{B}$. Taking the result obtained in Section 3 into account, we can assert that if the solution of such an electrostatic problem for the external region is represented in the form of the total scalar potential $\Phi_\Sigma = \Phi + \Psi$ expressed in terms of electrostatic multipole moments $Q_{i_1\ldots i_l}$, then the replacement

$$\Phi_\Sigma \to \tilde{\Phi}_\Sigma\,, \quad \mathbf{P} \to \mathbf{I}\,, \quad Q_{i_1\ldots i_l} \to \mathfrak{M}_{i_1\ldots i_l}\,, \quad l = 1, 2, \ldots \tag{40}$$

(where $\mathbf{P}$ is the polarization vector and $\tilde{\Phi}_\Sigma = \tilde{\Phi} + \tilde{\Psi}$ is the sum of the volume and surface scalar magnetic potentials) gives the external solution of the corresponding magnetostatic problem.

We now assume that the electric charge is distributed in ellipsoid (32) with some volume (ρ_L) and (or) surface (σ_{L+2}) density, such that the total charge Q of the ellipsoid is zero. The multipole representation of the potential of such a system of charges does not differ from the multipole representation of the potential of the system of bound charges, in which the volume, $\rho = -\operatorname{div}\mathbf{P}$, and surface, $\sigma = P_n$, charge densities are polynomials of respective degrees L and $L+2$. (Such densities are provided by the polarization vector $\mathbf{P}$ whose Cartesian components are polynomials of degree $L+1$.)

It follows from (35) and (39), according to (40), that an ellipsoidal magnet with bound magnetic charges (13) characterized by polynomial densities of the same degrees produces an external magnetic field with the scalar potential

$$\left.\begin{aligned} \tilde{\Phi}_{L-2}(\mathbf{r}) \\ \tilde{\Psi}_L(\mathbf{r}) \end{aligned}\right\} = \sum_{\varkappa=1}^{L} \mathrm{m}_{i_1\ldots i_\varkappa}\,\psi_{i_1\ldots i_\varkappa}(\mathbf{r}) \quad \begin{cases} L = 3, 4, \ldots, \\ L = 1, 2, \ldots, \end{cases} \tag{41}$$

where

$$\begin{aligned} \mathrm{m}_{j_1\ldots j_{2l+1}} &= \mathfrak{M}_{j_1\ldots j_{2l+1}} - (4l+1)!!\sum_{n=0}^{l-1}\frac{4n+3}{(2l+2n+3)!!} \\ &\quad\times \langle\!\langle \mathrm{m}_{j_1\ldots j_{2n+1}}\varkappa_{j_{2n+2}j_{2n+3}}\ldots\varkappa_{j_{2l}j_{2l+1}}\rangle\!\rangle\,, \\ \mathrm{m}_{j_1\ldots j_{2l}} &= \mathfrak{M}_{j_1\ldots j_{2l}} - (4l-1)!!\sum_{n=1}^{l-1}\frac{4n+1}{(2l+2n+1)!!} \\ &\quad\times \langle\!\langle \mathrm{m}_{j_1\ldots j_{2n}}\varkappa_{j_{2n+1}j_{2n+2}}\ldots\varkappa_{j_{2l-1}j_{2l}}\rangle\!\rangle\,. \end{aligned} \tag{42}$$

[7] The term *homoeoid* introduced by Thomson (Lord Kelvin) and Tait [19] to denote an infinitely thin layer formed by two similar and similarly oriented surfaces is used here, as in Routh's monograph [20], for an ellipsoidal simple layer if the surface density of sources is proportional to p.

[8] Here, $a_{(i)}$ is the ellipsoid semiaxis lying on the coordinate axis x_i. Nontensor subscripts are in parentheses.

Here, $\mathrm{m}_{i_1\ldots i_\varkappa}$ and $\mathfrak{M}_{i_1\ldots i_\varkappa}$ are respectively the partial (see below) and total magnetic multipole moments, and the absence of magnetic monopoles is taken into account.

