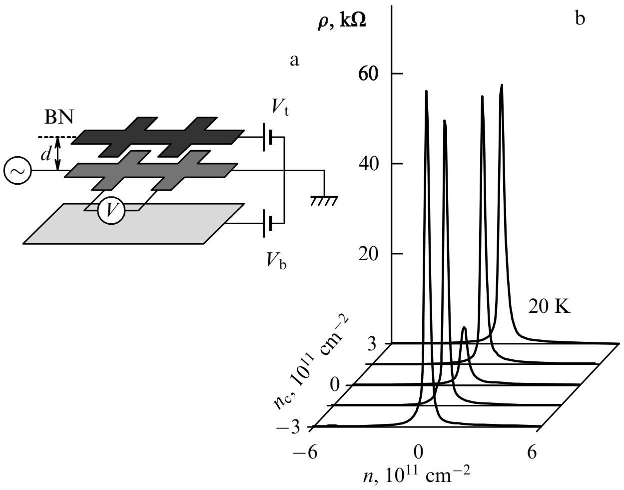

Figure 7. (a) Schematic diagram for the measurement in a structure comprising two separated graphene layers lying on an oxidized silicon substrate (the lower layer). There are boron nitride crystallites (not shown in the drawing) between all the layers. The thickness of the boron nitride layer between the graphene layers is $d \approx 10$ nm. (b) Dependence of the resistivity ρ on the carrier concentration n in the graphene layer under investigation for different carrier concentrations n_c in the control (upper) graphene layer.

magnetic field ($B < 0.1$ T) perpendicular to the graphene layer, which is more likely an indication of an interference effect rather than of the discovery of a forbidden band in graphene.

This behavior of the resistivity is indicative of a metal–insulator transition, demonstrating the Anderson localization on increasing $\rho > h/e^2$. In standard specimens on a silicon substrate, the metal–insulator transition is masked by inhomogeneities and the formation of 'pools' of electrons and holes near the electroneutrality point. The state of graphene inside each of them is far from electroneutrality point, and it remains metallic. Accordingly, the resistivity of the system is determined by semitransparent (owing to Klein tunneling [32, 33]) electron–hole transitions with a weak temperature dependence [34, 35].

The control graphene layer may effectively screen the fluctuation potential and suppress the emergence of electron and hole pools, making it possible to study the behavior of graphene in the vicinity of the electroneutrality point. This interpretation also favors the idea that the proximity of the minimal conductivity in traditional structures to the quantity $4e^2/h$ is due to the flow across the boundaries of the pools of electrons and holes. Therefore, an unusual situation, which is extrinsic to conventional metals and semiconductors, is realized in graphene, whereby localization results from a lowering of disordering rather than from its increasing.

In summary, it should be noted that the emergence of high-mobility graphene structures has led not only to a refinement of certain notions of the graphene physical properties, but also to their revision. At the same time, it is hard to overestimate the promise of the recently commenced work on layered structures consisting of two-dimensional crystals of boron nitride and graphene.

Graphene began its history at some point by separating from its three-dimensional progenitor — graphite. It is not unlikely that in the near future we will obtain a variety of new three-dimensional materials custom-made of different two-dimensional crystals and highly diversified in properties.

Acknowledgments. This work was supported by the Russian Foundation for Basic Research and the Programs of the Presidium of the Russian Academy of Sciences which are gratefully acknowledged.

References

1. Geim A K, Novoselov K S *Nature Mater.* **6** 183 (2007)
2. Castro Neto A H et al. *Rev. Mod. Phys.* **81** 109 (2009)
3. Das Sarma S et al. *Rev. Mod. Phys.* **83** 407 (2011)
4. Lozovik Yu E, Merkulova S P, Sokolik A A *Usp. Fiz. Nauk* **178** 757 (2008) [*Phys. Usp.* **51** 727 (2008)]
5. Morozov S V, Novoselov K S, Geim A K *Usp. Fiz. Nauk* **178** 776 (2008) [*Phys. Usp.* **51** 744 (2008)]
6. Dean C R et al. *Nature Phys.* **7** 693 (2011)
7. Elias D C et al. *Science* **323** 610 (2009)
8. Nair R R et al. *Small* **6** 2877 (2010)
9. Novoselov K S et al. *Proc. Natl. Acad. Sci. USA* **102** 10451 (2005)
10. Mayorov A S et al. *Nano Lett.* **11** 2396 (2011)
11. Beenakker C W J, van Houten H *Phys. Rev. Lett.* **63** 1857 (1989)
12. Elias D C et al. *Nature Phys.* **7** 701 (2011)
13. Novoselov K S et al. *Nature* **438** 197 (2005)
14. Bostwick A et al. *Science* **328** 999 (2010)
15. Khveshchenko D V *Phys. Rev. Lett.* **87** 246802 (2001)
16. Gorbar E V et al. *Phys. Rev. B* **66** 045108 (2002)
17. Drut J E, Lähde T A *Phys. Rev. Lett.* **102** 026802 (2009)
18. Mayorov A S et al. *Science* **333** 860 (2011)
19. McCann E, Fal'ko V I *Phys. Rev. Lett.* **96** 086805 (2006)
20. Lemonik Y et al. *Phys. Rev. B* **82** 201408(R) (2010)
21. Vafek O, Yang K *Phys. Rev. B* **81** 041401(R) (2010)
22. Abanin D A et al. *Science* **332** 328 (2011)
23. McEuen P L et al. *Phys. Rev. Lett.* **64** 2062 (1990)
24. Haug R J *Semicond. Sci. Technol.* **8** 131 (1993)
25. Tombros N et al. *Nature* **448** 571 (2007)
26. Sinova J et al. *Phys. Rev. Lett.* **92** 126603 (2004)
27. Kato Y K *Science* **306** 1910 (2004)
28. Wunderlich J et al. *Phys. Rev. Lett.* **94** 047204 (2005)
29. Imada M, Fujimori A, Tokura Y *Rev. Mod. Phys.* **70** 1039 (1998)
30. Evers F, Mirlin A D *Rev. Mod. Phys.* **80** 1355 (2008)
31. Ponomarenko L A et al. *Nature Phys.* **7** 958 (2011)
32. Katsnelson M I, Novoselov K S, Geim A K *Nature Phys.* **2** 620 (2006)
33. Cheianov V V, Fal'ko V I *Phys. Rev. B* **74** 041403(R) (2006)
34. Adam S et al. *Proc. Natl. Acad. Sci. USA* **104** 18392 (2007)
35. Cheianov V V et al. *Phys. Rev. Lett.* **99** 176801 (2007)

PACS numbers: **42.25.–p**, **42.30.–d**, 42.60.Jf
DOI: 10.3367/UFNe.0182.201204f.0442

Modern optics of Gaussian beams

V G Volostnikov

1. Introduction

A coherent light field, like any oscillatory process, is characterized by its amplitude and phase. The methods and means for analyzing light fields from intensity measurements underlie optical instruments, and from the physical standpoint the solution to any optical measurement problem

V G Volostnikov Samara Branch of the P N Lebedev Physical Institute, Russian Academy of Sciences, Samara, Russian Federation
E-mail: coherent@fian.smr.ru

Uspekhi Fizicheskikh Nauk **182** (4) 442 – 450 (2012)
DOI: 10.3367/UFNr.0182.201204f.0442
Translated by E N Ragozin; edited by A Radzig

involves establishing the relation between the energy and structural parameters of optical radiation.

