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Optics of anisotropic media

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Contents

1. **Introduction** **709**
2. **Plane waves in anisotropic media** **710**
   2.1 Plane waves in uniaxial anisotropic media; 2.2 Magnetic fields
3. **Wave reflection from an interface between anisotropic and isotropic media** **711**
   3.1 Nonspecularity of reflection. Transformation of modes; 3.2 Beam splitting with the help of a birefringent cone
4. **Calculation of the refraction at the interfaces of plane-parallel plates and of their transmission** **713**
   4.1 Wave reflection and refraction for incidence on the interface from inside an anisotropic medium; 4.2 General relations based on boundary conditions; 4.3 Wave reflection and refraction for incidence on the interface from a vacuum; 4.4 Reflection and transmission amplitudes of a plane-parallel anisotropic plate with thickness L
5. **D'yakonov surface waves (DSWs)** **716**
   5.1 Waves when the anisotropy axis is parallel to the interface; 5.2 Waves when the anisotropy axis is perpendicular to the propagation direction
6. **A possible experiment for the observation of D'yakonov surface waves** **717**
7. **Conclusion** **718**
8. **Appendices** **718**
   A. Plane waves in a biaxial anisotropic medium; B. Wave reflection from an interface on the side of an anisotropic medium; C. Formulas for reflection and refraction amplitudes at the interface outside of an anisotropic medium

References **720**

Abstract. A new effective analytical approach to describing electromagnetic waves in nonmagnetic anisotropic media is proposed. An analytical description of the refraction and reflection at an interface between isotropic and anisotropic media is demonstrated. Beam splitting upon reflection and refraction is reviewed, and surface wave generation is examined. D'yakonov surface waves and methods of their observation are discussed. Analytical and numerical calculations of the reflection and transmission of plane-parallel uniaxial plates are outlined.

1. Introduction

The description of electromagnetic waves in homogeneous anisotropic media has not changed since Fresnel's time more than 160 years ago. Here, for the first time since then, a new, radically different, approach is presented. We will not waste time describing the standard approach, which can be found in all the textbooks on electrodynamics or optics [1–13][1] containing chapters on anisotropic media; instead, we directly start with our approach.

The challenge is to describe the plane electromagnetic wave

$$\mathcal{E}\exp\left(\mathrm{i}\mathbf{kr}-\mathrm{i}\omega t\right) \tag{1}$$

propagating in an arbitrary direction. We will mainly consider a uniaxial medium, so the problem that faces us is to describe a plane wave with an arbitrary direction of propagation with respect to the anisotropy vector **a**. Describing the wave means finding its polarization vector $\mathcal{E}$ and the magnitude of the wave vector $\mathbf{k}(\omega)$ at a given frequency ω. In isotropic media, vector $\mathcal{E}$ can have an arbitrary direction in the plane perpendicular to the vector **k**, whose length is $k=|\mathbf{k}|=n\omega/c$, where c is the speed of light in vacuum, and n is the index of refraction of the medium. In an anisotropic medium, everything is slightly different.

An anisotropic medium is usually characterized by symmetric permittivity tensor components ε_{ij}. These components are usually treated as phenomenological parameters, so for defining the 3×3 tensor, in general, it is necessary to enter the 6 numbers. Actually, it is much easier. Anisotropy is always determined by a certain direction and a parameter that characterizes a peculiarity of the medium in this direction. In the case of a single preferred axis directed along a unit vector

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[1] In a recently published book by V A Aleshkevich [21] in the series 'University Course of General Physics', anisotropic media are also treated in the standard way. (*Note added in proof.*)

a, the permittivity tensor can be written as

$$\varepsilon_{ij} = \epsilon_1 \delta_{ij} + \epsilon' a_i a_j \,, \tag{2}$$

where a_j are the components of **a**, and ϵ' is a parameter indicating how strong the anisotropy manifests itself. As $\epsilon' \to 0$, the medium becomes isotropic.

The validity of such a representation of a uniaxial anisotropy has been proved by Fedorov [12]. This approach is fruitful not only in electrodynamics but also in the theory of elastic waves [14].

The paper outline is as follows. In the main part of the paper we will mostly present ideas and simple examples. The mathematics necessary for specific calculations is shifted to the Appendices. In Section 2, we find $\boldsymbol{\mathcal{E}}$ and $\mathbf{k}(\omega)$ in a uniaxial anisotropic medium. A description of biaxial media is given in Appendix A. In Section 3, the reflection and refraction of waves at the interface between isotropic and anisotropic media, the splitting of the waves upon reflection, the appearance of surface waves, and a possible device for a laboratory demonstration of the splitting are discussed. In Section 4, we show how to calculate the reflection and transmission of plane-parallel transparent anisotropic plates. Sections 5 and 6 are devoted to D'yakonov surface waves, and the results obtained by D'yakonov himself [15] are complemented. The conclusion (Section 7) summarizes all the results obtained.

2. Plane waves in anisotropic media

First of all, we recall how the wave equation is derived from the Maxwell equations, which in the absence of currents and charges are given by

$$-\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r}, t)\,, \quad \boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r}, t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r}, t)\,,$$
$$\boldsymbol{\nabla}\mathbf{B} = 0\,, \qquad \boldsymbol{\nabla}\mathbf{D} = 0\,, \tag{3}$$

where

$$\mathbf{B} = \mu\mathbf{H}\,, \qquad \mathbf{D} = \varepsilon\mathbf{E}\,, \tag{4}$$

and μ, ε are the magnetic and dielectric constants. In the following, we restrict ourselves to nonmagnetic media and take $\mu = 1$, and then equations (3) are simplified to

$$-\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, t) = \frac{1}{c}\frac{\partial}{\partial t}\mathbf{H}(\mathbf{r}, t)\,, \quad \boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r}, t) = \frac{1}{c}\frac{\partial}{\partial t}\varepsilon\mathbf{E}(\mathbf{r}, t)\,,$$
$$\boldsymbol{\nabla}\mathbf{H} = 0\,, \qquad \boldsymbol{\nabla}\varepsilon\mathbf{E} = 0\,. \tag{5}$$

Differentiating the second equation in set (5) over time and substituting it into the first equation yields the wave equation

$$-\boldsymbol{\nabla} \times \left[\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r}, t)\right] = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varepsilon\mathbf{E}(\mathbf{r}, t)\,. \tag{6}$$

Note (this is mentioned in no textbook) that this derivation is suitable only for homogeneous media. In the presence of an interface between two half-spaces, we get different equations in them, and transition across the border is governed by the boundary conditions that are dictated by the Maxwell equations themselves. By comparison, the Schrödinger wave equation in quantum mechanics is given once in the whole space, and boundary conditions are dictated by the Schrödinger equation itself.

2.1 Plane waves in uniaxial anisotropic media

In a uniaxial anisotropic medium, we choose the tensor ε in the form (2). Therefore, for a plane wave (1), we obtain

$$\varepsilon\boldsymbol{\mathcal{E}} = \epsilon_1\boldsymbol{\mathcal{E}} + \epsilon'\mathbf{a}(\mathbf{a}\boldsymbol{\mathcal{E}})\,, \tag{7}$$

and the last equation in set (5) is equivalent to

$$\epsilon_1\mathbf{k}\boldsymbol{\mathcal{E}} + \epsilon'(\mathbf{k}\mathbf{a})(\mathbf{a}\boldsymbol{\mathcal{E}}) = 0\,. \tag{8}$$

Substituting plane wave (1) into equation (6) leads to

$$k^2\boldsymbol{\mathcal{E}} - \mathbf{k}(\mathbf{k}\boldsymbol{\mathcal{E}}) = k_0^2\varepsilon\boldsymbol{\mathcal{E}}\,, \tag{9}$$

where $k_0 = \omega/c$, and substitution of formula (7) into (9) gives

$$k^2\boldsymbol{\mathcal{E}} - \mathbf{k}(\mathbf{k}\boldsymbol{\mathcal{E}}) - k_0^2\varepsilon\boldsymbol{\mathcal{E}}$$
$$\equiv (k^2 - k_0^2\epsilon_1)\boldsymbol{\mathcal{E}} - \mathbf{k}(\mathbf{k}\boldsymbol{\mathcal{E}}) - k_0^2\epsilon'\mathbf{a}(\mathbf{a}\boldsymbol{\mathcal{E}}) = 0\,. \tag{10}$$

To find $\boldsymbol{\mathcal{E}}$, we need to solve equation (10) with due account of Eqn (8).

The 3-dimensional vector $\boldsymbol{\mathcal{E}}$ can be represented by coordinates in some basis. If **k** is not parallel to **a**, we can use three independent vectors: **a**, $\boldsymbol{\kappa} = \mathbf{k}/k$, and

$$\mathbf{e}_1 = \mathbf{a} \times \boldsymbol{\kappa} \tag{11}$$

as a basis . In this basis (notice that it is not orthonormal), $\boldsymbol{\mathcal{E}}$ is presented as

$$\boldsymbol{\mathcal{E}} = \alpha\mathbf{a} + \beta\boldsymbol{\kappa} + \gamma\mathbf{e}_1\,, \tag{12}$$

with coordinates α, β, and γ being not independent, because of Eqns (8) and (10).

Substitution of expression (12) into equation (8) gives

$$\epsilon_1(k\beta + \alpha\mathbf{k}\mathbf{a}) + \epsilon'\mathbf{k}\mathbf{a}(\alpha + \beta\boldsymbol{\kappa}\mathbf{a}) = 0\,, \tag{13}$$

from which it follows that

$$\beta = -\frac{(1+\eta)\boldsymbol{\kappa}\mathbf{a}}{1+\eta(\boldsymbol{\kappa}\mathbf{a})^2}\,\alpha\,, \tag{14}$$

where $\eta = \epsilon'/\epsilon_1$. Substitution of formula (14) into expansion (12) yields

$$\boldsymbol{\mathcal{E}} = \alpha\left[\mathbf{a} - \boldsymbol{\kappa}\frac{(1+\eta)\boldsymbol{\kappa}\mathbf{a}}{1+\eta(\boldsymbol{\kappa}\mathbf{a})^2}\right] + \gamma\mathbf{e}_1 = \alpha\mathbf{e}_2 + \gamma\mathbf{e}_1\,, \tag{15}$$

which shows that the wave polarization vector $\boldsymbol{\mathcal{E}}$ lies in a plane of two independent vectors: $\mathbf{e}_1 = \mathbf{a} \times \boldsymbol{\kappa}$, and the vector orthogonal to it, namely

$$\mathbf{e}_2 = \mathbf{a} - \frac{1+\eta}{1+\eta(\boldsymbol{\kappa}\mathbf{a})^2}\,\boldsymbol{\kappa}(\boldsymbol{\kappa}\mathbf{a}) \equiv \mathbf{a} - \frac{\epsilon_2(\theta)}{\epsilon_1}\,\boldsymbol{\kappa}(\boldsymbol{\kappa}\mathbf{a})\,, \tag{16}$$

where the angle θ between vectors $\boldsymbol{\kappa}$ and **a**, and anisotropic dielectric constant

$$\epsilon_2(\theta) = \epsilon_1\,\frac{1+\eta}{1+\eta\cos^2\theta} \tag{17}$$

were introduced.

To find α and β coordinates, we substitute formula (15) into equation (10) and multiply it by $\mathbf{e}_1$. As a result, we obtain

$$(k^2 - k_0^2\epsilon_1)\gamma\mathbf{e}_1^2 = 0\,. \tag{18}$$

It follows, thence, that if $\gamma \neq 0$, then equation (18) can be satisfied only when

$$k^2 = k_0^2\epsilon_1\,. \tag{19}$$

Multiplying equation (10) by $\mathbf{a}$ and taking into account that

$$\mathbf{a}\mathbf{e}_2 = \frac{1-(\boldsymbol{\kappa}\mathbf{a})^2}{1+\eta(\boldsymbol{\kappa}\mathbf{a})^2}\,, \qquad \boldsymbol{\kappa}\mathbf{e}_2 = -\eta\boldsymbol{\kappa}\mathbf{a}\,\frac{1-(\boldsymbol{\kappa}\mathbf{a})^2}{1+\eta(\boldsymbol{\kappa}\mathbf{a})^2} = -\eta(\boldsymbol{\kappa}\mathbf{a})(\mathbf{a}\mathbf{e}_2)\,, \tag{20}$$

we arrive at

$$\big(k^2 - k_0^2\epsilon_2(\theta)\big)\alpha\mathbf{a}\mathbf{e}_2 = 0\,. \tag{21}$$

Therefore, if $\alpha \neq 0$ and $\mathbf{a} \neq \boldsymbol{\kappa}$, equation (21) can be satisfied only when

$$k^2 = k_0^2\epsilon_2(\theta)\,, \tag{22}$$

where $\epsilon_2(\theta)$ is given by relationship (17). Since the length of k is different for two polarization vectors, a single plain wave can exist only with a single polarization along either $\mathbf{e}_2$ or $\mathbf{e}_1$.

