METHODOLOGICAL NOTES

PACS numbers: 64.60.ae, 64.60.ah, 64.60.al, **64.60.–i**

Hybrid ramified Sierpinski carpet: percolation transition, critical exponents, and force field

A N Herega, N G Drik, A P Ugol'nikov

DOI: 10.3367/UFNe.0182.201205f.0555

Contents

1. **Introduction** **519**
2. **Percolation on a hybrid-ramified Sierpinski carpet** **519**
3. **Model of the force field of the Sierpinski square** **520**
4. **Conclusions** **520**

References **520**

Abstract. **This methodological note introduces the concept of and calculates percolation transition characteristics for a Sierpinski carpet with hybrid (finite–infinite) ramification. Recurrence formulas for calculating the force fields of Sierpinski prefractals of an arbitrary generation are obtained. The possibility of using the obtained results in the model of oscillatory interacting different-scale inner boundaries of a heterogeneous material is discussed.**

1. Introduction

Being a two-dimensional analog of the Cantor ternary set, a Sierpinski carpet, as is well known [1], can be constructed following a simple algorithm: each side of a square of a unit area is divided into three equal segments; the lines drawn through the ends of the segments parallel to the sides form nine small squares, the middle of which is removed. The procedure is repeated ad infinitum on each of the eight remaining squares [1, 2]. The set obtained represents a regular fractal with the self-similarity dimension $D = \ln 8/\ln 3 = 1.892789....$

Small squares (cells) obtained at any arbitrary step of this iterative procedure are considered to be connected if they share a common edge segment; in other words, the Sierpinski carpet is infinitely ramified, i.e., the problem of dividing it into parts can be performed by removing an infinite (countable) set of points. (We note that from the topological standpoint, the Sierpinski carpet is a one-dimensional object with a continual ramification index at each of its points.)

The specific value of ramification is not important, but "certain properties of fractals with finite and infinite ramification are essentially different" [2]. "For us, the most interesting property of such fractal lattices is that they allow a true percolation transition in contrast to lattices with finite ramification, on which the percolation path is destroyed if a finite number of nodes is removed" [2]. Parameters of the percolation transition on the Sierpinski carpet are studied in Ref. [3].

The proposed modification of the Sierpinski carpet consists in assuming that the cells having a common edge or vertex are connected. We refer to this analog of the known fractal as a Sierpinski carpet with hybrid ramification. Apparently, the modification of rules defining the connectedness leads to a change in percolation parameters of an infinite cluster of carpet cells.

2. Percolation on a hybrid ramified Sierpinski carpet

Following the algorithm described above, we divide any cell of the Sierpinski carpet at an arbitrary iteration step into nine squares and remove the middle one. We determine the probability p' that a cell belongs to a percolation cluster on the carpet, i.e., the probability that there is a 'flow' through the squares composing it, each belonging to an infinite cluster with the probability p. Since the renormalization group transformation [4] should in our case reflect the connectedness, the number of suitable combinations in the arrangement of squares in the cell is less than the combinatoric one. Therefore, the renormalization-group transformation for a carpet with hybrid ramification takes the form

$$p' = R(p) = p^8 + 8p^7(1-p) + 27p^6(1-p)^2 + 44p^5(1-p)^3 + 38p^4(1-p)^4 + 8p^3(1-p)^5 ,$$

with a nontrivial stationary point $p_c = 0.5093$, which defines the percolation threshold.

The index of the correlation length of the percolation system can be found from the relation $\nu = \ln b/\ln \lambda = 1.801$, where $b = 3$ is the number of squares along the cell side and

A N Herega Odessa State Academy of Construction and Architecture,
ul. Didrikhsona 4, 65029 Odessa, Ukraine
Tel. (1038-048) 798 37 35
E-mail: aherega@gmail.com
N G Drik Mechnikov Odessa National University,
ul. Dvoryanskaya 2, 65082 Odessa, Ukraine
Tel. (1038-048) 755 05 24
E-mail: dilatan@list.ru
A P Ugol'nikov Odessa State Academy of Refrigeration,
ul. Dvoryanskaya 1/3, 65082 Odessa, Ukraine
Tel. (1038-048) 723 64 12
E-mail: coalman6@gmail.com