Owing to the special nature of the optical range, it is not the complex amplitude of an optical signal that is amenable to measurement, but only its intensity, which is not the complete characteristic of a light field in the general case. Traditional interferometric techniques provide, in principle, a possibility of indirect phase measurements; however, in several problems it is impossible or hard to realize the interferometric principle for acquiring information about the complex amplitude or phase of the field. This situation takes place in astronomy, X-ray, and adaptive optics. In this connection, the quest for and investigation of field intensity–phase relations, which provide an answer to the question of how many intensity measurements should be made and of what type should they be so as to reconstruct the field itself or its certain characteristics, are relevant. On the other hand, there is a separate area of investigations in which the sought-after field is to be synthesized rather than reconstructed. This applies to the problems of radiation focusing on a domain with given spatial characteristics and to the task of the intracavity formation of a beam with a given output radiation structure. These problems are kindred to that of light field analysis: they all involve gaining information about the field from its energy characteristics. However, it is easy to see that they are significantly different as well: the physical realizability of a field with the intensity under analysis is inherent in the very formulation of the analysis problem, while the question of whether a field with a given intensity exists is one of the central ones in the synthesis problem. It nevertheless turned out that the results of investigations concerned with the analysis of light fields also open up fundamentally new possibilities for their synthesis in different physical situations. Recent years have seen important new findings, which have not been reflected in monographs. The central aim of this report is to fill in this gap.

This report outlines the results of an investigation into the so-called one-dimensional phase problem in optics. An explicit analytical relation was found between the intensity and the phase of a one-dimensional field in the Fresnel zone.

A two-dimensional problem in optics was also considered and shown to be radically different from the one-dimensional problem. It was established that the vector field of light energy flux generally comprises potential and vortical components; an explicit analytical relation was found between the two-dimensional phase and intensity distributions for vortex-free fields in the Fresnel zone; the vortical component was shown to obey a conservation law, specifically: the integral of the projection of the rotor of the light energy flux vector onto the direction of propagation is equal to zero for any plane in the Fresnel zone, and a relation was revealed between the vortical component of the light energy flux vector and wave-front dislocations.

Also obtained were explicit analytical dependences of the phase of a two-dimensional light field on its intensity as functions of certain parameters of the generating optical system.

The behavior of Gaussian beams under astigmatic action was investigated, too. It was theoretically shown that certain astigmatic optical systems accomplish the mutual transformation of Hermite–Gauss (HG) and Laguerre–Gauss (LG) beams. A parametric class of light beams — the generalized Hermite–Laguerre–Gauss (HLG) beams described by a complete system of parameter-dependent orthogonal functions — was discovered and experimentally realized, the known HG and LG beams being their special types. Optical systems that realize the HG-to-LG beam transformation and the results of the corresponding experiments are described below.

The question of the search for light fields that retain their structure in the course of propagation and focusing, correct to scale and rotation, is formulated and solved in the paraxial approximation. A total description of such light fields, which are termed spiral beams, is given, as are their propagation and rotation laws. The linkage between spiral beams and quantum mechanics is considered. Several ways of realizing spiral beams in experiment are suggested.

Methods for synthesizing light fields with a given intensity distribution that are structurally stable during propagation were considered. Proceeding from spiral beam optics, it was possible to obtain light fields whose intensity distribution is of the form resembling an arbitrary plane curve. The properties of spiral beams for closed curves were investigated. Such beams were found to exhibit characteristic quantization properties: first, the intensity distribution undergoes a radical change under similarity transformations of the corresponding curve and has the shape of this curve only for certain values of the similarity factor; second, the area under the beam's curve for the same values of the similarity factor is related to the Gaussian parameter by an integer-valued relation, and in this case the number of phase singularities of a spiral beam inside the curve is also quantized and their number is defined only by the area inside the curve rather than by its shape.

Also outlined are the results of the application of spiral beam optics to the synthesis problem of phase diffraction optical elements intended for the focusing of a light field on a plane curve and on a two-dimensional domain of a given shape. A new iterative method was proposed for the solution of this problem, which involves the employment of the near-field phase distribution of a spiral beam and its far-field intensity distribution for curves. Proposed for the focusing on a domain were the corresponding distributions of Fourier-invariant fields as the initial approximations in the synthesis of the corresponding phase diffraction optical elements. The results of numerical and natural experiments are presented.

2. Reconstruction of a one-dimensional coherent monochromatic field from measured intensities in the Fresnel zone

Let us elucidate the relationship between the intensity and the phase of a light field in the Fresnel zone. The equation which describes the Fresnel transformation has the form

$$F(x)=\sqrt{\frac{k}{2\pi l}}\,\exp\left(-\frac{1}{4}\,\pi\mathrm{i}\right)\int_a^b \exp\left[\frac{\mathrm{i}k}{2l}\,(x-\xi)^2\right]U(\xi)\,\mathrm{d}\xi\,. \tag{1}$$

Most works on the one-dimensional phase problem are concerned with algorithmic, purely numerical methods for reconstructing the object field from intensity measurements. On the other hand, it would be instructive to elucidate the physical aspect of the problem and its association with the mathematical formulation and, in particular, to derive explicit formulas expressing the intensity–phase relation. This formulation of the problem is all the more justified since the use of explicit formulas offers several advantages from the practical point of view: it shortens the computation time and permits, in principle, estimating the effect of intensity measurement uncertainty on the accuracy of phase reconstruction.

A similarly formulated problem for a parabolic approximation was studied in several papers (see Ref. [1] and references cited therein), which also suggest that deriving the exact solution requires knowledge of the phase derivative $\partial\varphi/\partial x$ at some point. It is stated simultaneously that finding the boundary condition for $\partial\varphi/\partial x$ from intensity measurements is unlikely.

We show below that it is possible to overcome this difficulty and to determine the field $F(x,l)$ and, hence, the object field $U(\xi)$ from the measurements of intensity $I(x,l)$ and its derivative $I_l(x,l)$ along the direction of field propagation in the Fresnel zone [1]. In the latter zone, $U(\xi) = F(x,0)$ and $F(x,l)$ are related by Eqn (1). It may be shown that the field $F(x,l)$ entering Eqn (1) obeys the parabolic equation [2]

$$\frac{\partial^2 F}{\partial x^2} + 2\mathrm{i}k\,\frac{\partial F}{\partial l} = 0\,. \tag{2}$$

Substituting $F(x,l)$ into Eqn (2) in the form $F(x,l) = \sqrt{I(x,l)}\,\exp\big(\mathrm{i}\varphi(x,l)\big)$ and separating the real and imaginary parts give the system of differential equations for the intensity and the phase of the field $F(x,l)$ in the Fresnel zone:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(I\,\frac{\partial\varphi}{\partial x}\right) + k\,\frac{\partial I}{\partial l} = 0\,,\\ &2I\,\frac{\partial^2 I}{\partial x^2} - \left(\frac{\partial I}{\partial x}\right)^2 - 4I^2\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2k\,\frac{\partial\varphi}{\partial l}\right] = 0\,. \end{aligned} \tag{3}$$

The former equation serves as the continuity equation for the flux $\mathbf{j} = (j_x, j_l) = (I\,\partial\varphi/\partial x,\ kI)$ and expresses the law of light energy conservation in differential form [1]. By integrating this equation, we obtain

$$I(x,l)\,\frac{\partial\varphi(x,l)}{\partial x} = -k\int_{x_0}^{x} I_l(t,l)\,\mathrm{d}t + c\,, \tag{4}$$

where $c = I(x_0)\,\partial\varphi/\partial x\,(x_0)$. Repeated integration yields an expression for the field phase in the Fresnel zone:

$$\varphi(x) = \varphi(a) - k\int_a^x \frac{\mathrm{d}t}{I(t)}\int_{x_0}^{t} I_l(\tau)\,\mathrm{d}\tau + c\int_a^x \frac{\mathrm{d}t}{I(t)}\,. \tag{5}$$

One can see from expression (4) that obtaining the phase requires knowledge of the boundary condition for $\partial\varphi/\partial x$ at some point x_0, the nonlinear character of the relation between $I(x,l)$ and $\varphi(x,l)$ in expression (4) making this issue quite significant. Let us show that the boundary condition for $\partial\varphi/\partial x$ may be found from intensity measurements.