Thus, in general, only two types of plane waves can propagate in any given direction. One wave has polarization vector $\mathbf{e}_1$, and the other $\mathbf{e}_2$. Only when propagating along the anisotropy vector ($\boldsymbol{\kappa} = \mathbf{a}$) can the plane wave, as in the isotropic case, have an arbitrary polarization perpendicular to the wave vector $\mathbf{k}$. Thus, $\mathbf{k}^2 = \epsilon_1 k_0^2$.

Generally, we call the wave mode with polarization $\boldsymbol{\mathcal{E}}_1 = \mathbf{e}_1$ 'transverse', because $\mathbf{e}_1 \perp \mathbf{k}$, and the wave mode with polarization along $\boldsymbol{\mathcal{E}}_2 = \mathbf{e}_2$ 'mixed', because this mode polarization vector, according to expressions (20), includes a longitudinal component, i.e. a component directed along the vector $\boldsymbol{\kappa}$. We believe that such a nomenclature is more meaningful than the meaningless common names 'ordinary' for a wave with polarization $\boldsymbol{\mathcal{E}}_1 = \mathbf{e}_1$, and 'extraordinary' for a wave with polarization $\boldsymbol{\mathcal{E}}_2 = \mathbf{e}_2$.

This part of our work is the most important. It shows a very elementary way to describing plane waves with an arbitrary direction of their propagation relative to the anisotropy axis, i.e. to finding wave-vector lengths and their vectors of linear polarization. With our description, we do not need such auxiliary notions like a ray or wave surface, wave-vector or dielectric constant ellipsoids. This is the first time it has become possible since Fresnel's time. It is just for the sake of the simplicity and beauty of this branch of physics that we decided to 'reinvent the wheel'.

2.2 Magnetic fields

Every electromagnetic wave, beside an electric field, contains the magnetic one. From the equation $\nabla\mathbf{H} = 0$, which is equivalent to $\mathbf{k}\mathbf{H} = 0$ for the plane wave, it follows that the field $\mathbf{H}$ is always orthogonal to $\mathbf{k}$. It is also orthogonal to $\boldsymbol{\mathcal{E}}$, which follows from the first equation in set (5). After substituting definition (1) into this equation, as well as the field $\mathbf{H}$ in the plane-wave form

$$\mathbf{H}(\mathbf{r}, t) = \boldsymbol{\mathcal{H}}\exp\left(\mathrm{i}\mathbf{k}\mathbf{r} - \mathrm{i}\omega t\right), \tag{23}$$

with the magnetic polarization vector directed along $\boldsymbol{\mathcal{H}}$, we arrive at

$$\boldsymbol{\mathcal{H}} = \frac{k}{k_0}\,\boldsymbol{\kappa}\times\boldsymbol{\mathcal{E}}\,. \tag{24}$$

For transverse and mixed modes in uniaxial media, respectively, we therefore obtain

$$\begin{aligned}\boldsymbol{\mathcal{H}}_1 &= \frac{k}{k_0}\,\boldsymbol{\kappa}\times\mathbf{e}_1 = \frac{k}{k_0}\,\boldsymbol{\kappa}\times[\mathbf{a}\times\boldsymbol{\kappa}]\,,\\ \boldsymbol{\mathcal{H}}_2 &= \frac{k}{k_0}\,\boldsymbol{\kappa}\times\mathbf{e}_2 = \frac{k}{k_0}\,\boldsymbol{\kappa}\times\mathbf{a}\,,\end{aligned} \tag{25}$$

and the total plane-wave field takes the form

$$\boldsymbol{\Psi}(\mathbf{r}, t) = \boldsymbol{\psi}_j\exp\left(\mathrm{i}\mathbf{k}_j\mathbf{r} - \mathrm{i}\omega t\right), \tag{26}$$

where $\boldsymbol{\psi}_j = \boldsymbol{\mathcal{E}}_j + \boldsymbol{\mathcal{H}}_j$, and j enumerates mode 1 or 2. In isotropic media, we can also choose, say, $\boldsymbol{\mathcal{E}} = \mathbf{a}\times\boldsymbol{\kappa}$ and $\boldsymbol{\mathcal{H}} = \boldsymbol{\kappa}\times[\mathbf{a}\times\boldsymbol{\kappa}]$. However, there $\mathbf{a}$ can have an arbitrary direction; therefore, the pair of orthogonal vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ can be rotated through any angle about the wave vector $\mathbf{k}$.

3. Wave reflection from an interface between anisotropic and isotropic media

Suppose that our space is split by the plane $z = 0$ into two half-spaces. The part for $z < 0$ is a uniaxial anisotropic medium, and the part for $z > 0$ is a vacuum with $\epsilon_1 = 1$, $\eta = 0$. We have two different wave equations in these parts, and the waves go from one medium to the other through the interface where they must obey boundary conditions imposed by the Maxwell equations.

Let us look for reflection of the two possible modes incident on the interface from within the anisotropic medium.

3.1 Nonspecularity of reflection. Transformation of modes

First, we note that reflection of the mixed mode is not, in general, specular. Indeed, since the direction of $\mathbf{k}$ after reflection changes, the angle θ between $\mathbf{a}$ and $\boldsymbol{\kappa}$ also changes, and $\mathbf{k}$, according to condition (22), changes too. However, the $\mathbf{k}_\parallel$ components parallel to the interface do not change, so the change in k means a change in the normal component $k_\perp$, and this testifies to nonspecularity of the wave reflection.

Let us calculate the change in $k_\perp$ for the incident mixed mode with the wave vector $\mathbf{k}_{2\mathrm{r}}$, where the subscript r means that mode 2 propagates to the right, toward the interface. For a given angle θ between $\mathbf{k}_{2\mathrm{r}}$ and $\mathbf{a}$, we can write

$$k_{2\mathrm{r}\perp} = \sqrt{\frac{\epsilon_1 k_0^2(1+\eta)}{1+\eta\cos^2\theta} - k_\parallel^2}\,; \tag{27}$$

however, the quantity $k_{2\mathrm{r}\perp}$ enters implicitly into $\cos\theta$, so for finding the explicit dependence of $k_{2r\perp}$ on $\mathbf{a}$, it is necessary to solve the equation

$$k_\parallel^2 + x^2 + \eta(k_\parallel\mathbf{l}\mathbf{a} + x\mathbf{n}\mathbf{a})^2 = k_0^2\epsilon_1(1+\eta)\,, \tag{28}$$

where x denotes $k_{2\mathrm{r}\perp}$, $\mathbf{n}$ is a unit vector of the normal directed toward the isotropic medium, and $\mathbf{l}$ is a unit vector along $\mathbf{k}_\parallel$, which, together with $\mathbf{n}$, constitutes the plane of incidence. The

solution of this equation assumes the form

$$x = \frac{1}{1+\eta(\mathbf{na})^2}\Big[-\eta k_\parallel(\mathbf{na})(\mathbf{la}) + \sqrt{\epsilon_1 k_0^2(1+\eta)\big(1+\eta(\mathbf{na})^2\big) - k_\parallel^2\big(1+\eta(\mathbf{la})^2+\eta(\mathbf{na})^2\big)}\Big]. \tag{29}$$

The sign chosen before the square root provides the correct asymptotics as $\eta \to 0$, equal to the isotropic value of $x \sim (\epsilon_1 k_0^2 - k_\parallel^2)^{1/2}$.

In general, vector $\mathbf{a}$ is representable as $\mathbf{a} = \alpha\mathbf{n} + \beta\mathbf{l} + \gamma\mathbf{t}$, where $\mathbf{t} = [\mathbf{nl}]$ is a unit vector perpendicular to the plane of incidence. The normal component $k_{2\mathrm{r}\perp}$ depends only on part of this vector, $\mathbf{a}' = \alpha\mathbf{n} + \beta\mathbf{l}$, which lies in the plane of incidence. If we denote $\alpha = |\mathbf{a}'|\cos\theta_a$, $\beta = |\mathbf{a}'|\sin\theta_a$, where $|\mathbf{a}'|$ is the projection of $\mathbf{a}$ onto the plane of incidence, and introduce new parameter $\eta' = \eta|\mathbf{a}'|^2 \leqslant \eta$, then formula (29) is simplified to

$$k_{2\mathrm{r}\perp} = \frac{1}{2(1+\eta'\cos^2\theta_a)}\Big[-\eta' k_\parallel \sin(2\theta_a) + 2\sqrt{\epsilon_1 k_0^2(1+\eta)(1+\eta'\cos^2\theta_a) - k_\parallel^2(1+\eta')}\Big]. \tag{30}$$

For the reflected mixed mode (mode 2, propagating to the left of the interface), equation (28) takes the form

$$k_\parallel^2 + x^2 + \eta(k_\parallel\mathbf{la} - x\mathbf{na})^2 = k_0^2\epsilon_1(1+\eta), \tag{31}$$

where $x = k_{2\mathrm{l}\perp}$, and its solution is written as

$$k_{2\mathrm{l}\perp} = \frac{1}{2(1+\eta'\cos^2\theta_a)}\Big[\eta' k_\parallel \sin(2\theta_a) + 2\sqrt{\epsilon_1 k_0^2(1+\eta)(1+\eta'\cos^2\theta_a) - k_\parallel^2(1+\eta')}\Big]. \tag{32}$$

It is evident that the difference between the normal wave-number components of the incident and reflected waves of mixed modes, $k_{2\mathrm{l}\perp} - k_{2\mathrm{r}\perp}$, is

$$k_{2\mathrm{l}\perp} - k_{2\mathrm{r}\perp} = \frac{\eta' k_\parallel \sin(2\theta_a)}{1+\eta'\cos^2\theta_a}. \tag{33}$$

In the following, we will present such differences in dimensionless variables:

$$\Delta_{22} \equiv \frac{k_{2\mathrm{l}\perp} - k_{2\mathrm{r}\perp}}{k_0\sqrt{\epsilon_1}} = \frac{\eta' q \sin(2\theta_a)}{1+\eta'\cos^2\theta_a}, \tag{34}$$

where $q^2 = k_\parallel^2/k_0^2\epsilon_1$. The reflection angle depends on the orientation of anisotropy vector $\mathbf{a}$ and can be larger than the specular one, when $\theta_a > 0$, or smaller, when $\theta_a < 0$.

In the case of the transverse incident mode, the length $k = |\mathbf{k}|$ of the wave vector, according to formula (19), does not depend on the orientation of $\mathbf{a}$; therefore, this wave is reflected specularly.

Generally, every incident mode after reflection creates another one, because it is impossible to satisfy the boundary conditions without another mode. Let us look at what the normal component of the wave vector of the other mode will be.

If the incident wave belongs to mode 2, the reflected transverse mode (mode 1 propagating to the left, away from the interface) will have $k_{1\mathrm{l}\perp} = (\epsilon_1 k_0^2 - k_\parallel^2)^{1/2}$. Therefore, according to (30), the difference $\Delta_{12} = (k_{1\mathrm{l}\perp} - k_{2\mathrm{l}\perp})/k_0\sqrt{\epsilon_1}$ is written out in the form

$$\Delta_{12} = \sqrt{1-q^2} - \frac{\eta' q \sin(2\theta_a) + 2\sqrt{(1+\eta)(1+\eta'\cos^2\theta_a) - q^2(1+\eta')}}{2(1+\eta'\cos^2\theta_a)}. \tag{35}$$

In the opposite case, when the incident mode is a transverse one, the reflected mixed mode will have $k_{2\mathrm{l}\perp}$ shown in (32). Therefore, the difference $\Delta_{21} = (k_{2\mathrm{l}\perp} - k_{1\mathrm{l}\perp})/k_0\sqrt{\epsilon_1} = -\Delta_{12}$.

Since reflection of mode 2 is, in general, nonspecular, it can happen that the wave vectors of reflected and transmitted waves will be arranged as shown in Fig. 1, and it follows that there are two critical angles for φ. The first critical angle, $\varphi_{\mathrm{c}1}$ ($q^2 = 1/\epsilon_1$), is the angle of total reflection. The transmitted wave becomes there evanescent. The totally reflected field contains two modes. At the second critical angle, $\varphi_{\mathrm{c}2}$, when q falls in the range

$$1 < q^2 < \frac{(1+\eta)(1+\eta'\cos^2\theta_a)}{1+\eta'}, \tag{36}$$

the reflected mode 1 also becomes evanescent. Together with the evanescent transmitted wave, mode 1 constitutes a surface wave, propagating along the interface. In this case, we have nonspecular total reflection of the single mode $\mathcal{E}_2$ and the surface wave tied to it.

