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Expanding Universe: slowdown or speedup?

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Abstract. **The kinematics and the dynamical interpretation of cosmological expansion are reviewed in a widely accessible manner with emphasis on the acceleration aspect. Virtually all the approaches that can in principle account for the accelerated expansion of the Universe are reviewed, including dark energy as an item in the energy budget of the Universe, modified Einstein equations, and, on a fundamentally new level, the use of the holographic principle.**

1. Introduction

There exist two fundamental reasons that render the answer to the question raised in the title of the article essential. First, in a very broad sense, the answer is necessary (but not sufficient) for predicting the final destiny of the Universe. Moreover, in a more narrow sense, knowledge of the current values of kinematical parameters (velocity, acceleration, etc.) is necessary for setting their initial values when solving the differential equations describing the dynamics of the Universe. Second, totally different cosmological models are required for describing the decelerating or accelerating expansion of the Universe. Customary components — nonrelativistic matter and radiation, on the one hand, and the general relativity (GR) theory, on the other — explain deceleration of the expansion. Explaining accelerated expansion requires either a radical change in the composition of the Universe or an even more drastic responsible decision, that the fundamental physics underlying our understanding of the world as a whole is erroneous. Experience tells us that the answer to such a question is not likely to be obtained soon. In the 80 years that have passed since the expansion of the Universe was discovered, we have still not established the source that created the initial field of velocities. The euphemism 'Big Bang' only serves as a cover for our lack of knowledge. The point is that what actually expands gives rise to heated disputes. "How can nothing expand?" [1]. Even more mysterious is how the expansion of the vacuum can speed up or slow down [2].

No single generation of homo sapiens has been able to resist the temptation of assuming that precisely it has finally revealed the true nature of the Universe. For example, owing to Hubble's discovery of the expansion of the Universe and to the new fundamental physical theories — primarily, the theory of relativity and quantum mechanics — the first half of the 20th century was marked by a new cosmological paradigm: the Big Bang model. The main source for this

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model is Hubble's law, discovered in the 1920s and confirmed by cosmological observations of all sorts. The Big bang model allowed explaining the thermal evolution of the Universe cooling down, predicted the existence of cosmic background radiation, and correctly described the relative contents of light elements and many other properties of the Universe.

By the end of the 20th century, the hope arose that the Big Bang model complemented by the inflation theory represented a correct model of the Universe, at least in the first approximation. However, this hope was not to be fulfilled. The change in the cosmological paradigm was due to observations that continued to be performed with ever increasing accuracy. Since the times of Hubble, cosmologists have tried to measure deceleration of the expansion, due to gravity. The confidence in revealing precisely this effect was so strong that the relevant parameter was called the deceleration parameter. However, in 1998, two independent collaborations [3, 4] studying distant supernovae presented convincing evidence that the expansion of the Universe was actually undergoing acceleration. It turned out that the decrease in luminosity was noticeably more rapid than previously considered on the basis of the Big Bang model. Such an additional fading signified that a given red shift had a certain additional distance corresponding to it. But it hence followed that the cosmological expansion was accelerating: in the past, the Universe expanded more slowly than now. The discovery of cosmological expansion is most likely one of the most important discoveries made, not only in contemporary cosmology but also, generally, in physics. A Universe expanding with acceleration serves as the most direct demonstration of our fundamental theories being either incomplete or, which is even worse, erroneous [5, 6].

The physical origin of cosmic expansion remains a supreme mystery. As noted above, if the Universe is only filled with matter and radiation, its expansion should slow down. If its expansion is speeding up, two possibilities remain open, either of which must lead to a revision of our principal physical ideas:

(1) up to 75% of the energy density of the Universe exists in the form of an unknown substance (conventionally called dark energy) with a large negative pressure providing accelerated expansion;

(2) the GR theory must be revised at cosmological scales.

It must be borne in mind that besides the two radical possibilities mentioned for making theory and observations consistent with each other, there also always exists an obvious conservative way of resolving the problem: more careful application of the available theoretical possibilities [7]. The phrase 'more careful application of the available theoretical possibilities' requires clarification. We consider an example. In the context of cosmological models based on a homogeneous and isotropic Universe, explaining the acceleration observed requires using a new form of matter with negative pressure, namely, dark energy (DE). According to an alternative explanation [7–10], accelerated expansion of the Universe is a consequence of its deviation from homogeneity. The mass density of the Universe is considered significantly inhomogeneous at scales inferior to the Hubble radius. To pass to an effectively homogeneous and isotropic Universe, it is necessary to average and/or smooth out the inhomogeneities up to a certain averaging scale, chosen accordingly. In such an averaged Universe, 'effective cosmological parameters' can be defined. It then turns out that the equations of motion thus obtained, generally speaking, differ from the equations with the same parameters in models based on the cosmological principle (a homogeneous and isotropic Universe). If the equations differ in a manner corresponding to an effective supplement of DE, then one can hope to achieve an explanation of the accelerated expansion of the Universe in the context of GR without having to rely on DE.

The first two possibilities are analyzed in the present article. We describe both the kinematics of cosmological expansion and the dynamic interpretation of this process. Most attention is given to the acceleration of cosmological expansion. We stress that the equivalence principle relates the expansion to the nature of gravity and the geometry of space–time, and therefore the actual value of this acceleration is extremely important for testing various cosmological models: the final destiny of the Universe significantly depends precisely on it.

2. Cosmography: the kinematics of expansion of the Universe

The scheme used in this section for describing the Universe is fully based on the cosmological principle and has been termed 'cosmography' [11]. The cosmological principle, asserting that at the Universe is homogeneous and isotropic at scales superior to several hundred megaparsecs, permits selecting a narrow class of homogeneous and isotropic models from the entire variety of models describing the Universe that can possibly be imagined. The most general space–time metric consistent with the cosmological principle is the Friedmann–Robertson–Walker (FRW) metric [1]

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right], \quad (2.1)$$

where $a(t)$ is a scale factor, r are coordinates of a point taking part in no other motion but global expansion of the Universe, and k is a parameter characterizing the spatial curvature of the Universe. In a certain narrow sense, the main task of cosmology consists in finding the dependence $a(t)$.

The cosmological principle permits constructing the metric of the Universe and taking the first steps toward interpretation of cosmological observations. Like kinematics, that is, the part of mechanics describing the motion of bodies regardless of the forces causing this motion, cosmography only represents the kinematics of cosmological expansion. Equations of motion (Einstein's equations) and an assumption concerning the matter composition of the Universe, which permits constructing the energy–momentum tensor, are necessary for constructing the key cosmological characteristic, the time dependence of the scale factor $a(t)$. The effectiveness of cosmography consists in providing the possibility of testing any cosmological models that do not contradict the cosmological principle. Modifications of GR or the introduction of new components (dark matter (DM), dark energy), would certainly alter the dependence $a(t)$, but have no effect on the relations between kinematical characteristics. The rate at which the Universe expands is determined by the Hubble constant $H(t) \equiv \dot{a}(t)/a(t)$, and it depends on time. A measure of this dependence is the

[1] We use the signature $(+,-,-,-)$ and the definition $R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$, $R_{\nu\mu} = R_{\nu\beta\mu}{}^{\beta}$, $R = g^{\mu\nu}R_{\mu\nu}$.

deceleration parameter $q(t)$. We determine $q(t)$ with the aid of the scale factor $a(t)$ expanded in a Taylor series in the vicinity of the current moment of time t_0:

$$a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \frac{1}{2}\,\ddot{a}(t_0)(t - t_0)^2 + \cdots . \qquad (2.2)$$

We represent relation (2.2) as

$$\frac{a(t)}{a(t_0)} = 1 + H_0(t - t_0) - \frac{q_0}{2}\,H_0^2(t - t_0)^2 + \cdots \qquad (2.3)$$

with the deceleration parameter

$$q(t) \equiv -\frac{\ddot{a}(t)\,a(t)}{\dot{a}^2(t)} = -\frac{\ddot{a}(t)}{a(t)}\,\frac{1}{H^2(t)} . \qquad (2.4)$$

As is shown below, the increase in the scale parameter accelerates if $q < 0$, while an accelerated expansion rate, $\dot{H} > 0$, corresponds to $q < -1$. In choosing the sign of the deceleration parameter, it seemed evident that gravity, the only force governing the dynamics of the Universe, slows its expansion. The natural desire to deal with a positive parameter actually predetermined the choice of sign. Subsequently, it turned out that the choice made did not correspond to the expansion dynamics observed and could rather serve as an example of historical curiosity.

For a more complete description of the kinematics of cosmological expansion, it is useful to consider an extended set of parameters [12–14]:

$$H(t) \equiv \frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t} ,$$

$$q(t) \equiv -\frac{1}{a}\,\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}\left(\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)^{-2} ,$$

$$j(t) \equiv \frac{1}{a}\,\frac{\mathrm{d}^3 a}{\mathrm{d}t^3}\left(\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)^{-3} ,$$

$$s(t) \equiv \frac{1}{a}\,\frac{\mathrm{d}^4 a}{\mathrm{d}t^4}\left(\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)^{-4} ,$$

$$l(t) \equiv \frac{1}{a}\,\frac{\mathrm{d}^5 a}{\mathrm{d}t^5}\left(\frac{1}{a}\,\frac{\mathrm{d}a}{\mathrm{d}t}\right)^{-5} .$$

We note that the last four parameters are dimensionless. The parameters whose expressions involve lower-order derivatives of a with respect to t can be expressed via the higher-order derivatives of a with respect to t. For example,

$$\frac{\mathrm{d}q}{\mathrm{d}\ln(1+z)} = j - q(2q+1) .$$

We expand the scale factor in a Taylor series in time, using the parameters introduced:

$$a(t) = a_0\left[1 + H_0(t - t_0) - \frac{1}{2}\,q_0 H_0^2(t - t_0)^2 + \frac{1}{3!}\,j_0 H_0^3(t - t_0)^3 + \frac{1}{4!}\,s_0 H_0^4(t - t_0)^4 + \frac{1}{5!}\,l_0 H_0^5(t - t_0)^5 + O\big((t - t_0)^6\big)\right] . \qquad (2.5)$$

In terms of the same parameters, the Taylor series for the red shift has the form

$$1 + z = \left[1 + H_0(t - t_0) - \frac{1}{2}\,q_0 H_0^2(t - t_0)^2 + \frac{1}{3!}\,j_0 H_0^3(t - t_0)^3 + \frac{1}{4!}\,s_0 H_0^4(t - t_0)^4 + \frac{1}{5!}\,l_0 H_0^5(t - t_0)^5 + O\big((t - t_0)^6\big)\right]^{-1} , \qquad (2.6)$$

$$z = H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right)H_0^2(t - t_0)^2 + \cdots . \qquad (2.7)$$

We now present a number of useful relations for the deceleration parameter:

$$q(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right) - 1 ,$$

$$q(z) = \frac{1+z}{H}\,\frac{\mathrm{d}H}{\mathrm{d}z} - 1 ,$$

$$q(z) = \frac{\mathrm{d}\ln H}{\mathrm{d}z}(1+z) - 1 ,$$

$$q(a) = -\left(1 + \frac{\mathrm{d}H/\mathrm{d}t}{H^2}\right) = -\left(1 + \frac{a\,\mathrm{d}H/\mathrm{d}a}{H}\right) ,$$

$$q = -\frac{\mathrm{d}\ln(aH)}{\mathrm{d}\ln a} .$$

For a one-component liquid of density ρ,

$$q(a) = -1 - \frac{a}{2\rho}\,\frac{\mathrm{d}\rho}{\mathrm{d}a} . \qquad (2.8)$$

The derivatives $\mathrm{d}H/\mathrm{d}z$, $\mathrm{d}^2H/\mathrm{d}z^2$, $\mathrm{d}^3H/\mathrm{d}z^3$, and $\mathrm{d}^4H/\mathrm{d}z^4$ can be expressed via the deceleration parameter q and the parameter j:

$$\frac{\mathrm{d}H}{\mathrm{d}z} = \frac{1+q}{1+z}\,H ,$$

$$\frac{\mathrm{d}^2H}{\mathrm{d}z^2} = \frac{j - q^2}{(1+z)^2}\,H ,$$

$$\frac{\mathrm{d}^3H}{\mathrm{d}z^3} = \frac{H}{(1+z)^3}(3q^2 + 3q^3 - 4qj - 3j - s) ,$$

$$\frac{\mathrm{d}^4H}{\mathrm{d}z^4} = \frac{H}{(1+z)^4}(-12q^2 - 24q^3 - 15q^4 + 32qj + 25q^2 j + 7qs + 12j - 4j^2 + 8s + l) .$$

For convenience, we present useful relations that allow passing from higher-order derivatives with respect to time to derivatives with respect to the red shift:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = (1+z)H\left[H + (1+z)\,\frac{\mathrm{d}H}{\mathrm{d}z}\right]\frac{\mathrm{d}}{\mathrm{d}z} + (1+z)^2H^2\,\frac{\mathrm{d}^2}{\mathrm{d}z^2} , \qquad (2.9)$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3} = -(1+z)H\left\{H^2 + (1+z)^2\left(\frac{\mathrm{d}H}{\mathrm{d}z}\right)^2 + (1+z)H\left[4\,\frac{\mathrm{d}H}{\mathrm{d}z} + (1+z)\,\frac{\mathrm{d}^2H}{\mathrm{d}z^2}\right]\right\}\frac{\mathrm{d}}{\mathrm{d}z} - 3(1+z)^2H^2 \times\left[H + (1+z)\,\frac{\mathrm{d}H}{\mathrm{d}z}\right]\frac{\mathrm{d}^2}{\mathrm{d}z^2} - (1+z)^3H^3\,\frac{\mathrm{d}^3}{\mathrm{d}z^3} , \qquad (2.10)$$

$$\frac{\mathrm{d}^4}{\mathrm{d}t^4} = (1+z)H\left[H^2 + 11(1+z)H^2\frac{\mathrm{d}H}{\mathrm{d}z} + 11(1+z)H\frac{\mathrm{d}H}{\mathrm{d}z} + (1+z)^3\left(\frac{\mathrm{d}H}{\mathrm{d}z}\right)^3 + 7(1+z)^2 H\frac{\mathrm{d}^2H}{\mathrm{d}z^2} + 4(1+z)^3 H\frac{\mathrm{d}H}{\mathrm{d}z}\frac{\mathrm{d}^2H}{\mathrm{d}z^2} + (1+z)^3H^2\frac{\mathrm{d}^3H}{\mathrm{d}z^3}\right]\frac{\mathrm{d}}{\mathrm{d}z} + (1+z)^2H^2\left[7H^2 + 22H\frac{\mathrm{d}H}{\mathrm{d}z} + 7(1+z)^2\left(\frac{\mathrm{d}H}{\mathrm{d}z}\right)^2 + 4H\frac{\mathrm{d}^2H}{\mathrm{d}z^2}\right]\frac{\mathrm{d}^2}{\mathrm{d}z^2} + 6(1+z)^3H^3\left[H + (1+z)\frac{\mathrm{d}H}{\mathrm{d}z}\right]\frac{\mathrm{d}^3}{\mathrm{d}z^3} + (1+z)^4H^4\frac{\mathrm{d}^4}{\mathrm{d}z^4}\,. \tag{2.11}$$

The derivatives of the squared Hubble parameter with respect to the red shift, $\mathrm{d}^{(i)}H^2/\mathrm{d}z^{(i)}$, $i = 1, 2, 3, 4$, can be expressed in terms of the cosmographic parameters:

$$\frac{\mathrm{d}(H^2)}{\mathrm{d}z} = \frac{2H^2}{1+z}(1+q)\,,$$
$$\frac{\mathrm{d}^2(H^2)}{\mathrm{d}z^2} = \frac{2H^2}{(1+z)^2}(1+2q+j)\,,$$
$$\frac{\mathrm{d}^3(H^2)}{\mathrm{d}z^3} = \frac{2H^2}{(1+z)^3}(-qj-s)\,,$$
$$\frac{\mathrm{d}^4(H^2)}{\mathrm{d}z^4} = \frac{2H^2}{(1+z)^4}(4qj+3qs+3q^2j-j^2+4s+l)\,.$$

The derivatives of the Hubble parameter with respect to time can also be expressed via the cosmographic parameters H, q, j, s, l:

$$\begin{aligned} \dot H &= -H^2(1+q)\,,\\ \ddot H &= H^3(j+3q+2)\,,\\ \dddot H &= H^4\left[s-4j-3q(q+4)-6\right],\\ \ddddot H &= H^5\left[l-5s+10(q+2)j+30(q+2)q+24\right]. \end{aligned} \tag{2.12}$$

It follows from (2.12) that the accelerated increase in the expansion rate, $\dot H > 0$, corresponds to $q < -1$.

From relation (2.8), the Hubble parameter can be seen to be related to the deceleration parameter by the integral relation

$$H = H_0 \exp\left\{\int_0^z \left[q(z')+1\right]\mathrm{d}\ln(1+z')\right\}.$$

Hence, it immediately follows that for constructing the main characteristic of the expanding Universe, $H(z)$, information is required on the dynamics of the cosmological expansion, which is encoded in the quantity $q(z)$.

3. Brief description of the dynamics of cosmological expansion

The dynamics of the Universe are described in the GR framework by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}^2}T_{\mu\nu}\,.$$

The energy–momentum tensor $T_{\mu\nu}$ describes the distribution of mass (energy) in space, while the components of the curvature tensor $R_{\mu\nu}$ and its trace R are expressed via the metric tensor $g_{\mu\nu}$ and its first and second derivatives. The Einstein equations are complicated nonlinear equations in general. The problem is simplified when the mass distributions considered exhibit particular symmetry properties encoded in the metric. For a homogeneous and isotropic universe described by the FRW metric, the Einstein equations reduce to a set of Friedmann equations:

$$\frac{\dot a^2}{a^2} = \frac{1}{3M_{\mathrm{Pl}}^2}\rho - \frac{k}{a^2}\,, \tag{3.1a}$$

$$\frac{\ddot a}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}(\rho+3p)\,. \tag{3.1b}$$

Here, ρ and p are the energy density and pressure of all the components present in the Universe at the moment of time considered and $M_{\mathrm{Pl}} = (8\pi G)^{-1/2}$ is the reduced Planck mass. These equations are not sufficient for a complete description of the dynamics of the Universe. A consequence of the Lorentz invariance of the energy–momentum tensor $T^{\mu}_{\nu,\mu} = 0$ is the conservation equation

$$\dot\rho + 3H(\rho+p) = 0\,, \tag{3.2}$$

which can be readily seen to represent the first law of thermodynamics for an ideal liquid with constant entropy, $\mathrm{d}E + p\,\mathrm{d}V = 0$. We note that this equation can be obtained from the Friedmann equations.

We now introduce the concept of the relative density of the ith component of energy density:

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\mathrm{c}}}\,, \qquad \rho_{\mathrm{c}} \equiv 3M_{\mathrm{Pl}}^2H^2\,, \qquad \Omega \equiv \sum_i \Omega_i\,. \tag{3.3}$$

Having determined the relative curvature density $\Omega_k \equiv -k/(a^2H^2)$, we can represent the first Friedmann equation as

$$\sum_i \Omega_i = 1\,.$$

To solve the Friedmann equations, it is necessary to determine the content of matter in the Universe and to construct the equation of state for each component. In the case of the simplest, linear parameterization, the equation of state has the form

$$p_i = w_i\,\rho_i\,. \tag{3.4}$$

Solving the Friedmann equations in the case of a one-component Universe that has a time-independent parameter of the equation of state ($w = w_i = \mathrm{const}$) and is spatially flat ($k = 0$), we obtain

$$a(t) \propto \left(\frac{t}{t_0}\right)^{2/[3(1+w)]}\,, \qquad \rho \propto a^{-3(1+w)}\,.$$

(We normalize the scale factor by the condition $a(t_0) = 1$.) These solutions exist only if $w \neq -1$. We deal with the last case specially. For a universe in which radiation (a relativistic gas of photons and neutrinos) is dominant, $w = 1/3$, and if matter dominates, then $w = 0$. As a consequence of such equations of state, we obtain the following in the case of matter:

$$a(t) \propto \left(\frac{t}{t_0}\right)^{2/3}, \qquad \rho \propto a^{-3}.$$

The last result can be explained as a simple consequence of the conservation of the number of particles. For radiation,

$$a(t) \propto \left(\frac{t}{t_0}\right)^{1/2}, \qquad \rho \propto a^{-4}.$$

This result is a consequence of the radiation energy density decreasing as a^{-3} owing to the increase in volume (expansion of the Universe), and as a^{-1} owing to the red shift. We note that it follows from Eqn (3.2) that $\rho = \text{const}$ for $w = -1$. In this case, the Hubble velocity remains constant, leading to an exponential increase in the scale factor,

$$a(t) \propto \exp(Ht).$$

Therefore, in the case of traditional cosmological components — matter and radiation ($w = 0$ and $w = 1/3$) — expansion of the Universe can only undergo deceleration, $\ddot{a} < 0$.

Using the definition of the deceleration parameter, we find that for a flat universe filled with a single component with the equation of state $p = w\rho$,

$$q = \frac{1}{2}(1 + 3w).$$

In the general case ($k = 0, \pm 1$, $\rho = \sum_i \rho_i$, and $p = \sum_i \rho_i w_i$), we obtain

$$q = \frac{\Omega}{2} + \frac{3}{2}\sum_i w_i \Omega_i. \qquad (3.5)$$

With (3.3), the last relation can be represented as

$$q = \frac{1}{2}\left(1 + \frac{k}{a^2 H^2}\right)\left(1 + 3\frac{p}{\rho}\right).$$

Because q is a slowly changing quantity (in the case of dominant matter, $q = 1/2$, and in the case of dominant DE in the form of the cosmological constant, $q = -1$), useful information is contained in its value averaged over time, which it is desirable to obtain without integrating the equations of motion for the scale factor. We verify that this is possible [15]. For this, we define the parameter $\bar{q}$ within the interval $[0, t_0]$ as

$$\bar{q}(t_0) = \frac{1}{t_0}\int_0^{t_0} q(t)\,\mathrm{d}t.$$

Substituting the definition of the deceleration parameter

$$q(t) = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right) - 1,$$

it is easy to obtain

$$\bar{q}(t_0) = -1 + \frac{1}{t_0 H_0}, \qquad (3.6)$$

or

$$t_0 = \frac{H_0^{-1}}{1 + \bar{q}}. \qquad (3.7)$$

As expected, the current age of the Universe is proportional to H_0^{-1}, but the proportionality coefficient is only determined by the average value of the deceleration parameter. It is useful to note that this purely kinematical result depends neither on the curvature of the Universe nor on the number of components filling the Universe, nor on the chosen version of gravity theory.

We now represent the results obtained above for the average deceleration parameter in a somewhat different form. In the case of a one-component flat universe, the Friedmann equations can be represented as

$$\rho = 3M_{\mathrm{Pl}}^2\frac{\dot{a}^2}{a^2}, \qquad (3.8)$$

$$p = -M_{\mathrm{Pl}}^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right). \qquad (3.9)$$

For a one-component universe with the equation of state $p = w\rho$, $w = \text{const}$, the equation for the scale factor

$$a\ddot{a} + \left(\frac{1 + 3w}{2}\right)\dot{a}^2 = 0$$

has the general solution

$$a(t) = a_0\left[\frac{3}{2}(1 + w)H_0 t\right]^{2/[3(1+w)]}. \qquad (3.10)$$

Hence, it immediately follows that

$$q = \frac{1 + 3w}{2} = \text{const}, \qquad t_0 = \frac{2H_0^{-1}}{3(1 + w)}. \qquad (3.11)$$

Therefore, in the case of radiation ($w = 1/3$), $q = 1$, and in the case of nonrelativistic matter ($w = 0$), $q = 1/2$. On the other hand, since in this case $q = \bar{q}$, the last relation can be rewritten as $t_0 = H_0^{-1}/(1 + \bar{q})$, which coincides with relation (3.7). The result in (3.11) can be represented as

$$T = \frac{H^{-1}}{1 + \bar{q}}, \qquad (3.12)$$

where T, H, and $\bar{q}$ are the age of the Universe, the Hubble parameter, and the average deceleration parameter. Because $\bar{q}$ is of the order of unity, it immediately follows from (3.10) that at any stage of evolution of the Universe, the Hubble time H_0^{-1} serves as a characteristic time scale.

Dynamic model-independent constraints on the kinematics of the Universe can further be obtained from the so-called energy conditions [16–20]. These conditions, based on quite general physical principles, impose restrictions on the components of the energy–momentum tensor $T_{\mu\nu}$. In choosing a model for the medium (a model, but not the equation of state!), these conditions can be transformed into inequalities restricting the possible values of pressure and density of the medium. In the Friedmann model, the medium is an ideal liquid, for which

$$T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu - p g_{\mu\nu}, \qquad (3.13)$$

where u_μ is the 4-velocity of the ideal liquid of energy density ρ and pressure p, which can be expressed via the scale factor and its derivatives,

$$\rho = 3M_{\mathrm{Pl}}^2\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right), \quad p = -M_{\mathrm{Pl}}^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right). \quad (3.14)$$

In GR, as in cosmology generally, significant importance is attributed to the energy conditions (see, e.g., Refs [16–19]), which in our case reduce to the conditions

$$\begin{aligned}
&\text{NEC} \Rightarrow \rho + p \geqslant 0\,,\\
&\text{WEC} \Rightarrow \rho \geqslant 0\,, \quad \rho + p \geqslant 0\,,\\
&\text{SEC} \;\; \Rightarrow \rho + 3p \geqslant 0\,, \quad \rho + p \geqslant 0\,,\\
&\text{DEC} \;\Rightarrow \rho \geqslant 0\,, \quad -\rho \leqslant p \leqslant \rho\,.
\end{aligned}$$

Here, NEC, WEC, SEC, and DEC correspond to the zero, weak, strong, and dominant energy conditions. Because these conditions do not require any definite equation of state for the substance filling the Universe, they impose very simple and model-independent constraints on the behavior of the energy density and pressure. Hence, the energy conditions provide one of the possibilities for explaining the evolution of the Universe on the basis of quite general principles. With expression (3.14), the energy conditions can be expressed in terms of the scale factor and its derivatives:

$$\begin{aligned}
&\text{NEC} \Rightarrow -\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \geqslant 0\,,\\
&\text{WEC} \Rightarrow \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \geqslant 0\,,\\
&\text{SEC} \Rightarrow \frac{\ddot{a}}{a} \leqslant 0\,,\\
&\text{DEC} \Rightarrow \frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \geqslant 0\,.
\end{aligned} \tag{3.15}$$

In the case of a flat universe, conditions (3.15) can be transformed into restrictions on the deceleration parameter q:

$$\begin{aligned}
&\text{NEC} \Rightarrow q \geqslant -1\,,\\
&\text{SEC} \Rightarrow q \geqslant 0\,,\\
&\text{DEC} \Rightarrow q \leqslant 2\,.
\end{aligned} \tag{3.16}$$

The weak condition WEC is not present among these conditions because it is always satisfied for arbitrary real $a(t)$.

Conditions (3.16), considered separately, essentially make both decelerated $(q > 0)$ and accelerated $(q < 0)$ expansion of the Universe possible. The meaning of the restrictions for NEC in (3.16) is quite clear. It follows from the second Friedmann equation that the inequality $\rho + 3p \leqslant 0$ is the condition for accelerated expansion of the Universe, i.e., accelerated expansion of the Universe is only possible if components exist with a large negative pressure, $p < -1/3\rho$. The energy requirements of SEC exclude the existence of such components. Consequently, in this case, $q \geqslant 0$. At the same time, the NEC and DEC conditions are compatible with the condition $p < -1/3\rho$, and they therefore allow the existence of states with $q < 0$.

To conclude this section, we turn to an interesting peculiarity of the dynamics of the expanding Universe. Owing to the Hubble law, galaxies lying on the Hubble sphere move away from us at the speed of light. The velocity of the Hubble sphere is the time derivative of the Hubble radius $R_{\mathrm{H}} = c/H$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(R_{\mathrm{H}}) = c\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right) = -\frac{c}{H^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = c(1+q)\,. \tag{3.17}$$

In a universe with decelerating expansion $(q > 0)$, the Hubble sphere, having the speed greater than the speed of light by cq, overtakes these galaxies. Therefore, the galaxies that were initially outside the Hubble sphere occur inside it. An observer at any point in the Universe sees an increasing number of galaxies. In a universe with accelerated expansion $(q < 0)$, the Hubble sphere, whose speed is less then the speed of light by cq, trails behind these galaxies. Hence, the galaxies that were initially inside the Hubble sphere occur outside it and are no longer available for observation. Should we consider them part of the physical reality? The difference between physics and metaphysics consists in the possibility of performing experimental tests of physical theories. Physics does not deal with objects that cannot be observed. However, the boundaries of physics are continuously extending to include more and more abstract concepts that were previously metaphysical: atoms, electromagnetic waves, black holes, and so on. This list can be continued.

We are apparently inhabitants of a universe with an accelerating expansion, in which, as in a universe with a decelerating expansion, there are galaxies so far away from us that no signal from them can be registered by an observer on Earth. If the cosmological expansion is accelerating, then we are moving away from these galaxies with a velocity greater than the speed of light. Therefore, if light from them has not reached us by now, then it will never reach us. Such galaxies cannot be observed now, nor will they ever be observable. These 'never observable galaxies' trace their origin back to the same Big Bang as our Milky Way. Should we consider them objects of physics or of metaphysics? Those who believe science fiction to be the realization of unlimited fantasy are quite mistaken. Science fiction is dull and lacks any flight of fantasy compared with cosmology.

Accelerated expansion of the Universe first appeared in cosmological models together with the inflation theory, which was developed in order to remove the numerous defects of the Big Bang model. To remove most of the defects of the Big Bang model, it turned out to be sufficient for the rapid accelerated expansion of the Universe to be exponential at the very beginning of its evolution, merely during a period of 10^{-35} s. The simplest way of achieving such a mode of expansion is to consider the dynamics of the Universe with a scalar field. Numerous versions of the inflation theory have been formulated, starting from models based on quantum gravity and the theory of high-temperature phase transitions with supercooling and exponential expansion in a state of a false vacuum. To illustrate the main idea of this theory, we consider a flat, homogeneous, and isotropic universe filled with a scalar field of potential $V(\varphi)$, independent of the coordinates. In this case, the first Friedmann equation (3.1a) becomes

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{2}\,\dot{\varphi}^2 + V(\varphi)\right).$$

Conservation equation (3.2) written for a scalar field takes the form of the Klein–Gordon equation on a nonstationary background:

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0\,. \tag{3.18}$$

In a rapidly expanding universe, the scalar field 'rolls' down very slowly, like a little ball in a viscous liquid where the effective viscosity is proportional to the rate of the expansion.