We define the differential operators

$$L \equiv \frac{\partial^2}{\partial z^2} + 2\mathrm{i}k\,\frac{\partial}{\partial l}\,, \quad L^* \equiv \frac{\partial^2}{\partial z^2} - 2\mathrm{i}k\,\frac{\partial}{\partial l} \tag{6}$$

and rewrite equation (2) in the form

$$LF(z,l) = L^*F^*(z,l) \equiv 0\,.$$

The action of operators L and L^* on $I(z,l)$ leads to the following result

$$\begin{aligned} LI(z,l) &= 2\,\frac{\partial}{\partial z}\left(F(z,l)\,\frac{\partial F^*}{\partial z}\,(z,l)\right),\\ L^*I(z,l) &= 2\,\frac{\partial}{\partial z}\left(F^*(z,l)\,\frac{\partial F}{\partial z}\,(z,l)\right). \end{aligned} \tag{7}$$

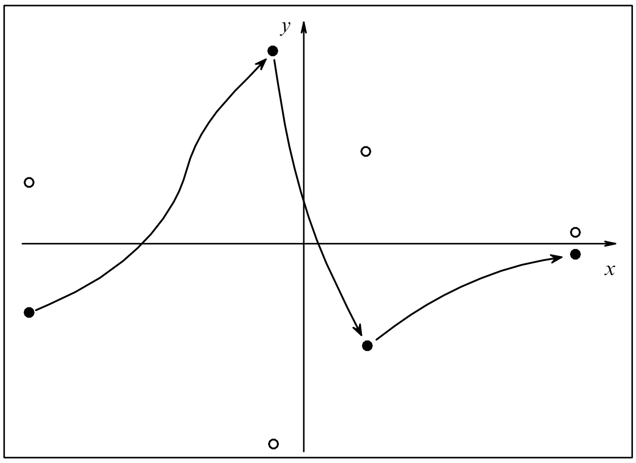

Figure 1. Zero pairs $(z_m, \bar{z}_m)$ of the analytic continuation of intensity $I(z,l)$. Black dots indicate the zeroes of field $F(z,l)$, for which the first of equalities (8) takes place.

Both differential relations (7) are proved by direct substitution with the use of Eqn (2), for instance:

$$\begin{aligned} LI &= F^*LF + FLF^* + 2\,\frac{\partial F}{\partial z}\,\frac{\partial F^*}{\partial z}\\ &= 2F\,\frac{\partial^2 F^*}{\partial z^2} + 2\,\frac{\partial F}{\partial z}\,\frac{\partial F^*}{\partial z} = 2\,\frac{\partial}{\partial z}\left(F\,\frac{\partial F^*}{\partial z}\right). \end{aligned}$$

Consequently, if z_1 and z_2 are the zeroes of function $F(z,l)$ for some fixed l, then

$$\int_{z_1}^{z_2} LI(z,l)\,\mathrm{d}z = 0\,, \quad \int_{\bar{z}_1^*}^{\bar{z}_2^*} L^*I(z,l)\,\mathrm{d}z = 0\,. \tag{8}$$

The use of equalities (8) permits determining all zeroes of the function F from magnitudes of the intensity $I(z,l)$ and its derivative $\partial I(z,l)/\partial l$ for some $l = l_0 = \mathrm{const}$ (Fig. 1).

Therefore, the problem of separating out the set of $F(z)$ zeroes from the set of $I(z)$ zeroes may be solved from the distributions of the intensity $I(x)$ and its derivative $I_l(x)$ in some fixed plane $l = \mathrm{const}$ with the aid of analytic continuation and the use of the properties of the functions $I(z)$ and $I_l(z)$ in the complex plane.

3. Reconstruction of a two-dimensional coherent monochromatic field from measured intensities in the Fresnel zone

The linkage between the intensity and the phase of a two-dimensional light field $F(x,y)$ is more poorly understood than its one-dimensional analog. In particular, the nature of the nonuniqueness of the problem's solution and the body of measurements required for its solution are not quite clear.

To substantively analyze the differences between the two- and one-dimensional phase problems, it is expedient to consider the two-dimensional version of a problem solved in the one-dimensional case.

In this section we shall study the two-dimensional problem, or the reconstruction of light field $F(x,y)$ at $l = \mathrm{const}$ from the measurements of intensity $I(x,y)$ and its derivative along the direction of propagation of the radiation $I_l(x,y)$ in the Fresnel zone. This formulation of the phase problem is of interest in the quality control of large-sized optics, the synthesis of optical elements and light fields, and several other applications.

It is well known that $U(\xi,\eta)$ and $I(x,y,l)$ in the two-dimensional case in the Fresnel domain are related by the expression

$$I(x,y,l) = F(x,y,l)\,\overline{F}(x,y,l)$$
$$= \left|\frac{k}{2\pi i l}\iint_\Omega \exp\left(\frac{ik}{2l}\left[(x-\xi)^2+(y-\eta)^2\right]\right)U(\xi,\eta)\,d\xi\,d\eta\right|^2, \tag{9}$$

where Ω is the $U(\xi,\eta)$ carrier, i.e. $U(\xi,\eta)=0$ when $(\xi,\eta)\notin\Omega$.

The amplitude of $F(x,y,l)$ from expression (9) satisfies the following quasioptical parabolic equation [1]

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+2ik\,\frac{\partial F}{\partial l}=0\,. \tag{10}$$

By analogy with the one-dimensional case, $F(x,y,l)$ may be represented in the form $F(x,y,l)=\sqrt{I(x,y,l)}\exp(i\varphi(x,y,l))$ and, on substituting it into equation (10), it is possible to separate the real and imaginary parts.

It is easy to verify that

$$\frac{I\nabla\varphi}{k}=\frac{\overline{F}\,\nabla F-F\nabla\overline{F}}{2ik}=\mathbf{j}=(j_x,j_y)\,, \tag{11}$$

where $\mathbf{j}$ is the vector component of the light energy flux in the (x,y) plane, and obtain the light energy conservation law, or the continuity equation which may be treated as the three-dimensional divergence of the vector $\mathbf{j}=(j_x,j_y,I)$ or as the equation for a flat field:

$$\frac{\partial\rho}{\partial l}+\nabla\mathbf{j}=0\,, \tag{12}$$

where $\rho=I(x,y,l)$, and $\mathbf{j}=(j_x,j_y)$. Substituting $F(x,y,l)=U(x,y,l)+iV(x,y,l)$ into expression (11) gives

$$\mathbf{j}=\frac{U\nabla V-V\nabla U}{k}=\frac{1}{k}\left(U\frac{\partial V}{\partial x}-V\frac{\partial U}{\partial x}\,,\ U\frac{\partial V}{\partial y}-V\frac{\partial U}{\partial y}\right). \tag{13}$$

One can see from expression (11) that the phase reconstruction problem in this formulation reduces to the reconstruction problem of the vector field from its rotor and divergence. The rotor of the flux is defined as

$$\mathrm{rot}_0\,\mathbf{j}=\frac{2}{k}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial y}-\frac{\partial U}{\partial y}\frac{\partial V}{\partial x}\right)=\frac{1}{k}\left(\frac{\partial I}{\partial x}\frac{\partial\varphi}{\partial y}-\frac{\partial I}{\partial y}\frac{\partial\varphi}{\partial x}\right). \tag{14}$$

Mention should be made of the fundamental difference between the two-dimensional phase problem and the similar one-dimensional one. The solution to the two-dimensional problem is similar to the solution of the one-dimensional problem only when $\mathrm{rot}_0\,\mathbf{j}(x,y)\equiv 0$. This is true, for instance, for fields $F(x,y,l)=F(g(x,y),l)$, where $g(x,y)$ is a real function. The physical meaning of the difference between the two- and one-dimensional cases may be treated as follows. As revealed by analysis of the similar one-dimensional problem, the structural and phase field properties may be directly and completely represented by its energy characteristics; the intensity distribution and the energy conservation law permitted obtaining explicit formulas for the phase of the field. As follows, for instance, from equation (12), the structural and phase properties of the field in the two-dimensional case are not necessarily representable directly by its energy characteristics.