Based on the results in Section 3, we can assume that multipole representation (41) of the scalar magnetic potential is caused not by bound magnetic charges but by the system of Ampère currents (14) characterized by the magnetization $\mathbf{I}$ with Cartesian components described by polynomials of degree $L+1$.

Finally, due to the universality of multipole expressions, we can assert that multipole representation (41) of the magnetic potential of an ellipsoid is valid not only for molecular currents (14) but also for conduction or convection currents for which the Cartesian components of vectors $\mathbf{j}(\mathbf{r})$ and $\mathbf{i}(\mathbf{r})/p$ are polynomials of respective degrees L and $L+2$, while the currents themselves, as any stationary currents in a bounded spatial region, obey conditions (1) and (2).

Thus, we obtained the multipole representation of the external scalar magnetic potential for electric currents in the ellipsoidal region whose densities were characterized above.

It would seem that a similar consideration is also applicable to multipole representation (37) of the potential produced by system of charges (37) if, of course, the total charge of the system is zero. Indeed, free charges with density (36) can be replaced by bound electric charges with the same density. In this case, the electrostatic potential retains form (37), only now both $L \geqslant 1$ and $\varkappa \geqslant 1$. Then, using transition rule (40), we find that the system of bound magnetic charges with the volume density $\tilde{\rho}_L^{\Gamma}$, which, like (36), is a polynomial harmonic function of degree L,[9] produces the scalar magnetic potential outside the ellipsoid with the multipole representation

$$\tilde{\Phi}_L^{\Gamma}(\mathbf{r}) \equiv \int \frac{\tilde{\rho}_L^{\Gamma}}{R}\,\mathrm{d}V = \sum_{\varkappa=1}^{L} \tilde{\mathrm{m}}_{i_1\ldots i_\varkappa}\,\varphi_{i_1\ldots i_\varkappa}(\mathbf{r})\,. \tag{43}$$

The *partial* magnetic multipole moments $\tilde{\mathrm{m}}_{i_1\ldots i_\varkappa}$ of volume currents in (43) can be finally expressed using the recurrent expressions

$$\begin{aligned}\tilde{\mathrm{m}}_{j_1\ldots j_{2l+1}} &= \widetilde{\mathfrak{M}}_{j_1\ldots j_{2l+1}} - (4l+1)!!\sum_{n=0}^{l-1}\frac{(4n+3)(4n+5)}{(2l+2n+5)!!}\\ &\quad\times\langle\!\langle \tilde{\mathrm{m}}_{j_1\ldots j_{2n+1}}\,\varkappa_{j_{2n+2}j_{2n+3}}\ldots\varkappa_{j_{2l}j_{2l+1}}\rangle\!\rangle\,,\quad l=0,1,2,\ldots,\\ \tilde{\mathrm{m}}_{j_1\ldots j_{2l}} &= \widetilde{\mathfrak{M}}_{j_1\ldots j_{2l}} - (4l-1)!!\sum_{n=1}^{l-1}\frac{(4n+1)(4n+3)}{(2l+2n+3)!!}\\ &\quad\times\langle\!\langle \tilde{\mathrm{m}}_{j_1\ldots j_{2n}}\,\varkappa_{j_{2n+1}j_{2n+2}}\ldots\varkappa_{j_{2l-1}j_{2l}}\rangle\!\rangle\,,\quad l=1,2,3,\ldots\end{aligned} \tag{44}$$

in terms of the total multipole moments $\widetilde{\mathfrak{M}}_{i_1\ldots i_\varkappa}$. Hence, as in the electrostatic case, an additional multipole representation of the magnetic potential exists (based on moments of a lower rank). However, it is not clear so far to which electric currents representation (43) corresponds.