In the slow-roll mode,

$$H\dot{\varphi} \gg \ddot{\varphi}\,, \quad V(\varphi) \gg \dot{\varphi}^2\,.$$

In this limit, the equations of motion take the form

$$3H\dot{\varphi} + V'(\varphi) = 0\,,$$

$$H^2 = \frac{V(\varphi)}{3M_{\mathrm{Pl}}^2}\,.$$

For definiteness, we consider the simplest model of a scalar field with a mass m and the potential energy $V(\varphi) = m^2\varphi^2/2$. Soon after the beginning of inflation, the relations $\ddot{\varphi} \ll 3H\dot{\varphi}$, $\dot{\varphi}^2 \ll m^2\varphi^2$ become valid. Therefore,

$$3\,\frac{\dot{a}}{a}\,\dot{\varphi} + m^2\varphi = 0\,,$$

$$H = \frac{\dot{a}}{a} = \frac{2m\varphi}{M_{\mathrm{Pl}}}\sqrt{\frac{\pi}{3}}\,.$$

Owing to the rapid growth of the scale factor and to the slow change in the field (large friction),

$$a \propto \exp(Ht)\,, \quad H = \frac{2m\varphi}{M_{\mathrm{Pl}}}\sqrt{\frac{\pi}{3}}\,.$$

For greater clarity, we find the equation of state of the scalar field in the slow roll mode. For a homogeneous scalar field in a potential $V(\varphi)$ in a locally Lorentzian reference frame, the nonzero components of the energy–momentum tensor are given by

$$T_{00} = \frac{1}{2}\,\dot{\varphi}^2 + V(\varphi) = \rho_\varphi\,, \quad T_{ij} = \left(\frac{1}{2}\,\dot{\varphi}^2 - V(\varphi)\right)\delta_{ij} = p_\varphi\delta_{ij}\,.$$

In the slow-roll mode, $\dot{\varphi}^2 \ll V(\varphi)$, whence $p_\varphi \approx -\rho_\varphi$. Therefore, the energy–momentum tensor in the slow-roll mode approximately coincides with the vacuum one, for which $p = -\rho$. Using (3.5) and taking into account that owing to the exponential growth of the scale factor, the inflation scenario ensures the case of flat space and consequently $\Omega = 1$, we obtain $q = -1$ during the period of inflationary expansion.

Thus, in a quite natural manner, the scalar field allows obtaining accelerated expansion of the Universe, at least at the earlier stages of its evolution. As the field decreases (slowly rolls down), the viscosity decreases, and the Universe exits from the inflation mode (exponential growth of the scale factor).

We note that a scalar field can provide not only an accelerated but also a decelerated expansion of the Universe. In the vicinity of the minimum of the inflation potential, the inflation conditions are certainly violated, and the Universe exits from the inflation mode. The scalar field starts to oscillate in the vicinity of the minimum. If the period of oscillations is assumed to be much smaller than cosmological time scales, which permits neglecting cosmological expansion in Eqn (3.18), it is not difficult to determine the effective equation of state in the vicinity of the minimum of the inflation potential. We represent Eqn (3.18) for the scalar field as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi\dot{\varphi}) - \dot{\varphi}^2 + \varphi V'_\varphi = 0\,.$$

In averaging over the period of oscillations, the first term vanishes; consequently,

$$\langle\dot{\varphi}^2\rangle \simeq \langle\varphi V'_\varphi\rangle\,.$$

The effective (averaged) equation of state is

$$w \equiv \frac{p}{\rho} \simeq \frac{\langle\varphi V'_\varphi\rangle - \langle 2V\rangle}{\langle\varphi V'_\varphi\rangle + \langle 2V\rangle}\,.$$

In the case of a quadratic potential $V \propto \varphi^2$, we obtain $w \simeq 0$, which corresponds to the equation of state of nonrelativistic matter.

We also note that models with scalar fields have become widespread in cosmology, and using them allows obtaining accelerated expansion, as well as more complex dynamics of the Universe. Besides, most models with a scalar field have a very good rationale in particle physics and in alternative theories.

4. Proofs of acceleration of the expansion of the Universe

Hubble's law says nothing about the value, the sign, or the actual possibility of inhomogeneous expansion of the Universe. The approximation in which Hubble's law holds is not sensitive to acceleration. To investigate the effects of nonlinearity, data for large red shifts are required. If any deviation from linearity happens to be observed, the possibility arises of determining the sign of the acceleration on the basis of the value and sign of the deviation. If the deviation revealed shows an increase in the distance for a given red shift, the acceleration is positive. The distance is estimated from the brightness of a source assuming the population of sources under consideration is represented by so-called candles — an ensemble of objects with practically the same luminosity. Therefore, the observed brightness of such objects depends only on the distance from the observer. Bursts of type-Ia supernovae (exploding white dwarfs) are an example of such objects. Because white dwarfs differ very little in mass, their luminosity is practically the same. An additional advantage consists in the enormous power ($\sim 10^{36}$ W) released in the explosion. Therefore, white dwarfs can be found at distances comparable to the size of the observable Universe.

If the intensity L of light emitted by an object (internal luminosity) is known, then, by measuring the intensity F of the light reaching us (observed flux), we can calculate the distance to the object. The distance d_L thus determined is called the photometric distance,

$$d_L^2 = \frac{L}{4\pi F}\,. \tag{4.1}$$

To determine the acceleration of the expansion of the Universe, it is necessary to express the photometric distance in terms of the red shift of the emission registered. Let E be the internal (absolute) luminosity of a source. An observer on Earth registers a flux of photons F. In an expanding universe, an increase in the wavelength of a photon (and, consequently, a decrease in its energy) during its motion from the source to the observer results in the effective (apparent) luminosity of the source $L = E/a(t)$. The conservation law for the energy emitted within an interval $\mathrm{d}t$ and absorbed within the interval

$\mathrm{d}t_0$ is

$$F 4\pi r^2 \,\mathrm{d}t_0 = E \,\mathrm{d}t = La(t)\,\mathrm{d}t\,, \tag{4.2}$$

where r is the comoving distance between the source and the observer at time moment t_0, which coincides with the physical distance in the case of normalization by $a(t_0) = 1$. Hence, the associated distance between the source and the observer does not change; the interval of conformal time $\mathrm{d}\eta = \mathrm{d}t/a$ between two light flares at a certain point of emission and the observer has the form

$$\frac{\mathrm{d}t}{a(t)} = \frac{\mathrm{d}t_0}{a_0}\,.$$

Therefore, it follows from (4.2) that

$$F = \frac{La^2(t)}{4\pi r^2}\,. \tag{4.3}$$

Comparing this expression with the definition of the photometric distance in (4.1), we find

$$d_L = \frac{r}{a(t)} = (1+z)r\,. \tag{4.4}$$

The physical meaning of the last result is simple. In the expanding Universe, the flux registered is suppressed by the factor $(1+z)^2$ owing to the photon wavelength increasing by the factor $1+z$ and to the time interval of the arrival of a fixed portion of energy also increasing by the same factor, $1+z$.

We determine the associated distance to a source of light observed at the present time as a function of its red shift. The trajectory of a photon is described by the equation $\mathrm{d}s^2 = 0$. We consider the radial trajectory with the observer at the origin of the reference frame. In the case of a spatially flat metric,

$$\mathrm{d}s^2 = a^2(t)(\mathrm{d}\eta^2 - \mathrm{d}r^2) = 0\,. \tag{4.5}$$

Taking into account that

$$\mathrm{d}\eta = \frac{\mathrm{d}\eta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}z}\,\mathrm{d}z = -\frac{\mathrm{d}z}{H(z)}\,, \tag{4.6}$$

we find

$$r(z) = \int_0^z \frac{\mathrm{d}z'}{H(z')}\,. \tag{4.7}$$

Hence, for a spatially flat universe,

$$d_L = (1+z)\int_0^z \frac{\mathrm{d}z'}{H(z')}\,. \tag{4.8}$$

In the general case,

$$d_L(z) = c(1+z)(1-\Omega_0)^{-1/2} H_0^{-1} S\left[(1-\Omega_0)^{1/2} H_0 \int_0^z \frac{\mathrm{d}z'}{H(z')}\right], \tag{4.9}$$

where

$$S(x) = \begin{cases} \sin x\,, & \Omega_0 > 1\,,\\ x\,, & \Omega_0 = 1\,,\\ \sinh x\,, & \Omega_0 < 1\,. \end{cases} \tag{4.10}$$

The quantities H_0 and $\Omega_0 \equiv \rho_0/\rho_{\mathrm{cr}}$ $(\rho_{\mathrm{cr}} \equiv 3M_{\mathrm{Pl}}^2 H_0^2)$ are related to the current time moment. For the multicomponent flat case,

$$d_L = \frac{1+z}{H_0}\int_0^z \frac{\mathrm{d}z'}{\sqrt{\sum_i \Omega_{0i}(1+z')^{3(1+w_i)}}}\,. \tag{4.11}$$

Relation (4.9) can be represented in terms of the deceleration parameter. In the flat case,

$$d_L(z) = (1+z)\int_0^z \frac{\mathrm{d}z'}{H(z')}$$
$$= (1+z)H_0^{-1}\int_0^z \mathrm{d}u\,\exp\left\{-\int_0^u \left[1+q(v)\right]\mathrm{d}\ln(1+v)\right\}. \tag{4.12}$$

We determine the modulus of the distance μ for a standard candle in the usual way:[2]

$$\mu(z) \equiv m_{\mathrm{B}}(z) - M_{\mathrm{B}} = 5\log\big(d_L\,[\mathrm{Mpc}]\big) + 25\,. \tag{4.13}$$

The quantity in (4.12) can be related to the history of the acceleration $q(z)$:

$$\mu(z) = 25 + 5\log\left(\frac{1+z}{H_0}\,[\mathrm{Mpc}]^{-1}\right)$$
$$\times \int_0^z \mathrm{d}u\,\exp\left[-\int_0^u \big(1+q(v)\big)\,\mathrm{d}\ln v\right], \tag{4.14}$$

where M_{B} and m_{B} are respectively the absolute and apparent star magnitudes of the source. Formula (4.14) is a fundamental relation linking the history of the deceleration parameter to measurements of type-Ia supernovae.

We note that only the FRW metric is used in relation (4.12). This means that the deceleration/acceleration dilemma can be investigated without assuming the validity of GR. However, the observation of supernovae does not permit directly determining the current deceleration parameter. To use these data, we must know $H(z)$ or $q(z)$, and for this, dynamic equations are required and the material composition of the Universe must be determined.

It is useful to present the expression for the photometric distance up to terms of the order of z^2:

$$d_L = \frac{z}{H_0}\left[1 + \left(\frac{1-q_0}{2}\right)z + O(z^2)\right], \tag{4.15}$$

where $q_0 = (1/2)\sum_i \Omega_i(1+3w_i)$ in the flat case. It follows from (4.15) that for small z, the photometric distance is linear in the red shift, and the proportionality coefficient is the inverse of the Hubble constant. For more distant cosmological objects, the photometric distance in the next order depends on the current value of the deceleration parameter q_0 or on the amount and type of components filling the Universe, which is equivalent. The expression for the photometric distance in the next order in the red shift can be represented as

$$d_L(z) = \frac{cz}{H_0}\left[1 + \frac{1}{2}(1-q_0)z - \frac{1}{6}(1 - q_0 - 3q_0^2 + j_0)z^2\right.$$
$$\left. + \frac{1}{24}(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10j_0q_0 + s_0)z^3 + O(z^4)\right]. \tag{4.16}$$

[2] In cosmology, type-Ia supernovae (SNe-Ia) are adopted as standard candles. If the luminosity at the maximum of brilliance is the same for all SNe-Ia, the distance to them can be determined in a gauge-invariant way.

We now briefly consider the method of testing cosmological models with the aid of bursts of SNe-Ia. We consider two supernovae in more detail [3, 4], one of which, 1992P, has the small red shift $z = 0.026$ and the star magnitude $m = 16.08$, and the second, 1997ap, has the small red shift $z = 0.83$ and the star magnitude $m = 24.32$. In the first case, because $z \ll 1$, we can write $d_L(z) \simeq z/H_0$. With the aid of (4.13), we find $M = -19.09$. The photometric distance for 1997ap can be obtained by substituting $m = 24.32$ and $M = -19.09$ in (4.13):

$$H_0 d_L \simeq 1.16 \quad \text{for} \quad z = 0.83\,. \tag{4.17}$$

On the other hand, using (4.11), we find the following for a universe filled with nonrelativistic matter:

$$H_0 d_L \simeq 0.95\,.$$

The last result clearly contradicts observations (4.17).

Thus, by registering bursts of any population of sources of radiation with the same internal luminosity (standard candles), it is possible to determine the velocity of expansion of the Universe at different moments of its evolution. Comparison of the obtained results with the predictions of theoretical models permits selecting the most appropriate models. Although conceptually the problem does not seem very complicated, its realization has encountered (and still encounters) many difficulties. We name only a small part of them. Bursts of supernovae are rare and random. To collect statistics, one must monitor a significant part of the sky. A burst lasts a limited time; therefore, it is necessary to notice a supernova as early as possible and to carry out observations of changes in its brightness. Naturally, the justification for using a type-Ia supernova as the standard candle still remains the most important question. In the early 1990s, two projects were implemented in the USA for revealing and analyzing bursts of SNe-Ia: the SuperNova Cosmology Project and the High-Z SuperNova Search. Precisely the results of these two projects [3, 4] led in 1998–1999 to the conclusion of the accelerated expansion of the Universe, which resulted in a total change in both modern cosmology and physics as a whole. During the past decade, the results in [3, 4] have been repeatedly tested with increasingly improved statistics. The main result remained intact: relatively recently (at $z \sim 0.5$), the Universe underwent a transition from decelerating to accelerating expansion.

We discuss in more detail the analysis performed in Ref. [21], in which the so-called golden set of SNE-Ia was used. The set contained 157 well-studied SNE-Ia with red shifts $0.1 < z < 1.76$. The analysis was based on relation (4.12) for the photometric distance in a spatially flat universe. In the case of a linear two-parameter expansion of the deceleration parameter

$$q(z) = q_0 + q_1 z\,, \tag{4.18}$$

the integral in (4.12) can be calculated analytically and the expression for the photometric distance becomes [22]

$$d_L(z) = \frac{1+z}{H_0}\exp(q_1) q_1^{q_0 - q_1} \times \left[\gamma\big(q_1 - q_0, (1+z)q_1\big) - \gamma(q_1 - q_0, q_1)\right], \tag{4.19}$$

where q_0 and q_1 are the values of $q(z)$ and $\mathrm{d}q(z)/\mathrm{d}z$ at $z = 0$ and γ is the incomplete gamma-function. Using (4.19), it is possible to obtain information on q_0 and q_1 and, consequently, on the global behavior of $q(z)$. A dynamic 'phase transition' occurs at $q(z_t) = 0$ or $z_t = -q_0/q_1$, which is equivalent.

Another parameterization that was used has the form

$$q(z) = q_0 + q_1\,\frac{z}{1+z}\,. \tag{4.20}$$

Parameterization (4.20) has the advantage of behaving well at large z, while the linear approximation leads to divergences. In this parameterization,

$$d_L(z) = \frac{1}{(1+z)H_0}\exp(q_1)\, q_1^{-(q_0+q_1)} \times \left[\gamma(q_1 + q_0, q_1) - \gamma\left(q_1 + q_0, \frac{q_1}{1+z}\right)\right]. \tag{4.21}$$

The parameter q_1 determines the correction to q_0 in the distant past: $q(z) = q_0 + q_1$ for $z \gg 0$. The likelihood function for parameters q_0 and q_1 can be determined with the aid of the χ^2 statistic:

$$\chi^2(H_0, q_0, q_1) = \sum_i \frac{\big(\mu_{p,i}(z_i; H_0, q_0, q_1) - \mu_{0,i}\big)^2}{\sigma^2_{\mu_{0,i}} + \sigma^2_v}\,, \tag{4.22}$$

where $\sigma_{\mu_{0,i}}$ is the dispersion of the distance modulus for the ith standard candle and σ_v is the dispersion of red shifts of supernovae caused by their peculiar velocity. The results obtained suggest that the Universe is presently undergoing expansion with acceleration ($q_0 < 0$), while earlier is was expanding with deceleration ($q_1 > 0$). These results were obtained for confidence intervals at levels of 99.2% and 99.8%. In the case of a linear expansion of the parameter q, transition from the decelerated expansion in the past to the current accelerated expansion occurred at the red shift $z_t = 0.46 \pm 0.13$. Regretfully, the importance of this result must not be overestimated: a linear approximation always leads to a transition when the parameters have opposite signs.

A subsequent, and statistically more reliable, analysis qualitatively confirmed all the results presented above. Naturally, the quantitative results depend on the sampling of supernovae used, but the main result remains the same: we live in a Universe undergoing accelerating expansion, and in the not-so-distant past it underwent a transition from decelerating to accelerating expansion.

Today, doubtless, the technique based on observation of bursts of supernovae is the leader. But it also has rivals. A promising and completely independent substitute (certainly not just a surrogate) is the observation of distances by the angular diameter $D_A(z)$ for a given set of distant objects. The combination of measurements of the Sunyaev–Zel'dovich effect (SZE) and of the surface brightness in the X-ray range allows determining the distances of the angular diameter of galactic clusters [23]. The Sunyaev–Zel'dovich effect is a small perturbation of the cosmic microwave background (CMB) spectrum, caused by the inverse Compton scattering of CMB photons passing through a population of hot electrons. Observations of temperature fluctuations in the CMB spectrum in galactic clusters, as well as observations in the X-ray range, permit obtaining the function $D_A(z)$ independently. Thus, $D_A(z)$ allows reconstructing the dynamics of the Universe in a way that is totally independent of $d_L(z)$.

For a spatially flat universe described by the FRW metric, the distance determined by the angular diameter has the form [24]

$$d_A(z) = \frac{1}{(1+z)H_0}\int_0^z \frac{du}{H(u)}$$
$$= \frac{1}{(1+z)H_0}\int_0^z \exp\left[-\int_0^u \big(1+q(u')\big)\,d\ln(1+u')\right] du'. \qquad (4.23)$$

In Ref. [24], a set of 38 measurements was considered of distances by the angular size for galactic clusters, which were obtained with the aid of the SZE/X-ray method, discussed in detail in Ref. [23]. The method is based on the method of maximum likelihood and on the χ^2 statistics,

$$\chi^2(z, p) = \sum_i \frac{\big(D_A(z_i, p) - D_{A_{0,i}}\big)}{\sigma^2_{D_{A0,i}} + \sigma^2_{stat}}, \qquad (4.24)$$

where $D_{A_{0,i}}$ is the observed distance determined by the angular size, $\sigma_{D_{A0,i}}$ is the root-mean-square deviation related to the uncertainty of the actual distance, and σ_{stat} is the contribution of different statistical errors. The value adopted for the Hubble parameter H_0 was $H_0^* = 80$ km (s Mpc)$^{-1}$. In the case of a linear parameterization, the results were as follows: this set of data is best approximated at $q_0 = -1.35$, $q_1 = 4.2$, and $z_t = 0.32$. This result confirms that the Universe is presently undergoing accelerated expansion ($q_0 < 0$) and that the accelerated expansion started at $z_t = 0.32$, while before this moment the expansion of the Universe was decelerating. In the case of another parameterization,

$$q(z) = q_0 + q_1 \frac{z}{1+z},$$

such calculations lead to the values $q_0 = -1.43$, $q_1 = 6.18$, and $z_t = 0.3$. In all cases, the results are in good agreement with the ones obtained using the SNE-Ia data.

Recently, it has been shown that the so-called luminous red galaxies (LRGs) provide an additional possibility of direct measurement of the expansion rate [25–27]. The idea of the method consists in reconstructing the Hubble parameter from the time derivative of the red shift,

$$H(z) = -\frac{1}{1+z}\frac{dz}{dt}. \qquad (4.25)$$

The derivative can be found from measurements of the 'difference in age' between two passively evolving galaxies with different, but close red shifts. The method was realized for $0.1 < z < 1.75$. We stress that this interval includes the transition region we are interested in, for which $z \sim 0.5$. The results of the data analysis in Ref. [28] are consistent with the data on supernovae and the angular diameter. In the nearest future, an array consisting of 2000 passively evolving galaxies in the range $0 < z < 1.5$ is expected to be obtained. From these observations, it will be possible to find 1000 values of $H(z)$ with an accuracy of 15%, if the age of the galaxies is determined with a 10% precision.

To conclude this section, we discuss a new promising possibility of investigating the history of the cosmological expansion of the Universe. We recall that initially it was cepheids, i.e., stars of intensities proportional to the periods of brilliance pulsation, that were adopted as standard candles. A classic example of a cepheid is the North Star, the star that is brightest and nearest to Earth, with variable brightness with a period of 3.97 days. Cepheids are good standard candles for galactic distances. They have permitted determining the size of our Galaxy and the distance to its nearest neighbor, the Andromeda galaxy. Investigation of the dynamics of the Universe requires passing to essentially new scales and, as a consequence, significantly more powerful standard candles. We recall that the cosmological principle, postulating the homogeneity and isotropy of the Universe (all the equations we use for the dynamics of the Universe are based on this principle) is applicable only at scales over 100 Mpc. Using much more powerful sources of radiation — of type-Ia supernovae — as standard candles would permit studying the history of the Universe much more profoundly. However, the possibilities of the new standard candles also turned out to be limited. At present, type-Ia supernovae have been observed only at $z < 2$, even though greater red shifts and, consequently, more powerful standard candles are required for a more reliable reconstruction of the history of cosmological expansion. There happen to exist objects with such properties that are at our disposal, such as the so-called gamma-ray bursts (GRBs). Gamma-ray bursts are large-scale energy releases of an explosive character observed in the hardest part of the electromagnetic spectrum with a duration from 3 to 100 s. The energy ($\sim 10^{54}$ erg) emitted in a GRB exceeds the energy released in an explosion of a supernova comparable to the rest mass of the Sun by an order of magnitude. Events giving rise to GRBs are so powerful that they can sometimes be observed by the naked eye, although they originate at a distance of several billion light years from Earth. The energy is released in the form of a collimated flux (jet). The existence of jets results in our seeing only a small part of all the bursts originating in the Universe. The distribution of GRB duration lengths exhibits a clearly pronounced bimodal character.

The mechanism of short GRBs is probably related to the fusion of neutron stars or of a neutron star and a black hole. Longer events are assumed to be related to the collapse into a black hole of the nucleus of a massive (> 25 solar masses) star with a large angular momentum — the so-called collapsar model. The possibility of using GRBs as a standard candle is due to the existence of the Amati relation (the idea of the conventional term 'Amati relation' is based on Ref. [29]), which relates the peak frequency of a burst and its total energy. The Amati relation is a direct analogue of the period–luminosity relation for cepheids. The large dispersion (Fig. 1) still limits the application of GRBs as standard candles but, nevertheless, the possibility of entering the region of substan-

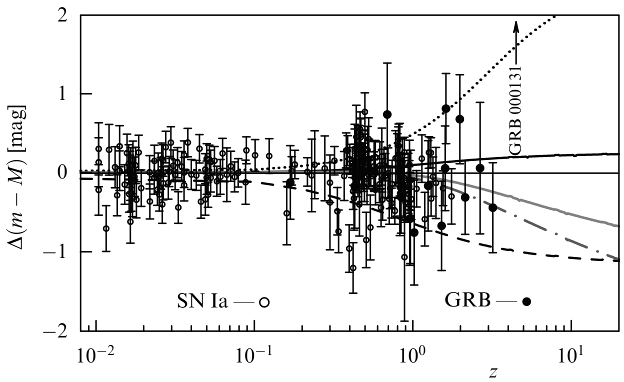

Figure 1. Data for 15 GRBs with known red shifts and collimation angles compared to the results of the standard cosmological model (curves).

tially larger red shifts makes this technique extremely attractive.

5. Dynamics of the scale factor in the standard cosmological model

We now consider the evolution of the deceleration parameter in the standard cosmological model (SCM). We recall that in the Big Bang model, the substances that filled up the Universe — matter and radiation — could only provide decelerating expansion. At present, DE is the dominant component of the Universe in the SCM; DE is a component with negative pressure. Precisely DE leads to the observed accelerating expansion of the Universe. We determine the red shift and transition time to the accelerating expansion, i.e., find the inflection point in the curve describing the time dependence of the scale factor (Fig. 2). In the SCM, the second Friedmann equation (3.1a) can be reduced to the form

$$\frac{\ddot{a}}{a} = \frac{1}{2} H_0^2 \left[2\Omega_{\Lambda 0} - \Omega_{\mathrm{m}0}(1+z)^3 \right], \tag{5.1}$$

where $\Omega_{\Lambda 0}$ and $\Omega_{\mathrm{m}0}$ are the present-day values of the relative DE density in the form of a cosmological constant and of the matter density. Hence, we obtain the red shift value at which the transition from decelerating to accelerating expansion took place as

$$z^* = \left(\frac{2\Omega_{\Lambda 0}}{\Omega_{\mathrm{m}0}} \right)^{1/3} - 1\,. \tag{5.2}$$

For the SCM parameters $\Omega_{\Lambda 0} \simeq 0.73$ and $\Omega_{\mathrm{m}0} \simeq 0.27$, we have $z^* \simeq 0.745$. We note that result (5.2) can be obtained using the fact that for a universe consisting of several components with the equations of state $p_i = w_i \rho_i$, deceleration parameter (3.5) is expressed as

$$q = \frac{1}{2} - \frac{3}{2} \frac{\Omega_{\Lambda 0}}{(1+z)^3 \Omega_{\mathrm{m}0} + \Omega_{\Lambda 0}} \tag{5.3}$$

(we recall that $\Omega = \sum_i \Omega_i = 1$ in the flat case). The condition $q = 0$ allows reproducing (5.2). We note the asymptotic forms of expression (5.3). For the early Universe $(z \to \infty)$ populated by components with positive pressure, $q(z \to \infty) = 1/2$, i.e., as expected, the expansion is decelerated, while in the distant future, with a dominant cosmological constant, the expansion becomes accelerated, $q(z \to -1) = -1$. The last result is a trivial consequence of the Universe expanding exponentially, $a \propto \exp(Ht)$, in the case of dominant DE in the form of the cosmological constant.

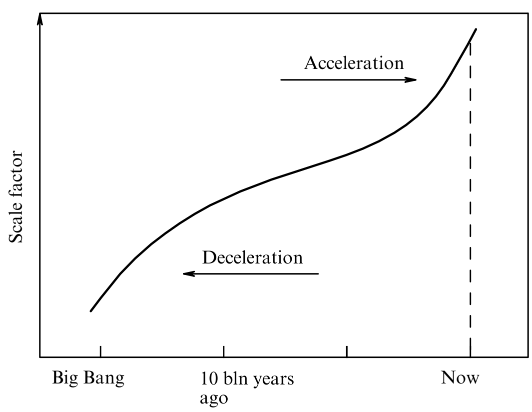

Figure 2. Time dependence of the scale factor in the SCM.

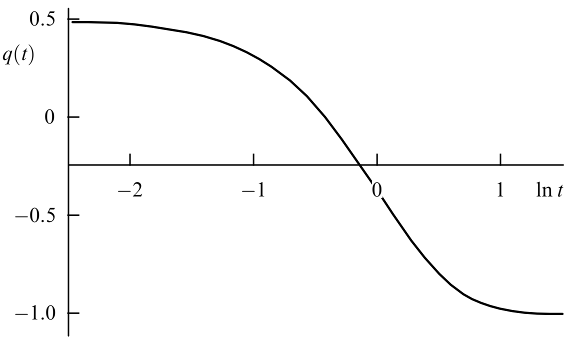

Figure 3. Time dependence of the deceleration parameter q in the SCM.

We note that the dependence $q(t)$ can be obtained directly from the definition $q = -\ddot{a}/aH^2$, using the SCM solution for the scale factor:

$$a(t) = A^{1/3} \sinh^{2/3}\left(\frac{t}{t_\Lambda}\right), \qquad A \equiv \frac{\Omega_{\mathrm{m}0}}{\Omega_{\Lambda 0}}, \quad t_\Lambda \equiv \frac{2}{3} H_0^{-1} \Omega_{\Lambda 0}^{-1/2}. \tag{5.4}$$

As a result, we obtain

$$q(t) = \frac{1}{2}\left[1 - 3\tanh^2\left(\frac{t}{t_\Lambda}\right)\right]. \tag{5.5}$$

The dependence $q(t)$ is presented in Fig. 3. We draw attention to the asymptotic behaviors as $t \to 0$ and $t \to \infty$ that correspond to the above asymptotic regimes as $z \to \infty$ and $z \to -1$.

We next determine the time corresponding to the transition to accelerating expansion. Inverting (5.4), we obtain

$$t(a) = \frac{2}{3} \Omega_{\Lambda 0}^{-1/2} H_0^{-1} \operatorname{arsinh}\left[\left(\frac{\Omega_{\Lambda 0}}{\Omega_{\mathrm{m}0}}\right)^{1/2} a^{3/2}\right]. \tag{5.6}$$

Passing from the red shift to the scale factor $a^* = (1+z^*)^{-1} = [\Omega_{\mathrm{m}0}/(2\Omega_{\Lambda 0})]^{1/3}$, we find

$$t^* \equiv t(a^*) = \frac{2}{3} \Omega_{\Lambda 0}^{-1/2} H_0^{-1} \sinh^{-1}\frac{1}{2} \simeq 5.25 \text{ bln years}\,. \tag{5.7}$$

In view of the physical significance of the result obtained, we present one more (apparently, the simplest) of its interpretations. If the quantity $aH = \dot{a}$ increases, then $\ddot{a} > 0$, corresponding to accelerating expansion of the Universe. In accordance with the first Friedmann equation,

$$\frac{aH}{H_0} = \sqrt{\frac{a^3 \Omega_{\Lambda 0} + \Omega_{\mathrm{m}0}}{a}} \simeq \sqrt{\frac{0.73 a^3 + 0.27}{a}}\,.$$

It is easy to show that this function starts to increase at $a^* \simeq 0.573$, which corresponds to $z^* = 0.745$. It is interesting to note that the transition to accelerating expansion of the Universe $(z \simeq 0.75)$ occurred significantly earlier than the transition to the DE-dominated stage $(z \simeq 0.4)$.

In the SCM, the role of DE is played by the cosmological constant with the equation of state $p_\Lambda = -\rho_\Lambda$, i.e., the parameter $w_\Lambda = -1$. A natural question arises: what is the limit value of this parameter such that the accelerated expansion is still realized at present? As we saw above, the condition for accelerated expansion is $\sum_i (\rho_i + 3p_i) < 0$. In the SCM, this condition transforms into

$$w_{\mathrm{DE}} < -\frac{1}{3}\,\Omega_{\mathrm{DE}}^{-1}\,, \qquad w_{\mathrm{DE}} < -0.46\,.$$

Naturally, a substance with such an equation of state differs from the cosmological constant, and it can be realized, for example, with the aid of scalar fields (see Section 6).

Using the SCM parameters, we now estimate the absolute value of the cosmological acceleration. Differentiating Hubble's law with respect to time, we obtain

$$\dot{V} = (\dot{H} + H^2)R\,. \tag{5.8}$$

The time derivative of the Hubble parameter is

$$\dot{H} = \frac{\ddot{a}a - \dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2\,.$$

Consequently,

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}(\rho_{\mathrm{m}} - 2\rho_{\Lambda})$$

$$= \frac{1}{3M_{\mathrm{Pl}}^2}\left(\rho_{\Lambda} - \frac{1}{2}\,\rho_{\mathrm{m}}\right) = H^2\left(\Omega_{\Lambda} - \frac{1}{2}\,\Omega_{\mathrm{m}}\right).$$

For the acceleration $\dot{V}$, we finally obtain the following analogue of Hubble's law:

$$\dot{V} = \tilde{H}R\,, \qquad \tilde{H} = H^2\left(\Omega_{\Lambda} - \frac{1}{2}\,\Omega_{\mathrm{m}}\right). \tag{5.9}$$

At present ($\Omega_{\mathrm{m}} = \Omega_{\mathrm{m}0}$, $\Omega_{\Lambda} = \Omega_{\Lambda 0}$), for example, at the distance $R = 1$ Mpc,

$$\dot{V} \simeq 10^{-11}\ \mathrm{cm\ s^{-2}}\,. \tag{5.10}$$

One of the possibilities of observing this effect is due to the red shift of any cosmological object slowly changing as a consequence of the acceleration (or deceleration) of the expansion of the Universe. We estimate the magnitude of this effect. From the definition

$$z = \frac{a(t_0)}{a(t)} - 1\,,$$

where t is the time of light emission and t_0 is the time of its registration, it follows that

$$\frac{\mathrm{d}z}{\mathrm{d}t_0} = \frac{\dot{a}(t_0)\,a(t) - a(t_0)\,\dot{a}(t)(\mathrm{d}t/\mathrm{d}t_0)}{a^2(t)}\,. \tag{5.11}$$

Taking into account that $\mathrm{d}t/\mathrm{d}t_0 = a(t)/a(t_0) = 1/(1+z)$, we obtain

$$\dot{z} = \frac{\dot{a}(t_0)}{a(t)} - \frac{a(t_0)\,\dot{a}(t)}{a^2(t)}\,\frac{1}{1+z}\,. \tag{5.12}$$

Hence, the rate with which the red shift changes in the case of light emitted at a time moment t and registered now at the time moment t_0 is determined by the relation

$$\dot{z} \equiv \frac{\mathrm{d}z}{\mathrm{d}t_0} = H_0(1+z) - H(t)\,.$$

In the SCM,

$$H = H_0\left[\Omega_{\mathrm{m}0}(1+z)^3 + \Omega_{\Lambda 0}\right]^{1/2}.$$

Therefore, for the red shift to change within the range Δt, we obtain

$$\Delta z = \dot{z}\Delta t = H_0\left\{1 + z - \left[\Omega_{\mathrm{m}0}(1+z)^3 + \Omega_{\Lambda 0}\right]^{1/2}\right\}. \tag{5.13}$$

We note that in the two limit cases, $\Omega_{\Lambda 0} = 1$, $\Omega_{\mathrm{m}0} = 0$ (accelerating expansion) and $\Omega_{\Lambda 0} = 0$, $\Omega_{\mathrm{m}0} = 1$ (decelerated expansion), Δz has different signs, as was expected. For the SCM parameters, the change in the red shift Δz and the velocity increment ΔV for a source with the red shift $z = 4$, within the observation interval $\Delta t_0 = 10$ years, amount to the following:

$$\Delta z \approx 10^{-9}\,, \qquad \Delta V = c\,\frac{\Delta z}{1+z} \approx 6\ \mathrm{cm\ s^{-1}}\,.$$

The result seems discouragingly small. But taking into account the rate with which the precision of observational cosmology progresses, we must not despair. We present the following example.