The complete definition of $\mathrm{rot}_0\,\mathbf{j}(x,y)$ in terms of the field intensities is impossible. It is nevertheless possible to prove several results characterizing the local and global properties of $\mathrm{rot}_0\,\mathbf{j}(x,y)$ [3].

Let $F(x,y)=\sqrt{I(x,y)}\exp(i\varphi(x,y))$ be the Fresnel transformation at $l=\mathrm{const}$ of some function with a finite carrier, with the scalar function $\mathrm{rot}_0\,\mathbf{j}(x,y)$ defined by equality (14). Then one has:

1) if $(x_0,\ y_0)$ is the intensity extremum point and $I(x_0,y_0)\neq 0$, then $\mathrm{rot}_0\,\mathbf{j}(x_0,y_0)=0$; if $I(x_0,y_0)=0$, then

$$\left|\mathrm{rot}_0\,\mathbf{j}(x_0,y_0)\right|=\frac{1}{k}\sqrt{\frac{\partial^2 I}{\partial x^2}\frac{\partial^2 I}{\partial y^2}-\left(\frac{\partial^2 I}{\partial x\,\partial y}\right)^2}\,(x_0,y_0)\,; \tag{15}$$

2) if $(x_0,\ y_0)$ is an isolated simple zero of function $F(x,y)$ and L is some contour which does not contain zeroes other than $(x_0,\ y_0)$, then

$$\oint_L\nabla\varphi\,d\mathbf{r}=2\pi\,\mathrm{sign}\,\mathrm{rot}_0\,\mathbf{j}(x_0,y_0)\,; \tag{16}$$

3) the following rotor 'conservation law' takes place:

$$\iint_{\mathrm{R}^2}\mathrm{rot}_0\,\mathbf{j}(x,y)\,dx\,dy=0\,. \tag{17}$$

In this section, therefore, the problem of the relation between the intensity and the phase of light field $F(x,y,l)$ was considered. In this case, we revealed a radical difference between the two- and one-dimensional phase problem solutions, which is due to the existence of the rotor of the light energy flux vector. The properties of the scalar function $\mathrm{rot}_0\,\mathbf{j}$, which generates this difference, were investigated.

4. Relation between the phase and intensity of the light field as functions of optical system parameters

In Section 3 we showed that passive measurements of the two-dimensional intensity distribution in the Fresnel zone do not permit deriving in general an explicit analytical relation between $I(x,y,l)$ and $\varphi(x,y,l)$. To state it in different terms, the information about the action of the field propagation operator $L=\partial^2/\partial x^2+\partial^2/\partial y^2+2ik\,\partial/\partial l$ on the intensity does not produce a result similar to that obtained for the operator $L=\partial^2/\partial x^2+2ik\,\partial/\partial l$ in the one-dimensional case. Now the question is in order: Are there operators which describe real physical situations and provide the solution of this problem for a two-dimensional light field?

Let us consider the transformation of an optical field by an optical system. To simplify calculations, in this section we put $x=x_1,\ y=x_2$. It is well known that the complex amplitude $F(x_1,x_2)$ in the image plane is a Fourier transform of the amplitude $U_0(\xi_1,\xi_2)$ at the exit pupil of an optical system:

$$F(x_1,x_2)=\iint_{\mathrm{R}^2}\exp\left[-i(x_1\xi_1+x_2\xi_2)\right]U_0(\xi_1,\xi_2)\,d\xi_1\,d\xi_2\,, \tag{18}$$

where $U_0(\xi_1,\xi_2)=P(\xi_1,\xi_2)f(\xi_1,\xi_2)$, $P(\xi_1,\xi_2)$ is the complex function of the pupil of the optical system, and $f(\xi_1,\xi_2)$ is the Fourier spectrum of the image.

It is evident that $F(x_1, x_2)$ and $I(x_1, x_2) = F(x_1, x_2) \times \overline{F}(x_1, x_2)$ depend on the parameters of the optical system, characterized by function $P(\xi_1, \xi_2)$.

Consider the problem of $F(x_1, x_2)$ reconstruction from the measurements of intensity $I(x_1, x_2)$ as a function of these parameters. By analogy with the aberration theory [1], we represent $P(\xi_1, \xi_2)$ in the form

$$P(\xi_1, \xi_2) = \chi(\xi_1, \xi_2) \exp\big(-\mathrm{i} W(\xi_1, \xi_2)\big), \tag{19}$$

where $\chi(\xi_1, \xi_2)$ is the characteristic pupil function Ω defined as

$$\chi(\xi_1, \xi_2) = \begin{cases} 1, & (\xi_1, \xi_2) \in \Omega, \\ 0, & (\xi_1, \xi_2) \notin \Omega, \end{cases}$$

$W(\xi_1, \xi_2) = W_{21}\xi_1^2 + W_{22}\xi_2^2$; here, W_{2n} stands for astigmatism.

Now if $W_{2n} = \alpha_{2n}$, the problem reduces to the reconstruction of $F(x_1, x_2)$ from the measurements of intensity $I(x_1, x_2)$ and its derivatives with respect to α_{2n} for $n = 1, 2$.

Upon similar transformations, equation (12) reduces to the following system of equations

$$\frac{\partial I}{\partial \alpha_{2n}} + 2\,\frac{\partial}{\partial x_n}\left(I\,\frac{\partial \varphi}{\partial x_n}\right) = 0, \quad n = 1, 2. \tag{20}$$

The system of equations (20) describes an optical system with cylindrical phase mask $\exp(-\mathrm{i}k\alpha_{2n}\xi_n^2)$ at the exit pupil of the optical system or an optical system with cylindrical defocusing of the illuminating beam and recording in the far-field radiation zone. We note that the problem in the one-dimensional case is completely similar to that considered in Section 2.

Therefore, the field variation under purposeful action on the field is more informative than its variation under natural propagation. This is supposedly an illustration of the well-known fact that an active experiment gives better results than passive observation.

The requisite astigmatic actions (20) on the light field may also be realized directly by employing specific diffraction elements [4].