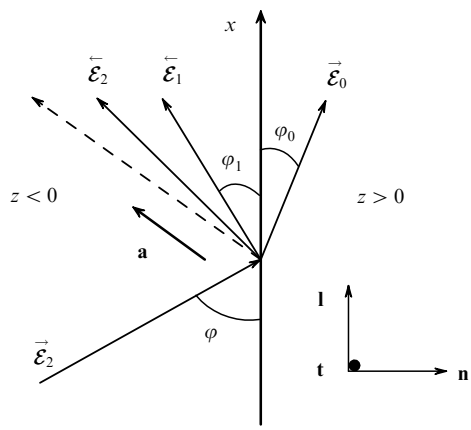

Figure 1. Arrangement of wave vectors of all the modes created by the incident wave of mode 2, i.e. of polarization $\overrightarrow{\mathcal{E}}_2$, when the anisotropy vector $\mathbf{a}$ has the direction shown here. The grazing angle of reflected mode 2, $\overleftarrow{\mathcal{E}}_2$, is less than the specular one (the specular direction is shown by the dashed arrow), and the grazing angle φ_1 of the reflected mode 1, $\overleftarrow{\mathcal{E}}_1$, is still lower. The grazing angle φ_0 of the transmitted wave $\overrightarrow{\mathcal{E}}_0$ is even lower than φ_1. It can be inferred that, at some critical value $\varphi = \varphi_{\mathrm{c}1}$, the angle φ_0 becomes zero, meaning that for $\varphi < \varphi_{\mathrm{c}1}$ the transmitted wave becomes evanescent and all the incident energy is totally reflected in the form of two modes. Moreover, there is a second critical angle $\varphi_{\mathrm{c}2}$, whereat $\varphi_1 = 0$. Below this angle for $\varphi < \varphi_{\mathrm{c}2}$, the mode $\overleftarrow{\mathcal{E}}_1$ also becomes evanescent. In this case, all the incident energy is totally reflected nonspecularly in the form of mode 2. At the same time, the two evanescent waves $\overrightarrow{\mathcal{E}}_0$ and $\overleftarrow{\mathcal{E}}_2$ combine into a surface wave propagating along the interface. The arrows over $\mathcal{E}$ show the directions of wave propagation with respect to the interface. In the figure is also shown the basis consisting of unit vector $\mathbf{n}$ along the normal (z-axis), unit tangential vector $\mathbf{l}$ (x-axis) which, together with $\mathbf{n}$, defines the plane of incidence, and vector $\mathbf{t}$ (y-axis) pointing toward the reader, which is normal to the plane of incidence.

3.2 Beam splitting with the help of a birefringent cone

Beam splitting at the interfaces of an anisotropic medium can be spectacularly demonstrated with the help of a birefringent cone, as illustrated in Fig. 2. In the geometrical optics approximation, a narrow incident beam of light, after refraction on the side surface of the cone, is split into two rays of two different modes, 1 and 2. Both modes are further split into two components upon reflection from the base of the cone. The four resulting beams, after refraction on the side surface, leave the cone and produce four bright spots on a vertical screen. Their positions and brightness depend on the direction of the anisotropy axis inside the cone and vary with cone rotation.

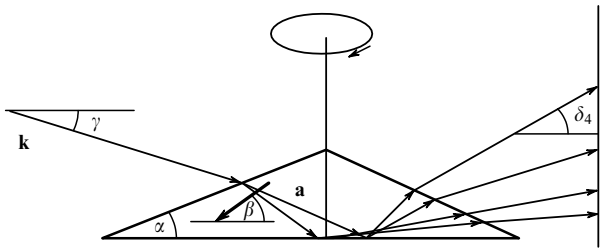

Figure 2. Demonstration of the light beam splitting in a birefringent cone. Bright spots on a vertical screen change their positions and brightness when the cone is rotated about a vertical axis.

The direct numerical calculations for parameters $\epsilon_1 = 1.6$, $\eta = 0.8$, with vector **a** in the figure plane, and for $\sin\alpha = 0.5$, $\sin\beta = 0.3$, and $\sin\gamma = 0.5$ show that outgoing beams moving upwards have directions characterized by $\tan\delta_1 = 0.2$, $\tan\delta_2 = 0.4$, $\tan\delta_3 = 0.6$, and $\tan\delta_4 = 0.7$, respectively.

4. Calculation of the refraction at the interfaces of plane-parallel plates and of their transmission

To describe the reflection and transmission of a plane-parallel anisotropic plate placed in an isotropic medium (for instance, a vacuum), it is necessary to know the reflections and refractions at interfaces from inside and outside the anisotropic medium, which are obtained by imposing boundary conditions stemming from the Maxwell equations. Knowledge of everything at the interfaces permits writing directly the reflection and transmission of the plate by the method which is explained in handbook [16] and will be briefly described below (the corresponding mathematical formulas are presented in the Appendices).

4.1 Wave reflection and refraction for incidence on the interface from inside an anisotropic medium

The wave function of the electromagnetic field in the full space may be represented as

$$\Psi(\mathbf{r}) = \Theta(z<0)\left(\exp(\mathrm{i}\vec{\mathbf{k}}_j\mathbf{r})\vec{\psi}_j + \sum_{j'=1,2}\exp(\mathrm{i}\overleftarrow{\mathbf{k}}_{j'}\mathbf{r})\overleftarrow{\psi}_{j'}\vec{\rho}_{j'j}\right) + \Theta(z>0)\exp(\mathrm{i}\mathbf{k}_0\mathbf{r})\left(\psi_{\mathrm{e}}\vec{\tau}_{\mathrm{e}j} + \psi_{\mathrm{m}}\vec{\tau}_{\mathrm{m}j}\right), \tag{37}$$

where $\boldsymbol{\psi} = \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{H}}$, Θ is a step function equal to unity when the inequality in its argument is satisfied, and to zero in the opposite case, half-space $z < 0$ is occupied by an anisotropic medium, and the half-space $z > 0$ is a vacuum. The arrows above the quantities mark the direction of wave propagation: $\vec{\psi}_j$ denotes the incident wave of mode j $(j = 1, 2)$, $\overleftarrow{\psi}_{j'}$ $(j' = 1, 2)$ denotes a reflected wave of mode j', $\vec{\mathbf{k}}_j = (\mathbf{k}_\parallel, k_{j\mathrm{r}\perp})$, $\overleftarrow{\mathbf{k}}_{j'} = (\mathbf{k}_\parallel, -k_{j'1\perp})$, $\mathbf{k}_0 = (\mathbf{k}_\parallel, (k_0^2 - k_\parallel^2)^{1/2})$, and $\psi_{\mathrm{e,m}}$, $\vec{\tau}_{\mathrm{e,m}j}$ are the refracted fields and refraction amplitudes of TE- and TM-modes, respectively, for the incident j-mode. To find the reflection $(\vec{\rho})$ and refraction $(\vec{\tau})$ amplitudes (the arrows over them point to the direction of propagation of the incident wave toward the interface), we need to impose boundary conditions on the field wave function (37).

4.2 General relations based on boundary conditions

Every incident wave field can be decomposed into TE- and TM-modes at the interface. In the TE-mode, the electric field is perpendicular to the plane of incidence, $\boldsymbol{\mathcal{E}} \propto \mathbf{t}$; therefore, the contribution of the jth mode to the TE-mode is $\boldsymbol{\mathcal{E}}_j\mathbf{t}$. In the TM-mode, the magnetic field polarization is perpendicular to the plane of incidence, $\boldsymbol{\mathcal{H}} \propto \mathbf{t}$; therefore, the contribution of the jth mode to the TM-mode is $\boldsymbol{\mathcal{H}}_j\mathbf{t}$. For a refracted field in the TE-mode, we may at once accept $\vec{\boldsymbol{\mathcal{E}}}_{\mathrm{e}} = \mathbf{t}$, $\vec{\boldsymbol{\mathcal{H}}}_{\mathrm{e}} = \boldsymbol{\kappa}_0 \times \mathbf{t}$, and for a refracted field in the TM-mode we accept $\vec{\boldsymbol{\mathcal{H}}}_{\mathrm{m}} = \mathbf{t}$, $\vec{\boldsymbol{\mathcal{E}}}_{\mathrm{m}} = -\boldsymbol{\kappa}_0 \times \mathbf{t}$.

4.2.1 Boundary conditions for TE-modes. In the TE-mode of an electromagnetic field for an incident j-mode, we derive the following three equations from the boundary conditions.

(1) Continuity of the electric field (the field vector is parallel to the interface) yields

$$\mathbf{t}\vec{\boldsymbol{\mathcal{E}}}_j + \mathbf{t}\overleftarrow{\boldsymbol{\mathcal{E}}}_1\vec{\rho}_{1j} + \mathbf{t}\overleftarrow{\boldsymbol{\mathcal{E}}}_2\vec{\rho}_{2j} = \mathbf{t}\vec{\boldsymbol{\mathcal{E}}}_{\mathrm{e}}\vec{\tau}_{\mathrm{e}j}. \tag{38}$$

(2) Continuity of the magnetic field component parallel to the interface and the plane of incidence yields

$$\mathbf{l}\vec{\boldsymbol{\mathcal{H}}}_j + \mathbf{l}\overleftarrow{\boldsymbol{\mathcal{H}}}_1\vec{\rho}_{1j} + \mathbf{l}\overleftarrow{\boldsymbol{\mathcal{H}}}_2\vec{\rho}_{2j} = \mathbf{l}[\boldsymbol{\kappa}_0 \times \mathbf{t}]\vec{\tau}_{\mathrm{e}j} \equiv -\kappa_{0\perp}\vec{\tau}_{\mathrm{e}j}. \tag{39}$$

(3) Continuity of the normal component of magnetic induction in the plane of incidence yields (for $\mu = 1$)

$$\mathbf{n}\vec{\boldsymbol{\mathcal{H}}}_j + \mathbf{n}\overleftarrow{\boldsymbol{\mathcal{H}}}_1\vec{\rho}_{1j} + \mathbf{n}\overleftarrow{\boldsymbol{\mathcal{H}}}_2\vec{\rho}_{2j} = \mathbf{n}[\boldsymbol{\kappa}_0 \times \mathbf{t}]\vec{\tau}_{\mathrm{e}j} \equiv \kappa_{0\parallel}\vec{\tau}_{\mathrm{e}j}. \tag{40}$$

The last equation is, in fact, not needed, because it coincides with equation (38).

4.2.2 Boundary conditions for TM-modes. In the TM-mode, we also have three equations following from the boundary conditions.

(1) Continuity of the magnetic field (the field vector is parallel to the interface) yields

$$\mathbf{t}\vec{\boldsymbol{\mathcal{H}}}_j + \mathbf{t}\overleftarrow{\boldsymbol{\mathcal{H}}}_1\vec{\rho}_{1j} + \mathbf{t}\overleftarrow{\boldsymbol{\mathcal{H}}}_2\vec{\rho}_{2j} = \vec{\tau}_{\mathrm{m}j}. \tag{41}$$

(2) Continuity of the electric field component parallel to the interface and the plane of incidence yields

$$\mathbf{l}\vec{\boldsymbol{\mathcal{E}}}_j + \mathbf{l}\overleftarrow{\boldsymbol{\mathcal{E}}}_1\vec{\rho}_{1j} + \mathbf{l}\overleftarrow{\boldsymbol{\mathcal{E}}}_2\vec{\rho}_{2j} = -\mathbf{l}[\boldsymbol{\kappa}_0 \times \mathbf{t}]\vec{\tau}_{\mathrm{m}j} \equiv \kappa_{0\perp}\vec{\tau}_{\mathrm{m}j}. \tag{42}$$

(3) Continuity of the normal component of field **D** in the plane of incidence yields

$$\mathbf{n}\varepsilon\vec{\mathcal{E}}_j + \mathbf{n}\varepsilon\overleftarrow{\mathcal{E}}_1\vec{\rho}_{1j} + \mathbf{n}\varepsilon\overleftarrow{\mathcal{E}}_2\vec{\rho}_{2j} = \mathbf{n}[\boldsymbol{\kappa}_0 \times \mathbf{t}]\vec{\tau}_{\mathrm{m}j} \equiv \kappa_{0\|}\vec{\tau}_{\mathrm{m}j}\,. \quad (43)$$

Again, we can neglect the last equation, because it coincides with Eqn (41). In the following, we will not show third equations like (40) and (43), because they are useless.