To elucidate this question, we consider simpler (partial) sources, which play a key role in the construction of multipole representations of the potentials of an ellipsoid [6–8]. A density given by a homogeneous harmonic polynomial (e.g., of degree ν) is called the partial density of a volume charge. The multipole representation of the corresponding *partial* potential is known [7]. If the total charge Q of the ellipsoid is zero, we can assume that we are dealing with bound charges. In turn, according to (40), the system of bound electric charges and its external potential can be replaced by the system of bound magnetic charges and its potential. Thus, the scalar partial potential of the magnetic field acquires the multipole representation[10]

$$\tilde{\Phi}^{(\nu)} = \tilde{\mathrm{m}}_{j_1\ldots j_\nu}\,\varphi_{j_1\ldots j_\nu} = \sum_{i+j+k=\nu}\frac{\nu!}{i!\,j!\,k!}\,\tilde{\mathrm{m}}_{ijk}\,\varphi_{ijk}\,,\qquad \nu\geqslant 1\,, \tag{45}$$

having the form of the total contraction of two symmetric tensors of rank ν. The expressions are presented here both in the usual tensor notation, where two repeated indices imply summation, and in the so-called *three-index notation*. The latter is applied only to symmetric tensors (or at least to the combination of indices in which a tensor is symmetric), which is illustrated with the example

$$\varphi_{klm} \equiv \varphi_{\underbrace{x\ldots x}_{k\text{ times}}\,\underbrace{y\ldots y}_{l\text{ times}}\,\underbrace{z\ldots z}_{m\text{ times}}}\,.$$

Partial magnetic potentials (45) can, of course, also be produced by electric currents. To find the answer to the question of how these partial currents 'look', we consider, as in Section 3, a nonferromagnetic magnet with the induced magnetization $\mathbf{I}$, but now we are dealing with an ellipsoidal body. If the Cartesian components of the vector $\mathbf{I}$ are homogeneous harmonic polynomials of degree $\nu+1$, then, as is shown in the Appendix, the density $\tilde{\rho}$ of the magnetic charge of the ellipsoid is also described by a homogeneous harmonic polynomial (of degree ν), i.e., is partial, and therefore the appearing field is characterized by potential (45). It is clear that the Ampère current $\mathbf{j}_{\mathrm{A}}$ in (14) corresponding to the same magnetization should be considered partial. It is shown in the Appendix that each Cartesian component of the vector $\mathbf{j}_{\mathrm{A}}$ is a homogeneous harmonic polynomial of degree ν.

Because of its universality, multipole representation (45) of the partial magnetic potential also retains its form for conduction currents. In this case, the density of the partial volume current of the ellipsoid is given by[11]

$$\frac{j_x^{(\nu)}}{a} = \sum_{k+l+m=\nu}\frac{\nu!}{k!\,l!\,m!}\,\alpha_{x;\,klm}\left(\frac{x}{a}\right)^k\left(\frac{y}{b}\right)^l\left(\frac{z}{c}\right)^m, \tag{46}$$

$$\frac{j_y^{(\nu)}}{b} = \sum_{k+l+m=\nu}\frac{\nu!}{k!\,l!\,m!}\,\alpha_{y;\,klm}\left(\frac{x}{a}\right)^k\left(\frac{y}{b}\right)^l\left(\frac{z}{c}\right)^m, \tag{47}$$

$$\frac{j_z^{(\nu)}}{c} = \sum_{k+l+m=\nu}\frac{\nu!}{k!\,l!\,m!}\,\alpha_{z;\,klm}\left(\frac{x}{a}\right)^k\left(\frac{y}{b}\right)^l\left(\frac{z}{c}\right)^m, \tag{48}$$

[9] Hereafter (including the Appendix), it is assumed that a harmonic function describing sources in the ellipsoid is harmonic in the coordinates x/a, y/b, and z/c.

[10] In this section, the partial magnetic multipole moments $\tilde{\mathrm{m}}_{ijk}$ of volume sources are indicated by a tilde, to distinguish them from the moments m_{ijk} of surface sources.