Let us select the phase function $W(\xi_1, \xi_2)$ entering expression (19) in the form

$$W(\xi_1, \xi_2) = T_0\left(\frac{T}{2\pi}\,\alpha\xi_1^2 + \xi_1\right) + T_0\left(\frac{T}{2\pi}\,\alpha\xi_2^2 + \xi_2\right), \tag{21}$$

where $T_0(x)$ is a T-periodic function of argument x, and α is a parameter. The phase element with the profile (21) may be represented as a Fourier series:

$$\exp\big(\mathrm{i}W(\xi_1, \xi_2)\big) = \sum_{m,n} c_m c_n \exp\left(\mathrm{i}m\alpha\xi_1^2 + \mathrm{i}m\,\frac{2\pi}{T}\,\xi_1\right) \times \exp\left(\mathrm{i}n\alpha\xi_2^2 + \mathrm{i}n\,\frac{2\pi}{T}\,\xi_2\right), \tag{22}$$

where

$$c_m = \frac{1}{T}\int_0^T \exp\left(-\frac{2\pi}{T}\,\mathrm{i}mx + \mathrm{i}T_0(x)\right) \mathrm{d}x.$$

It is easy to see that such a diffraction element operates as a system of off-axis astigmatic lenses with principal focal lengths $f_m = \pi/\alpha m\lambda$, $f_n = \pi/\alpha n\lambda$ in the (m, n)-th diffraction order (Fig. 2). The diffraction angles of order (m, n) are

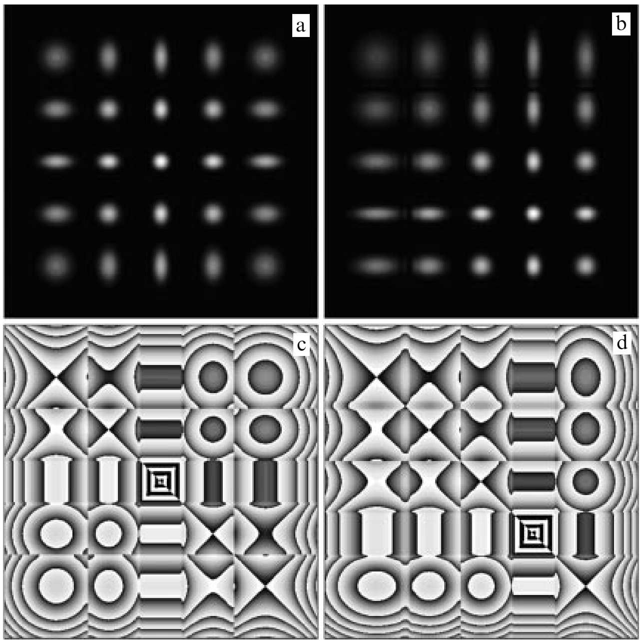

Figure 2. Diffraction pattern of a Gaussian beam: intensity (a, b) and phase (c, d) with diffraction element (21) without astigmatism (a, c), and with astigmatism $\pi(\xi_2^2 - \xi_1^2)/\lambda f_g$, $f_g = \pi/\alpha\lambda$ in figures (b, d).

$\beta_m = \arcsin(m\lambda/T)$ and $\beta_n = \arcsin(n\lambda/T)$, respectively, and the complex amplitude (19) in the image plane for phase function (21) takes on the form

$$\begin{aligned} F(x_1, x_2) &= \sum_{m,n} c_m c_n \iint_{\mathrm{R}^2} \exp(-\mathrm{i}x_1\xi_1 - \mathrm{i}x_2\xi_2)\, U(\xi_1, \xi_2) \\ &\quad \times \exp\left(\mathrm{i}m\alpha\xi_1^2 + \mathrm{i}m\,\frac{2\pi}{T}\,\xi_1\right) \exp\left(\mathrm{i}n\alpha\xi_2^2 + \mathrm{i}n\,\frac{2\pi}{T}\,\xi_2\right) \mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &= \sum_{m,n} c_m c_n F_{mn}\left(x_1 - \frac{2\pi}{T}\,m,\; x_2 - \frac{2\pi}{T}\,n\right). \end{aligned} \tag{23}$$

5. Transformation of Hermite–Gauss beams into Laguerre–Gauss beams

The phase problem in optics may be considered as the problem of the linkage between the structural and energy characteristics of a light field. In Sections 2–4 we investigated the relation between the intensity and phase of a light field which satisfies the quasioptical parabolic equation in the Fresnel zone.

Different modifications of this equation describe a broad class of phenomena in quantum mechanics and optics. It is evident that the fields which possess structural stability during propagation occupy a special place, and the intensity–phase relationship is characteristically embodied in them.

On the other hand, as noted in Section 4, an HG beam is transformed into an LG beam in the course of diffraction by an astigmatic diffraction element (Fig. 3). In this connection, the structurally stable solutions of the parabolic equation call for a closer examination.

HG beams are well-known families of the stable solutions of the parabolic equation in optics:

$$\mathcal{H}_{n,m}(x, y) = \exp(-x^2 - y^2)\,\mathcal{H}_n(\sqrt{2}x)\,\mathcal{H}_m(\sqrt{2}y), \quad n, m = 0, 1, \ldots, \tag{24}$$

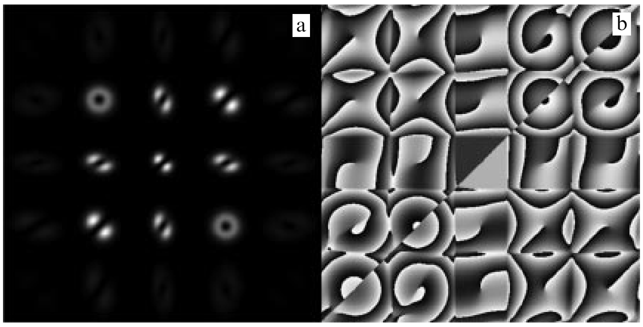

Figure 3. Pattern of Hermite–Gauss beam diffraction by diffraction element (21): intensity (a), and phase (b).

like LG beams:

$$\mathcal{L}_{n,\pm m}(x,y) = \exp(-x^2-y^2)(x \pm \mathrm{i}y)^m \mathcal{L}_n^m(2x^2+2y^2)\,, \quad n,\, m = 0, 1, \ldots\,, \tag{25}$$

and occupy a prominent place in the theory of resonators and light guides.

Therefore, the change in HG beams under defocusing reduces, correct to a quadratic phase factor, to only a change of scale.

On the other hand, the general form of astigmatism is described by the expression

$$\psi(\xi,\eta,a,\alpha) = a\left[(\xi^2-\eta^2)\cos 2\alpha + 2\xi\eta\sin 2\alpha\right]. \tag{26}$$

One can see from expression (26) that, unlike defocusing which is invariant under rotations, the form of the field

$$F(x,y,a) = \iint_{\mathrm{R}^2} \exp\left(-\mathrm{i}(x\xi+y\eta) + \mathrm{i}\psi(\xi,\eta,a,\alpha)\right) \times U(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{27}$$

depends on the rotation angle α in the propagation in 'astigmatic' space.

Let us now consider the transformation of HG beams under general astigmatic action (see also Ref. [1]):

$$F_{n,m}(x,y,a,\alpha) = \iint_{\mathrm{R}^2} \exp\left(-\mathrm{i}(x\xi+y\eta) + \mathrm{i}\psi(\xi,\eta,a,\alpha)\right) \times \mathcal{H}_{n,m}\left(\frac{\xi}{\rho}, \frac{\eta}{\rho}\right) \mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{28}$$

Of particular interest is a special case of transformation (28): for $a = 1/\rho^2$ and $\alpha = \pi/4$, HG beams go over to LG beams [4]:

$$\begin{aligned} &\iint_{\mathrm{R}^2} \exp\left[-\mathrm{i}(x\xi+y\eta) + \frac{2\mathrm{i}\xi\eta}{\rho^2}\right] \mathcal{H}_{n,m}\left(\frac{\xi}{\rho}, \frac{\eta}{\rho}\right) \mathrm{d}\xi\,\mathrm{d}\eta \\ &= \frac{\pi\rho^2}{\sqrt{2}}\,(-1)^{n+m} \exp\left(-\frac{1}{4}\,\mathrm{i}\rho^2 xy\right) \\ &\times \begin{cases} (2\mathrm{i})^n m!\,\mathcal{L}_{m,n-m}\left(\dfrac{\rho x}{2\sqrt{2}}, \dfrac{\rho y}{2\sqrt{2}}\right) & \text{for } n \geqslant m\,, \\ (2\mathrm{i})^m n!\,\mathcal{L}_{n,m-n}\left(\dfrac{\rho y}{2\sqrt{2}}, \dfrac{\rho x}{2\sqrt{2}}\right) & \text{for } n \leqslant m\,. \end{cases} \end{aligned} \tag{29}$$

For any fixed α, the set of fields is a full-value family of orthogonal, structurally stable beams, like the families of HG $\{\mathcal{H}_{n,m}(x,y), n,m=0,1,...\}$ and LG $\{\mathcal{L}_{n,m}(x,y), n, \pm m = 0,1,...\}$ modes. These fields, which were termed generalized Hermite–Laguerre–Gauss beams, were obtained experimentally for different α. These fields were sequentially realized in rotating the cylindrical lens about the optical axis by an angle α. The generalized beams obtained experimentally are exemplified in Fig. 4 for $n = 5$, $m = 4$.