So, there are four linear equations from which we can find four quantities: the two related to refracted TE and TM waves, and two others related to reflected waves of modes 1 and 2. This shows why the reflection of one mode generally produces a different mode.

The resulting linear equations are easily solved in the general case, and corresponding solutions are given in Appendix B. Here, we restrict ourselves to the particular case of normal incidence of the waves.

4.2.3 A particular case of normal incidence. In the case of normal incidence, wave reflection and refraction are especially simple, because there is no splitting upon reflection. We shall define the appropriate geometry by three basis vectors: **n**, **l**, and **t**, where **n** denotes the normal directed along the z-axis toward a vacuum, while vectors **l** and **t** lie on the interface and define the directions of x- and y-axes, respectively. The anisotropy vector **a** is supposed to lie in the (x, z) plane at angle θ with respect to **n**.

A plane wave propagating along **n** ($\boldsymbol{\kappa} = \mathbf{k}/k = \mathbf{n}$) can have only two types of polarization: a transverse mode with $\mathcal{E}_1 = \mathbf{e}_1 \equiv \mathbf{t}$ and $\mathcal{H}_1 = -n_1\mathbf{l}$, where $n_1 = \sqrt{\epsilon_1}$ [see expression (2)], or a mixed mode with $\mathcal{E}_2 = \mathbf{e}_2$ [see expression (16)] and $\mathcal{H}_2 = n_2(\theta)\,\mathbf{n} \times \mathbf{a}$, where $n_2(\theta) = \sqrt{\epsilon_2(\theta)}$ [see expression (17)].

Since there is no wave splitting at normal incidence, the boundary conditions are simplified. For mode 1, equations (38) and (39) are reduced to

$$1 + \vec{\rho}_{11} = \vec{\tau}_{\mathrm{e}1}\,, \qquad n_1(1 - \vec{\rho}_{11}) = \vec{\tau}_{\mathrm{e}1}\,, \quad (44)$$

and the transmitted wave has field polarization $\mathcal{E}_{\mathrm{e}1} = \mathbf{t}$, $\mathcal{H}_{\mathrm{e}1} = -\mathbf{l}$, identical to that of the incident field.

The solution of equation (44) is as follows:

$$\vec{\rho}_{11} = \frac{n_1 - 1}{n_1 + 1}\,, \qquad \vec{\tau}_{\mathrm{e}1} = \frac{2n_1}{n_1 + 1}\,, \qquad n_1 = \sqrt{\epsilon_1}\,. \quad (45)$$

With these formulas, we can immediately find the reflection and transmission of a plane-parallel plate with thickness D for an electromagnetic wave with polarization $\mathcal{E}_{\mathrm{e}1} = \mathbf{t}$ incident from a vacuum:

$$R_1 = -\vec{\rho}_{11}\,\frac{1 - \exp(2\mathrm{i}k_1 D)}{1 - \vec{\rho}_{11}^{\,2}\exp(2\mathrm{i}k_1 D)}\,,$$
$$T_1 = \exp(\mathrm{i}k_1 D)\,\frac{1 - \vec{\rho}_{11}^{\,2}}{1 - \vec{\rho}_{11}^{\,2}\exp(2\mathrm{i}k_1 D)}\,, \quad (46)$$

where $k_1 = k_0 n_1$. We see that the incident plane wave with linear polarization $\mathcal{E}_{\mathrm{e}1} = \mathbf{t}$ parallel to that of mode 1 inside the plate does not change polarization direction after transmission through the plate.

Now let us apply boundary conditions to the mixed mode $\mathcal{E}_2$ incident normally on the interface from inside the plate. The boundary conditions are then reduced to

$$\mathbf{la}\big(1 + \vec{\rho}_{22}(\theta)\big) = \vec{\tau}_{\mathrm{e}2}\,,$$
$$k_2(\theta)\,\mathbf{la}\big(1 - \vec{\rho}_{22}(\theta)\big) = k_0\vec{\tau}_{\mathrm{e}2}\,, \quad (47)$$

where the factor **la** appears because of the projection of the polarization vector onto the interface. Therefore, since $k_2 = k_0 n_2(\theta)$, and $n_2(\theta) = \sqrt{\epsilon_2(\theta)}$, the solutions of equations (47) are as follows:

$$\vec{\rho}_{22}(\theta) = \frac{n_2(\theta) - 1}{n_2(\theta) + 1}\,, \qquad \vec{\tau}_{\mathrm{e}2}(\theta) = \frac{2n_2(\theta)\,\mathbf{la}}{n_2(\theta) + 1}\,. \quad (48)$$

From the symmetry consideration, we can immediately find reflection and transmission amplitudes of the wave with unit polarization along **l**, which is incident on the interface from a vacuum:

$$\overleftarrow{\rho}_{\mathrm{e}2}(\theta) = \frac{1 - n_2(\theta)}{1 + n_2(\theta)}\,, \qquad \overleftarrow{\tau}_{\mathrm{e}2}(\theta) = \frac{2}{\mathbf{la}\big(1 + n_2(\theta)\big)}\,. \quad (49)$$

Whence, we can immediately find the reflection and transmission amplitudes for a plane-parallel plate of thickness L for a plane electromagnetic wave with polarization $\mathcal{E}_{\mathrm{e}2} = \mathbf{l}$ incident from a vacuum:

$$R_2(\theta) = -\vec{\rho}_{22}(\theta)\,\frac{1 - \exp\big(2\mathrm{i}k_2(\theta)L\big)}{1 - \vec{\rho}_{22}^{\,2}(\theta)\exp\big(2\mathrm{i}k_2(\theta)L\big)}\,,$$
$$T_2(\theta) = \exp\big(\mathrm{i}k_2(\theta)L\big)\,\frac{1 - \vec{\rho}_{22}^{\,2}(\theta)}{1 - \vec{\rho}_{22}^{\,2}(\theta)\exp\big(2\mathrm{i}k_2(\theta)L\big)}\,, \quad (50)$$

where $k_2(\theta) = k_0 n_2(\theta)$. We see that the incident wave with linear polarization $\mathcal{E}_{\mathrm{e}2} = \mathbf{l}$, which lies in the plane (**na**), does not change polarization direction after transmission through the plate.

Now we will consider transmission through the plate of a plane wave $\exp(\mathrm{i}k_0 z - \mathrm{i}\omega_0 t)\,\mathcal{E}_{\mathrm{e}}$ with intermediate polarization $\mathcal{E}_{\mathrm{e}} = \alpha\mathbf{t} + \beta\mathbf{l}$, where $|\alpha|^2 + |\beta|^2 = 1$. The transmitted electrical part of the wave, namely

$$\mathbf{E}_t(z, t) = \exp\big(\mathrm{i}k_0(z - L) - \mathrm{i}\omega_0 t\big)\big[\alpha T_1\mathbf{t} + \beta T_2(\theta)\,\mathbf{l}\big]\,, \quad (51)$$

will generally have elliptical polarization.

4.3 Wave reflection and refraction for incidence on the interface from a vacuum

We now consider the case where the half-space for $z < 0$ is a vacuum, and that for $z > 0$ is an anisotropic medium. The incident wave moves from the left in the vacuum. The wave function in the full space now has the form

$$\boldsymbol{\Psi}(\mathbf{r}) = \Theta(z < 0)\Big[\exp\big(\mathrm{i}\vec{\mathbf{k}}_0\mathbf{r}\big)\vec{\psi}_j + \exp\big(\mathrm{i}\overleftarrow{\mathbf{k}}_0\mathbf{r}\big)\sum_{j'=\mathrm{e,m}}\overleftarrow{\psi}_{j'}\vec{\rho}_{j'j}\Big]$$
$$+\,\Theta(z > 0)\Big[\exp\big(\mathrm{i}\vec{\mathbf{k}}_1\mathbf{r}\big)\vec{\psi}_1\vec{\tau}_{1j} + \exp\big(\mathrm{i}\vec{\mathbf{k}}_2\mathbf{r}\big)\vec{\psi}_2\vec{\tau}_{2j}\Big]\,, \quad (52)$$

where j, j' designate e or m for the TE- and TM-modes, respectively, and the term $\exp(\mathrm{i}\vec{\mathbf{k}}_0\mathbf{r})\vec{\psi}_j$ with the wave vector $\vec{\mathbf{k}}_0 = (\mathbf{k}_\|, k_{0\perp} = (k_0^2 - k_\|^2)^{1/2})$ describes the plain wave incident on the interface from the vacuum. In the TE-mode, factor $\vec{\psi}_{\mathrm{e}} = \vec{\mathcal{E}}_{\mathrm{e}} + \vec{\mathcal{H}}_{\mathrm{e}}$ contains $\vec{\mathcal{E}}_{\mathrm{e}} = \mathbf{t}$ and $\vec{\mathcal{H}}_{\mathrm{e}} = \vec{\boldsymbol{\kappa}}_0 \times \mathbf{t}$. In the

TM-mode, factor $\vec{\psi}_{\rm m} = \vec{\mathcal{E}}_{\rm m} + \vec{\mathcal{H}}_{\rm m}$ contains $\vec{\mathcal{E}}_{\rm m} = -\vec{\boldsymbol{\kappa}}_0 \times \mathbf{t}$ and $\vec{\mathcal{H}}_{\rm m} = \mathbf{t}$.

The reflected wave has the wave vector $\overleftarrow{\mathbf{k}}_0 = (\mathbf{k}_\parallel, -k_{0\perp})$ and fields $\overleftarrow{\mathcal{E}}_{\rm e} = \mathbf{t}$, $\overleftarrow{\mathcal{H}}_{\rm e} = \overleftarrow{\boldsymbol{\kappa}}_0 \times \mathbf{t}$, $\overleftarrow{\mathcal{H}}_{\rm m} = \mathbf{t}$, and $\overleftarrow{\mathcal{E}}_{\rm m} = -\overleftarrow{\boldsymbol{\kappa}}_0 \times \mathbf{t}$. The refracted field contains two wave modes with the wave vectors $\vec{\mathbf{k}}_1 = (\mathbf{k}_\parallel, k_{1\perp})$, $\vec{\mathbf{k}}_2 = (\mathbf{k}_\parallel, k_{2{\rm r}\perp})$ and electric fields $\vec{\mathcal{E}}_1 = \mathbf{e}_1 = \mathbf{a} \times \vec{\boldsymbol{\kappa}}_1$ and $\vec{\mathcal{E}}_2 = \mathbf{e}_2 = \mathbf{a} - \vec{\boldsymbol{\kappa}}_2[\mathbf{a} \times \vec{\boldsymbol{\kappa}}_2]\epsilon_2(\vec{\theta}_2)/\epsilon_1$. Here, $\boldsymbol{\kappa} = \mathbf{k}/k$, $k_{1\perp} = (\epsilon_1 k_0^2 - k_\parallel^2)^{1/2}$, and $k_{2{\rm r}\perp}$ is given by expression (30). For incident TE-mode, reflection ($\rho_{\rm ee}$, $\rho_{\rm me}$) and refraction ($\tau_{j{\rm e}}$) amplitudes ($j = 1, 2$) are found from boundary conditions

$$\mathbf{t}\vec{\mathcal{E}}_1\vec{\tau}_{1{\rm e}} + \mathbf{t}\vec{\mathcal{E}}_2\vec{\tau}_{2{\rm e}} = 1 + \vec{\rho}_{\rm ee}\,, \tag{53}$$

$$\mathbf{l}\vec{\mathcal{H}}_1\vec{\tau}_{1{\rm e}} + \mathbf{l}\vec{\mathcal{H}}_2\vec{\tau}_{2{\rm e}} = -\kappa_{0\perp}\big(1 - \vec{\rho}_{\rm ee}\big)\,, \tag{54}$$

$$\mathbf{t}\vec{\mathcal{H}}_1\vec{\tau}_{1{\rm e}} + \mathbf{t}\vec{\mathcal{H}}_2\vec{\tau}_{2{\rm e}} = \vec{\rho}_{\rm me}\,, \tag{55}$$

$$\mathbf{l}\vec{\mathcal{E}}_1\vec{\tau}_{1{\rm e}} + \mathbf{l}\vec{\mathcal{E}}_2\vec{\tau}_{2{\rm e}} = -\kappa_{0\perp}\vec{\rho}_{\rm me}\,. \tag{56}$$

The solution of these equations is elementary and is given in Appendix C.