At present, over 300 exoplanets (planets outside the Solar System) are known. The most successful method for revealing exoplanets consists in measuring the radial velocities of stars. A star with a planet undergoes oscillations of the 'to us–from us' velocity, which can be measured by observing the Doppler shift in the star spectrum. At first sight, this seems to be impossible. Due to the action of Earth, oscillations of the solar velocity amount to several centimeters per second with a period of one year, and in the case of Jupiter, to several meters per second. The thermal broadening of spectral lines of a star corresponds to the spread of velocities of several thousand kilometers per second. Hence, even in the case of Jupiter, it is necessary to measure shifts of spectral lines amounting to one thousandth part of their widths. This seems impossible, but this task has been fulfilled.

6. Dynamic forms of dark energy and the evolution of the Universe

The cosmological constant is only one possible realization of the hypothetical substance, DE, introduced in order to explain the accelerated expansion of the Universe. As we have seen, the parameter w in the equation of state $p = w\rho$ for such a substance must satisfy the condition $w < -1/3$ (in the absence of other components). Regretfully, the nature of DE is not known to us, which gives rise to an enormous number of hypotheses and candidates for the role of the fundamental component of the energy balance of the Universe. We have spoken many times of the rapid progress of observational cosmology during the past decade. But we are still unable to answer the question concerning the time evolution of the parameter w. If this parameter varies with time, we are compelled to search for an alternative to the cosmological constant. A great number of alternatives to $w = -1$ have been investigated in a short time. The scalar fields that formed the postinflational Universe are one of the main candidates for DE. The most popular version involves a scalar field φ with an appropriately chosen potential $V(\varphi)$. In these models, the scalar field, unlike the cosmological constant, is a dynamic variable, and the DE density is time-dependent. The models differ in the choice of the Lagrangian of the scalar field.

We start with probably the simplest DE model of such a type, which has been termed quintessence. We understand quintessence to be a scalar field φ in a potential $V(\varphi)$ minimally coupled to gravity, i.e., only subject to the influence of the curvature of space–time. Moreover, we restrict ourselves to dealing with the canonical form of kinetic energy. The action for such a field has the form

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, L = \int \mathrm{d}^4 x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \frac{\partial\varphi}{\partial x_\mu} \frac{\partial\varphi}{\partial x_\nu} - V(\varphi) \right], \tag{6.1}$$

where $g \equiv \det g_{\mu\nu}$. The equation of motion of the scalar field is found by varying the action with respect to the field,

$$\frac{1}{\sqrt{-g}}\, \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \frac{\partial\varphi}{\partial x_\nu} \right) = -\frac{\mathrm{d}V}{\mathrm{d}\varphi}\,. \tag{6.2}$$

In a flat Friedmann universe, i.e., for FRW metric (2.1), for a homogeneous field $\varphi(t)$ we obtain

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0\,, \tag{6.3}$$

where $V'(\varphi) \equiv \mathrm{d}V/\mathrm{d}\varphi$. Equation (6.3) is sometimes called the Klein–Gordon equation.

The energy–momentum tensor of the scalar field can be found by varying (6.1) with respect to the metric $g^{\mu\nu}$,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial\varphi}{\partial x_\mu} \frac{\partial\varphi}{\partial x_\nu} - g_{\mu\nu} L\,. \tag{6.4}$$

In the case of a homogeneous field $\varphi(t)$ in a locally Lorentzian frame, in which the metric $g_{\mu\nu}$ can be replaced by the Minkowski metric, we obtain the density and pressure of the scalar field as

$$\rho_\varphi = T_{00} = \frac{1}{2}\,\dot{\varphi}^2 + V(\varphi)\,, \quad p_\varphi = T_{ii} = \frac{1}{2}\,\dot{\varphi}^2 - V(\varphi)\,. \tag{6.5}$$

The Friedmann equations for a flat universe filled with a scalar field become

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2} \left[\frac{1}{2}\,\dot{\varphi}^2 + V(\varphi) \right], \tag{6.6a}$$

$$\dot{H} = -\frac{\dot{\varphi}^2}{2M_{\mathrm{Pl}}^2}\,. \tag{6.6b}$$

System (6.6a), (6.6b) should be supplemented by Klein–Gordon equation (6.3) for the scalar field. We note that it can be obtained from the conservation equation for the scalar field,

$$\dot{\rho}_\varphi + 3H(\rho_\varphi + p_\varphi) = 0\,, \tag{6.7}$$

by substituting expressions (6.5) for the energy density and pressure into it.

Using (6.5), we obtain the equation of state for the scalar field:

$$w_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\dot{\varphi}^2 - 2V}{\dot{\varphi}^2 + 2V}\,. \tag{6.8}$$

We see that the parameter of the equation of state of the scalar field, w_φ, is in the range

$$-1 \leqslant w_\varphi \leqslant 1\,. \tag{6.9}$$

The equation of state for the scalar field is conveniently represented as

$$w(x) = \frac{x-1}{x+1}\,, \qquad x \equiv \frac{(1/2)\dot{\varphi}^2}{V(\varphi)}\,. \tag{6.10}$$

The function $w(x)$ increases monotonically from its minimum value $w_{\min} = -1$ at $x = 0$ to its maximum asymptotic value $w_{\max} = 1$ as $x \to \infty$, which corresponds to $V = 0$. In the slow-roll limit, $x \ll 1$ ($\dot{\varphi}^2 \ll V(\varphi)$), the scalar field behaves like the cosmological constant, $w_\varphi = -1$. It is easy to see that in this case, $\rho_\varphi = \mathrm{const}$. In the other limit, $x \gg 1$ ($\dot{\varphi}^2 \gg V(\varphi)$) (hard matter), $w_\varphi = 1$. In this case, the energy density of the scalar field evolves as $\rho_\varphi \propto a^{-6}$. The intermediate situation, $x \sim 1$ and $p \sim 0$, corresponds to nonrelativistic matter.

Representing (6.7) in the integral form

$$\rho = \rho_0 \exp\left[-3 \int (1 + w_\varphi)\, \frac{\mathrm{d}a}{a} \right], \tag{6.11}$$

we find that in the general case, the energy density of the scalar field behaves as

$$\rho_\varphi \propto a^{-m}\,, \qquad 0 < m < 6\,. \tag{6.12}$$

The value $w_\varphi = -1/3$ is the boundary between the modes of decelerating and accelerating expansion of the Universe. Consequently, accelerating expansion is realized when $0 \leqslant m < 2$. A natural question arises: which potentials in the case of scalar fields can provide accelerating expansion of the Universe? The same question can be formulated somewhat differently: in which potentials can quintessence be considered DE? We address a simplified version of this question. We find the potential of the scalar field ensuring a power law of the scale factor increase:

$$a(t) \propto t^p\,. \tag{6.13}$$

For the expansion to be accelerated, the condition $p > 1$ must be satisfied. We recall that in the case of a universe filled with nonrelativistic matter, $p = 2/3$, and that if the universe is filled with radiation, then $p = 1/2$; therefore, in both these cases, the expansion is decelerating. Using the Friedmann equation, it is possible to express $V(\varphi)$ and $\dot{\varphi}$ in terms of H and $\dot{H}$, which allows obtaining a set of equations describing the parametric dependence of V on φ:

$$V = 3M_{\mathrm{Pl}}^2 H^2 \left(1 + \frac{\dot{H}}{3H^2} \right), \tag{6.14a}$$

$$\varphi = \int \mathrm{d}t\, \left(-M_{\mathrm{Pl}}^2 \dot{H} \right)^{1/2}. \tag{6.14b}$$

Eliminating time with the aid of the relation $\varphi/M_{\mathrm{Pl}} = \sqrt{2p}\, \ln t$, for power law (6.13), we find

$$V(\varphi) = V_0 \exp\left(-\sqrt{\frac{2}{p}}\, \frac{\varphi}{M_{\mathrm{Pl}}} \right). \tag{6.15}$$

This result signifies that under the condition $p > 1$, the scalar field in potential (6.15) can be interpreted as DE, i.e., it can provide accelerated expansion of the Universe. As we saw above, in the quintessence model, the required dynamic behavior can be achieved by a choice of the appropriate potential of the scalar field. The DE model realized with the aid of a scalar field by modification of the kinetic term has been termed the k-essence [30, 31].

We introduce the quantity

$$X \equiv \frac{1}{2} g^{\mu\nu} \frac{\partial\varphi}{\partial x_\mu} \frac{\partial\varphi}{\partial x_\nu}$$

and consider the action for a scalar field of the form

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, L(\varphi, X) \,, \tag{6.16}$$

where L is in general an arbitrary function of φ and X. The conventional action for the scalar field is

$$L(\varphi, X) = X - V(\varphi) \,. \tag{6.17}$$

We only deal with the subset of Lagrangian functions of the form

$$L(\varphi, X) = K(X) - V(\varphi) \,, \tag{6.18}$$

where $K(X)$ is a positive definite function of the kinetic energy X. To describe a homogeneous universe, we must choose $X = (1/2)\dot\varphi^2$. Using the standard definition (6.5), we find

$$p_\varphi = L(\varphi, X) = K(X) - V(\varphi) \,, \tag{6.19a}$$

$$\rho_\varphi = 2X \frac{\partial K(X)}{\partial X} - K(X) + V(\varphi) \,. \tag{6.19b}$$

Consequently, the equation of state for the k-essence is

$$w_\varphi = \frac{K(X) - V(\varphi)}{2X\,\partial K(X)/\partial X - K(X) + V(\varphi)} \,. \tag{6.20}$$

The main peculiarities of the k-essence can be demonstrated with the example of its simplified model [32], in which the Lagrangian function is $L = F(X)$. Such a model is called the purely kinetic k-essence. In this case,

$$\rho_\varphi = 2XF_X - F \,, \qquad F_X \equiv \frac{\partial F}{\partial X} \,, \tag{6.21a}$$

$$p = F \,, \tag{6.21b}$$

$$w_\varphi = \frac{F}{2XF_X - F} \,. \tag{6.21c}$$

The equations of motion for the field can be found either by writing the Euler–Lagrange equation for action (6.16) or by substituting density (6.21a) and pressure (6.21b) in the conservation equation for the k-essence. As a result, we obtain

$$F_X \ddot\varphi + F_{XX}\dot\varphi^2\ddot\varphi + 3HF_X\dot\varphi = 0 \,, \tag{6.22}$$

or in terms of the kinetic energy X,

$$(F_X + 2F_{XX}X)\dot X + 6HF_X X = 0 \,. \tag{6.23}$$

Equation (6.23) can be integrated:

$$XF_X^2 = ka^{-6} \,, \tag{6.24}$$

with a constant $k > 0$.

The solution $X(a)$ of (6.24) has an important property. The behavior of all the characteristics of the k-essence [ρ_φ, p_φ, w_φ; see (6.21a)–(6.21b)] as functions of the scale factor is fully determined by the function $F(X)$, and is independent of the evolution of other energy densities. The entire dependence of the k-essence on other components is only due to $a(t)$. But $a(t) \propto \rho_{\rm tot}$; therefore, $a(t)$ is determined by the dominant component of the energy density. Solutions of this sort have been called spectator solutions. Their existence permits coming closer to resolving the coincidence problem. It can be shown [33] that not only the purely kinetic k-essence but also its general version has this property.

The enormous number of cosmological observations at our disposal shows that the parameter w in the equation of state for DE lies in a narrow range in the vicinity of $w = -1$. Above, we examined the region $-1 \leqslant w < -1/3$. The lower boundary of the region, $w = -1$, corresponds to the cosmological constant, while the entire remaining interval can be realized with the aid of scalar fields with a canonical Lagrangian. We recall that the upper boundary $w = -1/3$ is related to the necessity of providing the observed accelerating expansion of the Universe. Is it possible to go beyond the limits of this interval?

This is a complicated question for the energy component, of which we know so little. In GR, limits are conventionally imposed on possible values of the components of the energy–momentum tensor by energy conditions (see Section 3). One of the simplest conditions of this type is $\rho + p \geqslant 0$. The physical motivation for this condition is to avoid the instability of the vacuum. Concerning the dynamics of the Universe, this condition requires the density of any admissible energy component not to increase with the expansion of the Universe. The cosmological constant, for which $\dot\rho_\Lambda = 0$, represents a limit case. Taking our ignorance concerning the nature of DE into account, it is reasonable to ask ourselves: Can this mysterious substance differ from the 'good' sources of energy known to us and violate the condition $\rho + p \geqslant 0$? Because DE must have a positive energy density (which is necessary for the Universe to be flat) and negative pressure (to provide the accelerated expansion of the Universe), this violation should lead to $w < -1$. Some time ago, such a component, which was called the phantom energy, attracted the attention of physicists [34–38]. The action for a phantom field φ minimally coupled to gravity differs from the canonical action for a scalar field by the sign of the kinetic term. The energy density and pressure of the phantom field are given by the expressions

$$\rho_\varphi = T_{00} = -\frac{1}{2}\dot\varphi^2 + V(\varphi) \,, \quad p_\varphi = T_{ii} = -\frac{1}{2}\dot\varphi^2 - V(\varphi) \,, \tag{6.25}$$

and the equation of state has the form

$$w_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\dot\varphi^2 + 2V(\varphi)}{\dot\varphi^2 - 2V(\varphi)} \,. \tag{6.26}$$

If $\dot\varphi^2 < 2V(\varphi)$, then $w_\varphi < -1$.

As an example, we consider the case where the Universe only contains nonrelativistic matter ($w = 0$) and a phantom field ($w_\varphi < -1$). The densities of these components evolve independently: $\rho_{\rm m} \propto a^{-3}$ and $\rho_\varphi \propto a^{-3(1+w_\varphi)}$. If matter domination terminates at a time moment $t_{\rm m}$, the solution for the scale factor at $t > t_m$ is expressed as

$$a(t) = a(t_m)\left[-w_\varphi + (1 + w_\varphi)\left(\frac{t}{t_m}\right)\right]^{2/[3(1+w_\varphi)]}. \qquad (6.27)$$

Hence, it immediately follows that if $w_\varphi < -1$, then, at the moment of time

$$t_{BR} = \frac{w_\varphi}{1 + w_\varphi}\, t_m\,,$$

the scale factor and a number of cosmological characteristics of the Universe (for instance, the scalar curvature and the energy density of the phantom field) become infinite. This catastrophe has been termed the Big Rip. The Big Rip is preceded by the so-called superacceleration mode. We explain the reason for the superacceleration mode to arise using a simple example. We consider the differential equation

$$\frac{dx}{dt} = Ax^2\,. \qquad (6.28)$$

If $A > 0$, Eqn (6.24) realizes a positive feedback. The rapid increase of the function $x(t)$ leads to the Big Rip (the function blows up) in a finite time. Indeed, the general solution of Eqn (6.28) has the form

$$x(t) = -\frac{1}{A(t + B)}\,, \qquad (6.29)$$

where B is the integration constant. At $t = -B$, the Big Rip occurs.

It is easy to see that model (6.28) represents a concrete version of the Friedmann equation for $w_\varphi < -1$. Because $\rho_\varphi \propto a^{-3(1+w_\varphi)}$, the first Friedmann equation can be represented as

$$\dot a = Aa^{-(3/2)(1+w_\varphi)+1}\,. \qquad (6.30)$$

For example, for $w_\varphi = -5/3$, Eqn (6.30) coincides precisely with (6.28).

7. $f(R)$-gravity, cosmology on a brane, and modified Newtonian dynamics

Although the SCM does explain the current accelerating expansion of the Universe and largely agrees with current observational data, the theoretical foundation of this model can be considered quite poor. Several proposals exist concerning dynamic alternatives for DE. We discussed them in Section 6. Regretfully, none of them can be considered to be completely without problems.

Another, more radical, approach is based on the assumption that no DE exists, while the acceleration is generated by the weakening of gravity at very large scales due to a modification of GR. In the framework of this broad approach, it is possible to single out three lines of research: $f(R)$-gravity, brane cosmology, and modified Newtonian dynamics (MOND). We briefly examine these three alternatives to the SCM from the standpoint of the issue we are interested in — decelerating or accelerating expansion of the Universe.

7.1 $f(R)$-gravity

The theory of $f(R)$-gravity is based on a direct generalization of the Einstein–Hilbert action by the replacement $R \to f(R)$. The new action is

$$S = -\frac{M_{Pl}^2}{2}\int d^4x\,\sqrt{-g}\,f(R)\,. \qquad (7.1)$$

The function $f(R)$, chosen as the generalization, depends only on the Ricci scalar R, while no other invariants, such as $R_{\mu\nu}R^{\mu\nu}$, are involved for the following reasons: the action $f(R)$ is sufficiently general to reflect the principal features of gravity, and at the same time it is sufficiently simple for calculations with it not to present technical problems. We note that the function $f(R)$ must satisfy stability conditions

$$f'(R) > 0\,, \quad f''(R) > 0\,, \qquad (7.2)$$

where the primes denote derivatives with respect to the Ricci scalar curvature R. The total action for the $f(R)$-gravity is

$$S = -\frac{M_{Pl}^2}{2}\int d^4x\,\sqrt{-g}\,f(R) + S_m(g_{\mu\nu}, \psi)\,, \qquad (7.3)$$

where ψ is the general notation for matter fields. After some transformations, variations with respect to the metric give

$$f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)\,f'(R) = \frac{T_{\mu\nu}}{M_{Pl}^2}\,, \qquad (7.4)$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}\,, \qquad (7.5)$$

∇_μ is the covariant derivative with the connection associated with the metric $g_{\mu\nu}$, and $\Box \equiv \nabla_\mu\nabla^\mu$.

Setting aside complications related to variation, we focus on field equations (7.4). These are partial differential equations of the fourth order with respect to the metric, because R already involves its second derivatives. When the action is linear in R, the fourth-order derivatives [the last two terms in the left-hand side of (7.4)] vanish, and the theory reduces to the standard GR.

We note that the trace of (7.4)

$$f'(R)R - 2f(R) + 3\Box f'(R) = \frac{T}{M_{Pl}^2}\,, \qquad (7.6)$$

where $T = g^{\mu\nu}T_{\mu\nu}$, relates R and T differentially, but not algebraically, as in GR, where $R = -(1/M_{Pl}^2)T$. This is a straightforward indication that the field equations of the $f(R)$-theory admit a broader class of solutions than GR does. As an illustration of this assertion, we recall that the Jebsen–Birkhoff theorem, asserting that the Schwarzschild solution is the only spherically symmetric vacuum solution, is not valid in the $f(R)$-theory. Without going into the details, we note that the equality $T = 0$ does not imply that R is equal to zero or even to a constant.

Equation (7.6) turns out to be very useful in considering various aspects of $f(R)$-gravity, especially the stability of solutions and the weak-field limit. For example, it is conveniently used in analyzing so-called maximum-symmetry solutions, i.e., those for which $R = \text{const}$. When $R = \text{const}$ and $T_{\mu\nu} = 0$, Eqn (7.6) reduces to

$$f'(R)R - 2f(R) = 0\,. \qquad (7.7)$$

For a given function $f(R)$, Eqn (7.7) is an algebraic equation for R. If $R = 0$ is a root of this equation, then (7.4) reduces to $R_{\mu\nu} = 0$ and the maximum-symmetry solution corresponds to the Minkowski space–time. If the root of Eqn (7.7) is $R = \text{const} = C$, then (7.4) reduces to $R_{\mu\nu} = (C/4)g_{\mu\nu}$, and the maximum-symmetry solution corresponds to the de Sitter or anti-de Sitter space (in GR, to the cosmological constant), depending on the sign of C.

We now pass directly to the description of the dynamics of the Universe in $f(R)$-cosmology. Substituting the FRW metric in (7.4) and using

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu} \tag{7.8}$$

as the energy–momentum tensor, we obtain

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2 f'}\left[\rho + \frac{1}{2}(Rf' - f) - 3H\dot{R}f''\right], \tag{7.9}$$

$$2\dot{H} + 3H^2 = -\frac{1}{M_{\mathrm{Pl}}^2 f'}\left[p + \dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + \frac{1}{2}(f - Rf')\right]. \tag{7.10}$$

As noted above, the main motivation for the transition to $f(R)$-gravity is that it leads to accelerating expansion of the Universe without the introduction of DE. Such a generalization of GR was first proposed by Starobinsky, who showed that introducing a certain function of the Ricci scalar curvature R into the Einstein–Hilbert action is equivalent to introducing a certain scalar field, the scalaron [39]. In [39], the exponential dependence of the scale factor on the cosmological time was also obtained, which serves as a standard example of the exponential expansion of the Universe.

The simplest way to verify that $f(R)$-gravity leads to the possibility of obtaining accelerated expansion of the Universe without involving DE is to introduce the effective energy density and effective pressure:

$$\rho_{\mathrm{eff}} = \frac{Rf' - f}{2f'} - \frac{3H\dot{R}f''}{f'}, \tag{7.11}$$

$$p_{\mathrm{eff}} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + (f - Rf')/2}{f'}. \tag{7.12}$$

In a spatially flat universe, ρ_{eff} must be nonnegative, as follows from (7.9) in the limit $\rho \to 0$. Then Eqns (7.9) and (7.10) become the standard Friedmann equations:

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\,\rho_{\mathrm{eff}}, \tag{7.13}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}(\rho_{\mathrm{eff}} + 3p_{\mathrm{eff}}). \tag{7.14}$$

In this case, the effective parameter w_{eff} in the equation of state is given by

$$w_{\mathrm{eff}} \equiv \frac{p_{\mathrm{eff}}}{\rho_{\mathrm{eff}}} = \frac{\dot{R}^2 f''' + 2H\dot{R}f'' + \ddot{R}f'' + (f - Rf')/2}{(Rf' - f)/2 - 3H\dot{R}f''}. \tag{7.15}$$

The denominator in (7.15) is strictly positive; therefore, the sign of w_{eff} depends on the numerator. In the general case, for a metric $f(R)$-model, in order to reproduce (imitate) the de Sitter equation of state (the cosmological constant) $w_{\mathrm{eff}} = -1$, the following condition must be satisfied:

$$\frac{f'''}{f''} = \frac{\dot{R}H - \ddot{R}}{\dot{R}^2}. \tag{7.16}$$

We consider two examples (regardless of their viability). The first is the function $f(R) \propto R^n$. We can readily calculate w_{eff} as a function of n if we assume a power-law dependence of the scale factor $a(t) = a_0(t/t_0)^\alpha$ [an arbitrary dependence of $a(t)$ leads to a time dependence of w_{eff}]. The result for $n \neq 1$ is

$$w_{\mathrm{eff}} = -\frac{6n^2 - 7n - 1}{6n^2 - 9n + 3}, \tag{7.17}$$

and in terms of n,

$$\alpha = \frac{-2n^2 + 3n - 1}{n - 2}. \tag{7.18}$$

A suitable choice of n yields the desired value of w_{eff}. For example, $n = 2$ leads to

$$w_{\mathrm{eff}} = -1, \qquad \alpha = \infty. \tag{7.19}$$

This result is to be expected if we consider quadratic corrections to the Einstein–Hilbert action that were used in the Starobinsky inflation scenario [39].

The second example is

$$f(R) = R - \frac{\mu^{2(n+1)}}{R^n}. \tag{7.20}$$

In this case, as before, assuming a power law for the scale factor, we obtain

$$w_{\mathrm{eff}} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}. \tag{7.21}$$

The condition of accelerated expansion $w_{\mathrm{eff}} < -1/3$ for the two models considered transforms into

$$n^2 \mp n - 1 > 0,$$

where the $-$ and $+$ signs respectively correspond to the first and second examples. In the second case, for $n = 1$, we find $w_{\mathrm{eff}} = -2/3$. We note that positive n implies the presence of terms inversely proportional to R, in contrast to the case considered above.

7.2 Brane cosmology

All models explaining the accelerated expansion of the Universe are united by a common idea, namely, the weakening of gravity in one way or another: with the aid of negative pressure in the SCM and in models with scalar fields; by transformation of the law of universal gravitation within modified gravity; and by the existence of cavities in models with inhomogeneities. An novel way of suppressing gravitation is realized in brane scenarios [40]. According to such a scenario, we live on a three-dimensional brane, which is immersed into the bulk of a higher dimension D (four-dimensional in the simplest version). All matter fields are restricted to the brane, while gravity propagates both on the brane and in the bulk (see [41]). The higher dimension of space in which gravity acts leads to its weakening, $F \propto R^{-(D-1)}$.

In this scenario, the equation of motion for a scalar field on a brane has the form

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0\,, \tag{7.22}$$

where

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\,\rho\left(1 + \frac{\rho}{2\sigma}\right) + \frac{\Lambda_4}{3} + \frac{\varepsilon}{a^4}\,, \tag{7.23}$$

ρ is simply the energy density of the scalar field, $\rho = \dot{\varphi}^2/2 + V(\varphi)$, and ε is an integration constant that transforms the gravity in the bulk into gravity on the brane.

The brane tension σ provides a relation between the four-dimensional Planck mass $M_{\rm Pl}$ and the five-dimensional Planck mass $M_{\rm Pl}^{(5)}$,

$$M_{\rm Pl} = \sqrt{6}\,\frac{M_{\rm Pl}^{(5)^3}}{\sqrt{\sigma}}\,. \tag{7.24}$$

The tension σ also relates the four-dimensional cosmological constant Λ_4 on the brane and the five-dimensional bulk cosmological constant $\Lambda_{\rm b}$,

$$\Lambda_4 = \frac{1}{M_{\rm Pl}^{(5)^3}}\left(\Lambda_{\rm b} + \frac{\sigma^2}{3M_{\rm Pl}^{(5)^3}}\right). \tag{7.25}$$

We note that (7.23) contains an additional term $\propto \rho^2/\sigma$, whose presence is related to the conditions on the bulk–brane boundary. Owing to this term, the damping experienced by the scalar field in rolling down the potential drastically increases. In this case, inflation can be realized for potentials that in the standard approach are too steep for the conditions of slow-roll inflation to be fulfilled. Indeed, the slow-roll parameters in brane models for $V/\sigma \gg 1$ are given by

$$\varepsilon \simeq 4\varepsilon_{\rm FRW}\left(\frac{V}{\sigma}\right)^{-1}, \qquad \eta \simeq 2\eta_{\rm FRW}\left(\frac{V}{\sigma}\right)^{-1}. \tag{7.26}$$

Hence, it follows that the condition of slow-roll inflation $(\varepsilon, \eta \ll 1)$ is achieved more easily than in the FRW cosmology, when the condition $V/\sigma \gg 1$ is satisfied. Consequently, inflation can take place in very steep quintessence potentials, such as $V \propto \exp(-\lambda\varphi)$, $V \propto \varphi^{-\alpha}$, and so on. This, in turn, gives rise to the hope that both inflation and quintessence can be caused by the same scalar field.

A radically different approach providing accelerated expansion of the Universe, was proposed in Refs [42–44].

The Dvali–Gabadadze–Porrati (DGP) model [45] differs radically from the Randall–Sundrum (RS) model considered above in that in the first model, both the bulk cosmological constant and the brane tension are set equal to zero and a curvature term is introduced into the action for the brane. In the DGP model, the action has the form

$$S = -\frac{M_{\rm Pl}^{(5)^3}}{2}\int_{\rm bulk} R - \frac{M_{\rm Pl}^2}{2}\int_{\rm brane} R + \int_{\rm brane} L_{\rm matter}\,. \tag{7.27}$$

The meaning of the term $\int_{\rm brane} R$ is that quantum effects related to matter fields probably lead to such a term in the Einstein action, as was first pointed out by Sakharov in his theory of induced gravity [46].

The Hubble parameter in a DGP brane world is

$$H = \sqrt{\frac{\rho_{\rm m}}{3M_{\rm Pl}^2} + \frac{1}{l_{\rm c}^2}} + \frac{1}{l_{\rm c}}\,, \tag{7.28}$$

where $l_{\rm c} = M_{\rm Pl}^2/M_{\rm Pl}^{(5)^3}$ is a new length scale determined by the four-dimensional and five-dimensional Planck masses $M_{\rm Pl}$ and $M_{\rm Pl}^{(5)}$. An important property of this model is that accelerating expansion of the Universe is not due to DE. Because gravitational interaction extends to the five dimensions of the spatial scale $R > l_{\rm c} = 2H^{-1}(1-\Omega_{\rm m})^{-1}$, the scale of these dimensions increases as the Universe expands, and the strength of gravitational interaction decreases, which would seem to be accelerating the expansion of the Universe.

In a more general class of brane models, including subclasses of both the RS and the DGP models, the action has the form

$$S = -\frac{M_{\rm Pl}^{(5)^3}}{2}\int_{\rm bulk}(R - 2\Lambda_{\rm b}) - \frac{1}{2}\int_{\rm brane}(M_{\rm Pl}^2 R - 2\sigma) + \int_{\rm brane} L_{\rm matter}\,. \tag{7.29}$$

If $\sigma = \Lambda_{\rm b} = 0$, action (7.29) transforms into action (7.27) of the DGP model, and if $M_{\rm Pl} = 0$, into the action of the RS model.