6. Fields with rotation and their properties

As a rule, the alteration of beams in their propagation and focusing is associated with the stretching–compressive deformations: converging and diverging beams. On the other hand, it is evident that even for a simple anisotropy of the beam phase the beam divergence (deformation) also becomes nonuniform. This brings up the legitimate question: Is there some analogy to the torsional strain in the case of a beam with nonuniform divergence? As shown in Section 2, generally the light energy flux consists of two components: divergent, and vortical. In a certain sense, the former component corresponds to stretching–compressive deformations, and the latter one to torsional strains.

In Section 5, we considered the links between HG and LG beams. A characteristic property of these beams is structure retention, correct to scale, in their propagation and focusing. Taking into consideration the vortical component of the light energy flux vector, the notion of structural stability of light fields may be extended. Specifically, this involves examining the question of whether there exist light fields which retain their structure, correct to scale and the character of rotation.

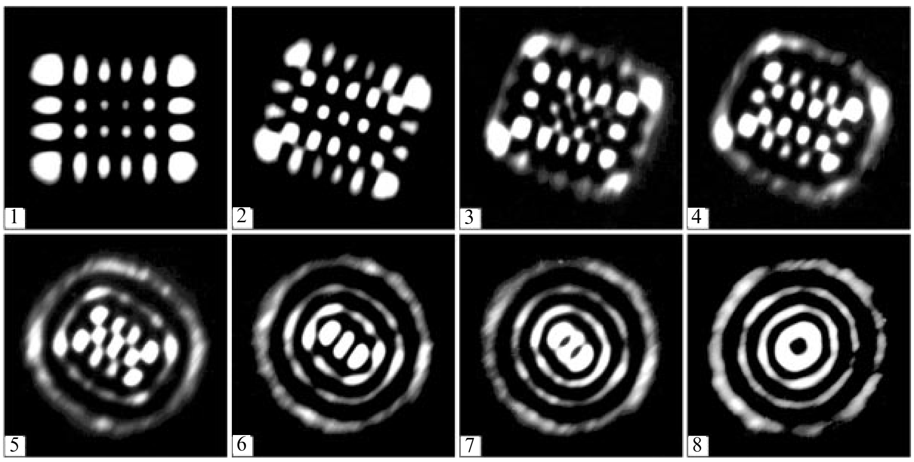

Figure 4. Hermite–Laguerre–Gauss beams with angle α varying from 0 to $\pi/4$.

In this case, the structural stability condition may be defined as follows:

$$I(x,y,l) = D(l) \times I_0\left(\frac{x\cos\theta(l)-y\sin\theta(l)}{d(l)}\,,\ \frac{x\sin\theta(l)+y\cos\theta(l)}{d(l)}\right), \quad (30)$$

where $\theta(l)$ stands for the intensity rotation in the propagation of field $F(x,y,l)$, and $d(l)>0$ is the intensity variation scaling. Let us define real variables by the equality $X+\mathrm{i}Y=(x+\mathrm{i}y)\exp(\mathrm{i}\theta(l))/d(l)$. The exponential decrease in the intensity at infinity (28) permits revealing the structure of the phase $\varphi_0(X,Y,l)$ and rewriting the representation for the light field in the form

$$F(x,y,l) = \frac{1}{d(l)}\,F_0(X,Y) \times \exp\left(\frac{1}{2}\,\mathrm{i}kd(l)\,d'(l)(X^2+Y^2)+\mathrm{i}\gamma(l)\right), \quad (31)$$

where $F_0(X,Y)=\sqrt{I_0(X,Y)}\,\exp\left(\mathrm{i}\varphi_0(X,Y,0)\right)$. The structural stability of the intensity thereby generates the structural stability of the phase.

On substituting expression (31) into the parabolic equation, we arrive at the equation for the function $F_0(X,Y)$:

$$\nabla^2F_0+4\mathrm{i}\theta_0\left(X\,\frac{\partial F_0}{\partial Y}-Y\,\frac{\partial F_0}{\partial X}\right)-4F_0(X^2+Y^2-\gamma_0)=0\,. \quad (32)$$

At $\theta_0=0$, equation (32) coincides with the stationary Schrödinger equation for a harmonic oscillator, and its solutions are well known. These are Hermite–Gauss functions $\mathcal{H}_{n,m}(X,Y)$, $\gamma_0=n+m+1$ for $n,m=0,\ 1,\ldots$, and Laguerre–Gauss functions $\mathcal{L}_{n,m}(X,Y)$, $\gamma_0=2n+|m|+1$ for $n,\pm m=0,1,\ldots$.

We shall seek the solutions of equation (32) in the form

$$F_0(X,Y)=\sum_{n,\pm m=0}^{\infty}c_{nm}\mathcal{L}_{n,m}(X,Y)\,. \quad (33)$$

Expansion (33) is always possible owing to the finiteness of the energy of field $F_0(X,Y)$ and the completeness of the system of functions $\{\mathcal{L}_{n,m}(X,Y),\ n,\pm m=0,1,\ldots\}$ in $L_2(\mathrm{R}^2)$. Substituting expression (33) into equation (32), we obtain

$$\sum_{n,m}c_{nm}\mathcal{L}_{n,m}(X,Y)(2n+|m|+\theta_0m-\gamma_0+1)=0\,,$$

and on the strength of the completeness of the system of LG functions, the following relation comes into play:

$$c_{nm}(2n+|m|+\theta_0m-\gamma_0+1)=0 \quad \text{for all } n,\ m\,.$$

Thus, the problem of the search for $F_0(X,Y)$ reduces to the determination of integers n, m from the equation

$$2n+|m|+\theta_0m=\gamma_0-1\,. \quad (34)$$

Thus, this completes the total description of rotating structurally stable solutions to the parabolic equation, which have come to be known as 'spiral light beams' [2].

The Schrödinger equation for the wave function of a charged particle with mass M and charge e in a uniform magnetic field H has the form

$$\nabla^2\psi+4\mathrm{i}\,\mathrm{sign}\,(eH)\,\frac{\partial\psi}{\partial\varphi}-4\psi\left(R^2-\frac{2cME_1}{\hbar|eH|}\right)=0\,,$$

where $E_1=E-p_z^2/2M$, E is the particle energy, and p_z is the component of the particle momentum along the field direction. One can see that the last equation is equivalent to Eqn (32).

7. Spiral beams with a given intensity distribution

It is well known from different works on the phase problem that the intensity–phase relations in the one- and two-dimensional cases are radically different. The physical aspects of this difference were considered in Sections 2 and 3, where we came to recognize that it is intimately concerned with the possibility of the occurrence of a vortical component of the light energy flux vector in the two-dimensional approach. A nonzero rotor of the light energy flux vector significantly complicates the intensity–phase relation in this case. On the other hand, this complexity also gives rise to new possibilities.