4.4 Reflection and transmission amplitudes of a plane-parallel anisotropic plate with thickness L

Now, when we understand what occurs at interfaces, we can build up [16] expressions for reflection and transmission matrices [$\hat{R}(L)$ and $\hat{T}(L)$, respectively] for a whole anisotropic plane-parallel plate of some thickness L, when the state of the incident wave is described by a general vector $|\vec{\xi}_0\rangle$. To do this, let us denote the state of the field of the modes $\mathbf{e}_1$ and $\mathbf{e}_2$ incident from inside the plate onto the second interface at $z = L$ by unknown two-dimensional vector $|\vec{\mathbf{x}}\rangle$ (106). If we were able to find $|\vec{\mathbf{x}}\rangle$, we could immediately write the state of a transmitted field

$$\vec{\hat{T}}(L)|\vec{\xi}_0\rangle = \vec{\hat{\mathcal{T}}}'|\vec{\mathbf{x}}\rangle\,, \tag{57}$$

where $\vec{\hat{\mathcal{T}}}'$ is the transmission matrix through the right interface of the section. The state of the field reflected from the whole plate assumes the form

$$\vec{\hat{R}}(L)|\vec{\xi}_0\rangle = \vec{\hat{\mathcal{R}}}|\vec{\xi}_0\rangle + \overleftarrow{\hat{\mathcal{T}}}'\overleftarrow{\hat{E}}(L)\vec{\hat{\mathcal{R}}}'|\vec{\mathbf{x}}\rangle\,, \tag{58}$$

where $\vec{\hat{\mathcal{R}}}$, $\vec{\hat{\mathcal{R}}}'$ are the reflection matrices from the left (from the vacuum side) and from the right (from the anisotropic medium) interfaces, respectively, $\overleftarrow{\hat{\mathcal{T}}}'$ is the transmission matrix through the left interface of the section, and $\overleftarrow{\hat{E}}(L)$, $\vec{\hat{E}}(L)$ designate diagonal matrices:

$$\overleftarrow{\hat{E}}(L) = \begin{pmatrix} \exp({\rm i}k_{1\perp}L) & 0 \\ 0 & \exp({\rm i}k_{2{\rm l}\perp}L) \end{pmatrix}, \qquad \vec{\hat{E}}(L) = \begin{pmatrix} \exp({\rm i}k_{1\perp}L) & 0 \\ 0 & \exp({\rm i}k_{2{\rm r}\perp}L) \end{pmatrix}, \tag{59}$$

which describe the propagation of two modes in the plate between two interfaces. Here, $k_{1\perp} = (\epsilon_1 k_0^2 - k_\parallel^2)^{1/2}$, while $k_{2{\rm r}\perp}$ and $k_{2{\rm l}\perp}$ are calculated according to expressions (29) or (30) and (32), respectively.

It is very easy to put together a self-consistent equation for the determination of vector $|\vec{\mathbf{x}}\rangle$:

$$|\vec{\mathbf{x}}\rangle = \vec{\hat{E}}(L)\vec{\hat{\mathcal{T}}}|\xi_0\rangle + \vec{\hat{E}}(L)\overleftarrow{\hat{\mathcal{R}}}'\overleftarrow{\hat{E}}(L)\vec{\hat{\mathcal{R}}}'|\vec{\mathbf{x}}\rangle\,. \tag{60}$$

The first term on the right-hand side describes the state of an incident wave transmitted through the first interface and propagated up to the second one. The second term describes the contribution of $|\vec{\mathbf{x}}\rangle$ itself to the state $|\vec{\mathbf{x}}\rangle$. After reflection from the second interface, the appropriate wave propagates to the left up to the first interface, and after reflection from it propagates back to the point $z = L$. The two terms on the right-hand side of expression (58) add together, giving rise to some new state. But we denoted it as $|\vec{\mathbf{x}}\rangle$, and this explains the derivation of equation (58).

From equation (58) we can directly find

$$|\vec{\mathbf{x}}\rangle = \Big[\hat{I} - \vec{\hat{E}}(L)\overleftarrow{\hat{\mathcal{R}}}'\overleftarrow{\hat{E}}(L)\vec{\hat{\mathcal{R}}}'\Big]^{-1}\vec{\hat{E}}(L)\vec{\hat{\mathcal{T}}}|\xi_0\rangle\,, \tag{61}$$

and substituting formula (61) into Eqns (57) and (58) gives

$$\vec{\hat{T}}(L) \equiv \begin{pmatrix} T_{\rm ee} & T_{\rm em} \\ T_{\rm me} & T_{\rm mm} \end{pmatrix} = \vec{\hat{\mathcal{T}}}'\Big[\hat{I} - \vec{\hat{E}}(L)\overleftarrow{\hat{\mathcal{R}}}'\overleftarrow{\hat{E}}(L)\vec{\hat{\mathcal{R}}}'\Big]^{-1}\vec{\hat{E}}(L)\vec{\hat{\mathcal{T}}}, \tag{62}$$

$$\vec{\hat{R}}(L) \equiv \begin{pmatrix} R_{\rm ee} & R_{\rm em} \\ R_{\rm me} & R_{\rm mm} \end{pmatrix} = \vec{\hat{\mathcal{R}}} + \overleftarrow{\hat{\mathcal{T}}}'\overleftarrow{\hat{E}}(L)\vec{\hat{\mathcal{R}}}'\Big[\hat{I} - \vec{\hat{E}}(L)\overleftarrow{\hat{\mathcal{R}}}'\overleftarrow{\hat{E}}(L)\vec{\hat{\mathcal{R}}}'\Big]^{-1}\vec{\hat{E}}(L)\vec{\hat{\mathcal{T}}}\,. \tag{63}$$

With these formulas at hand, we can easily calculate all the required reflectivities and transmissivities for plane-parallel plates:

$$\begin{pmatrix} |R_{\rm ee}|^2 & |R_{\rm em}|^2 \\ |R_{\rm me}|^2 & |R_{\rm mm}|^2 \end{pmatrix}, \qquad \begin{pmatrix} |T_{\rm ee}|^2 & |T_{\rm em}|^2 \\ |T_{\rm me}|^2 & |T_{\rm mm}|^2 \end{pmatrix} \tag{64}$$

for arbitrary parameters, arbitrary angles of incidence, arbitrary incident wave polarizations, and arbitrary directions of the anisotropy vector $\mathbf{a}$. In Fig. 3 we present, as an

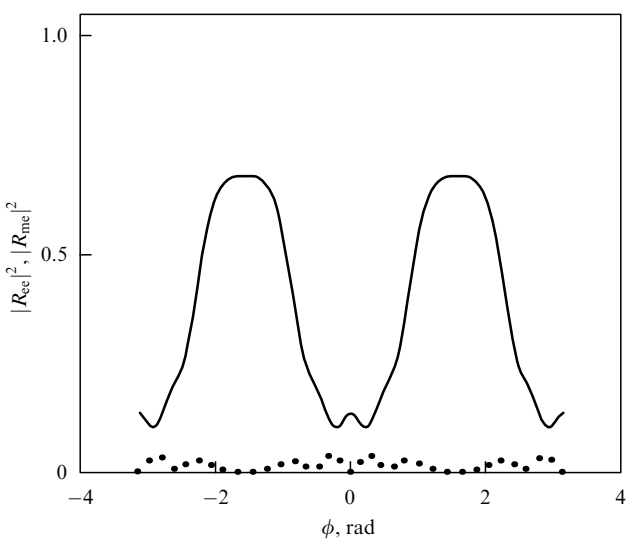

Figure 3. Dependences of reflectivities $|R_{\rm ee}|^2$ (solid curve) and $|R_{\rm me}|^2$ (dotted curve) of an anisotropic plate with $\epsilon_1 = 1.6$, $\eta = 0.8$ and dimensionless thickness $L\omega/c = 10$ on angle ϕ of the plate rotation around its normal, when the anisotropy vector $\mathbf{a}$ is parallel to the interfaces and at $\phi = 0$ is directed along $\mathbf{k}_\parallel$. The incidence angle θ is given by $\sin\theta = 0.9$.

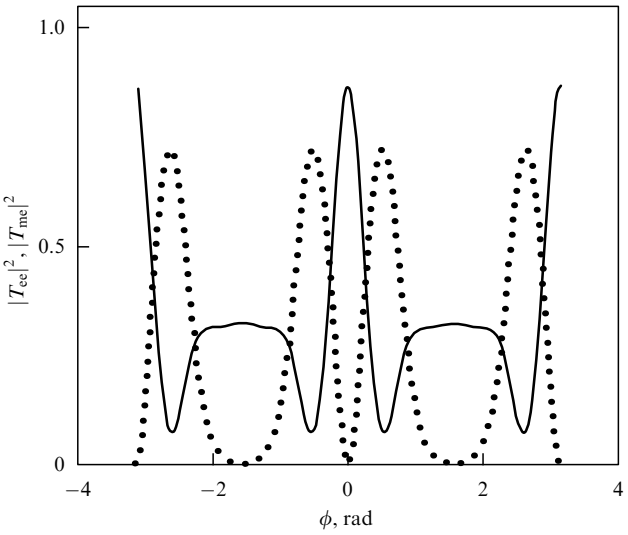

Figure 4. Dependences of transmissivities $|T_{ee}|^2$(solid curve) and $|T_{me}|^2$ (dotted curve) of an anisotropic plate on angle ϕ of the plate rotation around its normal, with all the parameters the same as those given in the caption to Fig. 3.

example, the reflectivities of a TE-mode wave from a plate of thickness L such that $L\omega/c = 10$. The anisotropy vector is parallel to the interfaces. Therefore, its orientation with respect to wave vector $\mathbf{k}_0$ of the incident wave varies with rotation of the plate by an angle ϕ around its normal. The transmissivities of the same plate, depending on the angle ϕ, are illustrated in Fig. 4.

5. D'yakonov surface waves (DSWs)

Above, we found that a surface wave can appear on the interface upon total reflection of a mixed mode at some angles of incidence. This surface wave, however, is coupled to the incident and reflected waves and does not occur without them. D'yakonov discovered in 1988 [15] (see also Refs [17–20]) that free surface waves, analogous to elastic Rayleigh waves on a free surface, can exist on the surface of a uniaxial anisotropic medium. We will derive relevant expressions for them with our tensor (2) and somewhat correct the previous results obtained by D'yakonov [15].

Let us again consider the space separated by a plane at $z = 0$ into two halves, as shown in Fig. 1. The left part ($z < 0$) corresponds to an anisotropic medium with dielectric constant (2), and the right part makes up an isotropic medium with dielectric constant $\epsilon = \epsilon_i$.

The surface wave is characterized by the wave function

$$\Psi(\mathbf{r}, t) = \Big\{\Theta(z<0)\left[\psi_1 \exp(p_1 z) + \psi_2 \exp(p_2 z)\right] + \Theta(z>0)\psi_\mathrm{i} \exp(-p_\mathrm{i} z)\Big\} \exp(\mathrm{i}\mathbf{k}_\parallel \mathbf{r}_\parallel - \mathrm{i}\omega t)\,, \quad (65)$$

where $\boldsymbol{\psi} = \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{H}}$, with

$$\boldsymbol{\mathcal{H}}_{1,2} = \frac{k_\parallel}{k_0}(\mathbf{l} \times \boldsymbol{\mathcal{E}}_{1,2} - \mathrm{i}q_{1,2}\,\mathbf{n} \times \boldsymbol{\mathcal{E}}_{1,2})\,, \qquad \boldsymbol{\mathcal{H}}_\mathrm{i} = \frac{k_\parallel}{k_0}(\mathbf{l} \times \boldsymbol{\mathcal{E}}_\mathrm{i} + \mathrm{i}q_\mathrm{i}\,\mathbf{n} \times \boldsymbol{\mathcal{E}}_\mathrm{i})\,, \quad (66)$$

and parameters $p_{1,2,\mathrm{i}}$ providing exponential decay of the surface wave away from the interface. In expressions (66) we also introduced dimensionless parameters $q_{1,2,\mathrm{i}} = p_{1,2,\mathrm{i}}/k_\parallel$. For $q_{1,\mathrm{i}}$, we have

$$q_{1,\mathrm{i}} = \sqrt{1 - \epsilon_{1,\mathrm{i}} z}\,, \quad (67)$$

where we resorted to the designation $z = k_0^2/k_\parallel^2$. Parameters $q_{1,\mathrm{i}}$ are positive and real, when $\epsilon_{1,\mathrm{i}} z < 1$.