Received 1 August 2011
Uspekhi Fizicheskikh Nauk **182** (5) 555 – 557 (2012)
DOI: 10.3367/UFNr.0182.201205f.0555
Translated by S D Danilov; edited by A M Semikhatov

$\lambda = (\mathrm{d}R/\mathrm{d}p)|_{p=p_c}$. The critical exponent of the order parameter β is determined from the equality $D = d - \beta/\nu$, where the dimension D of the percolation cluster can be approximated by that of the Sierpinski carpet; for the spatial dimension $d = 2$, $\beta = 0.193$. (To verify the values obtained, we note that in the case of the standard Sierpinski carpet, $\nu = 2.194$ and $\beta = 0.234$, according to our data, while $\nu = 2.13$ and $\beta = 0.27$, according to the results in Ref. [3].)

Other critical exponents can be found from the system of equalities for the two-exponent scaling [2]: the index of mean length of a finite cluster is $\gamma = \nu d - 2\beta = 3.216$, the critical exponent for the analog of specific heat is $\alpha = 2 - \nu d = -1.602$, and the index related to the largest size of finite clusters is $\Delta = \nu d - \beta = 1.809$.

3. Model of the force field of the Sierpinski square

We consider a 'wire' model of the Sierpinski carpet. Let the initial square frame be divided by four wires into nine equal squares. The procedure is repeated many times on each of the 8^m frames that are constructed in the subsequent step (the central ones are removed). We also assume that on each side of frames of an arbitrary 'generation', there are point sources distributed with a line density λ and generating a field with the strength $E \sim 1/r^2$. We find the analytic form of the force field generated by the multiscale network of the internal edges of the Sierpinski square of an arbitrary subdivision step m.

To simplify the expressions, we place the origin at the point where the field strength is sought, with axes aligned parallel to the square sides. We assume that the origin does not lie on any line containing edges of cells of the carpet of the mth generation. Let the carpet center be at the point $(\xi;\eta)$.

We set

$$E_x \equiv X_m(\xi;\eta)\,, \quad E_y \equiv Y_m(\xi;\eta)\,,$$

$$\xi(n,p) = \xi + (-1)^n p\,, \quad \eta(n,p) = \eta + (-1)^n p\,,$$

$$h = \frac{H}{3^m}\,, \quad m \in \mathrm{N}\,,$$

$$A(u;v) = \frac{k\lambda}{\sqrt{u^2+v^2}}\,, \quad B(u;v) = \frac{k\lambda v}{u\sqrt{u^2+v^2}}\,,$$

where k is a coefficient depending on the selected set of units. Then,

$$X_0(\xi;\eta) = \sum_{j=1}^{2}\sum_{i=1}^{2}(-1)^i \times \left[A\big(\xi(i,h);\,\eta(j,h)\big) - B\big(\xi(j,h);\,\eta(i,h)\big)\right],$$

$$Y_0(\xi;\eta) = X_0(\eta;\xi)\,,$$

$$\begin{aligned} X_m(\xi;\eta) = \sum_{i=1}^{2}\Bigg\{ & X_{m-1}\left(\xi;\eta\left(i,\frac{2H}{3}\right)\right) + X_{m-1}\left(\xi\left(i,\frac{2H}{3}\right);\eta\right) \\ & + \sum_{j=1}^{2}\Bigg\{X_{m-1}\left(\xi\left(j,\frac{2H}{3}\right);\,\eta\left(i,\frac{2H}{3}\right)\right) \\ & + (-1)^j\Bigg[A\left(\xi\left(j,\frac{H}{3}\right);\eta\left(i,\frac{H}{3}\right)\right) - A\left(\xi(j,H);\,\eta\left(i,\frac{H}{3}\right)\right) \\ & + B\left(\xi\left(i,\frac{H}{3}\right);\eta(j,H)\right) - B\left(\xi\left(i,\frac{H}{3}\right);\eta\left(j,\frac{H}{3}\right)\right)\Bigg]\Bigg\}\Bigg\}\,, \end{aligned}$$

$$Y_m(\xi;\eta) = X_m(\eta;\xi)\,.$$

For physical applications, it seems natural to place the origin at the center of the Sierpinski carpet and compute the field strength at an arbitrary point with coordinates $(x;\,y)$. Formally, for such a parallel translation it suffices to substitute $-x$ and $-y$ for ξ and η in functions X_m and Y_m.