From the results outlined in Section 6, it follows that in the two-dimensional case there exists a class of coherent light fields — spiral beams — of the form

$$F(x,y,l)=\frac{1}{\sigma}\exp\left(-\frac{x^2+y^2}{\rho^2\sigma}\right)f\left(\frac{x\pm\mathrm{i}y}{\rho\sigma}\right). \quad (35)$$

One can see from this representation that the class of these fields is rather broad, but the proof of the existence of a beam with given properties and the constructive method of its extraction from this class represents a nontrivial task. This section is concerned with the study of the possibilities for goal-seeking synthesis of light beams (35).

Let us consider several properties of the spiral beams of this class, which follow from representation (35) and are addressed in the subsequent discussion.

Property A. If $\mathcal{S}_n(z,\bar z)=\exp(-z\bar z/\rho^2)f_n(z/\rho)$ is some totality of spiral beams, their linear combination

$$\mathcal{S}(z,\bar z)=\sum_n c_n\mathcal{S}_n(z,\bar z)$$

is also a spiral beam. And, in general, if $\mathcal{S}_n(z,\bar z,a)=\exp(-z\bar z/\rho^2)f_n(z/\rho,\,a)$ stands for a parametric family of spiral beams, then

$$\mathcal{S}(z,\bar z)=\int\mathcal{S}(z,\bar z,a)\,\mathrm{d}a$$

is also a spiral beam.

Property B. If $\mathcal{S}_0(z,\bar z)=\exp(-z\bar z/\rho^2)f(z/\rho)$ defines some spiral beam, then

$$\mathcal{S}(z,\bar z)=\exp\left(-\frac{z\bar z}{\rho^2}\right)f\left(\frac{z\exp(-\mathrm{i}\alpha)}{\rho}\right)$$

likewise is a spiral beam possessing the same intensity distribution as $\mathcal{S}_0(z,\bar z)$ but turned by an angle α.

Property C. If $\mathcal{S}_0(z,\bar z)=\exp(-z\bar z/\rho^2)f(z/\rho)$ is some spiral beam, then

$$\mathcal{S}(z,\bar z)=\exp\left(-\frac{z\bar z-2z\bar z_0+z_0\bar z_0}{\rho^2}\right)f\left(\frac{z-z_0}{\rho}\right) \quad (36)$$

also represents a spiral beam possessing the same intensity distribution as $\mathcal{S}_0(z, \bar{z})$ but shifted to a point z_0. These results naturally generate the following question. Let there be some curve in a plane, specified in a complex parametric form $\zeta = \zeta(t)$, where parameter t varies over the interval $[0, T]$. Does there exist a spiral beam $\mathcal{S}(z, \bar{z}|\zeta(t), t \in [0, T])$ with the shape resembling this curve? It turns out that such a beam does exist and assumes the form

$$\begin{aligned}\mathcal{S}\big(z, \bar{z}|\zeta(t),\, t \in [0, T]\big) &= \exp\left(-\frac{z\bar{z}}{\rho^2}\right)\int_0^T \exp\left[-\frac{\zeta(t)\bar{\zeta}(t)}{\rho^2} + \frac{2z\bar{\zeta}(t)}{\rho^2}\right.\\ &\left. + \frac{1}{\rho^2}\int_0^t \big(\bar{\zeta}(\tau)\,\zeta'(\tau) - \zeta(\tau)\,\bar{\zeta}'(\tau)\big)\,\mathrm{d}\tau\right]\big|\zeta'(t)\big|\,\mathrm{d}t\,. \qquad (37)\end{aligned}$$

It is significant that for practical applications, i.e. in the representation of a plane curve in a discrete form, finite increments turn out to be more convenient than differentials, because in the former case there is no need to find derivatives — a none too pleasant task.

The beams for closed curves occupy a special place and deserve separate consideration. Let function $\zeta(t)$, $t \in [0, T]$ describe a closed curve without self-intersections. Without loss of generality, it may be assumed that the curve is traced counterclockwise with increasing t. We define $\zeta(t)$ for all real t by continuing it periodically beyond the segment $[0, T]$. Then, the functions $\zeta(t + a)$, $t \in [0, T]$ describe the same curve for different a. Do the spiral beams for the curves $\zeta(t + a)$ coincide for different a?

It will be shown that the beams constructed for closed curves exhibit characteristic quantization properties. This manifests itself in the fact that, first, the intensity distribution of these curves undergoes radical changes under the similarity transformation $\zeta(t) \to v\zeta(t)$, and possesses the topology of the curve $v\zeta(t)$ only for certain discrete values of parameter v. Second, the intensities of the beams constructed by the curves $v\zeta(t + a)$ for different a are the same only for these values of parameter v.

Let us find the condition whereby the spiral beams constructed for curves $\zeta(t)$ and $\zeta(t + a)$ coincide:

$$\big|\mathcal{S}\big(z, \bar{z}|\zeta(t),\, t \in [a, a + T]\big)\big|^2 \equiv \big|\mathcal{S}\big(z, \bar{z}|\zeta(t),\, t \in [0, T]\big)\big|^2\,.$$

We rewrite the above identity in the form

$$\begin{aligned}\exp\big(\mathrm{i}\Phi(a)\big)\,\mathcal{S}\big(z, \bar{z}|\zeta(t),\, t \in [a, a + T]\big)&\\ \equiv \mathcal{S}\big(z, \bar{z}|\zeta(t),\, t \in [0, T]\big)\,,& \qquad (38)\end{aligned}$$

where $\Phi(a)$ is some real function independent of z (otherwise, by canceling the Gaussian function from both sides of identity (38), we find that Φ is an analytical function of z and, consequently, may not be real for all z). Differentiating identity (38) with respect to a and making use of the periodicity of $\zeta(t)$, one obtains

$$\begin{aligned}&\exp\big(\mathrm{i}\Phi(a)\big)\,\mathcal{S}\big(z, \bar{z}|\zeta(t),\, t \in [a, a + T]\big)\\ &\quad\times\left[\mathrm{i}\Phi'(a) - \frac{\bar{\zeta}(a)\,\zeta'(a) - \zeta(a)\,\bar{\zeta}'(a)}{\rho^2}\right]\\ &\quad+ \exp\left[\mathrm{i}\Phi(a) - \frac{z\bar{z} - 2z\bar{\zeta}(a) + \zeta(a)\,\bar{\zeta}(a)}{\rho^2}\right]\\ &\quad\times\left[\exp\left(\frac{1}{\rho^2}\int_0^T(\bar{\zeta}\zeta' - \zeta\bar{\zeta}')\,\mathrm{d}\tau\right) - 1\right]|\zeta'(a)| = 0\,.\end{aligned}$$

Replacing the spiral beam in the first term in accordance with identity (38), canceling the Gaussian function, we rewrite this equation in a symbolic form

$$f(z)\,F_1(a) + \exp\left(\frac{2z\bar{\zeta}(a)}{\rho^2}\right)F_2(a) = 0\,,$$

where $f(z)$ is an integer analytical function, and $F_1(a)$ and $F_2(a)$ are some functions of a. This equality takes place for all z, a only when $F_1(a) = F_2(a) \equiv 0$ (when $f(z)$ has a zero, this follows immediately; the case where $f(z)$ has no zeroes is also simple). Therefore, one has

$$\Phi(a) = \frac{1}{\mathrm{i}\rho^2}\int_0^a(\bar{\zeta}\zeta' - \zeta\bar{\zeta}')\,\mathrm{d}\tau,\; \exp\left(\frac{1}{\rho^2}\int_0^T(\bar{\zeta}\zeta' - \zeta\bar{\zeta}')\,\mathrm{d}\tau\right) = 1$$

and, hence [4]

$$\frac{1}{\mathrm{i}\rho^2}\int_0^T\big(\bar{\zeta}(\tau)\,\zeta'(\tau) - \zeta(\tau)\,\bar{\zeta}'(\tau)\big)\,\mathrm{d}\tau = \frac{4S}{\rho^2} = 2\pi N\,,$$

where S is the area bounded by contour $\zeta(t)$.