[11] We note that the direct proof of the fact that current (46)–(48) produces potential (45) outside the ellipsoid proved to be, in accordance with the discussion in the Introduction, too cumbersome and is therefore not presented here. We also note that the expression for the simplest (partial) volume current in the ellipsoid and the multiple representation of its scalar potential can be found in [21].

where the polynomials are harmonic and $v \geqslant 1$. Current (46)–(48) should, in addition, satisfy conditions (1) and (2). The combination of coefficients $\alpha_{x;klm}$, $\alpha_{y;klm}$, and $\alpha_{z;klm}$ form a tensor of rank $v+1$ that is symmetric in all indices except one, which is separated by a semicolon. For symmetric indices, we hereafter use the three-index notation, the sum $k+l+m=v$ indicating the number of symmetric tensor indices.

If any Cartesian component $\mathbf{j}_L^{\Gamma}(x/a, y/b, z/c)$ of the density vector of a current circulating in the ellipsoid volume is a harmonic polynomial of degree L and conditions (1) and (2) are satisfied, then the equality

$$\mathbf{j}_L^{\Gamma}\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) = \sum_{v=1}^{L} \mathbf{j}^{(v)}\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) \tag{49}$$

holds. This is obvious because any harmonic polynomial is a sum of homogeneous harmonic polynomials. According to the superposition principle, current (49) corresponds to sum (45) of potentials given by (43). Thus, we found the expression for currents corresponding to magnetic potential (43).

It follows that the type of coordinate dependence of the current in (46)–(48) (homogeneity of degree v) satisfies the condition for the applicability of the Ferrers rule. We use this possibility.

Letting $\tilde{\Psi}$ denote the magnetic potential of surface currents, we can rewrite Ferrers rule (12) in the form

$$\tilde{\Psi}^{(v)} = \hat{F}^{(v)} \tilde{\Phi}^{(v)}, \tag{50}$$

where, according to (7), the surface (partial) current density is $\mathbf{i}^{(v)} = \mathbf{j}^{(v)} p$ and the operator is

$$\hat{F}^{(v)} = v + 2 - \mathbf{r}\,\frac{\partial}{\partial \mathbf{r}}\,.$$

Substituting expression (45) in (50), we obtain

$$\tilde{\Psi}^{(v)} = \sum_{i+j+k=v} \frac{v!}{i!\,j!\,k!}\,\tilde{\mathrm{m}}_{ijk}\,\hat{F}^{(v)} \varphi_{ijk}\,. \tag{51}$$

We now use the relation [6, 8]

$$\psi_{ijk} = \frac{1}{2v+3}\,\hat{F}^{(v)} \varphi_{ijk}\,, \qquad v = i+j+k\,,$$

expressing the tensor potential ψ_{ijk} on a homoeoid in terms of the tensor potential φ_{ijk} of an ellipsoid. Expression (51) then becomes

$$\tilde{\Psi}^{(v)} = (2v+3) \sum_{i+j+k=v} \frac{v!}{i!\,j!\,k!}\,\tilde{\mathrm{m}}_{ijk}\,\psi_{ijk}\,. \tag{52}$$

It remains to replace partial multipole moments $\tilde{\mathrm{m}}_{ijk}$ of volume sources (46)–(48) by partial moments m_{ijk} of surface currents $\mathbf{i}^{(v)} = \mathbf{j}^{(v)} p$. The corresponding expression derived in [7] for electric multipoles, which is of course also correct for magnetic multipoles in the corresponding notation, has the form $(2v+3)\,\tilde{\mathrm{m}}_{ijk} = \mathrm{m}_{ijk}$. Thus, we finally obtain

$$\tilde{\Psi}^{(v)} = \mathrm{m}_{j_1 \ldots j_v}\,\psi_{j_1 \ldots j_v} = \sum_{i+j+k=v} \frac{v!}{i!\,j!\,k!}\,\mathrm{m}_{ijk}\,\psi_{ijk}\,. \tag{53}$$

Now let the surface current in the ellipsoid be such that each Cartesian component of $\mathbf{i}/p$ is an arbitrary polynomial of degree $L > 0$, i.e.,

$$\mathbf{i}_L = \mathbf{j}_L\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) p\,. \tag{54}$$