To find q_2 for the field ψ_2, we need to solve the equation

$$1 - x^2 + \eta(\mathbf{la} - \mathrm{i}x\mathbf{na})^2 = z\epsilon_1(1+\eta)\,, \quad (68)$$

where x denotes q_2, $\mathbf{n}$ is a unit vector of the normal directed toward the isotropic medium, and $\mathbf{l}$ is a unit vector pointed along $\mathbf{k}_\parallel$, which together with $\mathbf{n}$ constitutes the plane of incidence. From this equation it is seen that q_2 can be real only if vector $\mathbf{a}$ is perpendicular to $\mathbf{n}$ or to $\mathbf{l}$. In the first case, the axis of anisotropy is parallel to the interface [15]:

$$\mathbf{a} = a_\mathrm{l}\,\mathbf{l} + a_\mathrm{t}\mathbf{t} = \cos\theta\,\mathbf{l} + \sin\theta\,\mathbf{t}\,, \quad (69)$$

and the solution to equation (68) is as follows:

$$x = q_2(\theta) = \sqrt{1 + \eta\cos^2\theta - \epsilon_1 z(1+\eta)}\,. \quad (70)$$

It is seen that $q_2(\theta)$ is a positive real number, when

$$z\epsilon_1\,\frac{1+\eta}{1+\eta\cos^2\theta} \equiv z\epsilon_2(\theta) < 1\,. \quad (71)$$

In the second case, one finds

$$\mathbf{a} = a_n\mathbf{n} + a_t\mathbf{t} = \cos\phi\,\mathbf{n} + \sin\phi\,\mathbf{t}\,, \quad (72)$$

and the solution to equation (68) is given by

$$q_2(\phi) = \frac{\sqrt{1 - \epsilon_1(1+\eta)z}}{\sqrt{1+\eta\cos^2\phi}}\,. \quad (73)$$

Below, we will show that in the second case free surface waves do not exist.

5.1 Waves when the anisotropy axis is parallel to the interface

When $\mathbf{a}$ is parallel to the interface, vectors $\boldsymbol{\mathcal{E}}_{1,2}$, according to Section 2.1, can be represented as

$$\boldsymbol{\mathcal{E}}_1 = -\frac{C_1}{k_\parallel}\,\mathbf{a} \times \mathbf{k} = C_1(\sin\theta\,\mathbf{n} + \mathrm{i}q_1\,\mathbf{a} \times \mathbf{n}) = C_1(\sin\theta\,\mathbf{n} + \mathrm{i}q_1\sin\theta\,\mathbf{l} - \mathrm{i}q_1\cos\theta\,\mathbf{t})\,, \quad (74)$$

$$\boldsymbol{\mathcal{E}}_2 = \tilde{C}_2\left[\mathbf{a} - \frac{\mathbf{k}(\mathbf{ak})}{k_\parallel^2}\,\frac{1+\eta}{1 - q_2^2 + \eta(\mathbf{la})^2}\right] = \tilde{C}_2\left[\mathbf{a} - (\mathbf{l} - \mathrm{i}q_2\mathbf{n})\,\mathbf{al}\,\frac{1}{1-q_1^2}\right] = C_2\big(\mathrm{i}q_2\cos\theta\,\mathbf{n} - q_1^2\cos\theta\,\mathbf{l} + (1-q_1^2)\sin\theta\,\mathbf{t}\big)\,, \quad (75)$$

where $C_2 = \tilde{C}_2/(1-q_1^2)$, $C_{1,2}$ are some complex coefficients, and we made use of relation (70).

On the basis $\mathbf{n}, \mathbf{l}, \mathbf{t}$, specified in Fig. 1, polarization $\boldsymbol{\mathcal{E}}_\mathrm{i}$ in an isotropic medium can be represented as

$$\boldsymbol{\mathcal{E}}_\mathrm{i} = \alpha\mathbf{n} + \beta\mathbf{l} + \gamma\mathbf{t} \quad (76)$$

with coordinates α, β, and γ. In view of equation $\epsilon_{\rm i}\nabla\mathcal{E}_{\rm i} = 0$, which is equivalent to

$$\mathrm{i}q_{\rm i}\alpha + \beta = 0\,, \tag{77}$$

vector (76) is reduced to the form

$$\mathcal{E}_{\rm i} = \alpha(\mathbf{n} - \mathrm{i}q_{\rm i}\mathbf{l}) + \gamma\mathbf{t}\,. \tag{78}$$

From the continuity of the **t**- and **l**-components of an electric field at the interface, we obtain two equations:

$$\begin{aligned} &\mathrm{i}C_1 q_1 \sin\theta - C_2 q_1^2 \cos\theta = -\mathrm{i}q_{\rm i}\alpha\,,\\ &-\mathrm{i}C_1 q_1 \cos\theta + C_2(1 - q_1^2)\sin\theta = \gamma\,. \end{aligned} \tag{79}$$

To derive another two equations, we need to use the continuity of tangential components of a magnetic field. Substituting expressions (74), (75), and (78) into (66) and neglecting common factor $k_\parallel/k_0$, we arrive at

$$\begin{aligned} \mathcal{H}_1 &= C_1\left[-\mathrm{i}q_1\cos\theta\,\mathbf{n} + q_1^2\cos\theta\,\mathbf{l} - (1 - q_1^2)\sin\theta\,\mathbf{t}\right],\\ \mathcal{H}_2 &= C_2(1 - q_1^2)(\sin\theta\,\mathbf{n} + \mathrm{i}q_2\sin\theta\,\mathbf{l} - \mathrm{i}q_2\cos\theta\,\mathbf{t})\,,\\ \mathcal{H}_{\rm i} &= \left(\gamma\mathbf{n} - \mathrm{i}q_{\rm i}\gamma\mathbf{l} - \alpha(1 - q_{\rm i}^2)\mathbf{t}\right). \end{aligned} \tag{80}$$

The continuity conditions for the **l**- and **t**-components give

$$\begin{aligned} &q_1^2 C_1\cos\theta + \mathrm{i}C_2 q_2(1 - q_1^2)\sin\theta = -\mathrm{i}q_{\rm i}\gamma\,,\\ &C_1\sin\theta + \mathrm{i}q_2 C_2\cos\theta = \alpha\varepsilon\,, \end{aligned} \tag{81}$$

where we introduced designation $\varepsilon = \epsilon_i/\epsilon_1$. Excluding γ and α from these equations, we obtain a homogeneous system of 2 equations for the unknowns $C_{1,2}$:

$$\begin{aligned} &q_1\cos\theta(q_1 + q_{\rm i})C_1 + \mathrm{i}C_2(q_2 + q_{\rm i})\epsilon_1 z\sin\theta = 0\,,\\ &\mathrm{i}C_1\sin\theta(q_1\varepsilon + q_{\rm i}) - C_2\cos\theta(\varepsilon q_1^2 + q_{\rm i}q_2) = 0\,. \end{aligned} \tag{82}$$

The system of the two linear equations (82) has a solution only if the determinant of its coefficients is equal to zero, which gives an equation for $z = k_0^2/k_\parallel^2$:

$$\begin{aligned} f(z) = {}& q_1(q_1 + q_{\rm i})(\varepsilon q_1^2 + q_{\rm i}q_2)\cos^2\theta\\ &- \epsilon_1 z(q_2 + q_{\rm i})(q_1\varepsilon + q_{\rm i})\sin^2\theta = 0\,. \end{aligned} \tag{83}$$

Solution of this equation defines the speed of the DSW: $c_{\rm D}(z) = c\sqrt{z}$.

We deduced equation (83) so scrupulously to show that result (83) slightly differs from the one presented by equation (8) of Ref. [15], and it is not reducible to Eqn (9) of the same paper, because the solution of equation (83) exists even for $\epsilon_{\rm i} < \epsilon_1$ (ϵ_1 is denoted $\epsilon_\perp$ in Ref. [15]). Moreover, it follows from Eqn (83) that the surface wave exists in a much wider range of angles θ than was found in Ref. [15]. For instance, from the dependence of ratio $v(\theta) = c_{\rm D}(z)/c$ on θ in Fig. 5 it is seen that the solution of Eqn (83) exists in the full range $0 < \theta < \pi/2$ for $\epsilon_1 = 1.6$, $\eta = 0.4$, and $\epsilon_{\rm i} = 1$.

5.2 Waves when the anisotropy axis is perpendicular to the propagation direction

When $\mathbf{a}\perp\mathbf{l}$, vectors $\mathcal{E}_{1,2}$, according to Section 2.1, and $\mathcal{H}_{1,2}$, according to formulas (66), can be represented as

$$\mathcal{E}_1 = -\frac{C_1}{k_\parallel}\,\mathbf{a}\times\mathbf{k} = C_1(\sin\phi\,\mathbf{n} + \mathrm{i}q_1\sin\phi\,\mathbf{l} - \cos\phi\,\mathbf{t})\,, \tag{84}$$

$$\mathcal{E}_2 = C_2\left[(1 - q_1^2 + q_2^2)\cos\phi\,\mathbf{n} + \mathrm{i}q_2\cos\phi\,\mathbf{l} + \sin\theta\,\mathbf{t}\right], \tag{85}$$

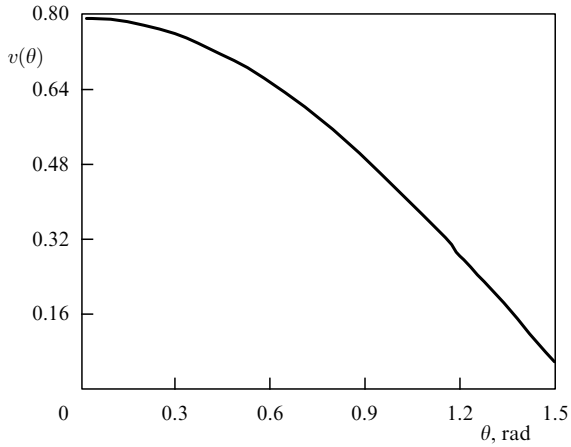

Figure 5. Dependence of $v(\theta) = c_{\rm D}(z)/c$ on angle θ (in rad) between anisotropy axis **a** and direction $\mathbf{k}_\parallel$ of the surface wave propagation. The curve was calculated for $\epsilon_1 = 1.6$, $\eta = 0.4$, and $\epsilon_{\rm i} = 1$.

$$\begin{aligned} \mathcal{H}_1 &= C_1\left[-\cos\phi\,\mathbf{n} - \mathrm{i}q_1\cos\phi\,\mathbf{l} - (1 - q_1^2)\sin\phi\,\mathbf{t}\right],\\ \mathcal{H}_2 &= C_2(1 - q_1^2)(\sin\phi\,\mathbf{n} + \mathrm{i}q_2\sin\phi\,\mathbf{l} - \cos\phi\,\mathbf{t})\,, \end{aligned} \tag{86}$$

and $\mathcal{E}_{\rm i}$, $\mathcal{H}_{\rm i}$ are the same as in formulas (78) and (80), respectively. After performing the same procedure as above, we obtain an equation for $z = k_0^2/k_\parallel^2$ in the form

$$f_1(z) = \left(\frac{\epsilon_{\rm i}}{\epsilon_1}\,q_2 + q_{\rm i}\right)\cos^2\phi + (q_2 + q_{\rm i})(1 - q_1 q_{\rm i})\sin^2\phi = 0\,, \tag{87}$$

which has no solution because all the terms in it are positive. Therefore, surface waves do not exist at such orientations of the anisotropy axis **a**, as was correctly pointed out in Ref. [15].

6. A possible experiment for the observation of D'yakonov surface waves

In the literature there are many reports on the experimental observation of DSWs (see, e.g., paper [20] and review [17]). We offer another layout of the experiment.

To observe D'yakonov surface waves, it is possible to use the experimental scheme shown in Fig. 6, which is different from the one used in paper [20]. A disc of a uniaxial crystal with anisotropy axis **a** parallel to the surface can be pivoted

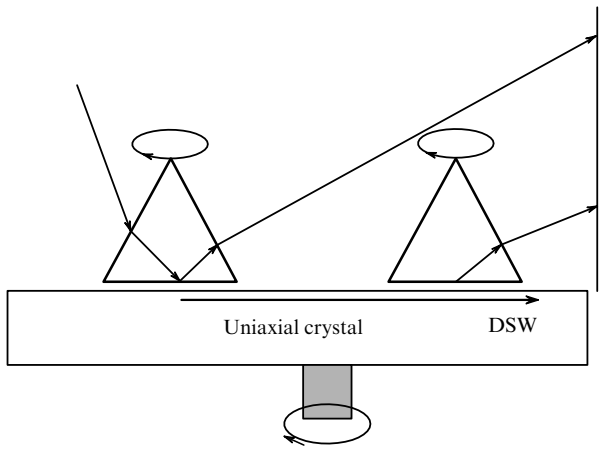

Figure 6. A possible experimental scheme for the excitation and recording of DSWs.

around its axis to change the angle between the direction of DSW propagation $\boldsymbol{\kappa} = \mathbf{k}_{\parallel}/k_{\parallel}$ and the vector $\mathbf{a}$. The DSW is excited at frustrated total reflection in an anisotropic cone, similar to that shown in Fig. 2 (here, for simplicity, we draw only one transmitted ray). Excitation takes place only when the speed of the incident or reflected wave inside the cone matches the speed of the DSW. Rotation of the cone around its axis permits some tuning of the speed.