As shown in [44], action (7.29) describes a universe with a period of accelerated expansion at the late stages of evolution, with the Hubble parameter determined by the relation

$$\frac{H^2(z)}{H_0^2} = \Omega_{\rm m}(1+z)^3 + \Omega_\sigma + 2\Omega_l \mp 2\sqrt{\Omega_l}\,\sqrt{\Omega_{\rm m}(1+z)^3 + \Omega_\sigma + \Omega_l + \Omega_{\Lambda_{\rm b}}}\,, \tag{7.30}$$

where

$$\Omega_l = \frac{1}{l_{\rm c}^2 H_0^2}\,, \qquad \Omega_{\rm m} = \frac{\rho_{0\rm m}}{3M_{\rm Pl}^2 H_0^2}\,, \qquad \Omega_\sigma = \frac{\sigma}{3M_{\rm Pl}^2 H_0^2}\,, \qquad \Omega_{\Lambda_{\rm b}} = -\frac{\Lambda_{\rm b}}{6H_0^2}\,. \tag{7.31}$$

The $\mp$ signs correspond to the two ways the brane can be immersed in the bulk. As in the DGP model, $l_{\rm c} \sim H_0^{-1}$ if $M_{\rm Pl}^{(5)} \sim 100$ MeV. In the case of small scales $r \ll l_{\rm c}$ and early times, GR is recovered within this approach, while in the case of large spatial scales $r \gg l_{\rm c}$ and late times, the brane effects start to play an important role. Indeed, with $M_{\rm Pl}^{(5)} = 0$ $(\Omega_l = 0)$, Eqn (7.30) reduces to the ΛCDM (Lambda-cold dark matter) model,

$$\frac{H^2(z)}{H_0^2} = \Omega_{\rm m}(1+z)^3 + \Omega_\sigma\,, \tag{7.32}$$

while in the case of $\sigma = \Lambda_{\rm b} = 0$, Eqn (7.30) reduces to the DGP model. An important feature of action (7.29) is that it leads to the effective equation of state $w_{\rm eff} \leqslant -1$. This is readily seen [44] from the expression for the current value of the effective equation of state:

$$w_0 = \frac{2q_0 - 1}{3(1-\Omega_{\rm m})} = -1 \pm \frac{\Omega_{\rm m}}{1-\Omega_{\rm m}}\sqrt{\frac{\Omega_l}{\Omega_{\rm m} + \Omega_\sigma + \Omega_l + \Omega_{\Lambda_{\rm b}}}}\,. \tag{7.33}$$

With the lower sign chosen, $w_0 < -1$.

It is interesting to note that in this model, the phase of accelerated expansion of the Universe is a transition phenomenon, which stops if the Universe returns to the matter-dominated phase.

A more detailed discussion of the models examined in this section can be found in review [47].

7.3 Modified Newtonian dynamics

The modified Newton dynamics is sometimes considered an alternative to the dark matter (DM) hypothesis. MOND [48] is such a modification of Newtonian physics that permits explaining flat rotational curves of galaxies without the need to use any assumptions whatsoever concerning DM. MOND assumes that Newton's second law $F = m\alpha$ (where α is acceleration) should be modified in the case of sufficiently small accelerations ($\alpha \ll \alpha_0$), such that

$$\mathbf{F} = m\boldsymbol{\alpha}\mu\left(\frac{\alpha}{\alpha_0}\right), \tag{7.34}$$

where $\mu(x) = x$ if $x \ll 1$ and $\mu(x) = 1$ if $x \gg 1$. It is easy to see that this law leads to a modification of the transitional formula for gravitational acceleration $\mathbf{F} = m\mathbf{g}_{\mathrm{N}}$, where $g_{\mathrm{N}} = GM/r^2$. The relation between the 'correct' and the Newtonian accelerations is

$$\alpha = \sqrt{\alpha_0 g_{\mathrm{N}}}\,. \tag{7.35}$$

For a rotating pointlike mass, $\alpha = v^2/r$ (this purely kinematical relation does not depend on the choice of dynamics). But α must be replaced in the last relation with the 'correct' acceleration. Hence,

$$v^4 = GM\alpha_0\,, \tag{7.36}$$

which means that in the case of sufficiently small accelerations, the rotational curves of an isolated body of mass M do not depend on the radial distance r at which the velocity is measured. In other words, this theory predicts not only flat rotational curves but also an infinite extension of the individual halo associated with a galaxy. This prediction may be a serious problem for MOND, because recent observations with the aid of galactic lensing have revealed the maximum extension of the halo to be about 0.5 Mpc. The value of α_0 required for an explanation of the observations

$$\alpha_0 \sim 10^{-8}\ \mathrm{cm\ s^{-2}}\,, \tag{7.37}$$

is of the same order of magnitude as cH_0. This supports the hypothesis that MOND may reflect the influence of cosmology on the local dynamics of particles.

Although the results of MOND are in good agreement with observations of individual galaxies, it is not clear whether they will be as successful in explaining the structures of clusters, for which strong gravitational lensing points to a higher concentration of masses at their centers than predicted by MOND. Another difficulty related to MOND is that it is problematic to incorporate it into a more general relativistic theory of gravitation. At present, it is not clear what theories such as MOND predict for complicated gravitational effects similar to gravitational lensing.

8. Dynamics of the Universe with interaction in the dark sector

In the SCM, DM and DE are considered independent components of the energy balance of the Universe. The role of DE is played by the cosmological constant already introduced by Einstein in constructing a stationary model of the Universe. The assumption that no interaction between components exists means that the energy densities of each of the components satisfy independent conservation equations:

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0\,.$$

A coupling of the components would lead to a modification of the evolution of the Universe. For instance, the energy density of the nonrelativistic component would not change according to the a^{-3} law, and the DE density (in the form of the cosmological constant) would not remain constant. On the one hand, such a modification of the theory opens new possibilities for resolving key problems of cosmology. For example, resolution of the coincidence problem [3] consists in the choice of interaction parameters ensuring that the condition

$$\frac{\Omega_{\mathrm{DE}}}{\Omega_{\mathrm{m}}} = O(1)$$

be satisfied whenever the condition $\ddot{a} > 0$ holds. On the other hand, introducing the interaction would lead to a modification of the relations between observed parameters. In particular, the fundamental relation between the photometric distance and the red shift, which to a significant extent serves as a basis for the proof of the accelerating expansion of the Universe, would be modified. This imposes strict constraints both on the form of the interaction and on its parameters.

A nonminimal coupling in the dark sector may significantly influence the history of the cosmological expansion of the Universe and the evolution of density perturbations, thus altering the growth rate of cosmological structures. The possibility of DM and DE interacting with each other is actively discussed in the literature [49–60]. With the aid of different (and independent) observational data, such as the data of WMAP (Wilkinson Anisotropy Probe), the results of observations of SNe-Ia and of baryon acoustic oscillations (BAO), a number of investigations have been performed to analyze the constraints on the intensity and form of the interaction in the dark sector.

8.1 Model of the Universe with a time-dependent cosmological constant

The simplest model involving interacting DM and DE is the cosmological model with a decaying vacuum. Actually, $\Lambda(t)$CDM cosmology represents one of those cases in which the parameter w for DE is equal to -1.

This model is based on the assumption that the role of DE is played by the physical vacuum, whose energy density is calculated on the background of a curved space. From the calculated density, the renormalized energy density of the physical vacuum in flat space is subtracted [61]. As a result, the effective energy density of the physical vacuum becomes dependent on the curvature of space–time and decreases from large initial values in the early Universe (with large curvatures) to nearly zero at present.

Owing to the Bianchi identity, the decay of the vacuum must be accompanied by the production of or increase in the

[3] The essence of the coincidence problem is that because the two densities decrease according to different laws during the entire history of the Universe, it is necessary to impose very precise conditions on their values in the early Universe in order to ensure that they have comparable values now.

mass of DM particles, which is a general property of models with a decaying vacuum or, in the more general case, of models with interacting DM and DE.

We pass to the description of the $\Lambda(t)$CDM model. In this section, we adopt the system of units in which the reduced Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} = 1$.

For the flat Universe described by the FRW metric, the first Friedmann equation (3.1b) and the conservation equation can be written as

$$\rho_{\rm tot} = 3H^2 \,, \qquad \dot\rho_{\rm tot} + 3H(\rho_{\rm tot} + p_{\rm tot}) = 0 \,,$$

where $\rho_{\rm tot} = \rho_{\rm m} + \rho_\Lambda$. Taking into account that $p_{\rm m} = 0$ and $p_\Lambda = -\rho_\Lambda$, we represent the conservation equation as

$$\dot\rho_{\rm m} + 3H\rho_{\rm m} = -\dot\rho_\Lambda \,. \tag{8.1}$$

In the right-hand side of (8.1), an additional summand has appeared that plays the role of the source — the decaying cosmological constant. The problem that arises in such an approach is that the same number of equations remain, but with an additional unknown function $\Lambda(t)$. The method described above for determining the density of the physical vacuum, although intuitively clear, encounters essential difficulties [62]. The phenomenological approach is therefore used more often in the literature. Below, we consider a simple, but analytically completely solvable model.

We consider the case where Λ depends on time as [63]

$$\Lambda = \sigma H \,. \tag{8.2}$$

It is interesting to note that with the choice $\sigma \approx m_\pi^3$ (where m_π is the energy scale of the vacuum condensate in quantum chromodynamics), relation (8.2) provides a value of Λ that is very close to the observed value.

For the components under consideration, the first Friedmann equation takes the form

$$\rho_\gamma + \Lambda = 3H^2 \,. \tag{8.3}$$

Equations (8.1) and (8.3), together with the equation of state

$$p_\gamma = w\rho_\gamma \equiv (\gamma - 1)\,\rho_\gamma$$

and the decay law of the cosmological constant in (8.1), fully determine the evolution of the scale factor. Combining these equations, we obtain the evolution equation in the form

$$2\dot H + 3\gamma H^2 - \sigma\gamma H = 0 \,.$$

When $H > 0$, this equation has the solution

$$a(t) = C\left[\exp\left(\frac{\sigma\gamma t}{2}\right) - 1\right]^{2/(3\gamma)} \,,$$

where C is one of two integration constants (the equation for the scale factor is of the second order), and the second integration constant is determined from the condition $a(t=0) = 0$. From this equation, we find the time-dependent densities of matter (radiation) and of the cosmological constant

$$\rho_\gamma = \frac{\sigma^2}{3}\left(\frac{C}{a}\right)^{3\gamma/2}\left[1 + \left(\frac{C}{a}\right)^{3\gamma/2}\right] \,, \tag{8.4}$$

$$\Lambda = \frac{\sigma^2}{3}\left[1 + \left(\frac{C}{a}\right)^{3\gamma/2}\right] \,. \tag{8.5}$$

In the framework of this scenario, we now analyze the history of the expansion of the Universe. In the radiation-dominated epoch, $\gamma = w + 1 = 4/3$, whence

$$a(t) = C\left[\exp\left(\frac{2}{3}\,\sigma t\right) - 1\right]^{1/2} .$$

For small times ($\sigma t \ll 1$), we return to the known dependence $a \propto t^{1/2}$,

$$a(t) = C\sqrt{\frac{2}{3}}\,\sigma t^{1/2} \,.$$

The densities ρ_γ in (8.4) and Λ in (8.5) transform in the radiation-dominated epoch as

$$\rho_\gamma \to \rho_{\rm r} = \frac{\sigma^2 C^4}{3}\,\frac{1}{a^4} + \frac{\sigma^2 C^2}{3}\,\frac{1}{a^2} \,,$$

$$\Lambda = \frac{\sigma^2}{3} + \frac{\sigma^2 C^2}{3}\,\frac{1}{a^2} \,.$$

In the limit $a \to 0$ ($t \to 0$),

$$\rho_{\rm r} = \frac{\sigma^2 C^4}{3}\,\frac{1}{a^4} = \frac{3}{4t^2} \,,$$

$$\Lambda = \frac{\sigma^2 C^2}{3}\,\frac{1}{a^2} = \frac{\sigma}{2t} \,.$$

Now we pass to the matter-dominated era. In this case, $\gamma = w + 1 = 1$ and

$$a(t) = C\left[\exp\left(\frac{1}{2}\,\sigma t\right) - 1\right]^{2/3} .$$

When $\sigma t \ll 1$, we return to the standard evolution law for matter

$$a(t) = C\left(\frac{\sigma}{2}\right)^{2/3} t^{2/3} \,.$$

For the densities ρ_γ and Λ in the matter-dominated epoch, we find

$$\rho_\gamma \to \rho_{\rm m} = \frac{\sigma^2 C^3}{3}\,\frac{1}{a^3} + \frac{\sigma^2 C^{3/2}}{3}\,\frac{1}{a^{3/2}} \,,$$

$$\Lambda = \frac{\sigma^2}{3} + \frac{\sigma^2 C^{3/2}}{3}\,\frac{1}{a^{3/2}} \,.$$

We note that in the limit of large times ($\sigma t \gg 1$), the scale factor increases exponentially,

$$a(t) = C\exp\left(\frac{\sigma}{3}\,t\right) .$$

In the same limit, the density of matter tends to zero, while the time-dependent function $\Lambda(t)$ becomes the 'true' cosmological constant.

In the matter-dominated epoch, Friedmann equation (8.3) can be represented as

$$H(z) = H_0\left[1 - \Omega_{\rm m0} + \Omega_{\rm m0}(1+z)^{3/2}\right] .$$

The last expression can be used for calculating the deceleration parameter ($q = (1+z)/H\,(\mathrm{d}H/\mathrm{d}z) - 1$),

$$q(z) = \frac{(3/2)\Omega_{\mathrm{m}0}(1+z)^{3/2}}{1-\Omega_{\mathrm{m}0}+\Omega_{\mathrm{m}0}(1+z)^{3/2}} - 1\,. \tag{8.6}$$

For the current deceleration parameter, we hence find

$$q(z=0) = \frac{3}{2}\,\Omega_{\mathrm{m}0} - 1\,.$$

Consequently, accelerating expansion of the Universe is realized under the condition

$$\Omega_{\mathrm{m}0} < \frac{2}{3}\,.$$

This condition is satisfied by the observed value $\Omega_{\mathrm{m}0} \approx 0.23$. It follows from (8.6) that the transition from the decelerating to accelerating expansion occurred when

$$z^* = \left(2\,\frac{1-\Omega_{\mathrm{m}0}}{\Omega_{\mathrm{m}0}}\right)^{2/3} - 1\,.$$

This value ($z^* \approx 1.2$) exceeds the corresponding value [although remaining the same order of magnitude, $O(1)$] in the SCM ($z^* \approx 0.75$), which is a consequence of the process of matter formation during the decay of the vacuum.

Thus, the model based on a time-dependent cosmological constant, in which the vacuum energy density depends linearly on the Hubble parameter, turns out to be quite viable. This model accurately reproduces the 'canonical' results relevant both to the radiation-dominated and matter-dominated eras. The current expansion of the Universe in this model is accelerating. Tests of the model using the most recent observational data on SNe-Ia give results (for example, $0.27 \leqslant \Omega_{\mathrm{m}0} \leqslant 0.37$) that are in good agreement with the current accelerated expansion.

Most likely, one of the most interesting properties of DM is its possible interaction with DE. Although most of realistic models (in particular, the SCM) assume that DM and DE are not coupled, there are no serious grounds for turning this assumption into a law. At present, active studies are under way of the possible consequences of the interaction of DM and DE [49–60, 64–66]. As we noted above, their interaction could at least mitigate certain burning cosmological problems, such as the coincidence problem. The DE density is approximately three times higher than the DM density. This relation can be explained if DM is somehow sensitive to DE.

We note that the possibility of DE, in the form of a scalar field interacting with DM underlies the scenario of warm inflation. This scenario, unlike that of cold inflation, does not assume the scalar field to be isolated (decoupled) from other fields during the inflation period. Therefore, instead of the Universe becoming overcooled during the inflation period, a certain amount of radiation is supported, which is sufficiently noticeable to be manifested in the postinflation dynamics.

The interaction of a quantum of the scalar field (inflaton) that caused the inflation with other fields implies that the inflationary evolution equation involves terms describing the dissipation of energy from the inflaton system to other particles. In Ref. [67], it was initially assumed that for a consistent description of the inflaton field with energy dissipation, the evolution equation must have the form of the Langevin equation, which involves a fluctuation–dissipation relation that unambiguously relates the fluctuation fields and the dissipated energy. Such an equation underlies the description of warm inflation.

Interaction between components in the Universe must be introduced so as to not violate the covariance of the energy–momentum tensor $T^{\mu\nu}_{\mathrm{tot}\,;\nu} = 0$; consequently, $T^{\mu\nu}_{\mathrm{m}\,;\nu} = -T^{\mu\nu}_{\mathrm{DE}\,;\nu} \neq 0$. The conservation equations in this case have the form

$$u_\nu T^{\mu\nu}_{\mathrm{m}\,;\mu} = -u_\nu T^{\mu\nu}_{\mathrm{DE}\,;\mu} = -Q\,, \tag{8.7}$$

where u_ν is the 4-velocity. For the FRW metric, Eqns (8.7) become

$$\dot\rho_{\mathrm{m}} + 3H\rho_{\mathrm{m}} = Q\,, \tag{8.8}$$

$$\dot\rho_{\mathrm{DE}} + 3H\rho_{\mathrm{DE}}(1+w_{\mathrm{DE}}) = -Q\,, \tag{8.9}$$

where ρ_{m} and ρ_{DE} are the DM and DE densities, w_{DE} is the parameter of the DE equation of state, and $H \equiv \dot a/a$ is the Hubble parameter. If $Q<0$, DM decays continuously into DE; if $Q>0$, the reciprocal process occurs:

$$Q<0\,,\quad \mathrm{DM} \to \mathrm{DE}\,,$$

$$Q>0\,,\quad \mathrm{DE} \to \mathrm{DM}\,.$$

We note that Eqns (8.8) and (8.9) satisfy the conservation equation

$$\dot\rho_{\mathrm{tot}} + 3H\rho_{\mathrm{tot}}(\rho_{\mathrm{tot}} + p_{\mathrm{tot}}) = 0\,, \tag{8.10}$$

where $\rho_{\mathrm{tot}} = \rho_{\mathrm{DE}} + \rho_{\mathrm{m}}$ is the total energy density.

The introduction of the interaction between DM and DE is effectively equivalent to altering the equation of state of the interacting components. Indeed, Eqns (8.8) and (8.9) can be written in the standard form of conservation equations for noninteracting components:

$$\dot\rho_{\mathrm{DE}} + 3H\rho_{\mathrm{DE}}(1+w_{\mathrm{DE,\,eff}}) = 0\,,$$

$$\dot\rho_{\mathrm{m}} + 3H\rho_{\mathrm{m,\,eff}} = 0\,,$$

where

$$w_{\mathrm{DE,\,eff}} = w_{\mathrm{DE}} - \frac{Q}{3H\rho_{\mathrm{DE}}}\,, \qquad w_{\mathrm{m,\,eff}} = \frac{Q}{3H\rho_{\mathrm{m}}} \tag{8.11}$$

play the role of effective equations of state of TE and TM.

Because we still know nothing of the nature of DE or DM, we cannot construct the interaction Q starting from first principles [68]. However, it follows from dimensional arguments that this quantity must be a function of one of the energy densities (or of their combination having the dimension of energy density) times a quantity of the dimension of inverse time. As the latter, it is natural to choose the Hubble parameter H.

In the literature, various expressions for Q can be found, the most commonly used of which are

$$Q = 3H\gamma\rho_{\mathrm{m}}\,,\quad Q = 3H\gamma\rho_{\mathrm{DE}}\,,\quad Q = 3H\gamma(\rho_{\mathrm{m}} + \rho_{\mathrm{DE}})\,. \tag{8.12}$$

As an example, we consider the simplest case of the following model:

$$\dot\rho_{\mathrm{m}} + 3H\rho_{\mathrm{m}} = \delta H\rho_{\mathrm{m}}\,,$$

$$\dot\rho_{\mathrm{DE}} + 3H(\rho_{\mathrm{DE}} + p_{\mathrm{DE}}) = -\delta H\rho_{\mathrm{m}}\,,$$

where δ is a dimensionless coupling constant. Integrating the last equations, we obtain

$$\rho_{\rm DE} = \rho_{\rm m0}\, a^{-[3(1+w_{\rm DE})+\delta]}\,,$$
$$\rho_{\rm m} = \frac{-\delta\rho_{\rm m0}}{3w_{\rm DE}+\delta}\, a^{-[3(1+w_{\rm DE})+\delta]} + \left(\rho_{\rm m0} + \frac{\delta\rho_{\rm m0}}{3w_{\rm DE}+\delta}\right) a^{-3}\,. \qquad (8.13)$$

We now substitute densities (8.13) in the first Friedmann equation (3.1b) and pass from the scale factor to the red shift; for $H^2(z)$, we can then write

$$H^2 = \frac{(1+z)^3 H_0^2}{3(3w_{\rm DE}+\delta)}\left[3w_{\rm DE}\Omega_{\rm DE}(1+z)^{3w_{\rm DE}+\delta} + \Omega_{\rm m}(3w_{\rm DE}+\delta)\right]. \qquad (8.14)$$

To find $q(z)$, we also use the formula

$$q(z) = \frac{1+z}{2H^2}\frac{{\rm d}H^2}{{\rm d}z} - 1\,. \qquad (8.15)$$

Using (8.14) and (8.15), we arrive at the dependence of the deceleration parameter on the red shift:

$$q(z) = -1 + \frac{3}{2}\,\frac{w_{\rm DE}\Omega_{\rm DE}\left[3(1+w_{\rm DE})+\delta\right](1+z)^{3w_{\rm DE}+\delta} + \Omega_{\rm m} + \delta/w_{\rm DE}}{3w_{\rm DE}\Omega_{\rm DE}(1+z)^{3w_{\rm DE}+\delta} + \Omega_{\rm m} + \delta/w_{\rm DE}}\,. \qquad (8.16)$$

If $w_{\rm DE} = -1$ and $\delta = 0$, Eqn (8.16) coincides with the expression for the deceleration parameter obtained in the SCM framework. We note that in this model, selecting suitable values of the parameter δ, it is possible to essentially mitigate or totally eliminate the problem of DM and DE densities being comparable. Indeed, using densities (8.13), we find the ratio $R = \rho_{\rm m}/\rho_{\rm DE}$:

$$R = -\frac{\delta}{3w_{\rm DE}+\delta} + \left(R_0 + \frac{\delta}{3w_{\rm DE}+\delta}\right) a^{3w_{\rm DE}+\delta}\,,$$

where $R_0 = \rho_{\rm m0}/\rho_{\rm D0}$ is the present ratio of densities. As is known, $R \sim a^{-3}$ in the SCM. We can reproduce this dependence by setting $\delta = 0$.

We now consider the inverse problem: instead of the interaction, we postulate the ratio

$$\frac{\rho_{\rm m}}{\rho_{\rm DE}} = f(a)\,, \qquad (8.17)$$

where $f(a)$ is any differentiable function of the scale factor. We then have

$$\rho_{\rm m} = \rho_{\rm DE} f(a)\,, \qquad (8.18)$$

$$\rho_{\rm DE} = \frac{\rho_{\rm m}}{f(a)}\,, \qquad (8.19)$$

$$\dot\rho_{\rm m} = \dot\rho_{\rm DE} f + \rho_{\rm DE} f'\dot a\,, \qquad (8.20)$$

$$\dot\rho_{\rm DE} = \frac{\dot\rho_{\rm m}}{f} - \frac{\rho_{\rm m}\dot a f'}{f^2}\,. \qquad (8.21)$$

Substituting expressions (8.18) and (8.20) in Eqn (8.8), we obtain

$$\dot\rho_{\rm DE} f + \rho_{\rm DE} f'\dot a + 3H\rho_{\rm DE} f = Q\,, \qquad (8.22)$$

where the prime is the derivative with respect to the scale factor. Substituting the expression for $\dot\rho_{\rm DE}$ found from (8.9) in (8.22), we have

$$Q = \frac{H\rho_{\rm DE} f}{1+f}\left(\frac{f'a}{f} - 3w_{\rm DE}\right). \qquad (8.23)$$

The first Friedmann equation becomes

$$3H^2 = \rho_{\rm DE} + \rho_{\rm m} = \rho_{\rm cr}\,, \qquad (8.24)$$

where $\rho_{\rm cr}$ is the critical density. Consequently, we can write

$$\Omega_{\rm DE} = \frac{\rho_{\rm DE}}{\rho_{\rm cr}} = \frac{1}{1+f}\,, \qquad (8.25)$$

$$\Omega_{\rm m} = \frac{\rho_{\rm m}}{\rho_{\rm cr}} = \frac{f}{1+f}\,. \qquad (8.26)$$

Now, substituting (8.26) in (8.23), we finally find

$$Q = H\rho_{\rm DE}\Omega_{\rm m}\left(\frac{f'a}{f} - 3w_{\rm DE}\right) = H\rho_{\rm m}\Omega_{\rm DE}\left(\frac{f'a}{f} - 3w_{\rm DE}\right). \qquad (8.27)$$

We note that if $f(a) \propto a^{\xi}$, then

$$Q = H\rho_{\rm m}\Omega_{\rm DE}(\xi - 3w_{\rm DE}) = H\rho_{\rm DE}\Omega_{\rm m}(\xi - 3w_{\rm DE})\,. \qquad (8.28)$$

From Eqn (8.10), we find

$$\frac{{\rm d}\ln\rho_{\rm tot}}{{\rm d}\ln a} = -3(1+w_{\rm eff})\,, \qquad (8.29)$$

where

$$w_{\rm eff} = \frac{p_{\rm tot}}{\rho_{\rm tot}} = \frac{\rho_{\rm DE} w_{\rm DE}}{\rho_{\rm DE}+\rho_{\rm m}} = \frac{w_{\rm DE}}{1+f} = \Omega_{\rm DE} w_{\rm DE}\,; \qquad (8.30)$$

consequently,

$$\frac{{\rm d}\ln\rho_{\rm tot}}{{\rm d}\ln a} = -3(1+\Omega_{\rm DE} w_{\rm DE})\,. \qquad (8.31)$$

From the last equation, we obtain

$$\rho_{\rm tot} = Ca^{-3}\exp\left(-\int 3\Omega_{\rm DE} w_{\rm DE}\,{\rm d}\ln a\right), \qquad (8.32)$$

where C is the integration constant, which can be determined if we require relation $\rho_{\rm tot}(a=1) = \rho_{\rm tot,0} = 3H_0^2 M_{\rm Pl}^2$ to hold. Using the expression for $\rho_{\rm tot}$, it is easy to find $E \equiv H/H_0$ from the Friedmann equations; this expression is used in testing cosmological models and searching for constraints on cosmological parameters. The expressions for $\rho_{\rm DE} = \Omega_{\rm DE}\rho_{\rm tot}$ and $\rho_{\rm m} = \Omega_{\rm m}\rho_{\rm tot}$ are also readily found from Eqns (8.25) and (8.32).

Finally, we note that with (8.17), we have

$$f_0 = f(a=1) = \frac{\rho_{\rm m0}}{\rho_{\rm DE0}} = \frac{\Omega_{\rm m0}}{1-\Omega_{\rm m0}}\,. \qquad (8.33)$$

We now let

$$f(a) = f_0\, a^{\xi}\,, \qquad (8.34)$$

where ξ is a constant and f_0 was defined above. Substituting (8.34) in (8.32) and requiring that the equality $\rho_{\rm tot}(a=1) = \rho_{\rm tot,0}$ hold, we determine the integration constant and find

$$\rho_{\rm tot} = \rho_{\rm tot,0}\, a^{-3}\left[\Omega_{\rm m0} + (1-\Omega_{\rm m0})a^{\xi}\right]^{-3w_X/\xi}. \qquad (8.35)$$

Substituting (8.35) in the Friedmann equation, we obtain

$$E^2 = \frac{H^2}{H_0^2} = a^{-3}\left[\Omega_{\mathrm{m0}} + (1-\Omega_{\mathrm{m0}})a^{\xi}\right]^{-3w_X/\xi}$$
$$= (1+z)^3\left[\Omega_{\mathrm{m0}} + (1-\Omega_{\mathrm{m0}})(1+z)^{-\xi}\right]^{-3w_X/\xi}. \quad (8.36)$$

Using the formulas obtained above, it is not difficult to find the expression for the deceleration parameter in this model:

$$q = 1 + \frac{3}{2}\left(w_{\mathrm{DE}}\Omega_{\mathrm{DE}} + w_{\mathrm{m}}\Omega_{\mathrm{m}}\right), \quad (8.37)$$

where the relative densities Ω_{DE} and Ω_{m} are given by

$$\Omega_{\mathrm{DE}} = \frac{\Omega_{\mathrm{m0}}}{\Omega_{\mathrm{m0}} + (1-\Omega_{\mathrm{m0}})a^{\xi}}, \quad \Omega_{\mathrm{m}} = \frac{(1-\Omega_{\mathrm{m0}})a^{\xi}}{\Omega_{\mathrm{m0}} + (1-\Omega_{\mathrm{m0}})a^{\xi}}. \quad (8.38)$$

Hence, we finally obtain

$$q = \frac{1}{2} + \frac{3}{2}\,\frac{w_{\mathrm{DE}}\Omega_{\mathrm{m0}}}{\Omega_{\mathrm{m0}} + \Omega_{\mathrm{m}}(1+z)^{\xi}}. \quad (8.39)$$

Because the DE density is not constant in this model, in contrast to the SCM, the condition $\xi < 3$ facilitates resolving the coincidence problem. In this model, the phase of accelerated expansion starts with the red shift z^* that is determined from the relation $q(z^*) = 0$:

$$z^* = \left[\frac{1-\Omega_{\mathrm{m0}}}{(1+3w_{\mathrm{DE}})\Omega_{\mathrm{m0}}}\right]^{1/\xi} - 1. \quad (8.40)$$

This expression is a natural generalization of expression (5.2) obtained in the SCM framework. The difference between the value in (8.40) and the corresponding SCM value depends on how large the difference is between the parameters ξ and w_{DE} and the respective values 3 and -1. When $\xi = 3$ and $w_{\mathrm{DE}} = -1$, expressions (5.2) and (8.40) coincide. We note that as $z \to \infty$, the expansion of the Universe decelerates ($q \to 1/2$), and $z \to -1$ corresponds to $q \to 1/2 - (3/2)w_{\mathrm{DE}}$. Hence, as was to be expected, the dynamics of this model does not differ asymptotically (for $z \to \infty$ and $z \to -1$) from the dynamics of a two-component universe filled with nonrelativistic matter and DE with the equation of state $p = w_{\mathrm{DE}}\,\rho$.

8.2 Cosmological models with a new type of interaction

In this section, we consider one more type of interaction, Q [69], whose sign (i.e., the direction of energy transfer) changes when the mode of decelerated expansion is replaced by the mode of accelerated expansion, and vice versa.

Recently, publications have appeared [70, 71], in which attempts, based on observational data, are made to determine not only the possibility itself of interaction existing in the dark sector but also its concrete form and sign. In this analysis, the whole set of red shifts z is divided into intervals, in each of which the function $\delta(z) = Q/(3H)$ is considered constant. This analysis has permitted determining that $\delta(z)$ most likely takes a zero value, $\delta = 0$, in the range of red shifts $0.45 \lesssim z \lesssim 0.9$. It turns out that this remarkable result gives rise to new problems. Indeed, as we noted in Section 8.1, the interactions considered in the literature are mainly of three types: $Q = 3H\gamma\rho_{\mathrm{m}}$, $Q = 3H\gamma\rho_{\mathrm{DE}}$, and $Q = 3H\gamma(\rho_{\mathrm{m}} + \rho_{\mathrm{DE}})$, and for a given model, the interaction is always either positive or negative, i.e. cannot change sign. A change of sign is only possible in the case $\gamma = f(t)$, where γ can change the sign of Q, or in the case $Q = 3H(\alpha\rho_{\mathrm{m}} + \beta\rho_{\mathrm{DE}})$, where α and β have different signs.

As noted in Ref. [70], the solution to this problem requires the introduction of a new type of interaction, capable of changing its sign during the evolution of the Universe.