In this case, the known expression (see, e.g., [24]) for the expansion of a polynomial in spherical functions (homogeneous harmonic polynomials) can be applied to each component of the current density (and therefore to the vector as a whole). This gives

$$\mathbf{j}_L = \sum_{k=1}^{L} \sum_{l=0}^{[k/2]} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^l \mathbf{j}^{(k-2l)}\,, \tag{55}$$

where $[k/2]$ is the integer part of $k/2$. Substituting (55) in (54), using coupling equation (32) applicable in this case, and taking into account that a sum of spherical functions of the same degrees is a spherical function of the same degree, we obtain the current density $\mathbf{i}_L$ and (according to the superposition principle) the potential $\tilde{\Psi}_L$:

$$\mathbf{i}_L = \sum_{v=1}^{L} \mathbf{i}^{(v)}\,, \qquad \tilde{\Psi}_L = \sum_{v=1}^{L} \tilde{\Psi}^{(v)}\,,$$

which is an independent confirmation of expression (41).

5. Conclusions

To summarize, we list some features of multipole representations found in our study.

The multipole representation of external volume ($\tilde{\Phi}$) and surface ($\tilde{\Psi}$) magnetic potentials of an ellipsoid is universal in the sense that the potentials $\tilde{\Psi}_{L+2}$ cannot be distinguished from the potentials $\tilde{\Phi}_L$ ($L = 1, 2, \ldots$), as follows from (41). Therefore, it is impossible to determine from the external scalar potential of an ellipsoid by which currents (surface currents with the density $\mathbf{i}_{L+2}$ or volume currents with the density $\mathbf{j}_L$, or a combination of the two) it is generated. At the same time, no volume current can reproduce the magnetic field of the surface current $\mathbf{i}_1$.

The universality is also manifested in the fact that the multipole representation for a fixed observation point is invariant on passing from one ellipsoid to any other confocal ellipsoid. This invariance follows from the similar invariance of the tensor potential of a homoeoid [6]. Hence, the external magnetic potential of the ellipsoid does not contain information on the size of ellipsoid (32) and the particular form (coefficients) of a polynomial describing sources, but depends on the observation point, the degree of the polynomial describing $\mathbf{j}$ or $\mathbf{i}/p$, and the size and orientation of the elliptic disc — the limit shape of the family of focused ellipsoids.

For potentials $\tilde{\Phi}_L^{\Gamma}$ of an ellipsoid in which the current density $\mathbf{j}_L$ is a harmonic polynomial in x/a, y/b, and z/c, in addition to the universal representation using the tensor potential of a homoeoid, another multipole representation exists that involves the tensor potential of the ellipsoid and is described by expressions (43) and (44). This specific representation is more 'economical' because it operates with multipoles with a maximum rank that is lower by two than that in the universal representation. The external potentials $\tilde{\Phi}_L^{\Gamma}$ are also invariant under the confocal transformation of ellipsoids.

The magnetostatic correspondences considered in this paper and magnetoelectrostatic analogies used here are, of course, not exhaustive. Other useful analogies can be found, e.g., in recent paper [25].

As regards the methodology, in our opinion, this paper is instructive in the following:

(i) the use of old results, which have lost their physical certainty but have retained their mathematical reliability, sometimes allows simplifying the derivation of the required result;

(ii) the universal formulas can be derived for a narrower subtype of physical systems.

Acknowledgments
The author thanks B M Bolotovskii and S P Efimov for their attention to this work and the useful discussions.

6. Appendix. Partial sources of the magnetic field of an ellipsoid

We consider a relation between the *partial* volume densities of charge and current sources of an ellipsoidal current. Magnetic charges and Ampère currents are related by the induced magnetization vector $\mathbf{I}$.