A second anisotropic cone, identical to the first one, detects the DSW, and the light transmitted into it through the small gap should be visible on a screen, as shown in Fig. 6.

7. Conclusion

In the case of uniaxial or biaxial anisotropic media, we used expressions for the permittivity tensor ϵ_{ij} in the form

$$\varepsilon_{ij} = \epsilon_1(\delta_{ij} + \eta a_i a_j)\,, \qquad \varepsilon_{ij} = \epsilon_1(\delta_{ij} + \eta_a a_i a_j + \eta_b b_i b_j)\,, \tag{88}$$

where ϵ_1 is a parameter of the isotropic part of the tensors, $\mathbf{a}$ and $\mathbf{b}$ are the unit vectors directed along axes of anisotropy, and η, $\eta_{a,b}$ are the respective anisotropy parameters. With such tensors, we can immediately find analytical expressions for the polarization vector $\boldsymbol{\mathcal{E}}$ and wave number $k(\omega)$ for a plain wave $\boldsymbol{\mathcal{E}}\exp(\mathrm{i}\mathbf{kr} - \mathrm{i}\omega t)$ with an arbitrary direction $\boldsymbol{\kappa} = \mathbf{k}/k$ of propagation. In the case of a uniaxial anisotropic medium, we found that only two modes of linear polarizations can propagate inside it. One is the transverse mode, with

$$\boldsymbol{\mathcal{E}}_1 = \mathbf{a} \times \boldsymbol{\kappa}\,, \qquad k_1 = \frac{\omega}{c}\sqrt{\epsilon_1}\,, \tag{89}$$

and the other is the mixed mode (it has a component of polarization parallel to the wave vector), with

$$\begin{gathered} \boldsymbol{\mathcal{E}}_2 = \mathbf{a} - \boldsymbol{\kappa}(\boldsymbol{\kappa}\mathbf{a})\,\frac{\epsilon_2(\theta)}{\epsilon_1}\,, \qquad k_2 = \frac{\omega}{c}\sqrt{\epsilon_2(\theta)}\,, \\ \epsilon_2(\theta) = \frac{\epsilon_1(1+\eta)}{1+\eta\cos^2\theta}\,, \qquad \cos\theta = \mathbf{a}\boldsymbol{\kappa}\,. \end{gathered} \tag{90}$$

Next, we considered reflection of the obtained plain waves from an interface between an isotropic and anisotropic media and revealed that reflection of every mode is accompanied by beam splitting, that the wave of mode 2 is, in general, reflected nonspecularly, and that, at some angles of incidence, reflection of mode 2 can create a surface wave, which is coupled with the incident and reflected waves of mode 2. The beam splitting upon reflection can be spectacularly demonstrated with the help of light transmission through an anisotropic cone.

After calculation of wave reflection from interfaces from inside and outside the anisotropic medium, we found an algorithm for calculating reflection and transmission amplitudes of plane-parallel plates without matching the wave field at the two interfaces. In the case of normal incidence on the plate of a plane wave with linear polarization, a transmitted wave generally has elliptical polarization. The form of the ellipse changes with rotation of the birefringent plate around its normal and at two distinct orthogonal directions the ellipse reduces to linear polarization identical to that of the incident wave.

We also considered the excitation of a free DSW on the surface of an anisotropic medium. We corrected an error in the derivation presented in Ref. [15] and showed that DSWs exist in a wider range of variation of dielectric constants and in a wider range of angles between the direction of surface wave propagation and the direction of anisotropy vector $\mathbf{a}$. We also proposed a new experimental scheme for observation of DSWs, shown in Fig. 6.

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We are thankful to Lukas Novotny, Svetlana Lukishova, and Sergio G Rodrigo from the Institute of Optics of the University of Rochester, New York, USA for taking interest in the problem and their encouragement.

8. Appendices

A. Plane waves in a biaxial anisotropic medium

A biaxial anisotropic medium is characterized by two unit vectors $\mathbf{a}$ and $\mathbf{b}$, and two anisotropy parameters ϵ'_a and ϵ'_a. Therefore, the tensor ε has the matrix elements

$$\varepsilon_{ij} = \epsilon_1\delta_{ij} + \epsilon'_a a_i a_j + \epsilon'_b b_i b_j\,, \tag{91}$$

and equations (7), (8), and (10) take the respective forms

$$\varepsilon\boldsymbol{\mathcal{E}} = \epsilon_1\boldsymbol{\mathcal{E}} + \epsilon'_a\mathbf{a}(\mathbf{a}\boldsymbol{\mathcal{E}}) + \epsilon'_b\mathbf{b}(\mathbf{b}\boldsymbol{\mathcal{E}})\,, \tag{92}$$

$$\boldsymbol{\kappa}\boldsymbol{\mathcal{E}} + \eta_a(\boldsymbol{\kappa}\mathbf{a})(\mathbf{a}\boldsymbol{\mathcal{E}}) + \eta_b(\boldsymbol{\kappa}\mathbf{b})(\mathbf{b}\boldsymbol{\mathcal{E}}) = 0\,, \tag{93}$$

$$(k^2 - k_0^2\epsilon_1)\boldsymbol{\mathcal{E}} - k^2\boldsymbol{\kappa}(\boldsymbol{\kappa}\boldsymbol{\mathcal{E}}) - k_0^2\epsilon_1\eta_a\mathbf{a}(\mathbf{a}\boldsymbol{\mathcal{E}}) - k_0^2\epsilon_1\eta_b\mathbf{b}(\mathbf{b}\boldsymbol{\mathcal{E}}) = 0\,. \tag{94}$$

In the last two equations, we introduce notations $\eta_a = \epsilon'_a/\epsilon_1$ and $\eta_b = \epsilon'_b/\epsilon_1$. For simplicity, we assume that $\mathbf{a}\perp\mathbf{b}$, introduce the orthonormal basis $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} = \mathbf{a}\times\mathbf{b}$, and on this basis represent the wave polarization vector as

$$\boldsymbol{\mathcal{E}} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \tag{95}$$

with coordinates α, β, γ which are not completely independent, because expansion (95) should satisfy equation (93). Substituting formula (95) into equation (93) gives

$$\alpha\mathbf{ka} + \beta\mathbf{kb} + \gamma\mathbf{kc} + \eta_a\mathbf{ka}\alpha + \eta_b\mathbf{kb}\beta = 0\,. \tag{96}$$

Therefore, one obtains

$$\gamma\boldsymbol{\kappa}\mathbf{c} = -\alpha\boldsymbol{\kappa}\mathbf{a}(1+\eta_a) - \beta\boldsymbol{\kappa}\mathbf{b}(1+\eta_b)\,. \tag{97}$$

From expression (95) we also find

$$\boldsymbol{\mathcal{E}}\mathbf{a} = \alpha\,, \qquad \boldsymbol{\mathcal{E}}\mathbf{b} = \beta\,. \tag{98}$$

Let us now substitute Eqn (98) and $\boldsymbol{\kappa}\boldsymbol{\mathcal{E}}$ from equation (93) into Eqn (94) and multiply the latter consecutively by $\mathbf{a}$ and $\mathbf{b}$. As a result, we obtain a system of two linear equations:

$$\begin{gathered} \Big(k^2\big[1+\eta_a(\boldsymbol{\kappa}\mathbf{a})^2\big] - k_0^2\epsilon_1(1+\eta_a)\Big)\alpha + \eta_b k^2(\boldsymbol{\kappa}\mathbf{a})(\boldsymbol{\kappa}\mathbf{b})\beta = 0\,, \\ \Big(k^2\big[1+\eta_b(\boldsymbol{\kappa}\mathbf{b})^2\big] - k_0^2\epsilon_1(1+\eta_b)\Big)\beta + \eta_a k^2(\boldsymbol{\kappa}\mathbf{a})(\boldsymbol{\kappa}\mathbf{b})\alpha = 0\,. \end{gathered} \tag{99}$$

The solution of this system exists if the determinant is equal to zero. This condition can be written out as

$$\big(k^2 - \epsilon_a(\theta_a)k_0^2\big)\big(k^2 - \epsilon_b(\theta_b)k_0^2\big) = \frac{\eta_a\eta_b k^2(\boldsymbol{\kappa}\mathbf{a})^2(\boldsymbol{\kappa}\mathbf{b})^2}{\big[1+\eta_a(\boldsymbol{\kappa}\mathbf{a})^2\big]\big[1+\eta_b(\boldsymbol{\kappa}\mathbf{b})^2\big]}\,, \tag{100}$$

where

$$\epsilon_{a,b}(\theta_{a,b}) = \frac{\epsilon_1(1+\eta_{a,b})}{1+\eta_{a,b}\cos^2\theta_{a,b}}\,, \quad \cos\theta_a = \boldsymbol{\kappa}\mathbf{a}\,, \quad \cos\theta_b = \boldsymbol{\kappa}\mathbf{b}\,. \tag{101}$$

The solution of Eqn (100) provides two different values of $k_{1,2}$, from which we find α, β. After substituting the latter coordinates into formula (97), we obtain the last coordinate γ. Thus, we find two different plain waves with wave vectors $\mathbf{k}_{1,2} = k_{1,2}\boldsymbol{\kappa}$ and linear polarizations $\boldsymbol{\mathcal{E}}_{1,2}$ (95).

B. Wave reflection from an interface on the side of an anisotropic medium

The exclusion of $\vec{\tau}_{\mathrm{e}j}$ from continuity conditions (38) and (39), and exclusion of $\vec{\tau}_{\mathrm{m}j}$ from analogous conditions (41) and (42) gives two equations for $\vec{\rho}_{1j}$, $\vec{\rho}_{2j}$, which are convenient to represent in the matrix form

$$\begin{pmatrix} \mathbf{l}\overleftarrow{\mathcal{H}}_1 + \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{E}}_1 & \mathbf{l}\overleftarrow{\mathcal{H}}_2 + \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{E}}_2 \\ \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{H}}_1 - \mathbf{l}\overleftarrow{\mathcal{E}}_1 & \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{H}}_2 - \mathbf{l}\overleftarrow{\mathcal{E}}_2 \end{pmatrix} \begin{pmatrix} \vec{\rho}_{1j} \\ \vec{\rho}_{2j} \end{pmatrix} = -\begin{pmatrix} \mathbf{l}\vec{\mathcal{H}}_j + \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_j \\ \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_j - \mathbf{l}\vec{\mathcal{E}}_j \end{pmatrix}. \tag{102}$$

The solution to this equation is written out as

$$\begin{pmatrix} \vec{\rho}_{1j} \\ \vec{\rho}_{2j} \end{pmatrix} = \frac{-1}{D} \begin{pmatrix} \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{H}}_2 - \mathbf{l}\overleftarrow{\mathcal{E}}_2 & -\mathbf{l}\overleftarrow{\mathcal{H}}_2 - \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{E}}_2 \\ -\kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{H}}_1 + \mathbf{l}\overleftarrow{\mathcal{E}}_1 & \mathbf{l}\overleftarrow{\mathcal{H}}_1 + \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{E}}_1 \end{pmatrix} \times \begin{pmatrix} \mathbf{l}\vec{\mathcal{H}}_j + \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_j \\ \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_j - \mathbf{l}\vec{\mathcal{E}}_j \end{pmatrix}, \tag{103}$$

where D is a determinant:

$$D = \left(\mathbf{l}\overleftarrow{\mathcal{H}}_1 + \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{E}}_1\right)\left(\kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{H}}_2 - \mathbf{l}\overleftarrow{\mathcal{E}}_2\right) - \left(\mathbf{l}\overleftarrow{\mathcal{H}}_2 + \kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{E}}_2\right)\left(\kappa_{0\perp}\mathbf{t}\overleftarrow{\mathcal{H}}_1 - \mathbf{l}\overleftarrow{\mathcal{E}}_1\right). \tag{104}$$