In Ref. [69], one such type of the interaction Q was proposed and its cosmological consequences were examined. It was noted in [69] that the range of red shifts within which the function $\delta(z)$ must change sign includes the moment at which expansion of the Universe stopped decelerating and started accelerating [see (5.2)]. Consequently, the simplest form of the interaction that can explain this is the interaction Q proportional to the deceleration parameter q:

$$Q = q(\alpha\dot{\rho} + 3\beta H\rho), \quad (8.41)$$

where α and β are dimensionless constants; the sign of Q changes as the Universe undergoes transition from the decelerating stage ($q > 0$) to the accelerating stage ($q < 0$). The authors of Ref. [69] also considered the cases

$$Q = q(\alpha\dot{\rho}_{\mathrm{m}} + 3\beta H\rho_{\mathrm{m}}), \quad (8.42)$$

$$Q = q(\alpha\dot{\rho}_{\mathrm{tot}} + 3\beta H\rho_{\mathrm{tot}}), \quad (8.43)$$

$$Q = q(\alpha\dot{\rho}_{\mathrm{DE}} + 3\beta H\rho_{\mathrm{DE}}). \quad (8.44)$$

In Ref. [72], the model of a universe with a decaying cosmological constant

$$\dot{\rho}_{\Lambda} = -Q$$

is considered. The Friedmann and Raychaudhuri equations then take the form

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\,\rho_{\mathrm{tot}} = \frac{1}{3M_{\mathrm{Pl}}^2}(\rho_{\Lambda} + \rho_{\mathrm{m}}), \quad (8.45)$$

$$\dot{H} = -\frac{1}{2M_{\mathrm{Pl}}^2}(\rho_{\mathrm{tot}} + p_{\mathrm{tot}}) = -\frac{\rho_{\mathrm{m}}}{2M_{\mathrm{Pl}}^2}. \quad (8.46)$$

In Sections 8.2.1–8.2.3, we follow [72] and consider cosmological models with the interaction as in (8.42)–(8.44).

8.2.1 The case $Q = q(\alpha\dot{\rho}_{\mathrm{m}} + 3\beta H\rho_{\mathrm{m}})$. We first consider the case where the interaction has form (8.42). Substituting this expression in conservation equation (8.8), we obtain

$$\dot{\rho}_{\mathrm{m}} = \frac{\beta q - 1}{1 - \alpha q}\,3H\rho_{\mathrm{m}}. \quad (8.47)$$

Substituting (8.47) in (8.42), we ultimately obtain

$$Q = \frac{\beta - \alpha}{1 - \alpha q}\,3qH\rho_{\mathrm{m}}. \quad (8.48)$$

From Eqn (8.46), we have

$$\rho_{\mathrm{m}} = -2M_{\mathrm{Pl}}^2\dot{H}. \quad (8.49)$$

Substituting (8.49) in Eqn (8.47), we find that

$$\ddot{H} = \frac{\beta q - 1}{1 - \alpha q}\,3H\dot{H}. \quad (8.50)$$

Thus, we have obtained a second-order equation for $H(t)$. We pass from the time derivative to the derivative with respect to the scale factor (the derivative is temporarily denoted by a prime). Equation (8.50) then takes the form

$$aH'' + \frac{a}{H}H'^2 + H' = \frac{\beta q - 1}{1 - \alpha q}\,3H' . \qquad (8.51)$$

This is a second-order equation for $H(a)$. We note that the deceleration parameter

$$q = -1 - \frac{\dot H}{H^2} = -1 - \frac{a}{H}H'$$

is also a function of H and H', and hence, if $\alpha \neq 0$, Eqn (8.51) is a transcendental differential equation of the second order, which has no analytic solutions. We therefore only consider the case $\alpha = 0$. Interaction (8.42) then becomes

$$Q = 3\beta q H \rho_{\rm m} .$$

When $\alpha = 0$, Eqn (8.51) has the solution

$$H(a) = C_{12}\Big[3C_{11}(1+\beta) - (2+3\beta)\,a^{-3(1+\beta)}\Big]^{1/(2+3\beta)} , \qquad (8.52)$$

where C_{11} and C_{12} are integration constants, which are determined below. We find the relative DM density

$$\Omega_{\rm m} \equiv \frac{\kappa^2 \rho_{\rm m}}{3H^2} = -\frac{2\dot H}{3H^2} = -\frac{2aH'}{3H} . \qquad (8.53)$$

Substituting (8.52) in (8.53), we obtain

$$\Omega_{\rm m} = \frac{2(1+\beta)}{2 + 3\beta - 3C_{11}(1+\beta)\,a^{3(1+\beta)}} . \qquad (8.54)$$

When the conditions $\Omega_{\rm m}(a=1) = \Omega_{\rm m0}$ and $H(a=1) = H_0$ are imposed, the integration constants become

$$C_{11} = \frac{\Omega_{\rm m0}(2+3\beta) - 2(1+\beta)}{3\Omega_{\rm m0}(1+\beta)} , \qquad (8.55)$$

$$C_{12} = H_0\big[3C_{11}(1+\beta) - (2+3\beta)\big]^{-1/(2+3\beta)} . \qquad (8.56)$$

Substituting expressions (8.55) and (8.56) in (8.52), we finally find

$$E \equiv \frac{H}{H_0} = \left\{1 - \frac{2+3\beta}{2(1+\beta)}\,\Omega_{\rm m0}\Big[1 - (1+z)^{3(1+\beta)}\Big]\right\}^{1/(2+3\beta)} . \qquad (8.57)$$

This model involves two free parameters, $\Omega_{\rm m0}$ and β. We note that when $\beta = 0$, expression (8.57) reduces to $E(z) = [\Omega_{\rm m0}(1+z)^3 + (1-\Omega_{\rm m0})]^{1/2}$, which is equivalent to the ΛCDM model. Using the relation

$$q(z) = -\frac{1+z}{E(z)}\frac{\rm d}{{\rm d}z}\left(\frac{1}{E(z)}\right) - 1 ,$$

we find the dependence of the deceleration parameter on the red shift in this model:

$$q(z) = -1 + \frac{3}{2}\,\Omega_{\rm m0}\,\frac{(1+z)^{3(1+\beta)}}{E^{2+3\beta}} . \qquad (8.58)$$

The effective parameter of the equation of state is known to be of the form

$$w_{\rm eff} \equiv \frac{p_{\rm tot}}{\rho_{\rm tot}} = \frac{2q-1}{3} .$$

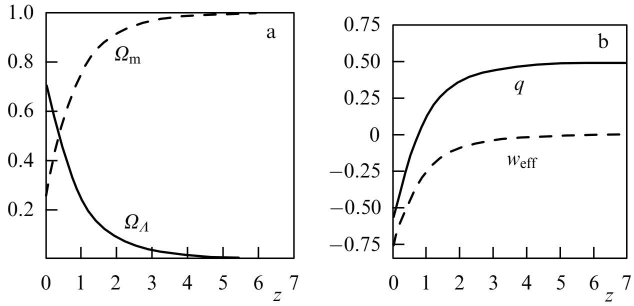

Figure 4. $\Omega_{\rm m}$, Ω_Λ, q, and $w_{\rm eff}$ as functions of the red shift z for $\Omega_{\rm m0} = 0.2738$ and $\beta = -0.010$ in the case where $Q = 3\beta q H\rho_{\rm m}$.

Figure 4 shows the dependences of some cosmological parameters on the red shift z. The free parameters $\Omega_{\rm m0}$ and β are chosen so as to provide the best agreement with observations. It can be shown that the transition from decelerated ($q > 0$) to accelerated ($q < 0$) expansion occurred in this model at $z_{\rm t} = 0.7489$. The parameter β was chosen negative; consequently, DM decays into DE when $z > z_{\rm t}$, and DE decays into DM when $z < z_{\rm t}$. When $z = z_{\rm t}$, no interaction in the dark sector exists in the Universe.

8.2.2 The case $Q = q(\alpha\dot\rho_{\rm tot} + 3\beta H\rho_{\rm tot})$. We now consider case (8.43). Acting as in Section 8.2.1, we obtain

$$Q = 3M_{\rm Pl}^2 q H^3\left(2\alpha\,\frac{\dot H}{H^2} + 3\beta\right) . \qquad (8.59)$$

Substituting expressions (8.49) and (8.59) in Eqn (8.8) and passing to the derivative with respect to the scale factor, we find

$$aH'' + \frac{a}{H}H'^2 + (4 + 3\alpha q)\,H' + \frac{9\beta q H}{2a} = 0 . \qquad (8.60)$$

As in Section 8.2.1, we have obtained a second-order differential equation for the function $H(a)$. An analytic solution can be found only if $\alpha = 0$:

$$H(a) = C_{22}\,a^{-3(2-3\beta+r_1)/8}\left(a^{3r_1/2} + C_{21}\right)^{1/2} , \qquad (8.61)$$

where C_{21} and C_{22} are integration constants and $r_1 \equiv \sqrt{4 + \beta(4+9\beta)}$. Substituting (8.61) in (8.53), we obtain

$$\Omega_{\rm m} = \frac{1}{4}\left[2 - 3\beta + \left(\frac{2C_{21}}{a^{3r_1/2} + C_{21}} - 1\right) r_1\right] . \qquad (8.62)$$

As usual, the integration constants are determined from the conditions $\Omega_{\rm m}(a=1) = \Omega_{\rm m0}$ and $H(a=1) = H_0$:

$$C_{21} = -1 + \frac{2r_1}{2 - 3\beta - 4\Omega_{\rm m0} + r_1} , \qquad C_{22} = H_0(1 + C_{21})^{-1/2} . \qquad (8.63)$$

We finally obtain

$$E \equiv \frac{H}{H_0} = (1+z)^{3(2-3\beta+r_1)/8}\left[\frac{(1+z)^{-3r_1/2} + C_{21}}{1 + C_{21}}\right]^{1/2} . \qquad (8.64)$$

This model also involves two free parameters, $\Omega_{\rm m0}$ and β. Using the condition $0 \leqslant \Omega_{\rm m} \leqslant 1$ for $a \to 0$, we obtain from (8.62) that $\beta \geqslant 0$. The best agreement of this model with observational data is achieved for $\Omega_{m0} = 0.2701$ and $\beta = 0$. This means that this interaction model is less consistent with observations than the ΛCDM model. The reader can find a detailed discussion of the issue in the work by the author of the model [72]. The transition from the phase of decelerated expansion $(q > 0)$ to the phase of accelerated expansion $(q < 0)$ occurs at $z_t = 0.7549$.

8.2.3 The case $Q = q(a\dot{\rho}_\Lambda + 3\beta H \rho_\Lambda)$. We finally consider case (8.44). Performing computations similar to those in Sections 8.2.1 and 8.2.2, we obtain

$$Q = \frac{3\beta q H \rho_\Lambda}{1 + \alpha q} \,. \tag{8.65}$$

Taking Eqn (8.49) into account, we find

$$\rho_\Lambda = 3M_{\rm Pl}^2 H^2 - \rho_{\rm m} = M_{\rm Pl}^2 (3H^2 + 2\dot{H}) \,. \tag{8.66}$$

Consequently, the equation for the Hubble parameter in terms of the scale factor is

$$aH'' + \frac{a}{H} H'^2 + \left(4 + \frac{3\beta q}{1 + \alpha q}\right) H' + \frac{9\beta q H}{2a(1 + \alpha q)} = 0 \,. \tag{8.67}$$

The analytic solution is successfully found in the case $Q = 3\beta q H \rho_\Lambda$:

$$H(a) = C_{32}\, a^{-3(2-5\beta+r_2)/[4(2-3\beta)]} (a^{3r_2/2} + C_{31})^{1/(2-3\beta)} \,, \tag{8.68}$$

where C_{31} and C_{32} are integration constants and $r_2 \equiv [(2-\beta)^2]^{1/2} = |2-\beta|$. Substituting (8.68) in (8.53), we obtain

$$\Omega_{\rm m} = \frac{1}{2(2-3\beta)} \left[2 - 5\beta + \left(\frac{2C_{31}}{a^{3r_2/2} + C_{31}} - 1\right) r_2\right] . \tag{8.69}$$

Assuming that $\Omega_{\rm m}(a=1) = \Omega_{\rm m0}$ and $H(a=1) = H_0$, we find

$$C_{31} = -1 + \frac{2r_2}{2 - 5\beta + r_2 + 2\Omega_{\rm m0}(3\beta - 2)} \,,$$
$$C_{32} = H_0 (1 + C_{31})^{1/(3\beta-2)} \,. \tag{8.70}$$

Finally,

$$E \equiv \frac{H}{H_0} = (1+z)^{3(2-5\beta+r_2)/[4(2-3\beta)]} \times \left[\frac{(1+z)^{-3r_2/2} + C_{31}}{1 + C_{31}}\right]^{1/(2-3\beta)} . \tag{8.71}$$

This model also has two free parameters, $\Omega_{\rm m0}$ and β. The maximum likelihood method for these parameters and the given model yields $\Omega_{\rm m0} = 0.2717$ and $\beta = 0.0136$ [72]. The analysis of observational data carried out in [72] also testifies in favor of $\beta > 0$, which contradicts the models considered in Sections 8.2.1 and 8.2.2.

The dependences of the deceleration parameter and of the effective parameter of the equation of state are practically coincident with those considered in Section 8.2.1. It is not difficult to show that the transition from the phase of decelerated $(q > 0)$ to the phase of accelerated $(q < 0)$ expansion occurs at $z_t = 0.7398$. Because the value of β obtained from observations is greater than zero, DE decays into DM $(Q > 0)$ at $z > z_t$ and the reciprocal process $(Q < 0)$ takes place at $z < z_t$.

9. State-determining parameters for interacting dark energy

We consider a universe filled with two components: nonrelativistic matter with negligible pressure, $p_{\rm m} \ll \rho_{\rm m}$, and DE with the equation of state $p_{\rm DE} = w\rho_{\rm DE}$, where $w < 0$. The DE decays into DM in accordance with (8.8) and (8.9). For convenience, we represent the interaction between the components as $Q = -3\Pi H$, where Π is a new quantity of the dimension of pressure.

It is easy to show that the Hubble parameter and its first time derivative and, consequently, the deceleration parameter do not depend explicitly on the existence or absence of the interaction between the components; $q = -1 - (\dot{H}/H^2) = (1 + 3w\Omega_{\rm DE})/2$. Nevertheless, differentiating $\dot{H}$, we obtain

$$\frac{\ddot{H}}{H^3} = \frac{9}{2}\left(1 + \frac{p_{\rm DE}}{\rho}\right) + \frac{9}{2}\left[w(1+w)\,\frac{\rho_{\rm DE}}{\rho} - w\,\frac{\Pi}{\rho} - \frac{\dot{w}}{3H}\,\frac{\rho_{\rm DE}}{\rho}\right] . \tag{9.1}$$

Unlike H and $\dot{H}$, the second derivative $\ddot{H}$ does depend on the interaction between the components. Consequently, to have the possibility of distinguishing between models of different types of interaction or between models with or without the interaction, it is necessary to analyze the cosmological dynamics of the Universe with parameters that depend on $\ddot{H}$.

In Refs [73, 74], two new parameters (so-called determinants of state) were introduced, which involve the second derivative of the Hubble parameter with respect to time:

$$r = \frac{\dddot{a}}{aH^3} \,, \qquad s = \frac{r-1}{3(q-1/2)} \,. \tag{9.2}$$

For the interaction in the dark sector, these parameters are

$$r = 1 + \frac{9}{2}\,\frac{w}{1+\kappa}\left[1 + w - \frac{\Pi}{\rho_{\rm DE}} - \frac{\dot{w}}{3wH}\right] , \tag{9.3}$$

where $\kappa \equiv \rho_{\rm m}/\rho_{\rm DE}$ and

$$s = 1 + w - \frac{\Pi}{\rho_{\rm DE}} - \frac{\dot{w}}{3wH} \,. \tag{9.4}$$

For models without interaction, i.e., if $\Pi = 0$, the expressions for r and s reduce to the expressions that were studied in Refs [73, 74].

We show how the determinants of state allow analyzing cosmological models in which the dominant components interact with each other. We note that the third derivative of the scale factor is typically necessary for the description of any variations in the general equation of state of the cosmic environment [13]. This becomes evident from the general relation [74]

$$r - 1 = \frac{9}{2}\,\frac{\rho + P}{P}\,\frac{\dot{P}}{\dot{\rho}} \,, \tag{9.5}$$

where P is the total pressure of the cosmic environment. In the case under consideration, $P \approx p_{\rm DE}$. Because

$$\frac{\rm d}{{\rm d}t}\left(\frac{P}{\rho}\right) = \frac{\dot{\rho}}{\rho}\left(\frac{\dot{P}}{\dot{\rho}} - \frac{P}{\rho}\right) , \tag{9.6}$$

it is obvious that in accordance with (8.8) and (8.9), the interaction term in $\dot{P} \approx \dot{p}_{\rm DE} = \dot{w}\rho_{\rm DE} + w\dot{\rho}_{\rm DE}$ additionally changes the time dependence of the common parameter of state P/ρ.

In cosmological models with $\kappa \sim 1$ at late stages of the evolution of the Universe, the coincidence problem can be resolved rigorously. At any rate, this ratio must change slowly at times of the order of H^{-1}. In terms of κ, Eqns (8.8) and (8.9) become

$$\dot{\kappa} = -3H\left[\left(\frac{\rho_{\mathrm{DE}} + \rho_{\mathrm{m}}}{\rho_{\mathrm{m}}\,\rho_{\mathrm{DE}}}\right)\Pi - w\right]\kappa\,. \tag{9.7}$$

In what follows, we consider the solution of Eqn (9.7) in some cosmological models.

9.1 Scale solutions

It was shown in Ref. [75] that scale solutions, i.e., solutions of the form $\rho_{\mathrm{m}}/\rho_{\mathrm{DE}} \propto a^{-\xi}$, where ξ is a constant parameter in the range $[0, 3]$, can be obtained when DE decays into DM. These solutions are of interest because they allow resolving the coincidence problem [76]. Indeed, the model with $\xi = 3$ reduces to the ΛCDM model with $w = -1$ and $\Pi = 0$. In the case of $\xi = 0$, the ratio of densities is frozen, $\kappa = \mathrm{const}$, and therefore the coincidence problem does not arise [77]. It can be shown that if $w = \mathrm{const}$, then the interaction that permits obtaining scale solutions can be found as

$$\frac{\Pi}{\rho_{\mathrm{DE}}} = \left(w + \frac{\xi}{3}\right)\frac{\kappa_0(1+z)^{\xi}}{1+\kappa_0(1+z)^{\xi}}\,, \tag{9.8}$$

where κ_0 denotes the current ratio of energy densities. Substituting (9.8) in expressions (9.3) and (9.4), we obtain the determinants of state

$$r = 1 + \frac{9}{2}\,\frac{w}{1+\kappa_0(1+z)^{\xi}}\left[1 + w - \left(w + \frac{\xi}{3}\right)\frac{\kappa_0(1+z)^{\xi}}{1+\kappa_0(1+z)^{\xi}}\right], \tag{9.9}$$

$$s = 1 + w - \left(w + \frac{\xi}{3}\right)\frac{\kappa_0(1+z)^{\xi}}{1+\kappa_0(1+z)^{\xi}}\,. \tag{9.10}$$

Figure 5 presents the function $r(s)$ for different values of ξ. The smaller ξ is, the lower the corresponding curve in the $s-r$ region lies and the less important the coincidences problem becomes. We note that in no way do these curves differ qualitatively from those constructed for models with noninteracting components (Fig. 6) [73]. For comparison, we note that the ΛCDM model ($\Pi = 0$, $w = -1$) corresponds to the point $s = 0$, $r = 1$ (not shown in the figure). For models with scale solutions, we find the current values for the determinants of state:

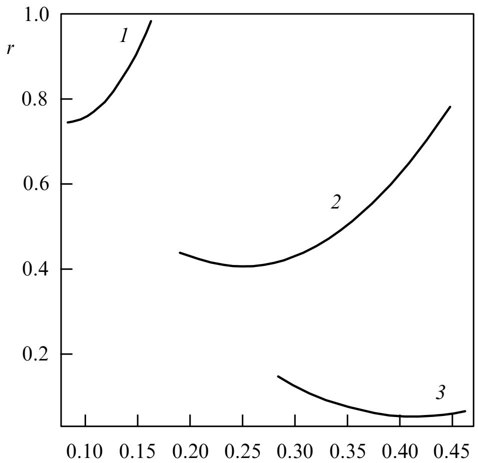

Figure 5. The $r(s)$ space in the red-shift interval $[0, 6]$ for the following values of the parameter ξ: 2.5 (curve *1*), 1.5 (curve *2*), and 0.5 (curve *3*) for $w = -0.95$ and $\kappa_0 = 3/7$.

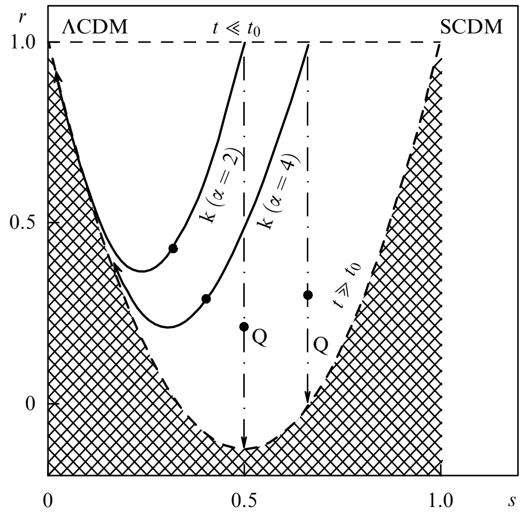

Figure 6. Determinants of state (r, s) for different forms of DE. In models involving quintessence (Q) ($w = \mathrm{const} \neq -1$), the value of s is constant, $s = 1 + w$, while r asymptotically decreases, tending to $r(t \gg t_0) \simeq 1 + (9w/2)(1+w)$. The results are presented for two models with quintessence, in which $w_{\mathrm{Q}} = -0.25, -0.5$; two models with k-essence (k), in which the potential of the scalar field has the form $V(\phi) \propto \phi^{-\alpha}$, $\alpha = 2, 4$; the ΛCDM model ($r = 1$, $s = 0$); and the SCDM (Standard Cold Dark Matter) model in the absence of Λ ($r = 1$, $s = 1$). The shaded region is forbidden for the models with quintessence and k-essence considered in Ref. [73]. The dots show the modern values of determinants of state (r, s) for Q- and k-models at $\Omega_{\mathrm{m}0} = 0.3$.

$$r_0 = 1 + \frac{9}{2}\,\frac{w}{1+\kappa_0}\,s_0\,, \qquad s_0 = 1 + w - \left(w + \frac{\xi}{3}\right)\frac{\kappa_0}{1+\kappa_0}\,. \tag{9.11}$$

Taking into account that

$$q_0 = \frac{1}{2}\,\frac{1+\kappa_0+3w}{1+\kappa_0}\,, \tag{9.12}$$

and introducing

$$q_{0\Lambda} \equiv q_0(w=-1) = -\frac{1}{2}\,\frac{2-\kappa_0}{1+\kappa_0} \Leftrightarrow \frac{3}{2}\,\frac{\kappa_0}{1+\kappa_0} = 1 + q_{0\Lambda}\,, \tag{9.13}$$

we can classify different models by their dependence $s_0(q_0)$, which has the form

$$s_0 = \frac{2}{3}\left[(q_0 - q_{0\Lambda}) + \left(\frac{\xi}{3} - 1\right)(1 + q_{0\Lambda})\right]. \tag{9.14}$$

The difference $q_0 - q_{0\Lambda}$ in the right-hand side of (9.14) shows by how much the value of w in the model considered differs from the similar value in the SCM, and the term

$(\xi/3 - 1)(1 + q_{0\Lambda})$ corresponds to the deviation from $\xi = 3$. For models with $w = -1$, i.e., models with $q_0 = q_{0\Lambda}$, we have

$$s_0 = \frac{2}{3}\left(\frac{\xi}{3} - 1\right)(1 + q_{0\Lambda}) .$$

The value $\xi = 3$ corresponds to the ΛCDM model with $s_0 = 0$. If $\kappa_0 = 3/7$, we have $s_0(\xi = 1) = -0.2$, but at the same time $s_0(\xi = 0) = -0.3$. Similar arguments are also valid for other values of w. Thus, with the aid of the parameter s_0, it is possible to distinguish between cosmological models with scale solutions that have the same deceleration parameter.

10. Holographic dynamics: an entropic acceleration

In Sections 2–9, we described the dynamics of the Universe based on GR and on some of its generalizations. We now show that there exists an essentially different approach, which permits both reproducing the achievements of the traditional description and resolving a number of problems it encountered.

The traditional standpoint was to assume that the dominant part of the degrees of freedom of our world is composed of fields filling up space. But it was gradually realized that such an assumption hinders the construction of the theory of quantum gravity: for that theory to make sense, all the integrals involved in the theory have to be cut off at small distances. As a consequence, our world must be described on a three-dimensional lattice with a period of the order of the Planck length. Recently, some physicists have adhered to an even more radical point of view: a complete description of Nature requires, instead of a three-dimensional, only a two-dimensional lattice placed at the spatial boundary of our world. Such an approach is based on the so-called holographic principle [78–85]. The name is related to the optical hologram, which represents a two-dimensional record of a three-dimensional object.

The holographic principle consists of two main assertions:

(1) all the information contained in a certain region of space can be 'encoded' (represented) on the boundary of this region;

(2) the theory on the boundaries of the region must not involve more than one degree of freedom per Planck area,

$$N \leqslant \frac{Ac^3}{G\hbar} . \tag{10.1}$$

The central part in the holographic principle is the assumption that all the information on the Universe can be written on a certain two-dimensional surface — a holographic screen. This approach leads to the possibility of a new interpretation of cosmological acceleration and to a completely new perception of gravity. Gravity is understood to be an entropic force caused by a change in information related to the positions of material bodies. To be more precise, the amount of information related to matter and its position is measured in terms of entropy. The relation between entropy and information is that a change in information I represents a negative change in entropy S, $\Delta I = -\Delta S$. The change in entropy due to a displacement of matter leads to a so-called entropic force, which has the form of the gravitational force. The origin of the entropic force is, consequently, due to the universal tendency of any microscopic theory to maximize entropy. Dynamics can be constructed in terms of the change in entropy, and it is independent of the details of the microscopic theory. For instance, no field associated with the entropic force exists. Entropic forces are typical for such macroscopic systems as colloids or biophysical systems. Large colloidal molecules suspended in a thermal environment of smaller particles are subject to entropic forces. Another phenomenon governed by entropic forces is osmosis.

Probably the best known example of the manifestation of an entropic force is the elasticity of a polymeric molecule. An individual polymeric molecule can be represented as an aggregation of many monomers of fixed length. Each monomer can rotate freely around its point of attachment and choose any spatial direction. Each such configuration has the same energy. When a polymer molecule is placed into a thermal reservoir, it tends to form a ring, because such a configuration is entropically preferable: there are many more configurations in which a polymer molecule is short than when it is stretched out. The statistical tendency to pass into the maximum-entropy state translates into a macroscopic force; in the case being considered, into the elastic force.

We now consider a small area of the holographic screen and a particle of mass m approaching it. According to the holographic principle, the particle affects the amount of information (and, consequently, the entropy) stored on the screen. It is natural to assume that the change in entropy in the vicinity of the screen depends linearly on the displacement Δx,

$$\Delta S = 2\pi k_{\mathrm{B}} \frac{mc}{\hbar} \Delta x . \tag{10.2}$$

The factor 2π is introduced for convenience. To understand why the quantity ΔS is proportional to the mass, we imagine that the particle has split into two or more particles of smaller masses. Each of the particles then carries its own change in entropy in a displacement by Δx. Because entropy is additive and mass can also be considered additive in the cosmological context, entropy is naturally proportional to mass. Owing to the first law of thermodynamics, the entropic force related to a change in information satisfies the relation

$$F \Delta x = T \Delta S . \tag{10.3}$$

If the gradient of entropy is known [it can be found using (10.2)], knowing the temperature of the screen allows calculating the entropic force. As is known, an observer moving with acceleration a feels the temperature [86] (the Unruh temperature)

$$k_{\mathrm{B}} T = \frac{1}{2\pi} \frac{\hbar}{c} a . \tag{10.4}$$

We assume that the total energy of the system E is uniformly distributed over all N bits of information on the holographic screen. The temperature is then determined as the average energy per bit,

$$E = \frac{1}{2} N k_{\mathrm{B}} T . \tag{10.5}$$

Formulas (10.1)–(10.5) allow constructing holographic dynamics and, as a special case, the dynamics of the Universe, without involving gravity. It was E Verlinde who came up with such an interpretation of the holographic principle [87].

The ideology presented above is to a significant extent based on the successful description of the physics of black holes [88–91]. At first sight, there seems to be nothing in common between the extremely rarified Universe with the density $\rho \sim 10^{-29}$ g cm^{-3} and a 'typical' black hole of a stellar mass with the critical density $\rho \sim 10^{14}$ g cm^{-3} ($M = 10 M_{\odot}$). However, the situation changes drastically in the case of transition to more massive black holes. Because $r_g \propto M$, the critical density of a black hole is $\rho \propto M^{-2}$. The mass of black holes in the nuclei of galaxies can reach values up to $10^{10} M_{\odot}$, while their gravitational radius ($r_g \sim 3 \times 10^{15}$ cm) can be approximately five times greater than the size of the Solar System. In this case, the critical density is $\rho \sim 10^{-4}$ g cm^{-3}, which is nearly an order of magnitude less than the density of air (1.3×10^{-3} g cm^{-3}). We estimate how close the parameters of the observable Universe are to the parameters of a black hole. We estimate the mass of the observable Universe, assuming the Hubble radius H^{-1} to be its radial size. The mass contained inside the Hubble sphere is

$$M_{\text{univ}} = \frac{4\pi}{3} R_{\text{H}}^3 \rho \,. \tag{10.6}$$

For the flat part of the Universe confined by the Hubble radius (as we have repeatedly noted, the deviations from planarity are small), it is reasonable to choose ρ as critical density (3.3), $\rho = 3M_{\text{Pl}}^2 H^2$; then relation (10.6) becomes

$$M_{\text{univ}} = \frac{R_{\text{H}}}{2G} \,.$$

Substituting this value of mass in the expression for the gravitational radius of the Universe, $r_g = 2GM_{\text{univ}}$, we find

$$r_g = R_{\text{H}} \,. \tag{10.7}$$

For many reasons, result (10.7) must only be treated as an estimate; however, this estimate clearly favors the use of the arguments developed for black holes in the observable Universe. We note that in the case of the Sun, the ratio of the physical radius $R_{\odot} = 695{,}500$ km to the gravitational radius $r_{g\odot} = 3$ km amounts to more than five orders of magnitude.

Identifying the holographic screen with the Hubble sphere of the radius $R = H^{-1}$ (which is valid for a flat universe) and applying the holographic principle, we try to reproduce the Friedmann equations by using neither Einstein's equations nor Newton's dynamics. A holographic screen of the area $A = 4\pi R^2$ carries the (maximum) information of $N = 4\pi R^2 / L_{\text{Pl}}^2$ bits. A change in the amount of information $\mathrm{d}N$ in the time $\mathrm{d}t$ related to expansion of the Universe $R \to R + \mathrm{d}R$ is expressed as

$$\mathrm{d}N = \frac{\mathrm{d}A}{L_{\text{Pl}}^2} = \frac{8\pi R}{L_{\text{Pl}}^2} \, \mathrm{d}R \,,$$

where $c = k_{\text{B}} = 1$. A change in the Hubble radius leads to a change in the Hawking temperature ($T = \hbar/(2\pi R)$),

$$\mathrm{d}T = -\frac{\hbar}{2\pi R^2} \, \mathrm{d}R \,.$$

From the equal-distribution law, it follows that

$$\mathrm{d}E = \frac{1}{2} N \, \mathrm{d}T + \frac{1}{2} T \, \mathrm{d}N = \frac{\hbar}{L_{\text{Pl}}^2} \, \mathrm{d}R = \frac{\mathrm{d}R}{G} \,, \tag{10.8}$$

where $L_{\text{Pl}}^2 = \hbar G / c^3$. The quantity $\mathrm{d}R$ can be represented as

$$\mathrm{d}R = -H\dot{H}R^3 \, \mathrm{d}t \,. \tag{10.9}$$

On the other hand, the energy flux through the Hubble sphere can be calculated if the energy–momentum tensor of the substance filling the Universe is known. Considering this substance to be an ideal liquid and taking into account that $T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu}$, we obtain

$$\mathrm{d}E = A(\rho + p) \, \mathrm{d}t \,. \tag{10.10}$$

Comparing (10.8) and (10.10), we find

$$\dot{H} = -\frac{1}{2M_{\text{Pl}}^2} (\rho + p) \,.$$

It is well known that the system of equations

$$\dot{H} = -\frac{1}{2M_{\text{Pl}}^2} (\rho + p) \,, \qquad \dot{\rho} + 3H(\rho + p) = 0 \tag{10.11}$$

is equivalent to system (3.1a), (3.1b). Our goal is therefore achieved. We note that back in 1995, a more general problem was resolved [92]: the Einstein equations were obtained based on thermodynamic arguments. This important result follows from the fact that entropy is proportional to the area of the horizon and from the assumption that relation (10.4) holds for each accelerating observer existing within his own causal horizon, if the temperature T is understood to be the Unruh temperature.

