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***METHODOLOGICAL NOTES***

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# Structure of the Maxwell equations in the region of linear coupling of electromagnetic waves in weakly inhomogeneous anisotropic and gyrotropic media

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## Contents



**Abstract. Linear coupling of electromagnetic waves in weakly inhomogeneous non-one-dimensional media is considered as a manifestation of the polarization degeneracy of the Maxwell equations. It is shown that the presence of two polarization-degenerate normal waves imposes strong constraints on the dielectric tensor components near the interaction region. As a result, the possible types of linear wave coupling and the corresponding wave equations admit a universal classification, which is independent of the way in which the linear medium is modeled.**

## 1. Introduction

Linear coupling of electromagnetic waves in weakly inhomogeneous media is a very important fundamental process in plasma physics, crystal optics, electrodynamics of metamaterials, etc. In each of these fields, the problem of linear coupling is considered independently, with the specific properties of the dielectric response in a given medium and, not infrequently, problem geometry taken into account [1–8]. In this paper, we aim to look at the problem from a different angle. Namely, we consider a linear nondissipative medium with a dielectric permittivity tensor of the general form in order to clarify the conditions that the dielectric response must satisfy to ensure the existence of effectively interacting modes in an unbounded weakly inhomogeneous medium. We proceed from the assumption that linear coupling between two vector electromagnetic waves can occur only in the vicinity of polarization degeneracy points with two linearly independent solutions of the Maxwell equations for a single wave vector $\mathbf{k}(\omega)$. This condition, which can actually be regarded as the definition of linear coupling, is in itself sufficient to impose strong constraints on the dielectric tensor components and offer a universal classification of the possible types of linear wave coupling independent of a concrete medium model. Moreover, such an approach permits overcoming a number of difficulties encountered in the theory of linear coupling of electromagnetic waves in the case of multidimensional strongly anisotropic and gyrotropic media.

The modern coupling theory extensively uses the interacting wave approximation [1, 4, 6, 8–12]:

$$\hat{D}_1 E_1 = \eta E_2 \,, \qquad \hat{D}_2 E_2 = \eta^* E_1 \,, \tag{1}$$

where $\hat{D}_1$ and $\hat{D}_2$ are operators describing the propagation of geometric optical modes with complex amplitudes $E_1$ and $E_2$ in a weakly inhomogeneous medium whose properties only slightly change over distances of the order of the wavelength, and $\eta$ is the mode coupling constant in the interaction region. The propagation of the modes outside this region is described by the equations $\hat{D}_1 E_1 = 0$ and $\hat{D}_2 E_2 = 0$. From the physical standpoint, these equations describe the propagation of two modes separated in the $\mathbf{r}$- or $\mathbf{k}$-space by a nontransparent layer whose thickness is proportional to $|\eta|$. Wave operators in a weakly inhomogeneous medium can be associated with their Fourier transforms $D_1(\omega, \mathbf{k})$ and $D_2(\omega, \mathbf{k})$, while the original system of coupled waves corresponds to the disper-

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sion relation

$$D_1(\omega, \mathbf{k})\, D_2(\omega, \mathbf{k}) = |\eta|^2 \,. \quad (2)$$

This relation defines two modes propagating independently of each other in the framework of the geometric optics approximation. Far from the interaction region, these modes continuously pass into 'decoupled' modes satisfying dispersion relations $D_1(\omega, \mathbf{k}) = 0$ and $D_2(\omega, \mathbf{k}) = 0$. The geometric optics approximation is violated in the bounded regions of space where wave numbers of both modes become close to each other, such that the medium is no longer smooth at the beat wavelength scales, i.e.,

$$|\mathbf{k}_1 - \mathbf{k}_2|\, L \ll 2\pi \,,$$

where $L$ is the medium inhomogeneity scale. This condition is realized only in the vicinity of the afore-defined polarization degeneracy points of vector wave equations.[1] Inside this region, the field must be sought by solving the original wave equations (1) without using the geometric optics approximation. Far from the region of interest, these exact solutions are asymptotically given by linear combinations of geometric optics modes (2) with definite linear couplings between their amplitudes. This means that the radiation incident in the form of a single mode passes through the interaction region and/or reflects from it in the form of two coherent modes propagating independently far from the interaction region but having a fixed amplitude ratio depending on the conditions under which the wave passes through the region where the geometric optics conditions are violated. It is this process that is basically meant by linear wave coupling. The pictorial representation of this process is the 'tunneling' of radiation through a nontransparency zone, equivalent to the transition of a quantum mechanical particle through a one-dimensional potential barrier [16].