Let the Cartesian components of the magnetization be homogeneous harmonic polynomials (with respect to the coordinates $\bar{x} = x/a$, $\bar{y} = y/b$, and $\bar{z} = z/c$) of degree $\nu + 1$:

$$I_x = \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!\,l!\,m!}\, \varkappa_{x;\,klm}\, \bar{x}^k \bar{y}^l \bar{z}^m \,, \tag{A.1}$$

$$I_y = \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!\,l!\,m!}\, \varkappa_{y;\,klm}\, \bar{x}^k \bar{y}^l \bar{z}^m \,, \tag{A.2}$$

$$I_z = \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!\,l!\,m!}\, \varkappa_{z;\,klm}\, \bar{x}^k \bar{y}^l \bar{z}^m \,. \tag{A.3}$$

The polynomial coefficients $\varkappa_{x;\,klm}$, $\varkappa_{y;\,klm}$, and $\varkappa_{z;\,klm}$ in (A.1)–(A.3) form a tensor of rank $\nu + 2$, which is symmetric in all its indices in the three-index notation, except for the first index (separated from the other by a semicolon). The harmonicity of the polynomials is provided in this case by the irreducibility of the tensor $\varkappa$ in the symmetric indices, i.e., by the relations

$$\varkappa_{x;\,k+2,\,l,\,m} + \varkappa_{x;\,k,\,l+2,\,m} + \varkappa_{x;\,k,\,l,\,m+2} = 0 \,, \tag{A.4}$$

$$\varkappa_{y;\,k+2,\,l,\,m} + \varkappa_{y;\,k,\,l+2,\,m} + \varkappa_{y;\,k,\,l,\,m+2} = 0 \,, \tag{A.5}$$

$$\varkappa_{z;\,k+2,\,l,\,m} + \varkappa_{z;\,k,\,l+2,\,m} + \varkappa_{z;\,k,\,l,\,m+2} = 0 \,, \tag{A.6}$$

expressing in the three-index notation that the contraction of $\varkappa$ in any pair of symmetric indices vanishes.

The chosen magnetization type corresponds, according to Poisson, to the volume density of a (bound) magnetic charge:

$$\begin{aligned}\tilde{\rho} = -\operatorname{div}\mathbf{I} = &- \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{(k-1)!\,l!\,m!}\, \frac{\varkappa_{x;\,klm}}{a}\, \bar{x}^{k-1} \bar{y}^l \bar{z}^m \\ &- \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!(l-1)!m!}\, \frac{\varkappa_{y;\,klm}}{b}\, \bar{x}^k \bar{y}^{l-1} \bar{z}^m \\ &- \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!\,l!\,(m-1)!}\, \frac{\varkappa_{z;\,klm}}{c}\, \bar{x}^k \bar{y}^l \bar{z}^{m-1} \,.\end{aligned}$$

The respective changes of summation indices $k \to k+1$, $l \to l+1$, and $m \to m+1$ in the first, second, and third sums gives

$$\tilde{\rho} = \sum_{k+l+m=\nu} \frac{\nu!}{k!\,l!\,m!}\, \alpha_{klm} \left(\frac{x}{a}\right)^k \left(\frac{y}{b}\right)^l \left(\frac{z}{c}\right)^m ,$$

where three-index components of the rank-ν tensor α symmetric in all its indices are given by

$$\frac{\alpha_{klm}}{\nu+1} \equiv -\frac{\varkappa_{x;\,k+1,\,l,\,m}}{a} - \frac{\varkappa_{y;\,k,\,l+1,\,m}}{b} - \frac{\varkappa_{z;\,k,\,l,\,m+1}}{c} \,. \tag{A.7}$$