The derivation of the Friedmann equations from the holographic principle is, doubtless, an important result; however, it only reproduces the well-known. A natural question arises: Is it possible, on the basis of this principle, to develop a new approach to describing the dynamics of the Universe? If yes, then is it possible in the framework of the new approach to overcome irresolvable difficulties of the traditional approach? We start with the logical scheme underlying the holographic dynamics of the Universe.

The most important observation in cosmology after Hubble's discovery is the 1998 discovery of the accelerated expansion of the Universe. An enormous number of subsequent observations have confirmed this fact. However, attempts are still being made to disprove this conclusion, mainly on the basis of extremely artificial ad hoc assumptions that contradict the cosmological principle. If the accelerated expansion is considered a firmly established fact, this phenomenon must be interpreted invoking a minimum number of additional assumptions. At present, the most popular approach consists in including an additional term, called dark energy, in the Friedmann equation. Including DE leads to the fruitful standard cosmological model. But the nature of DE remains unknown. Holographic dynamics is an alternative method for describing the evolution of the Universe. Additional terms that are related to the contribution of surface terms and are neglected in the traditional approach are introduced into the GR equations in the framework of this method. Such an ideology corresponds to the spirit of the holographic principle based on the special role attributed to the surface. In this scenario, the concept of DE is totally absent. It is replaced by the entropic force acting on the horizon and leading to the accelerated expansion of the Universe. We discuss this scenario in greater detail.

In constructing any field theory, the standard approach consists in obtaining the equations of motion from a certain effective action, which corresponds to the energy scale characteristic of the phenomenon described by this theory.

GR is not an exception to this rule. The GR equations of motion are obtained by varying the action with respect to the dynamic variables involved in the effective action. A standard device that allows calculating integrals obtained in varying the action is integration by parts, which allows isolating total derivatives containing the variation of the dynamic variable. Here, the boundary conditions are assumed to be set: consequently, the variations of dynamic derivatives vanish and the total derivatives do not affect the equations of motion. In holographic physics, the main action occurs precisely on the surface, and the assumption concerning the fixed boundary conditions does not quite seem justified. Naturally, the conclusion arises that the contribution of the degrees of freedom of the holographic screen must be taken into account in the form of surface terms.

We show that considering boundary terms in the Einstein–Hilbert action is equivalent to introducing a nonzero energy–momentum tensor into the standard Einstein equations. The action for the gravitational field has the form

$$S_{\mathrm{EH}} = -\frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4 x\,\sqrt{-g}\,R\,. \tag{10.12}$$

Varying this action with respect to the metric $g_{\mu\nu}$ in a compact region Ω gives

$$\delta\int_\Omega \mathrm{d}^4x\,\sqrt{-g}\,R = \int_\Omega \mathrm{d}^4x\,\sqrt{-g}\Big[\left(g^{\mu\nu}\nabla^2\delta g_{\mu\nu} - \nabla^\mu\nabla^\nu\delta g_{\mu\nu}\right) - \left(R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R\right)\delta g_{\mu\nu}\Big]\,, \tag{10.13}$$

where $\nabla^2 = g^{\alpha\beta}\,\nabla_\alpha\nabla_\beta$. The total derivative in (10.13) can be represented in the form of a contribution of the boundary of the region $\partial\Omega$, where these contributions do not vanish. To obtain the Einstein equations from the principle of least action without boundary terms, it is necessary to add a certain functional to the Einstein–Hilbert action for compensating the contribution of total derivatives in (10.13). We let $S_{\mathrm{boundary}}[g]$ denote this functional. The total action then takes the form

$$S = -\frac{M_{\mathrm{Pl}}^2}{2}\int_\Omega \mathrm{d}^4x\,\sqrt{-g}\,R + S_{\mathrm{boundary}}[g] + S_{\mathrm{source}}\,, \tag{10.14}$$

where S_{source} includes possible sources of the gravitational field related to matter fields. The variation of the functional $S_{\mathrm{boundary}}[g]$ is then expressed as

$$\delta S_{\mathrm{boundary}}[g] = -\frac{M_{\mathrm{Pl}}^2}{2}\int_\Omega \mathrm{d}^4x\,\sqrt{-g}\,\left(g^{\mu\nu}\nabla^2\delta g_{\mu\nu} - \nabla^\mu\nabla^\nu\delta g_{\mu\nu}\right). \tag{10.15}$$

In the context of holographic physics, where the decisive role is played by the boundaries, the functional $S_{\mathrm{boundary}}[g]$ can be interpreted as the action of holographic DE. The Einstein equations involving the contribution of matter fields and boundary terms, have the form

$$R^{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}R = \frac{1}{M_{\mathrm{Pl}}^2}\left(T^{\mu\nu}_{\mathrm{source}} + T^{\mu\nu}_{\mathrm{boundary}}\right), \tag{10.16}$$

$$T^{\mu\nu}_{\mathrm{source}} = -\frac{2}{\sqrt{-g}}\,\frac{\delta S_{\mathrm{source}}}{\delta g_{\mu\nu}}\,,\quad T^{\mu\nu}_{\mathrm{boundary}} = -\frac{2}{\sqrt{-g}}\,\frac{\delta S_{\mathrm{boundary}}}{\delta g_{\mu\nu}}\,.$$

We note that in flat space, with $g_{\mu\nu} = 0$, the boundary action vanishes, $S_{\mathrm{boundary}}[g] = 0$, and the Einstein equations take the standard form. We also note that such a modification of the Einstein equations does not destroy the general structure of the equations, which implies the conservation of the structure of the Friedmann equations.

The appearance of additional terms in the Einstein equations requires a 'holographic correction' of the Friedmann equations. The simplest way to implement such a correction is to supplement the second Friedmann equation with an entropic force [93]. The structure of the entropic term can be restored from dimension arguments:

$$\frac{\ddot a}{a} = -\frac{1}{6M_{\mathrm{Pl}}^2}(\rho + 3p) + \frac{a_{\mathrm{e}}}{L_{\mathrm{b}}}\,, \tag{10.17}$$

where $a_{\mathrm{e}} = F_{\mathrm{e}}/m$ is the acceleration caused by the entropic force and L_{b} is the length scale determined by the position of the holographic screen. In the choice of the Hubble sphere as a holographic screen, this length scale coincides with the Hubble radius, $L_{\mathrm{b}} = R_{\mathrm{H}} = H^{-1}$. The entropic acceleration can be expressed via the temperature of the holographic screen T_{b}:

$$a_{\mathrm{e}} = \frac{F_{\mathrm{e}}}{m} = T_{\mathrm{b}}\,\frac{\Delta S}{\Delta x}\,\frac{1}{m} = 2\pi T_{\mathrm{b}}\,. \tag{10.18}$$

Is this scheme consistent with at least the main observations? As we have already seen, the absolute value of cosmological acceleration of the Hubble sphere in the SCM is $\dot V \simeq 4\times10^{-10}$ m s^{-2}. We estimate this quantity in the holographic approach.

It is natural to assign the Unruh temperature $T_{\mathrm{b}} = T_{\mathrm{U}}$ to the holographic screen. We relate this temperature to the Hawking radiation temperature

$$T_{\mathrm{H}} = \frac{\hbar}{8\pi k_{\mathrm{B}} G M} = \frac{\hbar g}{2\pi k_{\mathrm{B}}}\,, \tag{10.19}$$

where g is the acceleration of free fall onto the Hubble sphere. Comparing (10.19) with the expression for the Unruh temperature

$$T_{\mathrm{U}} = \frac{\hbar a}{2\pi k_{\mathrm{B}}}\,, \tag{10.20}$$

we obtain

$$T_{\mathrm{H}} = T_{\mathrm{U}}(a = g)\,. \tag{10.21}$$

Hence, the Unruh radiation temperature coincides with the Hawking radiation temperature, but it depends not on the surface gravity (the free-fall acceleration) but on the acceleration of the reference frame. Owing to the equivalence principle, the free-fall acceleration on the Hubble sphere is equal to the acceleration of the reference frame, which we can find. Considering the Hubble sphere as an analog of the horizon for black-hole events, we can represent (10.19) as

$$T_{\mathrm{H}} = \frac{\hbar}{k_{\mathrm{B}}}\,\frac{H}{2\pi} \sim 3\times10^{-30}\ \mathrm{K}\,. \tag{10.22}$$

Comparing (10.22) with (10.20), we find that the acceleration of the Hubble sphere is

$$a_{\mathrm{H}} = H \simeq 10^{-9}\ \mathrm{m\ s^{-2}}\,. \tag{10.23}$$

This result is in good qualitative agreement with the result obtained in the SCM.

Naturally, the merit of any new model consists not so much in reproducing the results of preceding models (this is just the mandatory program), as in resolving the problems that were irresolvable for the preceding models. From this standpoint, the holographic dynamics is promising because it gives rise to hope to resolve the two main issues of the SCM: the cosmological constant and coincidence problem. We note that the entropic interpretation of gravity, as well as the holographic principle, gave rise to great interest in the physical community; however, at present, this approach has not been universally accepted. For example, in [94, 95], contradictions are presented that are related to the entropic interpretation of gravity.

We start with the first problem. The cosmological constant problem consists in the enormous difference (120 orders of magnitude) between the observed DE density in the form of the cosmological constant and its 'expected' value. The expectations are based on quite natural assumptions concerning the cutoff parameter of the integral that represents the density of zero-point vacuum oscillations. The holographic principle allows replacing 'natural assumptions' by more rigorous quantitative estimates.

10.1 Universe with holographic dark matter

In any effective quantum field theory defined in a spatial region of a characteristic size L and using an ultraviolet cutoff Λ, the entropy of the system has the form $S \propto \Lambda^3 L^3$. For example, fermions situated at the nodes of a spatial lattice of characteristic size L and period Λ^{-1} are in one of the $2^{(L\Lambda)^3}$ states. Consequently, the entropy of such a system is $S \propto \Lambda^3 L^3$. In accordance with the hololgraphic principle, this quantity should satisfy the inequality [96]

$$L^3 \Lambda^3 \leqslant S_{\rm BH} \equiv \frac{1}{4}\,\frac{A_{\rm BH}}{l_{\rm Pl}^2} = \pi L^2 M_{\rm Pl}^2\,, \tag{10.24}$$

where $S_{\rm BH}$ is the entropy of a black hole and $A_{\rm BH}$ is the surface area of a black hole event, which in the simplest case coincides with the surface of a sphere of the radius L. Relation (10.24) shows that the value of the infrared (IR) cutoff cannot be chosen independently of the value of the ultraviolet (UV) cutoff.

We have obtained an important result [96]: in the framework of holographic dynamics, the value of the IR cutoff is strictly related to the value of the UV cutoff. In other words, physics at small UV scales depends on the physical parameters at large IR scales. For instance, when inequality (10.24) becomes an exact equality,

$$L \sim \Lambda^{-3} M_{\rm Pl}^2\,. \tag{10.25}$$

Effective field theories with UV cutoff (10.25) necessarily involve numerous states whose gravitational radius exceeds the size of the region within which the theory is defined. In other words, for any cutoff parameter, a sufficiently large volume exists in which the entropy in quantum field theory exceeds the Bekenstein limit. To verify this, we note that the effective quantum field theory is usually required to be capable of describing the system at the temperature $T \leqslant \Lambda$. For $T \gg 1/L$, this system has the thermal energy $M \sim L^3 T^4$ and entropy $S \sim L^3 T^3$. Condition (10.24) is satisfied for $T \leqslant (M_{\rm Pl}^2/L)^{1/3}$, which corresponds to the gravitational radius $r_{\rm g} \sim L(LM_{\rm Pl}) \gg L$. To overcome this difficulty, an even stricter constant is proposed in [96] for the IR cutoff, $L \sim \Lambda^{-1}$, which excludes all states that are within the limits of their gravitational radii. Taking into account that

$$\rho_{\rm vac} \approx \frac{\Lambda^4}{16\pi^2}\,, \tag{10.26}$$

we can rewrite condition (10.24) as

$$L^3 \rho_\Lambda \leqslant L M_{\rm Pl}^2 \equiv 2M_{\rm BH}\,, \tag{10.27}$$

where $M_{\rm BH}$ is the mass of the black hole of the gravitational radius L. By the order of magnitude, the total energy contained in a region of size L does not therefore exceed the mass of the black hole of the same size. This consequence is surprisingly in agreement with expression (10.7) under the condition that the IR cutoff scale is identical to the Hubble radius H^{-1}; moreover, such a choice must be logical in the context of cosmology.

In the cosmological context we are interested in, if the total energy contained in a region of size L is postulated to not exceed the mass of the black hole of the same size, i.e.,

$$L^3 \rho_\Lambda \leqslant M_{\rm BH}\,, \tag{10.28}$$

we reproduce the relation between small and large scales in a natural way. If inequality (10.28) were violated, the Universe would only be composed of black holes. Applying this relation to the Universe as a whole, it is natural to identify the IR scale with the Hubble radius H^{-1}. For the upper boundary of the energy density, we then obtain

$$\rho_\Lambda \sim L^{-2} M_{\rm Pl}^2 \sim H^2 M_{\rm Pl}^2\,. \tag{10.29}$$

The quantity ρ_Λ is conventionally called the holographic DE. We let $\rho_{\rm DE}$ denote the DE density in what follows. Taking into account that

$$M_{\rm Pl} \simeq 1.2 \times 10^{19}\ {\rm GeV}\,, \qquad H_0 \simeq 1.6 \times 10^{-42}\ {\rm GeV}\,,$$

we find

$$\rho_{\rm DE} \sim 10^{-46}\ {\rm GeV}^4\,. \tag{10.30}$$

The last quantity is in good agreement with the observed value of DE density $\rho_{\rm DE} \simeq 3 \times 10^{-47}\ {\rm GeV}^4$. Therefore, in the framework of holographic dynamics, no problem with the cosmological constant exists.

Although the value obtained for the DE density is correct, a problem with the equation of state arises in choosing the Hubble radius as the IR scale: in this case, the holographic DE does not account for the accelerating expansion of the Universe [97], as can be readily verified using the following simple arguments.

We consider a universe consisting of holographic DE with the density given by relation (10.29), and of matter. Then $\rho = \rho_{\rm DE} + \rho_{\rm m}$. From the first Friedmann equation, it follows that $\rho \propto H^2$. If $\rho_{\rm DE} \propto H^2$, the dynamic behavior of the holographic DE and the behavior of DM coincide; consequently, no expansion is possible. If matter was dominant at the beginning of the Universe, the DE introduced with the aid of (10.29) is actually a spectator solution, because it reproduces the dynamics of the dominant component. The existence of spectator solutions in the framework of holographic models gives hope that the coincidence problem will be resolved, but it contradicts the observed dynamics of the Universe.

For the holographic DE to account for the accelerating expansion of the Universe, we try to use spatial scales differing from the Hubble radius as the IR cutoff. The first thing that suggests itself is to replace the Hubble radius by the particle horizon R_{p}:

$$R_{\text{p}} = a \int_0^t \frac{\text{d}t}{a} = a \int_0^a \frac{\text{d}a}{Ha^2} \,. \tag{10.31}$$

Regretfully, such a replacement does not yield the desired result. To verify this, we once more start with the model of a matter-dominated Universe. Owing to the causality principle, the influence of gravity cannot extend to regions separated by a distance greater than the causal horizon. Consequently, the vacuum energy density entering the Friedmann equations cannot be an arbitrary function of L. In the matter-dominated epoch, the causal horizon (horizon of particles) increases as $a^{3/2}$; therefore, $\rho_{\text{DE}}(L) \sim a^{-3}$. Because $\rho(t) \sim a(t)^{-3(1+w)}$, it immediately follows that for the holographic DE with the IR cutoff on the horizon of particles, $w = 0$, and we once more arrive at an explicit contradiction with observations.

We represent the holographic DE density as [97]

$$\rho_{\text{DE}} = 3b^2 M_{\text{Pl}}^2 L^{-2} \,. \tag{10.32}$$

The coefficient $3b^2$ $(b > 0)$ is introduced for convenience, and we understand M_{Pl} in what follows to be the reduced Planck mass, $M_{\text{Pl}}^{-2} = 8\pi G$. In the epoch when this type of holographic DE is dominant, the first Friedmann equation, using (10.32), can be rewritten as

$$R_{\text{p}} H = b \,; \tag{10.33}$$

this immediately implies that

$$\frac{1}{Ha^2} = b \frac{\text{d}}{\text{d}a} \left(\frac{1}{Ha} \right), \tag{10.34}$$

which allows finding the dependence of this type of holographic DE on the scale factor

$$\rho_{\text{DE}} = 3\alpha^2 M_{\text{Pl}}^2 a^{-2(1+1/b)} \tag{10.35}$$

and obtaining the parameter of the equation of state

$$w = -\frac{1}{3} + \frac{2}{3b} > -\frac{1}{3} \,. \tag{10.36}$$

We next verify that the problem revealed pertains to the case of a universe with dominant holographic DE. Replacing $L \to R_{\text{h}}$ in (10.32), we obtain the first Friedmann equation in the form

$$R_{\text{h}} H = b \,, \tag{10.37}$$

which again leads to relation (10.34), whence

$$H^{-1} = \alpha a^{1+1/b} \,, \tag{10.38}$$

where α is a constant. Because $R_{\text{h}} = bH^{-1}$ and $\rho_\Lambda = 3b^2 M_{\text{Pl}}^2 R_{\text{h}}^{-2}$,

$$\rho_{\text{DE}} = 3\alpha^2 M_{\text{Pl}}^2 a^{-2(1+1/b)} \,. \tag{10.39}$$

How can we find the parameter w in the equation of state if the dependence $\rho = f(a)$ is known? A possible strategy is as follows. We use the conservation equation

$$\dot{\rho} + 3H\rho(1+w) = 0 \,. \tag{10.40}$$

In the case $\rho = f(a)$, it takes the form

$$f'(a)\dot{a} + 3Hf(a)(1+w) = 0 \,, \tag{10.41}$$

whence

$$w = -\frac{1}{3} \frac{f'(a)}{f(a)} a - 1 = -\frac{1}{3} \frac{\text{d} \ln f(a)}{\text{d}a} a - 1 \,. \tag{10.42}$$

It can be readily verified that in the known cases of matter $\rho \propto a^{-3}$, radiation $\rho \propto a^{-4}$, and the cosmological constant $\rho = \text{const}$, we obtain correct values: $w = 0$, $1/3$, and -1, respectively. Equation (10.42) implies the expression for w in (10.36). This type of energy does not therefore merit the title of 'dark energy', because it does not achieve the main goal of dark energy, accounting for the accelerating expansion of the Universe. The roots of the difficulties are related to the derivative $\text{d}/\text{d}a\,(H^{-1}/a) > 0$, which follows from (10.34). To obtain an accelerating Universe, it is necessary to 'slow down' the growth of the IR cutoff scale. It turns out [98] that this can be achieved by substituting the event horizon for the particle horizon. We recall that the size of the spatial region in which the signals emitted at a time moment t certainly reach an observer at rest in some distant future is

$$R_{\text{e}} = a(t) \int_t^\infty \frac{\text{d}t'}{a(t')} \,. \tag{10.43}$$

Once again, assuming DE to be dominant and solving the first Friedmann equation as

$$\int_a^\infty \frac{\text{d}a}{Ha^2} = \frac{b}{Ha} \,, \tag{10.44}$$

we find

$$\rho_{\text{DE}} = 3b^2 M_{\text{Pl}}^2 R_{\text{e}}^{-2} = 3\alpha^2 M_{\text{Pl}}^2 a^{-2(1-1/b)} \,, \tag{10.45}$$

or $w = -1/3 - 2/3b < -1/3$. We have obtained a component that behaves like DE, i.e., accounts for the accelerated expansion of the Universe. If $b = 1$, this component behaves like the cosmological constant. If $b < 1$, then $w < -1$. In the traditional approach, this value of w corresponds to the phantom model.

In the foregoing, we have examined the properties of holographic DE in two limit cases: dominant matter and dominant DE. We next discuss the general situation, i.e., the dynamics of the Universe when the ratio between the densities of both components is arbitrary [99]. For simplicity, we restrict ourselves to the case of a flat universe and use the radius of the event horizon R_{e} as the IR cutoff scale. Introducing the relative density of the holographic DE as

$\Omega_{\rm DE} = \rho_{\rm DE}/\rho_{\rm cr}$ ($\rho_{\rm cr} = 3M_{\rm Pl}^2 H^2$), we represent (10.32) with $L = R_{\rm e}$ as

$$HR_{\rm e} = \frac{b}{\sqrt{\Omega_{\rm DE}}} \,. \qquad (10.46)$$

Naturally, if $\Omega_{\rm DE} = 1$ and the substitution $R_{\rm e} \to R_{\rm h}$ is performed, Eqn (10.46) transforms into (10.37). Taking the time derivatives of both sides of (10.43), we obtain

$$\dot{R}_{\rm e} = HR_{\rm e} - 1 = \frac{b}{\sqrt{\Omega_{\rm DE}}} - 1 \,. \qquad (10.47)$$

It follows from definition (10.32) that

$$\frac{{\rm d}\rho_{\rm DE}}{{\rm d}t} = -6b^2 M_{\rm Pl}^2 R_{\rm e}^{-3} \dot{R}_{\rm e} = -2H\left(1 - \frac{\sqrt{\Omega_{\rm DE}}}{b}\right)\rho_{\rm DE} \,. \qquad (10.48)$$

By the energy conservation law,

$$\frac{\rm d}{{\rm d}a}(a^3 \rho_{\rm DE}) = -3a^2 p_{\rm DE} \,, \qquad (10.49)$$

$$p_{\rm DE} = -\frac{1}{3}\frac{{\rm d}\rho_{\rm DE}}{{\rm d}\ln a} - \rho_{\rm DE} \,. \qquad (10.50)$$

Consequently, the equation of state has the form

$$w_{\rm DE} = \frac{p_{\rm DE}}{\rho_{\rm DE}} = -\frac{1}{3}\frac{{\rm d}\ln \rho_{\rm DE}}{{\rm d}\ln a} - 1 = -\frac{1}{3}\left(1 + \frac{2}{b}\sqrt{\Omega_{\rm DE}}\right) . \qquad (10.51)$$

We used the relation ${\rm d}\ln a = H\,{\rm d}t$. This result can be obtained without calculating the pressure p_Λ with the aid of relation (10.41). The expression obtained for $w_{\rm DE}$ is a consequence of the definition of holographic DE in (10.32); therefore, it does not depend on the other energy components. It follows from this result that $w_{\rm DE} \simeq -1/3$ if other energy components are dominant and $w_{\rm DE} = -(1/3)(1 + 2/b)$ if DE is dominant. The last result coincides with expression (10.42) obtained for a universe filled only with holographic DE.

At first sight, we have reached our objective. The holographic DE with density (10.32), on the one hand, provides agreement between the observed density and the theoretical estimate and, on the other hand, leads to an equation of state that generates accelerating expansion of the Universe. But holographic DE with the event horizon as the IR cutoff scale still does not resolve all the problems related to the causality principle: according to the definition of the event horizon, the dynamics of holographic DE depends on the future evolution of the scale factor. It is difficult to consider such a dependence to be causal.

In search of ways out of the impasse, we once again turn to the cosmological constant problem, which consists in the abyss between theoretical estimates and the observed value of DE density. The simplest version of DE is the cosmological constant, which is related to the vacuum mean of quantized fields and can be measured with the aid of gravitational experiments. Consequently, the cosmological constant problem is a quantum gravity problem. Although no complete theory of quantum gravity exists, unification of quantum mechanics and GR may shed light on this problem.

From the first days of quantum mechanics, the concept of measurements [real and thought (gedankenexperiment)] played a fundamental role in understanding the physical reality. GR asserts that the laws of classical physics can be verified with unlimited accuracy. The relation revealed above between the macroscopic (IR) and microscopic scales dictates the necessity of a more profound analysis of the measurement process. The uncertainty relation, together with GR, produces the fundamental space–time scale — the Planck length $L_{\rm Pl} \sim 10^{-33}$ cm.

The existence of a fundamental length influences the process of measurement in a critical manner [100]. We assume that a fundamental length $L_{\rm f}$ exists. Because the space–time coordinate system must be physically reasonable, it has to be attached to physical bodies. Therefore, postulating the fundamental length is equivalent to imposing restrictions on the realizability of precise coordinate systems. In terms of experiments with light signals, this means, for example, that the time required for a light signal to travel from body A to body B and back, measured by clocks in the system of A, is subject to uncontrollable fluctuations. Fluctuations in experiments with light signals should be considered indications of fluctuations of the metric, i.e., the gravitational field. Therefore, postulating the existence of a fundamental length is equivalent to postulating fluctuations of the gravitational field.

A direct consequence of the existence of quantum fluctuations of the metric [101–104] is the following conclusion, related to the problem of measuring distances in Minkowski space: the distance t (we recall that we use the system in which $c = \hbar = 1$, whence $L_{\rm Pl} = t_{\rm Pl} = M_{\rm Pl}^{-1}$) cannot be measured with an accuracy exceeding [101]

$$\delta t = \beta t_{\rm Pl}^{2/3} t^{1/3} \,, \qquad (10.52)$$

where β is a coefficient of the order of unity. Following [96], we can consider result (10.52) as a relation between the UV and IR scales in the framework of the effective quantum field theory satisfying the entropic peculiarities of black holes. Indeed, rewriting relation (10.29) in terms of length and performing the substitution $\Lambda \to \delta t$, we reproduce (10.52), but in the holographic interpretation.

Relation (10.52), together with the quantum mechanical energy–time uncertainty relation, allows estimating the energy density of quantum fluctuations of Minkowski space–time. In accordance with (10.52), we can regard a region of volume t^3 as composed of cells of volume $\delta t^3 \sim t_{\rm Pl}^2 t$. Consequently, each such cell represents a minimally detectable unit of space–time for the scale t. If the age of the region chosen is t, its existence, in accordance with the time–energy uncertainty principle, cannot be realized with an energy less than $\sim t^{-1}$. We thus arrive at the conclusion: if the lifetime (age) of a certain spatial region of linear size t is equal to t, there exists a minimal cell of volume δt^3 whose energy cannot be less than

$$E_{\delta t^3} \sim t^{-1} \,. \qquad (10.53)$$

It immediately follows from (10.52) and (10.53) that in accordance with the energy–time uncertainty principle, the energy density of metric (quantum!) fluctuations in Minkowski space is [102, 104]

$$\rho_{\rm q} \sim \frac{E_{\delta t^3}}{\delta t^3} \sim \frac{1}{t_{\rm Pl}^2 t^2} \,. \qquad (10.54)$$

It is essential that the dynamic behavior of the density of metric fluctuations (10.54) coincides with that of holographic

DE introduced in (10.29) and (10.32), although the derivations of these expressions are based on absolutely different physical principles. The holographic DE density was obtained from entropic constraints (the holographic principle), while the energy density of metric fluctuations in Minkowski space is only related to their quantum nature, namely, to the uncertainty principle.

Relation (10.54) allows introducing an alternative model of holographic DE [105–108], in which the age of the universe T is used as the IR scale. In such a model,

$$\rho_{\mathrm{q}} = \frac{3n^2 M_{\mathrm{Pl}}^2}{T^2}\,, \qquad (10.55)$$

where n is a free parameter of the model, and the numerical coefficient is introduced for convenience. Energy density (10.55) with $T \sim H_0^{-1}$, where H_0 is the current value of the Hubble parameter, leads to the observed value of the DE density for a coefficient n of the order of unity. In the SCM, where $H_0 \simeq 72$ km (s Mpc)$^{-1}$, $\Omega_{\mathrm{DE}} \simeq 0.73$, and $T \simeq 13.7$ billion years, we find that $n \simeq 1.15$.

We now answer the key question: does the holographic energy density in form (10.55) result in accelerating expansion of the Universe? For simplicity, we consider a universe in which other energy components are absent. In this case, the first Friedmann equation becomes

$$H^2 = \frac{\rho_{\mathrm{q}}}{3M_{\mathrm{Pl}}^2}\,. \qquad (10.56)$$

The age of the Universe T involved in (10.55) is related to the scale factor as

$$T = \int_0^a \frac{\mathrm{d}a'}{Ha'}\,. \qquad (10.57)$$

The solution of Eqn (10.56) with energy density (10.55) has the form

$$a = \left[n(H_0 t + \alpha)\right]^n. \qquad (10.58)$$

The integration constant can be determined from the condition $a_0 = 1$. Calculating the second derivative of the scale factor, we can show that the accelerating expansion of the Universe is ensured by the condition $n > 1$. We note that the value $n \simeq 1.15$ obtained above and leading to the observed value of DE density satisfies this condition. The same condition can be obtained from the DE conservation equation, which is readily reduced to the form

$$w_{\mathrm{q}} = -1 - \frac{\dot{\rho}}{3H\rho}\,. \qquad (10.59)$$

Using (10.55) and (10.56), we represent (10.59) as

$$w_{\mathrm{q}} = -1 + \frac{2}{3n}\,. \qquad (10.60)$$

As has been repeatedly noted, accelerated expansion of the Universe requires $w < -1/3$, which is equivalent to the condition $n > 1$ obtained above.

As in the preceding model of holographic DE, we now pass to the more general case: besides DE, the universe contains matter with a density ρ_{m}. This universe is described by the Friedmann equation

$$H^2 = \frac{1}{3M_{\mathrm{Pl}}^2}(\rho_{\mathrm{q}} + \rho_{\mathrm{m}})\,. \qquad (10.61)$$

Passing to the relative densities

$$\Omega_{\mathrm{m}} = \frac{\rho_{\mathrm{m}}}{3H^2 M_{\mathrm{Pl}}^2}\,, \qquad \Omega_{\mathrm{q}} = \frac{\rho_{\mathrm{q}}}{3H^2 M_{\mathrm{Pl}}^2} = \frac{n^2}{T^2 H^2}\,,$$

we represent Friedmann equation (10.61) in the form

$$\frac{\mathrm{d}\Omega_{\mathrm{q}}}{\mathrm{d}\ln a} = \left(3 - \frac{2}{n}\sqrt{\Omega_{\mathrm{q}}}\right)(1 - \Omega_{\mathrm{q}})\Omega_{\mathrm{q}}\,. \qquad (10.62)$$

Equation (10.62) can be integrated:

$$\frac{1}{n}\ln a + c_0 = -\frac{1}{3n-2}\ln\left(1 - \sqrt{\Omega_{\mathrm{q}}}\right) - \frac{1}{3n+2}\ln\left(1 + \sqrt{\Omega_{\mathrm{q}}}\right) + \frac{1}{3n}\ln\Omega_{\mathrm{q}} + \frac{8}{3n(9n^2-4)}\ln\left(\frac{3n}{2} - \sqrt{\Omega_{\mathrm{q}}}\right). \qquad (10.63)$$

The integration constant can be determined from the condition that $\Omega_{\mathrm{q}} \simeq 0.73$ at $a = 1$.