Therefore, the beam intensity is independent of the origin of integration a only for curves whose area satisfies the quantization condition

$$S = \frac{1}{2}\,\pi\rho^2 N, \quad \text{where} \quad N = 1, 2, \ldots\,. \qquad (39)$$

The closed curves which satisfy equality (39) will be referred to as N-quantized curves, and the spiral beams for such curves are called N-quantized beams.[1]

A strictly defined number of optical vortices inside the domain bounded by the generating curve correspond to a quantized beam, which depends on the domain area and not on its shape (Fig. 5) [3, 4]. Hence it follows that, with an increase in domain area, say, from $S = (1/2)\,\pi\rho^2 N$ to $S = (1/2)\,\pi\rho^2(N + 1)$, an increase in the number of zeroes occurs inside the domain due to the entry of one zero from

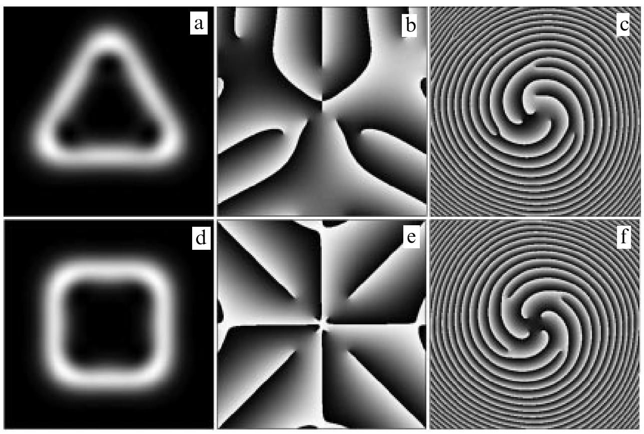

Figure 5. Intensity (a, d), phase (b, e), and phase outside the beam waist (c, f) for a spiral beam in the form of the boundary of a triangle (a–c) and a square (d–f).

[1] If we refer to the quantum-mechanical analogy noted in Section 6, the wave functions of a ground-state particle in a constant magnetic field correspond to the spiral beams with $\theta_0 = \pm 1$, $\gamma_0 = 1$. In this case, condition (39) matches the quantized magnetic flux through the contour $\zeta(t)$: $\Phi = (2\pi\hbar c/|e|)\,N$ (see also Ref. [2]).

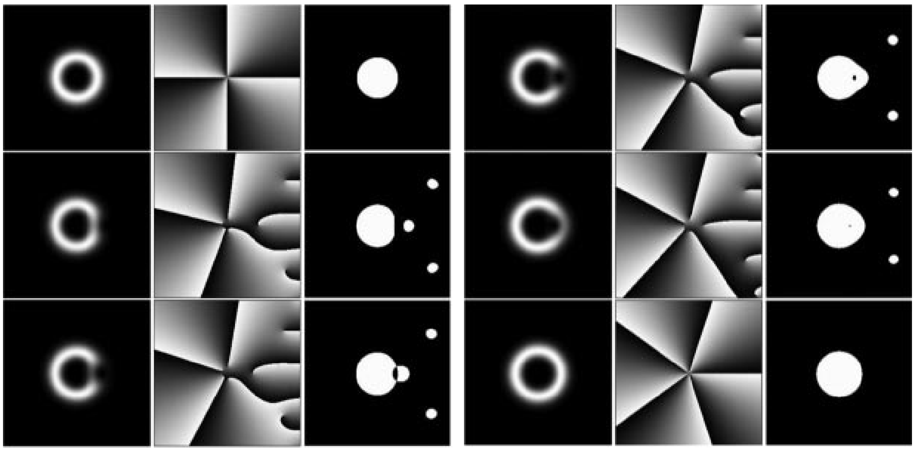

Figure 6. Spiral beam evolution under variation of the radius of the generating circumference.

the outside. One can see the process of zero penetration inside the contour in Fig. 6 which shows the evolution of a spiral beam for a circumference $\zeta(t) = R\exp(\mathrm{i}t),\ t \in [0, 2\pi]$ for $2R^2/\rho^2 \in [4.0, 5.0]$. The zero entry zone is determined, as discussed above, by the integration origin.

8. Conclusions

The aim of our conclusion consists in providing a summarized and comparative analysis of the new results presented and in formulating some incompletely resolved problems.

The generalized Hermite–Laguerre–Gauss beams found in the investigation of astigmatic transformations of Gaussian beams make up a parametric family in which the previously known HG and LG beams are special representatives corresponding to two certain parameter values. Furthermore, the astigmatic transformations of Gaussian beams permitted proposing a new approach to the synthesis of phase elements intended for the formation of light fields in the form of arbitrarily shaped domains. This problem is now at the research stage, and the level of its solution is still far from the results obtained for light fields in the form of curves. In our view, the reason lies with two interrelated circumstances: first, a domain, unlike a curve, is not an ordered set. Second, light fields in the form of domains contain phase singularities of both signs and are not structurally stable in the Fresnel zone. That is why the synthesis of the appropriate phase elements is complicated by several factors: the domain shape, the synthesis technique, etc.

In recent years, the term singular optics has been used in reference to light fields with wavefront dislocations, or optical vortices. Fields of this kind, which are formed and observed both in linear and nonlinear optical media, are the subject of rather intensive investigation, and the development of adequate theoretical and experimental approaches for the exploration of fields with optical vortices is, therefore, a topical task.

Of course, any coherent light field may be formally represented as a superposition of the known HG and LG beams, but this approach proves to be nonoptimal for the analysis and synthesis of fields with phase singularities.

Vortical light fields which retain, correct to scale and kind of rotation, their structure in the course of propagation, or spiral beams, which are the concern of Section 7, are peculiar 'vortical modes' in the class of fields with phase singularities, and deserve special consideration as a subject of coherent optics.

In our opinion, this is due to the following main reasons. First, spiral beams, despite the fact that they differ greatly in the shape of intensity distributions, are described by explicit analytical expressions, which makes them efficient instruments in the study of the laws of formation and transformation of light fields with phase singularities of a general kind.

Second, in quantum mechanics there is a direct analog to spiral beams — the wave functions of a charged particle in a uniform magnetic field — and the laws of spiral beam transformation have their representation in the theory of coherent states. It is quite possible that these analogies will be mutually beneficial, both for quantum mechanics and for optics. Lastly, the flexibility of variation of spiral beam intensity distributions with the retention of structural stability in the beam propagation and focusing is of interest for laser technologies and the development of specific atomic traps, whereas the nonzero angular momentum of these beams opens new possibilities for manipulating microobjects.

References

1. Volostnikov V G, Preprint No. 93 (Moscow: Lebedev Physics Institute, 1990); *J. Sov. Laser Res.* **11** 601 (1990)
2. Abramochkin E G, Volostnikov V G *Usp. Fiz. Nauk* **174** 1273 (2004) [*Phys. Usp.* **47** 1177 (2004)]
3. Abramochkin E G, Volostnikov V G *Sovremennaya Optika Gaussovykh Puchkov* (Modern Optics of Gaussian Beams) (Moscow: Fizmatlit, 2010)
4. Volostnikov V G, Thesis for the Degree of Doctor of Physicomathematical Sciences (Saratov: Saratov State Univ., 1997)