Substitution of these expressions into formulas (38) and (41) gives wave refraction amplitudes $\vec{\tau}_{\mathrm{e,m}j}$:

$$\begin{pmatrix} \vec{\tau}_{\mathrm{e}j} \\ \vec{\tau}_{\mathrm{m}j} \end{pmatrix} = \begin{pmatrix} \mathbf{t}\vec{\mathcal{E}}_j \\ \mathbf{t}\vec{\mathcal{H}}_j \end{pmatrix} + \begin{pmatrix} \mathbf{t}\overleftarrow{\mathcal{E}}_1 & \mathbf{t}\overleftarrow{\mathcal{E}}_2 \\ \mathbf{t}\overleftarrow{\mathcal{H}}_1 & \mathbf{t}\overleftarrow{\mathcal{H}}_2 \end{pmatrix} \begin{pmatrix} \vec{\rho}_{1j} \\ \vec{\rho}_{2j} \end{pmatrix}. \tag{105}$$

The most general case. Above, we considered the case where the incident wave has polarization vector $\mathbf{e}_j$ with a unit amplitude. (Let us recall that vectors $\mathbf{e}_j$ cannot be normalized to unity.) To find later reflections from plane-parallel plates, we will consider a more general case where the incident wave has both modes with amplitudes $x_{1,2}$. To find amplitudes of reflected and transmitted waves in the general case, it is convenient to represent the state of the incident wave in the form of the two-dimensional vector

$$|\vec{\mathbf{x}}\rangle = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \end{pmatrix}. \tag{106}$$

Then, the states of reflected and refracted waves are also described by two-dimensional vectors, which can be represented as

$$|\overleftarrow{\boldsymbol{\psi}}\rangle = \begin{pmatrix} \overleftarrow{\psi}_1 \\ \overleftarrow{\psi}_2 \end{pmatrix} = \vec{\hat{\mathcal{R}}}'|\vec{\mathbf{x}}\rangle\,, \qquad |\vec{\boldsymbol{\psi}}_0\rangle = \begin{pmatrix} \vec{\psi}_{\mathrm{e}} \\ \vec{\psi}_{\mathrm{m}} \end{pmatrix} = \vec{\hat{\mathcal{T}}}'|\vec{\mathbf{x}}\rangle\,, \tag{107}$$

where $\vec{\hat{\mathcal{R}}}'$ and $\vec{\hat{\mathcal{T}}}'$ are two-dimensional matrices

$$\vec{\hat{\mathcal{R}}}' = \begin{pmatrix} \vec{\rho}_{11} & \vec{\rho}_{12} \\ \vec{\rho}_{21} & \vec{\rho}_{22} \end{pmatrix}, \qquad \vec{\hat{\mathcal{T}}}' = \begin{pmatrix} \vec{\tau}_{\mathrm{e}1} & \vec{\tau}_{\mathrm{e}2} \\ \vec{\tau}_{\mathrm{m}1} & \vec{\tau}_{\mathrm{m}2} \end{pmatrix}. \tag{108}$$

We introduced the prime here and below to distinguish wave refraction and reflection from inside the medium, and the similar matrices obtained for incident waves outside the medium.

These formulas will be used later for calculating the wave reflection and transmission amplitudes of plane-parallel anisotropic plates. We have two interfaces for a plate; therefore, we also need to describe wave reflection and refraction at the left interface from inside the plate. They can be easily found from symmetry considerations. Their representation is obtained from formulas (103)–(105) by reversing the top arrows and changing the sign of $\kappa_{0\perp}$. After this action, we find

$$\overleftarrow{\hat{\mathcal{R}}}' = \begin{pmatrix} \overleftarrow{\rho}_{11} & \overleftarrow{\rho}_{12} \\ \overleftarrow{\rho}_{21} & \overleftarrow{\rho}_{22} \end{pmatrix}, \qquad \overleftarrow{\hat{\mathcal{T}}}' = \begin{pmatrix} \overleftarrow{\tau}_{\mathrm{e}1} & \overleftarrow{\tau}_{\mathrm{e}2} \\ \overleftarrow{\tau}_{\mathrm{m}1} & \overleftarrow{\tau}_{\mathrm{m}2} \end{pmatrix}. \tag{109}$$

Reflection from outside the medium is to be considered separately.

Energy conservation law. It is always necessary to ensure the correctness of the formulas obtained. One of the best controls is the check of energy conservation law. One should always check whether the energy density flux of an incident wave along the normal to the interface is equal to the sum of energy density fluxes of reflected and refracted waves, and the correct definition of the energy fluxes is most important. In isotropic media, it is possible to define energy flux along a vector $\mathbf{n}$ as

$$\mathbf{Jn} = \frac{\mathbf{kn}}{k}\,\frac{c}{\sqrt{\epsilon}}\,\frac{\epsilon\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2}{8\pi}\,, \tag{110}$$

or

$$\mathbf{Jn} = c\,\frac{\mathbf{n}[\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{H}}]}{4\pi}\,. \tag{111}$$

In isotropic media, both definitions are equivalent, because $\boldsymbol{\mathcal{H}} = \mathbf{k}\times\boldsymbol{\mathcal{E}}/k_0$ and $\mathbf{k}\boldsymbol{\mathcal{E}} = 0$. The first definition seems even more preferable, since the second one can be written even for stationary fields, where there is no energy flux.

In anisotropic media, only the second definition is valid, and because the field $\boldsymbol{\mathcal{E}}$ in mode 2 is not orthogonal to $\mathbf{k}$, the direction of the energy density flux is determined not only by the wave vector, but also by the direction of the field $\boldsymbol{\mathcal{E}}$ itself.

C. Formulas for reflection and refraction amplitudes at the interface outside an anisotropic medium

Exclusion of $\vec{\rho}_{\mathrm{ee}}$ and $\vec{\rho}_{\mathrm{me}}$ leads to the matrix equation

$$\begin{pmatrix} \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_1 - \mathbf{l}\vec{\mathcal{H}}_1 & \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_2 - \mathbf{l}\vec{\mathcal{H}}_2 \\ \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_1 + \mathbf{l}\vec{\mathcal{E}}_1 & \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_2 + \mathbf{l}\vec{\mathcal{E}}_2 \end{pmatrix} \begin{pmatrix} \vec{\tau}_{1\mathrm{e}} \\ \vec{\tau}_{2\mathrm{e}} \end{pmatrix} = \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix}, \tag{112}$$

and its solution is given by

$$\begin{pmatrix} \vec{\tau}_{1e} \\ \vec{\tau}_{2e} \end{pmatrix} = \frac{1}{D_e} \begin{pmatrix} \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_2 + \mathbf{l}\vec{\mathcal{E}}_2 & -\kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_2 + \mathbf{l}\vec{\mathcal{H}}_2 \\ -\kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_1 - \mathbf{l}\vec{\mathcal{E}}_1 & \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_1 - \mathbf{l}\vec{\mathcal{H}}_1 \end{pmatrix} \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix}, \tag{113}$$

where D_e is the determinant:

$$D_e = \left(\kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_1 - \mathbf{l}\vec{\mathcal{H}}_1\right)\left(\kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_2 + \mathbf{l}\vec{\mathcal{E}}_2\right) - \left(\kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_2 - \mathbf{l}\vec{\mathcal{H}}_2\right)\left(\kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_1 + \mathbf{l}\vec{\mathcal{E}}_1\right). \tag{114}$$

Substitution of $\vec{\tau}_{je}$ into boundary conditions (53) and (55) yields

$$\begin{pmatrix} \vec{\rho}_{ee} \\ \vec{\rho}_{me} \end{pmatrix} = \begin{pmatrix} \mathbf{t}\vec{\mathcal{E}}_1 & \mathbf{t}\vec{\mathcal{E}}_2 \\ \mathbf{t}\vec{\mathcal{H}}_1 & \mathbf{t}\vec{\mathcal{H}}_2 \end{pmatrix} \begin{pmatrix} \vec{\tau}_{1e} \\ \vec{\tau}_{2e} \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{115}$$

In the case of the incident TM-mode, we have boundary conditions

$$\mathbf{t}\vec{\mathcal{H}}_1\vec{\tau}_{1m} + \mathbf{t}\vec{\mathcal{H}}_2\vec{\tau}_{2m} = 1 + \vec{\rho}_{mm}, \tag{116}$$

$$\mathbf{l}\vec{\mathcal{E}}_1\vec{\tau}_{1m} + \mathbf{l}\vec{\mathcal{E}}_2\vec{\tau}_{2m} = \kappa_{0\perp}\left(1 - \vec{\rho}_{mm}\right), \tag{117}$$

$$\mathbf{t}\vec{\mathcal{E}}_1\vec{\tau}_{1m} + \mathbf{t}\vec{\mathcal{E}}_2\vec{\tau}_{2m} = \vec{\rho}_{em}, \tag{118}$$

$$\mathbf{l}\vec{\mathcal{H}}_1\vec{\tau}_{1m} + \mathbf{l}\vec{\mathcal{H}}_2\vec{\tau}_{2m} = \kappa_{0\perp}\vec{\rho}_{em}. \tag{119}$$

Excluding $\vec{\rho}_{me}$ and $\vec{\rho}_{mm}$ leads to

$$\begin{pmatrix} \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_1 + \mathbf{l}\vec{\mathcal{E}}_1 & \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_2 + \mathbf{l}\vec{\mathcal{E}}_2 \\ \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_1 - \mathbf{l}\vec{\mathcal{H}}_1 & \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_2 - \mathbf{l}\vec{\mathcal{H}}_2 \end{pmatrix} \begin{pmatrix} \vec{\tau}_{1m} \\ \vec{\tau}_{2m} \end{pmatrix} = \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix}; \tag{120}$$

therefore,

$$\begin{pmatrix} \vec{\tau}_{1m} \\ \vec{\tau}_{2m} \end{pmatrix} = \frac{1}{D_m} \begin{pmatrix} \kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_2 - \mathbf{l}\vec{\mathcal{H}}_2 & -\kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_2 - \mathbf{l}\vec{\mathcal{E}}_2 \\ -\kappa_{0\perp}\mathbf{t}\vec{\mathcal{E}}_1 + \mathbf{l}\vec{\mathcal{H}}_1 & \kappa_{0\perp}\mathbf{t}\vec{\mathcal{H}}_1 + \mathbf{l}\vec{\mathcal{E}}_1 \end{pmatrix} \begin{pmatrix} 2\kappa_{0\perp} \\ 0 \end{pmatrix}, \tag{121}$$

where $D_m = -D_e$ (114). Substitution of $\vec{\tau}_{jm}$ into formulas (116) and (118) gives

$$\begin{pmatrix} \vec{\rho}_{em} \\ \vec{\rho}_{mm} \end{pmatrix} = \begin{pmatrix} \mathbf{t}\vec{\mathcal{E}}_1 & \mathbf{t}\vec{\mathcal{E}}_2 \\ \mathbf{t}\vec{\mathcal{H}}_1 & \mathbf{t}\vec{\mathcal{H}}_2 \end{pmatrix} \begin{pmatrix} \vec{\tau}_{1m} \\ \vec{\tau}_{2m} \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{122}$$

In the general case, when the incident wave has an amplitude $\vec{\xi}_e$ in the TE-mode, and an amplitude $\vec{\xi}_m$ in the TM-mode, the state of the incident wave can be described by two-dimensional vector

$$|\vec{\xi}_0\rangle = \begin{pmatrix} \vec{\xi}_e \\ \vec{\xi}_m \end{pmatrix}, \tag{123}$$

and the states of reflected and transmitted waves can be represented as

$$|\overleftarrow{\xi}_0\rangle = \begin{pmatrix} \overleftarrow{\xi}_e \\ \overleftarrow{\xi}_m \end{pmatrix} = \vec{\hat{\mathcal{R}}}|\vec{\xi}_0\rangle, \qquad |\vec{\xi}\rangle = \begin{pmatrix} \vec{\xi}_1 \\ \vec{\xi}_2 \end{pmatrix} = \vec{\hat{\mathcal{T}}}|\vec{\xi}_0\rangle, \tag{124}$$

where $\vec{\hat{\mathcal{R}}}$ and $\vec{\hat{\mathcal{T}}}$ are the two-dimensional matrices

$$\vec{\hat{\mathcal{R}}} = \begin{pmatrix} \vec{\rho}_{ee} & \vec{\rho}_{em} \\ \vec{\rho}_{me} & \vec{\rho}_{mm} \end{pmatrix}, \qquad \vec{\hat{\mathcal{T}}} = \begin{pmatrix} \vec{\tau}_{1e} & \vec{\tau}_{1m} \\ \vec{\tau}_{2e} & \vec{\tau}_{2m} \end{pmatrix}. \tag{125}$$

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