We note that the totally symmetric tensor α is irreducible. Indeed, its contraction with respect to any two indices is zero. The three-index notation of this statement has the form

$$\alpha_{k+2,\,l,\,m} + \alpha_{k,\,l+2,\,m} + \alpha_{k,\,l,\,m+2} = 0$$

and follows from the equality

$$\begin{aligned}&\frac{1}{a}\left(\varkappa_{x;\,k+3,\,l,\,m} + \varkappa_{x;\,k+1,\,l+2,\,m} + \varkappa_{x;\,k+1,\,l,\,m+2}\right) \\ &\quad + \frac{1}{b}\left(\varkappa_{y;\,k+2,\,l+1,\,m} + \varkappa_{y;\,k,\,l+3,\,m} + \varkappa_{y;\,k,\,l+1,\,m+2}\right) \\ &\quad + \frac{1}{c}\left(\varkappa_{z;\,k+2,\,l,\,m+1} + \varkappa_{z;\,k,\,l+2,\,m+1} + \varkappa_{z;\,k,\,l,\,m+3}\right) = 0 \,,\end{aligned}$$

which is satisfied because of (A4)–(A6).

The irreducibility of α implies the harmonicity of the polynomial $\tilde{\rho}(x/a,\, y/b,\, z/c)$ and proves that $\tilde{\rho} = \tilde{\rho}^{(\nu)}$, i.e., the volume density of the bound magnetic charge under study is the partial density.

The magnetization in (A.1)–(A.3) corresponds to the volume density $\mathbf{j}_{\mathrm{A}} = \mathrm{c}\operatorname{rot}\mathbf{I}$ of Ampère currents. In particular,

$$\begin{aligned}\frac{(j_{\mathrm{A}})_x}{\mathrm{c}} = \frac{\partial I_z}{\partial y} - \frac{\partial I_y}{\partial z} = &\sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!\,(l-1)!\,m!}\, \frac{\varkappa_{z;\,klm}}{b}\, \bar{x}^k \bar{y}^{l-1} \bar{z}^m \\ &- \sum_{k+l+m=\nu+1} \frac{(\nu+1)!}{k!\,l!\,(m-1)!}\, \frac{\varkappa_{y;\,klm}}{c}\, \bar{x}^k \bar{y}^l \bar{z}^{m-1} \,.\end{aligned}$$

Changing the summation indices as $l \to l+1$ and $m \to m+1$ in the first and second sums allows combining them. This gives

$$\frac{(j_{\mathrm{A}})_x}{\mathrm{c}} = \sum_{k+l+m=\nu} \frac{\nu!}{k!\,l!\,m!}\, \beta_{x;\,klm} \left(\frac{x}{a}\right)^k \left(\frac{y}{b}\right)^l \left(\frac{z}{c}\right)^m ,$$

where

$$\frac{\beta_{x;\,klm}}{\nu+1} \equiv \frac{\varkappa_{z;\,k,\,l+1,\,m}}{b} - \frac{\varkappa_{y;\,k,\,l,\,m+1}}{c} \,.$$

It is easy to see that the irreducibility of the tensor $\varkappa$ in symmetric indices of the y and z components of the tensor leads to the irreducibility of the x component of β. Indeed,

$$\begin{aligned}&\frac{1}{\nu+3}\left(\beta_{x;\,k+2,\,l,\,m} + \beta_{x;\,k,\,l+2,\,m} + \beta_{x;\,k,\,l,\,m+2}\right) \\ &\quad = \frac{1}{b}\left(\varkappa_{z;\,k+2,\,l+1,\,m} + \varkappa_{z;\,k,\,l+3,\,m} + \varkappa_{z;\,k,\,l+1,\,m+2}\right) \\ &\quad - \frac{1}{c}\left(\varkappa_{y;\,k+2,\,l,\,m+1} + \varkappa_{y;\,k,\,l+2,\,m+1} + \varkappa_{y;\,k,\,l+1,\,m+3}\right) = 0 \,.\end{aligned}$$

Obviously, the y and z components have similar properties.

Thus, the density $\mathbf{j}_A$ of the Ampère current corresponding to magnetization (A.1)–(A.3) is the partial current density, i.e., $\mathbf{j}_A = \mathbf{j}^{(v)}$.