We analyze the dynamics of the relative DE density is two cases: for dominant matter and for dominant DE. In the first case, it follows from (19.62) that

$$\Omega_{\mathrm{q}} \approx c_1 a^3\,. \qquad (10.64)$$

The rapid increase in the relative part of DE, independent of n, leads to the DE-dominated epoch. In this period,

$$\Omega_{\mathrm{q}} \approx 1 - c_2 a^{-(3n-2)/n}\,. \qquad (10.65)$$

The equation of state for DE can be obtained using relation (10.59):

$$w_{\mathrm{q}} = -1 + \frac{2}{3n}\sqrt{\Omega_{\mathrm{q}}}\,. \qquad (10.66)$$

In the early Universe, in the matter-dominated epoch, $\Omega_{\mathrm{q}} \to 0$ and $w_{\mathrm{q}} \to -1$, i.e., the holographic DE in the model behaves in this epoch like the cosmological constant. In the later DE-dominated epoch, when $\Omega_{\mathrm{q}} \to 1$, equation of state (10.66) naturally transforms into relation (10.60). We note that the destiny of a universe filled with matter and holographic DE with density (10.55) is the accelerated expansion according to a power-law dependence of the scale factor on time in (10.58). Thus, the DE holographic model in which the age of the Universe is chosen to be the IR cutoff scale permits the following:

(1) obtaining the observed value of the DE density;

(2) accounting for the accelerating expansion of the Universe at the later stages of its evolution;

(3) resolving contradictions related to the causality principle.

However, we must not jump to the final conclusions. On the one hand, the first success in applying the holographic principle gave rise to hopes of constructing a correct description of the dynamics of the Universe on its basis, in which a whole range of problems inherent to the traditional approach would be absent. On the other hand, this very success became a source of optimism that, from our standpoint, is unjustified. We believe that titles such as "solution to the problem of dark energy" [109] serve as a manifestation of some sort of 'holographic extremism'. We must bear in mind the phrase by Winston Churchill: "Success is the ability to go from one failure to another with no loss of enthusiasm." Holographic dynamics is one of the youngest sectors of theoretical physics. Along this line of research, physicists have not experienced sufficient failure to aspire to ultimate success.

11. Transition acceleration

Unlike fundamental theories, physical models only reflect the current state of our understanding of a process or phenomenon for the description of which they were developed. The efficiency of a model is to a significant extent determined by its flexibility, i.e., its ability to update when new information appears. Precisely for this reason, the evolution of any broadly applied model is accompanied by numerous generalizations aimed at resolving conceptual problems, as well as a description of the ever increasing number of observations. In the case of the SCM, these generalizations can be divided into two main classes. The first includes replacing the cosmological constant with more complicated dynamic forms of DE, for which the possibility of their interaction with DM must be taken into account. Generalizations pertaining to the second class are of a more radical character and claim to introduce one more substitution for the cosmological paradigm. The ultimate goal of these generalizations (explicit or implicit) consists in the renunciation of dark components by means of modifying Einstein's equations and, as a consequence, Newton's equations. The generalizations of both the first and the second classes can be demonstrated using the example of a phenomenon that has been termed 'transition acceleration'.

As we saw in Section 5, a characteristic feature of the dependence of the deceleration parameter q on the red shift z in the SCM is that it monotonically tends to its limit value $q(z) = -1$ as $z \to -1$. Physically, this means that when DE became the dominant component (at $z \sim 1$), the Universe in the SCM was doomed to experience eternal accelerating expansion.

In what follows, we consider some cosmological models involving dynamic forms of DE that lead to transition acceleration, and we also discuss what the observational data say about the modern rate of expansion of the Universe.

11.1 Theoretical premises

Barrow [110] was among the first to indicate that transition acceleration is possible in principle. He showed that within quite sound scenarios that allow explaining the current accelerated expansion of the Universe, the possibility was not excluded of a return to the era of dominant nonrelativistic matter and, consequently, to decelerated expansion. Therefore, the transition to accelerating expansion does not necessarily mean eternal accelerating expansion.

To demonstrate this, we follow Ref. [110] and consider a homogeneous and isotropic flat universe filled with nonrelativistic matter and a scalar field with a potential $V(\varphi)$. We consider the scalar field potential

$$V(\varphi) = V_{\mathrm{p}}(\varphi) \exp(-\lambda\varphi) . \tag{11.1}$$

In some versions of low-energy limits of string theory, $V_{\mathrm{p}}(\varphi)$ is a polynomial. An exponential potential with a small minimum was first proposed in Ref. [111]. This minimum was provided on the exponential background with the aid of a polynomial $V_{\mathrm{p}}(\varphi)$ of the simplest form:

$$V_{\mathrm{p}}(\varphi) = (\varphi - \varphi_0)^2 + A . \tag{11.2}$$

In this case, the potential takes the form

$$V(\varphi) = \exp(-\lambda\varphi)\big(A + (\varphi - \varphi_0)^2\big) . \tag{11.3}$$

For potential (11.3) to be consistent with string theory potentials, the constant parameters A and φ_0 must be of the order of unity (in Planck units). In these models of quintessence, accelerated expansion of the Universe at the later stages of evolution is realized without any initial fitting of parameters, i.e., no fine tuning problem exists.

The accelerated expansion starts when the field rolls down with $\varphi = \varphi_0 + [1 \pm (1 - \lambda^2 A)^{1/2}]/\lambda$ to a local minimum, which appears owing to the quadratic factor in (11.3), where $\lambda^2 A \leqslant 1$. While the field is in the state of a false vacuum, its kinetic energy is negligible ($\varphi \approx \mathrm{const}$), and the subsequent dominance of ρ_φ is due to the nearly constant potential energy, which triggers the period of accelerated expansion of the Universe that never finishes.

In Ref. [110], this scenario was shown to be neither the only one possible nor the most probable one.

Transition vacuum dominance appears in two cases. When $A\lambda^2 < 1$, the field φ reaches the local minimum with a kinetic energy sufficient for overcoming the local minimum, and it continues to roll down the exponential part of the potential to the region where $\varphi \gg \varphi_0$. The kinetic energy is therefore determined not by the initial conditions but by the parameters of the potential. Transition acceleration also originates when the condition $A\lambda^2 > 1$ is satisfied. Because the increase in A is proportional to λ^{-2}, the potential loses its local minimum and flattens out near the inflection point. This suffices to cause the period of accelerated expansion of the Universe; however, the field never stops rolling down the potential, and after a certain time, matter again starts dominating in the Universe with the dependence $a(t) \propto t^{2/3}$.

Thus, the Universe stops experiencing accelerating expansion and returns to the mode of decelerating expansion. Moreover, as shown in [110], for the well-grounded family of potentials of form (11.2), transition acceleration is more probable than the eternal accelerating expansion.

11.2 Models involving transition acceleration

To demonstrate explicitly that transition acceleration is an essential feature of various cosmological models, we briefly consider some of them.

In [110], Barrow considered a model of the Universe in which the role of DE was played by a scalar field in potential (11.3). As an example, we discuss some cosmological models where decelerating expansion replaces accelerating expansion.

11.2.1 Scalar field, multidimensional cosmology, and transition acceleration. It is shown in Ref. [112] that transition acceleration can also be realized in an exponential potential; moreover, the case of a multidimensional (d-dimensional) cosmological model is considered. The action in this model has the form

$$S = \int \mathrm{d}^d x \sqrt{-g} \left[-\frac{R}{2M_{\mathrm{Pl}\,d}^2} + \frac{1}{2} g^{\mu\nu} \partial_\mu\varphi\, \partial_\nu\varphi - V_0 \exp(-\lambda\varphi) \right] , \tag{11.4}$$

where $M^2_{\mathrm{Pl}\,d} = 8\pi G_d$, G_d is the gravitational constant in d-dimensional space, $V_0 > 0$, and $\lambda > 0$ (the case $\lambda < 0$ is related to $\lambda > 0$ by the substitution $\varphi \to -\varphi$). Following [112], we consider the flat FRW metric for a flat universe ($k = 0$),

$$ds^2 = dt^2 - a^2(t)\, dx^i\, dx^i\,, \qquad i = 1, \ldots, d-1\,. \tag{11.5}$$

In this case, the action becomes

$$S = \int d^d x \left[(d-1)(d-2)a^{d-3}\dot{a}^2 + a^{d-1}\left(\frac{1}{2}\,\dot{\varphi}^2 - V(\varphi)\right)\right]. \tag{11.6}$$

In variables u, v, we have

$$\varphi = M_{\mathrm{Pl}\,d}\sqrt{\frac{d-2}{d-1}}\,(v-u)\,, \qquad a^{d-1} = \exp(v+u)\,, \tag{11.7}$$

whence

$$S = \int d^{d-1}x\, dt\, \exp(u+v) \times \left\{\frac{2(d-2)}{M^2_{\mathrm{Pl}\,d}(d-1)}\,\dot{u}\dot{v} - V_0 \exp\left[-2\alpha(v-u)\right]\right\}, \qquad \alpha \equiv \frac{M_{\mathrm{Pl}\,d}}{2}\sqrt{\frac{d-2}{d-1}}\,\lambda\,. \tag{11.8}$$

We now pass to a new time variable τ:

$$\frac{d\tau}{dt} = M_{\mathrm{Pl}\,d}\sqrt{\frac{(d-1)V_0}{2(d-2)}}\,\exp\left[\alpha(u-v)\right], \tag{11.9}$$

whence

$$S = \frac{1}{M_{\mathrm{Pl}\,d}}\sqrt{\frac{2(d-2)V_0}{d-1}} \times \int d^{d-1}x\, d\tau\, \exp(u+v)\exp\left[\alpha(u-v)\right](u'v'-1)\,. \tag{11.10}$$

Using (11.7)–(11.9), it is possible to show [112] that for $\alpha < 1$, the general solution of (11.10) has the form

$$\begin{aligned} ds^2 &= \frac{2(d-2)}{M^2_{\mathrm{Pl}\,d}(d-1)V_0}\exp\left(\frac{4\alpha^2\tau}{w}\right) \times \frac{\left[1 + m\exp(-2w\tau)\right]^{2\alpha/(1-\alpha)}}{\left[1 - m\exp(-2w\tau)\right]^{2\alpha/(1+\alpha)}}\, d\tau^2 \\ &\quad - \exp\left(\frac{4\tau}{sw}\right)\left[1 + m\exp(-2w\tau)\right]^{2/[s(1-\alpha)]} \times \left[1 - m\exp(-2w\tau)\right]^{2/[s(1+\alpha)]}\, dx^i\, dx^i\,, \\ \varphi &= M_{\mathrm{Pl}\,d}\sqrt{\frac{d-2}{d-1}}\left\{\frac{2\alpha\tau}{w} - \frac{1}{1+\alpha}\log\left[1 - m\exp(-2w\tau)\right] + \frac{1}{1-\alpha}\log\left[1 + m\exp(-2w\tau)\right]\right\}, \end{aligned} \tag{11.11}$$

where $s = d-1$ and m is an integration constant. The asymptotic forms of this solution are

$$\begin{aligned} &a \sim t^{1/(d-1)}\,, \qquad \varphi = -M_{\mathrm{Pl}\,d}\sqrt{\frac{d-2}{d-1}}\,\log t \quad \text{for } t \approx 0\,, \\ &a \sim t^{4/[(d-2)M^2_{\mathrm{Pl}\,d}\lambda^2]}\,, \qquad \varphi = \frac{2}{\lambda}\log t \quad \text{for } t \gg 1\,. \end{aligned} \tag{11.12}$$

At the first stages of evolution, the equation of state is extremely rigid, $p = \rho$; at the later stages, it is not difficult to obtain

$$p = \omega\rho\,, \qquad \omega = \frac{d-2}{2(d-1)}\,M^2_{\mathrm{Pl}\,d}\lambda^2 - 1\,. \tag{11.13}$$

In accordance with Eqn (11.12), accelerating expansion continues eternally if $\lambda < 2/(M_{\mathrm{Pl}\,d}\sqrt{d-2})$. We show that if the condition

$$\frac{2}{M_{\mathrm{Pl}\,d}\sqrt{d-2}} < \lambda < 2M^{-1}_{\mathrm{Pl}\,d}\sqrt{\frac{d-1}{d-2}}$$

is satisfied, then solution (11.11) with $m > 0$ has a period of transition acceleration. We find $da/dt = (da/d\tau)(d\tau/dt)$ using (11.11); it can be shown that $\dot{a}$ is proportional to the positive definite quantities m and τ. Setting $|m| = 1$, which can be achieved by a shift along the τ axis, we find

$$\ddot{a} = -(\text{positive})\left\{\left[(d-1)\alpha^2 - 1\right]Z^2 - 2(d-2)\,\mathrm{sign}\,(m)\,\alpha Z + d - 1 - \alpha^2\right\}, \tag{11.14}$$

where $Z \equiv \cosh(2w\tau)$. If $(d-1)\alpha^2 < 1$ [which corresponds to $\lambda < 2/(M_{\mathrm{Pl}\,d}\sqrt{d-2})$], we obtain the eternal accelerating expansion for any m because the first summand dominates at later stages of evolution. We note that this solution is an attractor. If $(d-1)\alpha^2 > 1$ at late times, the solution always corresponds to decelerating expansion of the Universe. When $(d-1)\alpha^2 > 1$ and $m < 0$, the right-hand side of (11.14) is negative definite, which corresponds to decelerating expansion of the Universe during the entire period of evolution. Finally, if $(d-1)\alpha^2 > 1$ and $m > 0$, the solution always provides a period of transition acceleration [113]. Indeed, the equation $\ddot{a} = 0$ has two roots

$$Z_\pm = \frac{(d-2)\alpha \pm \sqrt{d-1}\,(1-\alpha^2)}{(d-1)\alpha^2 - 1} \tag{11.15}$$

defined in the interval $\tau_-(\alpha) < \tau < \tau_+(\alpha)$ corresponding to accelerating expansion. The boundaries of the interval, $\tau_\pm$, are real and positive because $Z_\pm > 1$ within the interval $\alpha \in (1/\sqrt{d-1}, 1)$. In the limit $\alpha \to 1$, the roots Z_+ and Z_- coincide and the duration of accelerating expansion tends to zero. In the opposite limit, $\alpha \to 1/\sqrt{d-1}$, we obtain $Z_- = d/(2\sqrt{d-1})$, $Z_+ \to \infty$, which corresponds to an infinite period of accelerating expansion. In the case of higher space dimensions,

$$Z_\pm = \frac{1}{\alpha} \pm \frac{1-\alpha^2}{\alpha^2\sqrt{d}} + O\left(\frac{1}{d}\right),$$

whence it follows that the length of the transition acceleration period in a universe with a large number of spatial dimensions decreases.

11.2.2 Transition acceleration in models with several scalar fields. Cosmological models with the phase of accelerating expansion unlimited in time are in contradiction with string theory because an event horizon exists in these models, which makes their formulation in terms of the *S*-matrix impossible [114–116].

The problem of the cosmological event horizon is the main technical problem in high-energy physics because relativistic quantum theory is defined in terms of a set of scattering amplitudes, called the *S*-matrix. One of the fundamental assumptions of quantum relativistic theories and string theories is that the in- and out-states are infinitely separated in time, and therefore behave like free noninteracting states.

But the existence of an event horizon assumes a finite Hawking temperature, and therefore the conditions for defining the *S*-matrix can no longer by fulfilled. The absence of the *S*-matrix is a formal mathematical problem that arises both in string theory and in the theory of elementary particles.

Models with transition acceleration allow avoiding this contradiction, because they account for accelerated expansion now (and in the recent past) and do not lead to contradictions in the future. In Ref. [117], the dynamics of a homogeneous and isotropic universe is considered in which the role of DE is played (generally speaking, alternately) by two scalar fields φ and ψ. We write the equations of motion for this system

$$\begin{aligned}\dot\rho_{\rm b} &= -3H\gamma_{\rm b}\,\rho_{\rm b}\,,\\ \ddot\varphi &= -3H\dot\varphi - \partial_\varphi V\,,\\ \ddot\psi &= -3H\dot\psi - \partial_\psi V\,,\end{aligned}\tag{11.16}$$

and the first Friedmann equation

$$H^2 = \frac{1}{3M_{\rm Pl}^2}(\rho_{\rm b}+\rho_{\rm Q}) - \frac{k}{a^2}\,.$$

Here, the dot indicates the derivative with respect to time t, the index b corresponds to background components such as DM (m) or radiation (r), and the index Q stands for dark energy in the form of two scalar fields. It is also convenient to define

$$p_{\rm b,Q} = (\gamma_{\rm b,Q}-1)\,\rho_{\rm b,Q};$$

then $\gamma_{\rm m}=1$ and $\gamma_{\rm r}=4/3$, where $\gamma_i \equiv 1+w_i$ for any ith component. The energy density and pressure of the fields of quintessence in a potential $V(\varphi,\psi)$ have the form

$$\begin{aligned}\rho_{\rm Q} &= \frac12\,\dot\varphi^2 + \frac12\,\dot\psi^2 + V(\varphi,\psi)\,,\\ p_{\rm Q} &= \frac12\,\dot\varphi^2 + \frac12\,\dot\psi^2 - V(\varphi,\psi)\,.\end{aligned}$$

As previously, the possibility of transition acceleration depends critically on the shape of the potential. We consider several possibilities of obtaining transition acceleration. In Ref. [117], a large number of examples is considered, including the case where both fields interact via their potential $V(\varphi,\psi)$, and where one of the fields is free. Here, we only discuss models with potentials depending on two fields. A straightforward generalization of potential (11.3) to the case of the minimal coupling between the two scalar fields allows obtaining transition acceleration.

We consider the potential

$$V(\varphi,\psi) = M^4\exp(-\lambda\varphi)\big[P_0 + f(\psi)(\varphi-\varphi_{\rm c})^2 + g(\psi)\big]\,,\tag{11.17}$$

where $\varphi_{\rm c}$, λ, and P_0 are some constants. The additionally introduced scalar field ψ controls the existence or absence of a minimum in the potential with respect to φ. The main idea is that, on the one hand, a minimum initially exists in the potential for the scalar field φ that accounts for accelerating expansion of the Universe and, on the other hand, evolution of ψ results in the vanishing of this minimum and in the Universe returning to the mode of decelerating expansion. The model in [111] is restored if $f\equiv 1$ and $g\equiv 0$. The position of the minimum of potential (11.17) is given by

$$\varphi_\pm = \varphi_{\rm c} + \frac1\lambda\left(1 \pm \sqrt{1-\lambda^2\,\frac{P_0+g(\psi)}{f(\psi)}}\right).\tag{11.18}$$

The function g ($g>0$) responsible for the mass of the scalar field usually has the form $g\propto\psi^2$, but because it does not influence the dynamics of the system, we set $g\equiv 0$ for simplicity.

The minimum in (11.18) vanishes when the condition

$$f(\psi) < \lambda^2 P_0 \equiv f(\psi_{\rm c})\tag{11.19}$$

is satisfied. We note that potential (11.17) can be represented as

$$V(\varphi,\psi) = \frac{M^4}{\lambda^2}\exp(-\lambda\varphi)\big[f(\psi_{\rm c}) + f(\psi)(\lambda\varphi-\lambda\varphi_{\rm c})^2\big]\,.\tag{11.20}$$

We assume that f is a positive, continuous, and monotonic function at $\psi>0$ and/or $\psi<0$. From condition (11.19), it is not difficult to see that if f decreases (increases), then accelerating expansion occurs at ψ smaller than $\psi_{\rm i}$ (larger than $\psi_{\rm c}$) ($\psi_{\rm i}$ is the initial value of the scalar field) because minimum (11.18) exists. Transition acceleration can be achieved with different functions f, for example, $f = 1+\alpha\psi^p$, $f=\tanh(\alpha\psi^p)$, where $p,\alpha\geqslant 0$, and $f=\exp(\alpha\psi^p)$, $f=\cosh(\alpha\psi^p)$ with arbitrary p and α. During the evolution of the Universe, the scalar field φ rolls down the potential and dominates in the exponential part, and hence $M_{\rm Pl}^2 m_\varphi^2 \sim M_{\rm Pl}^2 m_\psi^2 \sim V \sim \dot\varphi^2 \sim M_{\rm Pl}^2 H^2$ with $\Omega_{\rm Q} = 4/\lambda^2$ and $w_{\rm Q}=1/3$ during the radiation-dominated era and with $\Omega_{\rm Q}=3/\lambda^2$ and $w_{\rm Q}=0$ when matter dominates as φ approaches the minimum. When the field φ is at minimum (11.18) and oscillates around it, the Universe experiences accelerated expansion with $V\gg\dot\varphi^2$ and therefore $w_{\rm Q}\simeq -1$, while the condition $\psi\leqslant\psi_{\rm c}$ ($\psi\geqslant\psi_{\rm c}$) is satisfied for ψ if f increases (decreases). When minimum (11.18) vanishes and the field φ starts to roll down freely, taking large values (gaining speed), the expansion of the Universe becomes decelerated, $q>0$.

As a concrete example, we consider a very simple case:

$$f(\psi) = \psi^2\,.\tag{11.21}$$

Here, the minimum in φ vanishes if $-\psi_{\rm c}\leqslant\psi\leqslant\psi_{\rm c}$, where $\psi_{\rm c}\equiv\lambda\sqrt{P_0}$. As in model (11.3), it is possible to obtain $\lambda\gtrsim 9$, which is consistent with the restrictions imposed by cosmological observations, if $\varphi_{\rm i}$ is fixed, and the value of $\varphi_{\rm c}$ must be

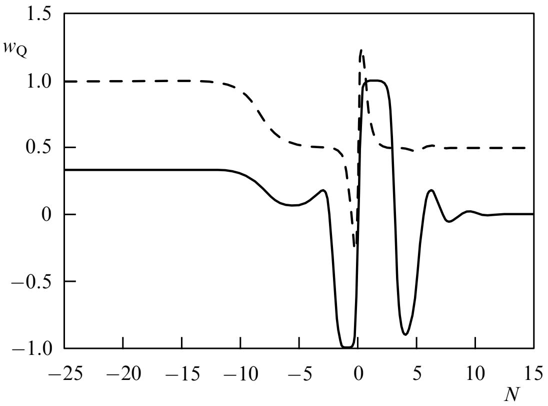

Figure 7. Dependences of the parameter of the equation of state w_Q (solid curve) and of the deceleration parameter q (dashed curve) for the model whose results are presented in Fig. 8. Here, $w_{Q,0} \simeq -0.491$, $q_0 = 0.013$, and $w_{\rm eff} \simeq -0.874$.

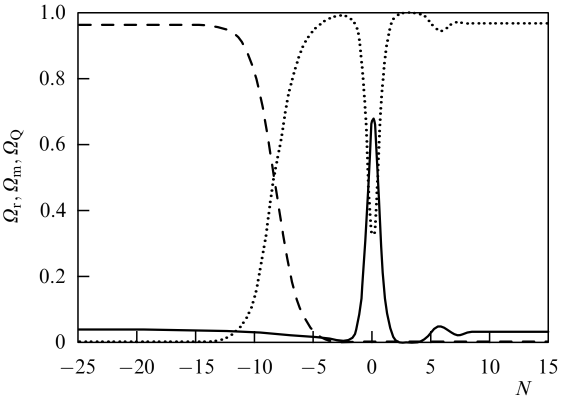

Figure 8. Dependence of the relative density on the logarithm of the scale factor $\ln a = N$ for radiation Ω_r (dashed curve), matter Ω_m (dotted curve), and quintessence Ω_Q (solid curve) in the case of potential (11.20) for $\lambda = 10$, $P_0 = 0.164$, and $\varphi_c = 23.8$ ($\varphi_i = 0$ and $\psi_i = 5$). The modern value of the quintessence relative density is $\Omega_{Q,0} \simeq 0.661$. When the age of the Universe reaches $t_{\rm end} \simeq 0.996\, t_0$, where $t_0 \simeq 0.912 H_0^{-1}$, its accelerating expansion stops.

subjected to fine tuning in order to make the quintessence dominate at the present moment.

Figure 7 shows the parameter of the equation of state, w_Q, whose value at present is $w_{Q,0} \simeq -0.491$, $w_{\rm eff,0} = -0.874$. When the field φ is at the minimum, $w_Q \simeq -1$. The deceleration parameter q also presented in Fig. 7 indicates that the Universe has already passed the stage of transition acceleration and its expansion is decelerating, $q_0 \simeq 0.013$. Accelerating expansion ($q < 0$) starts at $z \simeq 0.658$ and terminates at $z \simeq 0.0035$, when the age of the Universe is $t_{\rm end}/t_0 \simeq 0.996$ (at present, $H_0 t_0 \simeq 0.912$).

11.2.3 Decaying dark energy as a scalar field. Cosmological models in which DE decays lead to numerous possibilities, one of which is the existence of transition acceleration — the period when accelerating expansion is superseded by decelerated one. The dynamics of such a universe, evidently, differ from the evolution in the SCM framework. The model of decaying DE discussed here is a prototype of the standard inflationary model, in which the field that caused inflationary expansion of the Universe experiences decay.

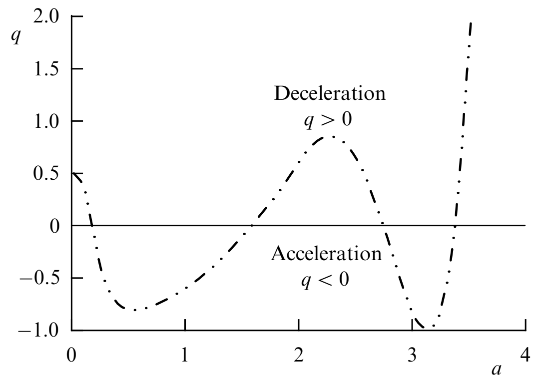

Figure 9. Dependences of the deceleration parameter q on the scale factor for a universe filled with matter and DE in the form of a scalar field in the potential $V(\varphi) = V_0 \cos(\varphi m/\sqrt{V_0})$ at $m = 0.74$, $V_0 = 150$, $\varphi(0) = 0.23$, and $\varphi'(0) = 0$. The present moment corresponds to $a = 1$. The parameters of the potential have been chosen so as to make the value of q the same as in the SCM, $q(1) \approx -0.6$.

We consider the model of a scalar field with a potential that takes both positive and negative values [118]:

$$V(\varphi) = V_0 \cos\frac{\varphi}{f}, \qquad f = \frac{\sqrt{V_0}}{m}. \tag{11.22}$$

The assumption that the potential of a scalar field can take negative values is very intriguing, but we must bear in mind that besides the fundamental possibility of explaining the observed transition acceleration, this model is not based on any observational data. Nevertheless, such effective potentials often arise in supergravity and M-theory [118]. We show that it is possible to obtain transition acceleration in the framework of this model. We also note that in cosmology with negative potentials, the evolution of the Universe differs essentially from its evolution in the SCM. For example, when potential energy (11.22) taking a negative value dominates in the Friedmann equation,

$$\left(\frac{\dot a}{a}\right)^2 = \frac{1}{3M_{\rm Pl}^2}\left(\rho + \frac{1}{2}\dot\varphi^2 + V(\varphi)\right), \tag{11.23}$$

even the collapse of a spatially flat universe becomes possible (which in principle cannot exist in the SCM), but this can happen only at the later stages of the evolution of the Universe, much later than the moment when the phase of transition acceleration comes to an end. The moment at which the Universe starts to collapse can be preceded by several stages of transition acceleration (Fig. 9), which become more frequent as the moment of collapse approaches.

11.2.4 Transition acceleration in a universe with interacting components. We consider a spatially flat universe consisting of three components: DE, DM, and baryons. The first Friedmann equation for such a universe has the form

$$3M_{\rm Pl}^2 H^2 = \rho_{\rm DE} + \rho_{\rm m} + \rho_{\rm b}, \tag{11.24}$$

where, as usual, $\rho_{\rm DE}$ is the DE density, $\rho_{\rm m}$ is the DM energy density, $\rho_{\rm b}$ is the baryon energy density, and $H = \dot a/a$ is the Hubble parameter. The equation of state for DE has the form $p_{\rm DE} = w\rho_{\rm DE}$.

In the case of interaction, the conservation equations become

$$\dot{\rho}_{\rm DE} + 3H(1+w)\,\rho_{\rm DE} = -Q\,,\qquad \dot{\rho}_{\rm m} + 3H\rho_{\rm m} = Q\,, \tag{11.25}$$

where Q characterizes the interaction. The conservation equation for the baryon component is

$$\dot{\rho}_{\rm b} + 3H\rho_{\rm b} = 0 \Rightarrow \rho_{\rm b} = \rho_{\rm b0}\left(\frac{a_0}{a}\right)^3. \tag{11.26}$$

The total density is $\rho = \rho_{\rm m} + \rho_{\rm b} + \rho_{\rm DE}$. Without loss of generality, we assume that the energy density $\rho_{\rm m}$ is expressed as

$$\rho_{\rm m} = \tilde{\rho}_{\rm m0}\left(\frac{a_0}{a}\right)^3 f(a)\,, \tag{11.27}$$

where $\tilde{\rho}_{\rm m0}$ and a_0 are constants and $f(a)$ is an arbitrary differentiable function of the scale factor. From (11.25) and (11.27), we obtain

$$Q = \rho_{\rm m}\,\frac{\dot{f}}{f} = \tilde{\rho}_{\rm m0}\left(\frac{a_0}{a}\right)^3 \dot{f}. \tag{11.28}$$

Let

$$f(a) = 1 + g(a)\,. \tag{11.29}$$

In the absence of interaction, $f(a) = 1$, and therefore the function $g(a)$ is responsible for the interaction. Then, taking into account that

$$\dot{f} = \dot{g} = \frac{{\rm d}g}{{\rm d}a}\,\dot{a}\,, \tag{11.30}$$

we obtain

$$Q = \tilde{\rho}_{\rm m0}\,\frac{{\rm d}g}{{\rm d}a}\,\dot{a}\left(\frac{a_0}{a}\right)^3. \tag{11.31}$$

This means that

$$\rho_{\rm m} = \tilde{\rho}_{\rm m0}(1+g)\left(\frac{a_0}{a}\right)^3, \tag{11.32}$$

where $\rho_{\rm m0} = \rho_{\rm m}(a_0)$ if the interaction exists and $\tilde{\rho}_{\rm m0} = \tilde{\rho}_{\rm m}(a_0)$ in the absence of interaction. The two initial values of the DM density are related as

$$\rho_{\rm m0} = \tilde{\rho}_{\rm m0}(1 + g_0)\,, \tag{11.33}$$

where $g_0 \equiv g(a_0)$. As can be seen from (11.28), when $Q > 0$, DE decays into DM, ${\rm d}g/{\rm d}a > 0$. When ${\rm d}g/{\rm d}a < 0$, the decay proceeds in the opposite direction. From Eqns (11.25) and (11.31), we obtain

$$\dot{\rho}_{\rm DE} + 3H(1+w)\,\rho_{\rm DE} = -\tilde{\rho}_{\rm m0}\,\frac{{\rm d}g}{{\rm d}a}\,\dot{a}\left(\frac{a_0}{a}\right)^3. \tag{11.34}$$

When $w = {\rm const}$, the solution of Eqn (11.34) has the form

$$\rho_{\rm DE} = (\rho_{\rm m0} + \tilde{\rho}_{\rm m0}g_0)\left(\frac{a_0}{a}\right)^{3(1+w)} - \tilde{\rho}_{\rm m0}\left(\frac{a_0}{a}\right)^3 g + 3w\tilde{\rho}_{\rm m0}a_0^3 a^{-3(1+w)}\int_{a_0}^{a} {\rm d}a\, g a^{3w-1}\,. \tag{11.35}$$

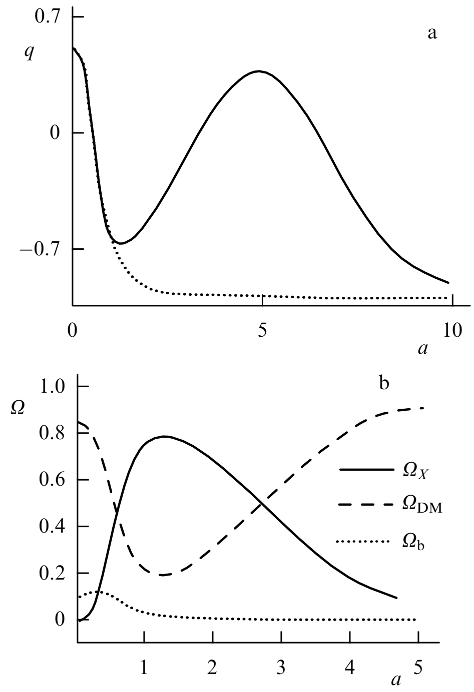

Figure 10. (a) Dependence $q(a)$ of the deceleration parameter on the scale factor in the model with interacting DE and DM (solid curve) for $n = 7$, $\sigma = 1.5$, and in the SCM (dotted curve). (b) Dependences of relative densities on the scale factor for $n = 7$ and $\sigma = 1.5$.