4. Conclusions

A possible application of the results obtained is the description of oscillatory interaction for multiscale internal boundaries in a heterogeneous material. The statistical self-similarity in the arrangement of the material internal boundaries leads to the formation of energy conditions favoring the emergence of larger-scale boundaries. In turn, the deformation fields associated with these boundaries act on smaller-scale heterogeneities and induce their further growth. This happens synchronously across all scales [5–7]. The simplest model analogs of such networks of internal boundaries can be the fractals like the Sierpinski carpet or the Menger sponge, modified with the help of an affine map.

A formal stochastic model for the interaction of structure heterogeneities of various scale levels was explored in [7, 8]. The structure of the material was modeled by an open dynamical system with three interacting scale levels of heterogeneities, and its evolution was described by the system of bilinear iterative equations

$$\begin{cases} x_{n+1} = x_n - k_{xy}\,p x_n^2 + k_{yx}\,q y_n^2 + x_{\mathrm{in}}\,, \\ y_{n+1} = y_n + k_{xy}\,p x_n^2 - (k_{yx} + k_{yz})\,q y_n^2 + k_{zy} r z_n^2\,, \\ z_{n+1} = z_n + k_{yz}\,q y_n^2 - (k_{zy} + k_{\mathrm{out}})\,r z_n^2\,, \end{cases}$$

where x, y, z are the dynamical variables defining the potential energy of the level, and x_{in} defines the energy of external action. The coefficients k_{ij} describe the fraction of energy transferred between heterogeneities of different scales, the coefficients p, q, and r describe its fraction spent to restructuring, $\{k_{ij}\}$ and $\{p,q,r\} \in (0,1)$, and $\{x,y,z\} \in R^+$.

The character of the evolution of this system as a function of energy supply was studied in Refs [8, 9]. It is shown there that for small values of the parameter x_{in} the system has a stationary state, while as in x_{in} increases, two variants of the system evolution become possible: the first one follows the Feigenbaum scenario, and in the second one, a situation analogous to the Hopf bifurcation occurs after a periodic behavior. The further increase in x_{in} leads to a bifurcation that leads to a chaotic behavior.

The coefficients in the above system of equations should characterize specific features of the material structure. The estimation of their numerical values in Refs [8, 9] relies on maximally general assumptions stemming from the analysis of the physical situation. Their more precise definition can presumably be found by using information on the structure and properties of internal boundaries.

References

1. Bozhokin S V, Parshin D A *Fraktaly i Mul'tifraktaly* (Fractals and Multifractals) (Izhevsk: RKhD, 2001)
2. Sokolov I M *Usp. Fiz. Nauk* **150** 221 (1986) [*Sov. Phys. Usp.* **29** 924 (1986)]
3. Ben-Avraham D, Havlin S, Movshovitz D *Philos. Mag. B* **50** 297 (1984)
4. Reynolds P J, Stanley H E, Klein W *Phys. Rev. B* **21** 1233 (1980)

5. Panin A E et al. *Strukturnye Urovni Plasticheskoi Deformatsii i Razrusheniya* (Structural Levels of Plastic Deformation and Break-up) (Ed. V E Panin) (Novosibirsk: Nauka, 1990)
6. Olemskoi A I, Sklyar I A *Usp. Fiz. Nauk* **162** (6) 29 (1992) [*Sov. Phys. Usp.* **35** 455 (1992)]
7. Vyrovoi V N, Gerega A N, Korobko O A, in *Trudy IPME NAN Ukrainy* (Ed. V F Evdokimov) (Kiev: IPME NAN Ukrainy, 2010) p. 253
8. Bekker M, Herega A, Lozovskiy T *Adv. Dynam. Syst. Appl.* **4** (2) 179 (2009)
9. Aslanov A M, Herega A N, Lozovski T L *Zh. Tekh. Fiz.* **76** (6) 134 (2006) [*Tech. Phys.* **51** 812 (2006)]