References

1. Frenkel Ya I *Sobranie Izbrannykh Trudov* (Selected Works) Vol. 1 *Elektrodinamika* (*Obshchaya Teoriya Elektrichestva*) (Electrodynamics (The General Theory of Electricity)) (Moscow – Leningrad: Izd. AN SSSR, 1956)
2. Frenkel Ya I *Elektrodinamika* (Electrodynamics) Vol. 2 (Leningrad – Moscow: ONTI, 1935)
3. Ferrers N M *Quart. J. Pure Appl. Math.* **14** (53) 1 (1877)
4. Muratov R Z *Potentsialy Ellipsoida* (Potentials of an Ellipsoid) (Moscow: Atomizdat, 1976)
5. Medvedev B V *Nachala Teoreticheskoi Fiziki* (The Fundamentals of Theoretical Physics) (Moscow: Nauka, 1977)
6. Efimov S P, Muratov R Z *Astron. Zh.* **67** 302 (1990) [*Sov. Astron.* **34** 152 (1990)]
7. Efimov S P, Muratov R Z *Astron. Zh.* **67** 314 (1990) [*Sov. Astron.* **34** 157 (1990)]
8. Muratov R Z *Astron. Zh.* **70** 1271 (1993) [*Astron. Rep.* **37** 641 (1993)]
9. Poisson S D *Mémoir. l'Acad. Sci. Paris* **5** 247 (1821 – 1822)
10. Mattis D C *The Theory of Magnetism*: *An Introduction to the Study of Cooperative Phenomena* (New York: Harper & Row, 1965) [Translated into Russian (Moscow: Mir, 1967)]
11. Whittaker E T *A History of the Theories of Aether and Electricity* Vol. 1 *The Classical Theories* (London: T. Nelson, 1951) [Translated into Russian (Izhevsk: RKhD, 2001)]
12. Maxwell J C *A Treatise on Electricity and Magnetism* Vol. 2 (London: Oxford Univ. Press, 1891) [Translated into Russian (Moscow: Nauka, 1989)]
13. Ampère A-M *J. Physique* **93** 447 (1821) [Translated into Russian: *Elektrodinamika* (Electrodynamics) (Moscow: Izd. AN SSSR, 1954) p. 283]
14. Ampère A-M *Mémoir. l'Acad. Sci.* **6** 175 (1827) [Translated into Russian: *Elektrodinamika* (Electrodynamics) (Moscow: Izd. AN SSSR, 1954) p. 7]
15. Tamm I E *Osnovy Teorii Elektrichestva* (Fundamentals of the Theory of Electricity) (Moscow: Nauka, 1989) [Translated into English (Moscow: Mir Publ., 1979)]
16. Muratov R Z *Zh. Tekh. Fiz.* **67** (4) 1 (1997) [*Tech. Phys.* **42** 325 (1997)]
17. Efimov S P *Teor. Mat. Fiz.* **39** 219 (1979) [*Theor. Math. Phys.* **39** 425 (1979)]
18. Muratov R Z *Zh. Tekh. Fiz.* **72** (4) 6 (2002) [*Tech. Phys.* **47** 380 (2002)]
19. Thomson W, Tait P G *Treatise on Natural Philosophy* Pt. 2 (Cambridge: The Univ. Press, 1912)
20. Routh E J *A Treatise on Analytical Statics* Vol. 2 (Cambridge: The Univ. Press, 1922)
21. Muratov R Z, Shkuratnik V L *Zh. Tekh. Fiz.* **75** (8) 1 (2005) [*Tech. Phys.* **50** 961 (2005)]
22. Landau L D, Lifshitz E M *Elektrodinamika Sploshnykh Sred* (Electrodynamics of Continuous Media) (Moscow: Nauka, 1992) [Translated into English (Oxford: Pergamon Press, 1984)]
23. Stratton J A *Electromagnetic Theory* (New York: McGraw-Hill, 1941) [Translated into Russian (Moscow – Leningrad: GITTL, 1948)]
24. Hobson E W *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge: The Univ. Press, 1931) [Translated into Russian (Moscow: IL, 1952)]
25. Agre M Ya *Usp. Fiz. Nauk* **181** 173 (2011) [*Phys. Usp.* **54** 167 (2011)]