We rewrite the second Friedmann equation in terms of $g(a)$:

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left\{\tilde{\rho}_{\rm m0}(1+g)\left(\frac{a_0}{a}\right)^3 + \rho_{\rm b0}\left(\frac{a_0}{a}\right)^3 + (1+3w)\left[(\rho_{\rm m0} + \tilde{\rho}_{\rm m0}g_0)\left(\frac{a_0}{a}\right)^{3(1+w)} - \tilde{\rho}_{\rm m0}\left(\frac{a_0}{a}\right)^3 g + 3w\tilde{\rho}_{\rm m0}a_0^3 a^{-3(1+w)}\int_{a_0}^{a}{\rm d}a\, g a^{3w-1}\right]\right\}. \tag{11.36}$$

To solve Eqn (11.36), it is necessary to define the function $g(a)$. Because the nature of neither DE nor DM is known, it is impossible to indicate the form of $g(a)$ starting from first principles; we therefore introduce the interaction in this model such that the dynamics of the model be consistent with observational data.

We consider the interaction for which the function $g(a)$ is represented as $g(a) = a^n \exp(-a^2/\sigma^2)$, where n is a natural number and σ is a positive real number. The existence of transition acceleration implies that the DE density starts to decrease, i.e., its decay occurs, ${\rm d}g/{\rm d}a > 0$. This condition requires n and σ to satisfy the inequality $n\sigma^2 > 2$.

In Fig. 10, the dependences of the relative densities on the scale factor are shown for $n = 7$ and $\sigma = 1.5$. The model considered permits ensuring transition acceleration for a certain choice of the interaction parameters, but it is indistinguishable from the SCM for large (as well as small) values of the scale factor.

11.2.5 Transition acceleration in a universe with a decaying cosmological constant. As a simple example of transition acceleration, we consider the model with a decaying cosmological constant:

$$\dot{\rho}_{\rm m} + 3\,\frac{\dot{a}}{a}\,\rho_{\rm m} = -\dot{\rho}_{\Lambda}\,, \tag{11.37}$$

where $\rho_{\rm m}$ and ρ_{Λ} are the densities of the DM energy and of the cosmological constant Λ. At early stages of the expansion of the Universe, when ρ_{Λ} is quite small, such a decay does not influence cosmological evolution in any way. At later stages, as the DE contribution increases, the influence of its decay ever more alters the standard dependence of the DM energy density $\rho_{\rm m} \propto a^{-3}$ on the scale factor a. We assume the deviation to be described by a function $\epsilon(a)$ of the scale factor. Then

$$\rho_{\rm m} = \rho_{{\rm m},0}\, a^{-3+\epsilon(a)}\,, \tag{11.38}$$

where $a = a_0 = 1$ in the present epoch. Other fields of matter (radiation, baryons) evolve independently and are conserved. Hence, the DE density has the form

$$\rho_{\Lambda} = \rho_{\rm m0}\int_a^1 \frac{\epsilon(\tilde{a}) + \tilde{a}\epsilon'\ln\tilde{a}}{\tilde{a}^{4-\epsilon(a)}}\,{\rm d}\tilde{a} + X\,, \tag{11.39}$$

where the prime denotes the derivative with respect to the scale factor and X is the integration constant. If radiation is neglected, the first Friedmann equation takes the form

$$H = H_0\left[\Omega_{{\rm b},0}\, a^{-3} + \Omega_{\rm m0}\varphi(a) + \Omega_{X,0}\right]^{1/2}. \tag{11.40}$$

The function $\varphi(a)$ is then written as

$$\varphi(a) = a^{-3+\epsilon(a)} + \int_a^1 \frac{\epsilon(\tilde{a}) + \tilde{a}\epsilon'\ln\tilde{a}}{\tilde{a}^{4-\epsilon(a)}}\,{\rm d}\tilde{a}\,, \tag{11.41}$$

where $\Omega_{X,0}$ is the relative contribution of the constant X to the common relative density. To proceed, it is necessary to make some assumptions concerning the concrete form of $\epsilon(a)$. We follow the original work [119] and consider the simplest case

$$\epsilon(a) = \epsilon_0 a^{\xi} = \epsilon_0(1+z)^{-\xi}\,, \tag{11.42}$$

where ϵ_0 and ξ can take both positive and negative values. It follows from (11.39) that

$$\rho_{\Lambda} = \rho_{\rm m0}\epsilon_0\int_a^1 \frac{1+\ln(\tilde{a}^{\xi})}{\tilde{a}^{4-\xi-\epsilon_0\tilde{a}^{\xi}}}\,{\rm d}\tilde{a} + X\,. \tag{11.43}$$

We note that the case $\epsilon_0 = 0$ corresponds to the SCM, i.e., $X \equiv \rho_{\Lambda}$.

Using the formulas presented above, it is not difficult to also obtain the dependences for the relative densities:

$$\Omega_{\rm b}(a) = \frac{a^{-3}}{A + a^{-3} + B^{-1}\varphi(a)}\,, \tag{11.44a}$$

$$\Omega_{\rm m}(a) = \frac{a^{-3+\epsilon(a)}}{D + Ba^{-3} + \varphi(a)}\,, \tag{11.44b}$$

$$\Omega_{\Lambda}(a) = \frac{D + \varphi(a) - a^{-3+\epsilon(a)}}{D + Ba^{-3} + \varphi(a)}\,, \tag{11.44c}$$

where $A = \Omega_{X,0}/\Omega_{{\rm b},0}$, $B = \Omega_{{\rm b},0}/\Omega_{\rm m0}$, and $D = \Omega_{X,0}/\Omega_{\rm m0}$.

Within this simple model, it is practically possible to obtain any dynamics of the Universe with the aid of an appropriate choice of the parameters ϵ_0 and ξ. In the context of this paper, the case of immediate interest is where $\epsilon_0 > 0$ and ξ takes large positive values ($\xi \gtrsim 0.8$). The solid curve in Fig. 11 shows the dependence of the deceleration parameter for $\xi = 1.0$ and $\epsilon_0 = 0.1$. We note that for these parameters, the expansion of the Universe is accelerating at present, when $a \sim 1$, but the dominance of DE is not eternal, unlike in the SCM, and when $a \gg 1$, the Universe will enter a new era of dominance of nonrelativistic matter. Such a form of dynamic behavior is not peculiar to most of the models with $\Lambda(t)$ or models with interacting quintessence, discussed in the literature, but it is characteristic of the so-called thawing [120–122] and hybrid [123] potentials that follow from string theory or M-theory [114, 115, 124] (also see [125]).

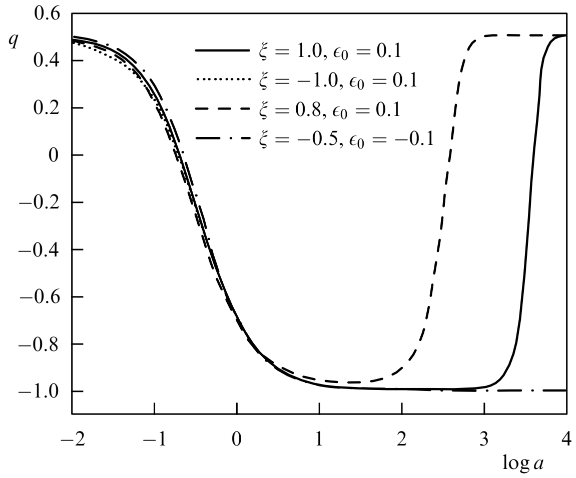

Figure 11. Deceleration parameter as a function of $\log a$ for the chosen values of ϵ_0 and ξ.

To better represent the phenomenon of transition acceleration, we find the explicit form of the deceleration parameter $q = -a\ddot{a}/\dot{a}^2$ in this model:

$$q(a) = \frac{3}{2}\,\frac{\Omega_{{\rm b},0}a^{-3} + \Omega_{\rm m0}a^{\epsilon(a)-3}}{\Omega_{{\rm b},0}a^{-3} + \Omega_{\rm m0}\varphi(a) + \Omega_{X,0}} - 1\,. \tag{11.45}$$

The parameter q is represented as a function of $\log a$ for different values of ξ and ϵ_0 in Fig. 11. We note that in the distant past ($\alpha \ll 1$), the deceleration parameter $q(a) \to 1/2$, which corresponds to a matter-dominated universe. However, for certain values of ξ, a long (but finite, in contrast to the case of the SCM) era of accelerated expansion sets in. In the distant future ($\alpha \gg 1$), the Universe again returns to decelerated expansion ($q > 0$).

11.3 Transition acceleration in models with holographic dark energy

In the literature, models are usually considered in which the required dynamics of the Universe is ensured by some (single) form of DE. The holographic approach discussed in Section 10 leads to serious restrictions imposed on the dynamics of the Universe. To explain the observed dynamics of the Universe, the action of the gravitational field, in addition to the fields of ordinary matter (both dark and baryon), is conventionally supplemented with either the cosmological constant, to which the role of the physical vacuum is attributed in the SCM, or more complicated dynamical objects: scalar fields, k-essence, and so on. In the context of holographic cosmology, these second summands are usually neglected, and only the contribution of boundary terms can be taken into account. Nevertheless, in our opinion, no theoretical premises exist for such restrictions.

In this section, we consider a holographic model in which both volume and surface terms are involved. The role of the first is played by a homogeneous scalar field in an exponential potential interacting with DM. Holographic DE in form (10.62) plays the role of a boundary term. It is shown that in this model, the phase of transition expansion of the evolution period exists during which accelerating expansion of the Universe is replaced by decelerating expansion, after which the Universe once again enters the stage of (already eternal) accelerating expansion. For simplicity, we ignore the holographic contribution of the scalar field or suppose it to be fully taken into account by the DE contribution of form (10.62).

To describe the dynamic properties of such a universe, it is convenient to pass to the dimensionless variables

$$x = \frac{\dot{\varphi}}{\sqrt{6} M_{\rm Pl} H}\,, \qquad y = \frac{1}{M_{\rm Pl} H}\sqrt{\frac{V(\varphi)}{3}}\,, \qquad z = \frac{1}{M_{\rm Pl} H}\sqrt{\frac{\rho_{\rm m}}{3}}\,, \qquad u = \frac{1}{M_{\rm Pl} H}\sqrt{\frac{\rho_{\rm q}}{3}}\,. \tag{11.46}$$

Evolution of the scalar field is described by the Klein–Gordon equation, which in the case of a scalar field with matter has the form

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{{\rm d}V}{{\rm d}\varphi} = -\frac{Q}{\dot{\varphi}}\,. \tag{11.47}$$

We consider the case where the interaction parameter Q is a linear combination of the scalar field and DE energy densities,

$$Q = 3H(\alpha\rho_{\varphi} + \beta\rho_{\rm m})\,, \tag{11.48}$$

where α and β are constant parameters. For this model, independently of the form of the scalar field potential $V(\varphi)$, the system of dynamic equations takes the form

$$\begin{aligned} x' &= \frac{3x}{2}\, g(x,z,u) - 3x + \sqrt{\frac{3}{2}}\lambda y^2 - \gamma\,, \\ y' &= \frac{3y}{2}\, g(x,z,u) - \sqrt{\frac{3}{2}}\lambda x y\,, \\ z' &= \frac{3z}{2}\, g(x,z,u) - \frac{3}{2}\, z + \gamma\,\frac{x}{z}\,, \\ u' &= \frac{3u}{2}\, g(x,z,u) - \frac{u^2}{n}\,, \end{aligned} \tag{11.49}$$

where

$$g(x,z,u) = 2x^2 + z^2 + \frac{2}{3n}\,u^3\,, \quad \lambda \equiv -\frac{1}{V}\frac{{\rm d}V}{{\rm d}\varphi}\, M_{\rm Pl}\,, \tag{11.50}$$

$$Q = 9H^3 M_{\rm Pl}^2\left[\alpha(x^2+y^2) + \beta z^3\right]\,, \qquad \gamma = \frac{\alpha(x^2+y^2)+\beta z^3}{x}\,. \tag{11.51}$$

As has already been noted, we consider the simplest case of an exponential potential:

$$V = V_0 \exp\left(\sqrt{\frac{2}{3}}\,\frac{\mu\varphi}{M_{\rm Pl}}\right)\,, \tag{11.52}$$

where μ is a constant. With (11.51), system of equations (11.49) becomes

$$\begin{aligned} x' &= \frac{3x}{2}\left[g(x,z,u) - \frac{\alpha(x^2+y^2)+\beta z^2}{x^2}\right] - 3x - \mu y^2\,, \\ y' &= \frac{3y}{2}\, g(x,z,u) + \mu x y\,, \\ z' &= \frac{3z}{2}\left[g(x,z,u) + \frac{\alpha(x^2+y^2)+\beta z^2}{z^2}\right] - \frac{3}{2}\, z\,, \\ u' &= \frac{3u}{2}\, g(x,z,u) - \frac{u^2}{n}\,. \end{aligned} \tag{11.53}$$

The deceleration parameter is expressed as

$$q = -1 + \frac{3}{2}\left(2x^2 + z^2 + \frac{2}{3n}\,u^3\right)\,. \tag{11.54}$$

We note that cosmological parameters do not depend explicitly on the type of the interaction, but only determine the dynamics of the variables. This fact essentially complicates the analysis of system of equations (11.53). We make some comments concerning this system of equations. First, we consider the case $y = 0$, which corresponds to a free scalar field; here, it is not difficult to obtain some constraints on the interaction parameters, which are consequences of the requirement that the energy density be real:

$$2\sqrt{\frac{\beta}{\alpha}} > 1 + \alpha + \beta\,, \tag{11.55}$$

which, in turn, requires that the conditions $0 > \beta > \alpha$ and $|\alpha| + |\beta| < 1$ be satisfied. Such an unstable critical point of system of equations (11.53) corresponds to a universe filled with DM. The case of several similar points is also possible, but is of no interest. In conclusion, we note that it is possible to show that any of these critical points also exists within the interval $x_0 < 0$. When $z \neq 0$ and for the constraints imposed in (11.55), we obtain

$$x_{\rm c} = \left[\left(a + \sqrt{\frac{\beta}{\alpha}}\right)^{1/2} + a\right] z_{\rm c}\,, \tag{11.56}$$

where $a = \left[2\sqrt{\beta/\alpha} - (1+\alpha+\beta)\right]^{1/4}$.

11.3.1 The case $Q = 0$. We next consider the case of the uncoupled scalar field and DM in more detail. The critical points of system (11.53) for $\alpha = \beta = 0$ are presented in Table 1. The phase space induced by system of equations (11.53) contains six physically admissible critical points, the last of which is an attractor. The first critical point $(1, 0, 0, 0)$ is unstable and corresponds to a universe dominated by a scalar field with an extremely rigid equation of state ($w_{\varphi} = 1$); the second, also unstable, corresponds to the period of evolution during which the scalar field is dynamically equivalent to the cosmological constant.

The next point $(0, 0, 1, 0)$ is of no interest: it corresponds to a universe only consisting of DM, and it is also unstable. The fourth critical point $(0, 0, 0, 1)$, which corresponds to a universe consisting only of holographic DE of form (10.62), was discussed in detail previously. The sixth and last critical point is of physical interest; it is an attractor and corresponds to a universe filled with a scalar field and holographic DE. This critical point is fully determined by the scalar field

Table 1. Critical points of autonomous system of equations (11.53).

Coordinates (x_c, y_c, z_c, u_c)	Type of critical point	q	w_φ	w_{tot}
$(1, 0, 0, 0)$	Unstable	2	1	1
$(0, 1, 0, 0)$	Unstable	-1	-1	-1
$(0, 0, 1, 0)$	Unstable	$\frac{1}{2}$	—	0
$(0, 0, 0, 1)$	Stable	$-1 + \frac{1}{n}$	—	$-1 + \frac{2}{3n}$
$\left(-\frac{3}{2\mu}, \frac{3}{2\mu}, \sqrt{1 - \frac{3}{2\mu^2}}, 0\right)$	Unstable	$\frac{1}{2}$	—	0
$(x_*, y_*, 0, u_*)$	Attractor	$q_* < 0$	w_φ^*	w_{tot}^*

potential μ and the value of n:

$$x_* = \frac{2}{3n\mu} u_* , \qquad y_* = \sqrt{1 - \left(1 + \frac{4}{9n^2\mu^2}\right) u_*^2} ,$$
$$z_* = 0 , \qquad u_* = \frac{3}{2n\mu^2} \left(-1 + \sqrt{1 + \frac{4n^2\mu^4}{9}}\right) . \tag{11.57}$$

The fact that $x_* \propto u_*$ is characteristic of so-called spectator solutions [126]. We also note that a so-called background interaction exists between the scalar field and DE, which is a consequence of the dynamics of the scalar field being affected by the holographic DE, which, having a negative pressure, influences the expansion rate of the Universe — the Hubble parameter present in the equation of motion for the scalar field.

At the attractor point, the DM density vanishes. For this model to be consistent with observations, it is necessary to rigorously set the initial conditions for the accelerating expansion of the Universe to start before this phase of evolution is reached.

11.3.2 The case $Q = 3H\alpha\rho_\varphi$. In the example presented in Section 11.3.1, no transition acceleration arises. To explain this possible phenomenon, we consider a model in which the scalar field interacts with DM. In this section, we examine the case with the interaction parameter $\beta = 0$ in (11.48). Figure 12 shows the dependences of Ω_q, Ω_m, and Ω_φ for $\alpha = 0.005$, $\mu = -5$, and $n = 3$.

From the form of the equations and the character of the interaction, it is not difficult to understand that neither the nature nor the position of the critical points found above change when the interaction is switched on. The interaction only affects the behavior of dynamic variables that corresponds to different trajectories in the phase space between the critical points. This corresponds to the fact that the interaction parameters are only present in the time derivatives of the Hubble parameter of the second and higher orders.

For the given values of interaction parameters, transition acceleration starts nearly in the present era. As in the standard cosmological models, where DE is realized in the form of scalar fields, the dominance of the latter under certain conditions ($\dot\varphi \approx V(\varphi)$) gives rise to the phase of accelerating expansion of the Universe. As the Universe expands, the contribution of Ω_q increases, which results in the background (space) changing more rapidly than the field. This is equivalent to the scalar field becoming asymptotically free.

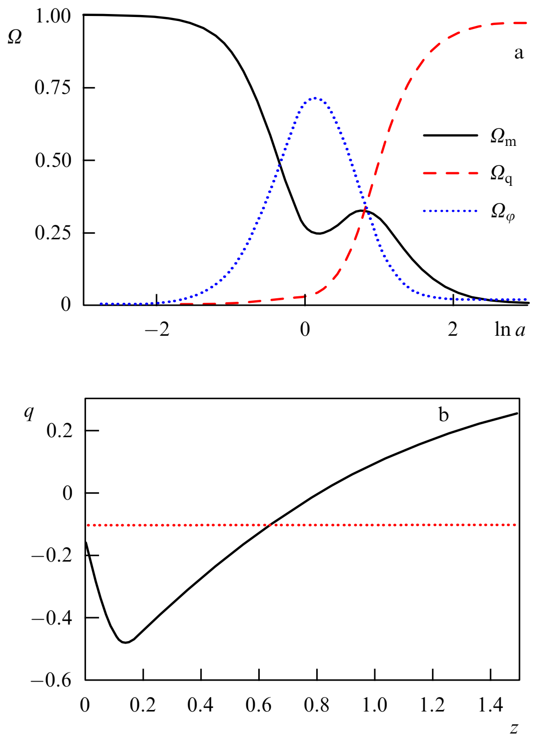

Figure 12. (a) Relative densities of the scalar field Ω_φ (dotted curve), holographic DE Ω_q (dashed curve), and Ω_m (solid curve) as functions of the logarithm of the scale factor $N = \ln a$ for $n = 3$, $\alpha = 0.005$, and $\mu = -5$. (b) Dependence of the deceleration parameter $q(z)$ for the same values of n, α, and μ.

Such a field is known to have a super-rigid equation of state, and it makes the Universe decelerate its expansion. Subsequently, however, when the contribution of Ω_q increases so much that the scalar field can no longer hinder the expansion of the Universe, its expansion again starts to accelerate.

11.4 Transition acceleration: hints at its observation

Based on independent observational data, including SNe-Ia brilliance curves, data on the CSF temperature anisotropy, and BAO measurements, it was shown in [127] that the acceleration with which the Universe expands has reached its maximum value and is decreasing at present (Fig. 13). In terms of the deceleration parameter, this means that this parameter has reached its minimum and is increasing at present. Hence, the main result of the analysis in Ref. [127]

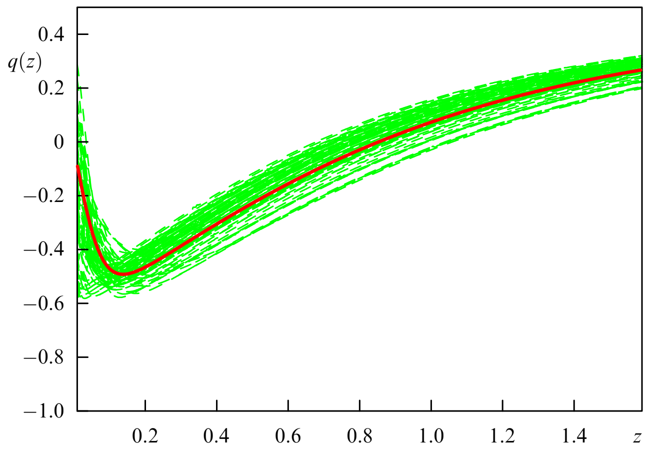

Figure 13. Deceleration parameter $q(z)$ restored from independent observational data, including SNe-Ia brilliance curves, data on the CSF temperature anisotropy, and baryon acoustic oscillations, using parameterization (11.59). The bold curve shows the best fit at a confidence level of 1σ [127].

is that the SCM is not the only explanation of observational data (although it is the simplest), and the accelerated expansion of the Universe in which DE presently dominates is merely a transition phenomenon.

We note that it is also shown in Ref. [127] that using the Chevallier–Polarski–Linder (CPL) parameterization

$$w(z) = w_0 + \frac{w_a z}{1+z} \tag{11.58}$$

for the parameter of the equation of state does not allow unambiguously combining the data obtained from observations of close supernovae, such as SNE-Ia, and of the CMB anisotropy. A possible resolution of this contradiction is to renounce this parameterization and adopt another one. In Ref. [127], a parameterization was proposed that is capable of uniting these arrays of data:

$$w(z) = -\frac{1 + \tanh\left[(z - z_t)\Delta\right]}{2}\,. \tag{11.59}$$

In this approximation, $w = -1$ at earlier times of the evolution of the Universe, and w increases to its maximum value $w \sim 0$ at small z.

Figure 13 shows the dependence of the deceleration parameter q restored using parameterization (11.59).

In 2010, in the framework of the Supernova Cosmology Project (SCP), the most recent array was published [128] of data on bursts of supernovae, which includes 557 events, making it the largest present-day body of data. Moreover, the array of data on supernovae with small red shifts ($z < 0.3$) has been significantly enlarged.

At present, there are already several studies [129, 130] in which these observations are analyzed in order to check the hypothesis of transition acceleration.

All the authors agree that the final answer can only be given by repeated, more precise, observations. Moreover, it seems that to obtain consistent results, the entire technique of data handling has to be corrected. For example, as shown in [129, 130] (Fig. 14), there are contradictions between the data obtained from observations of SNE-Ia and BAO at small red shifts and CSF observations at large z.

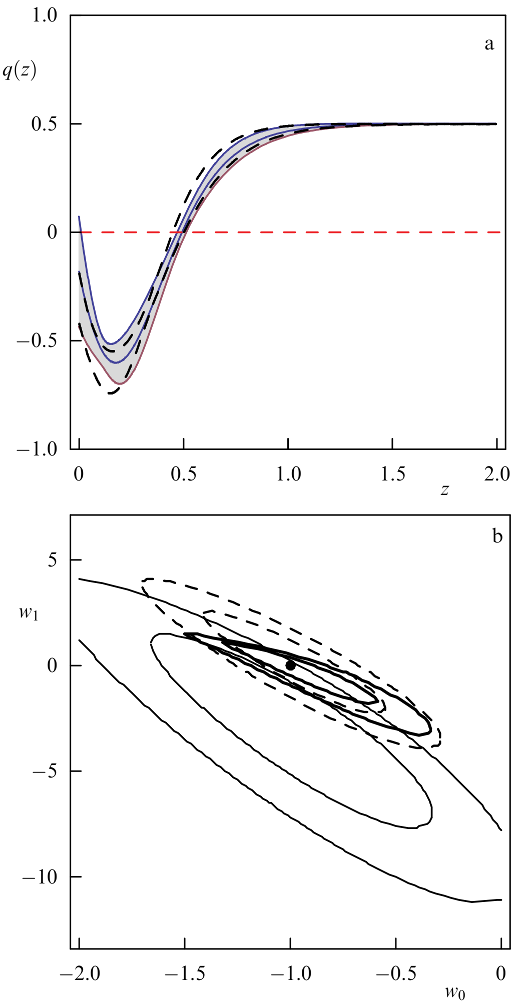

Figure 14. (a) Deceleration parameter $q(z)$ restored at a 2σ confidence level from the results of Union2+BAO observations. The shaded region and the region between the two dashed curves respectively correspond to the existence and absence of systematic errors in the observations of SNE-Ia. (b) Confidence regions of 68.3% and 95% for w_0 and w_1 in the CPL parameterization, $w = w_0 + w_1 z/(1+z)$. The Union2S results are shown with the dashed line, Union2S + BAO with the bold solid line, and Union2S + BAO + CMB with the thin solid line. The point $w_0 = -1$, $w_1 = 0$ corrsponds to the spatially flat ΛCDM model.

The contradiction consists in the fact that the analysis of two separate series of data yields opposite results. For example, when only the SNE-Ia and BAO data are used, the probability that the acceleration of the expansion of the Universe has already reached its maximum ($z \sim 0.3$) and is at present starting to decrease turns out to be quite high. However, if these data are supplemented with the CSF observations, the results of the analysis change substantially and no deviations from the ΛCDM model are revealed.

Therefore, restoration of the DE evolutionary dependence and the answer to the question of whether the expansion of our Universe will decelerate or the accelerating expansion will go on forever (as in the SCM) depends strongly on the data obtained from observations of SNE-Ia, their quality, the technique of reconstructing cosmological parameters [such as $q(z)$, $w(z)$, and Ω_{DE}], and the actual parameterization of the equation of state. For a detailed answer to this question, we must wait for more precise observational data and find methods of their analysis that are less model-dependent.

12. Conclusion

An astonishing evolution of our perception of the Universe has taken place during the past century: from a static system, which only consisted of a single galaxy, up to the Universe, which has been expanding for billions of years starting from an initial singularity and containing several hundred billion galaxies. Although the first ideas of the structure of the Universe originated along with homo sapiens, the main volume of cosmological knowledge was obtained in the 20th century, termed the golden age of cosmology. On the one hand, GR provided the conceptual basis of the modern picture of the world. On the other hand, the appearance of ever more powerful telescopes, functioning in different parts of the electromagnetic wave spectrum, has permitted significantly extending the scale of observations (both terrestrial and cosmic) and, thus, more reliably establishing the applicability of the models used. The transition to digital encoding of the information obtained with telescopes has turned out to be a no less revolutionary step than the actual invention of the telescope. The huge volume of information obtained was transformed into the Standard Cosmological Model. According to this model, we live in a spatially flat Universe experiencing accelerating expansion, three quarters of the Universe are composed of a new form of energy (dark energy) and one fifth of a new form of matter (dark matter), with small admixtures of ordinary (baryon) matter and of a relativistic component (photons, neutrinos).

Only a few decades ago, cosmology could be defined as a science studying two quantities: the current rate of expansion of the Universe H_0 and the deceleration parameter q_0. Until the late 1990s, precisely these two quantities were considered to determine the ultimate destiny of the Universe. Starting from 1998, this idea has undergone a total change. We now know these quantities with a reasonable accuracy, but it turned out that they have little to do with the destiny of the Universe. Unexpectedly, a third quantity appeared, which, as it now seems, governs everything. Dark energy (in the form of the cosmological constant or in various dynamic forms generated by scalar fields) has finally balanced the energy budget by making the total density of the energy content of the Universe equal to the critical density predicted by the inflation theory. The new energy component has a negative pressure and, as a consequence, causes accelerating expansion of the Universe. Dark energy affects both the past and the future evolution of the Universe. While this future of dark energy in the form of the cosmological constant is quite transparent, representing a monotonic accelerating expansion, a whole ensemble of different scenarios is plausible in the case of dynamic forms of dark energy: the Big Rip, the Big Whimper, the Big Decay, the Big Crunch, the Big Brunch, the Big Splat, etc.

We emphasize once again that the new cosmology is based on the accelerated expansion of the Universe discovered at the end of the 20th century, although the Nobel prize awarded in 2011 for the discovery of the accelerating expansion of the Universe formally rendered this effect (if we are allowed to use such a prosaic term for this sublime phenomenon) a reliably established observational fact.

To summarize the current state of affairs, we must honestly admit: we still poorly understand why the expansion of the Universe is accelerating. The remaining doubts can be dispersed by a cross-check of the expansion kinematics, based on different physical mechanisms. If the fundamental conclusion that the expansion of the Universe is accelerating turns out to be wrong, it will be difficult to count on the results obtained by different methods (with the exception of quite improbable chance coincidence) to be in agreement with each other. Until now, the most impressive aspect of cosmological acceleration has consisted in different strategies for its investigation leading to the same conclusion that the Universe has entered the period of accelerating expansion. However, history teaches us not to let down our guard. By the end of the 20th century, we were already under the impression not only that we had acquired an understanding of the fundamentals of the structure of the Universe but also that we could predict distant stages of its evolution. For example, the authors of review [131] boldly claimed: "With our current understanding of physics and astrophysics many issues of the final destiny of the Universe can be qualitatively resolved. Our goal is to continue developing the qualitative theory of the future." The authors of the review traced the development of several dozen of evolutionary stages of the Universe right up to fantastic times of $\sim 10^{100-200}$ years in length. The classification they created was much superior in precision to the one said to be ridiculed by A Averchenko in the following: "The history of fam inhabitants was dark and incomprehensible, but scientists nevertheless divided it into three periods." Such a harmonious picture was destroyed like a house of cards by the force of observations that in 2011 merited a Nobel Prize.

The following has been attributed to A Eddington: "Don't believe the results of experiments until they're confirmed by theory." This comment is not theoretical self-conceit but an understanding that science is not only a set of facts but also their explanation. GR permits an accelerated expansion of a universe filled with a substance with negative pressure, but it still does not provide a profound understanding of this phenomenon. Further investigation of the kinematics and dynamics of the expansion of the Universe will provide work for both experimental cosmologists (observers) and theoreticians for a long time.

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