We are aware of three cases allowing a rigorous substantiation of the coupled wave model: one-dimensionally inhomogeneous media [1, 2, 4], weakly anisotropic media [5, 6, 8], and conversion of electromagnetic modes into quasielectrostatic ones, e.g., plasma waves [4, 7, 17]. Clearly, Eqns (1) in inhomogeneous media describe the same radiation tunneling process through a nontransparency region as in the one-dimensional case. Taking additional dimensions into account in the wave operators only yields a more exact value of the coupling constant (the effective width of the nontransparency region), which is actually computed in the framework of geometric optics. But the Maxwell equations in weakly inhomogeneous anisotropic and gyrotropic media do not always lead to relations similar to Eqns (1) for interacting waves. Formally, the cause is the geometric optical coupling constant $\eta \to 0$ for a sufficiently wide circle of problems. In this case, the coupling constant must be substituted by a differential operator that accounts for the essentially non-one-dimensional character of the linear wave coupling in free space [18–21]. We note that the wave coupling condition itself does not change, because the effective interaction is possible only in the vicinity of polarization degeneracy points. The method described in this paper allows thoroughly studying the conditions of applicability of the widely used approximation (1) to the Maxwell equations in weakly inhomogeneous media and demonstrating the limitations of this model in many cases important for applications.

We note that in numerous publications, coupled wave equations (1) are considered as the starting point for investigations of the influence of three-dimensional inhomogeneity on the linear coupling of electromagnetic waves in the media where these equations can strictly speaking be inapplicable [10, 22–29]. The approach used in all these studies first yields a dispersion relation describing geometric optical modes, which is thereafter used to restore the wave equations describing mode coupling. This approach is not technically flawless even if it gives correct results because the same dispersion equation may correspond to various differential problems. Non-one-dimensional inhomogeneous problems are especially fraught with errors. This is why the present analysis is focused on the search for degeneracy points of the original Maxwell vector equations rather than a simpler scalar dispersion relation.

## 2. Dielectric permittivity tensor at the polarization degeneracy point

It this section, we consider an inhomogeneous medium specified by a dielectric permittivity tensor $\varepsilon_{ij}(\omega)$ in a Cartesian coordinate system $x_1, x_2, x_3$. For a plane electromagnetic waves propagating in this medium,

$$\tilde{\mathbf{E}} = \mathbf{E} \exp\left(\mathrm{i}\mathbf{k}\mathbf{r} - \mathrm{i}\omega t\right),$$

the Maxwell equations can be represented as a system of linear algebraic equations for the Cartesian components of the electric field $E_j$:

$$(k^2 \delta_{ij} - k_i k_j - k_0^2 \varepsilon_{ij})\, E_j = 0 \,, \quad (3)$$

where $k_0 = \omega/c$ is the vacuum value of the wave vector corresponding to the given radiation frequency, $k = |\mathbf{k}|$, $\delta_{ij}$ is the Kronecker symbol, where indices $i$ and $j$ range from 1 to 3 and denote projections of vector values onto the corresponding coordinate axes, and the repeated indices imply summation. The condition for the existence of a nontrivial solution of system (3) gives rise to the dispersion equation

$$\det\left(k^2 \delta_{ij} - k_i k_j - k_0^2 \varepsilon_{ij}\right) = 0 \,,$$

relating $\mathbf{k}$ and $\omega$ medium eigenmodes. We assume polarization degeneracy, i.e., the existence of two linearly independent vectors $\mathbf{E}$ [solutions of the system of wave equations (3)] for certain $\mathbf{k}(\omega)$ satisfying the dispersion relation. We show in what follows that this condition imposes rather strong constraints on the dielectric tensor components, the possible

[1] For qualitative understanding, such points are convenient to represent as a result of the intersection of two 'noninteracting' dispersion relations $D_1(\omega, \mathbf{k}^0) = 0$ and $D_2(\omega, \mathbf{k}^0) = 0$. Indeed, it is easy to see that given a small enough interaction parameter $|\eta| L \ll 2\pi\, |\partial D(\omega, \mathbf{k}^0)/\partial \mathbf{k}|$, the region of violated geometric optics is localized in the vicinity of the common root $\mathbf{k}^0$ of 'noninteracting' dispersion relations. In certain classic studies, the transformation point is defined as the branching point of the solution of the dispersion relation [4, 7]; however, this representation depends on the parameterization of the dispersion relation and therefore lacks universality. For example, in the well-known case of coupling between ordinary and extraordinary waves in a cold plasma inhomogeneous along $x$, the branching point $n_x^2(x)$ at a constant wave vector across the inhomogeneity corresponds to the turn of the extraordinary wave rather than a transformation in the vicinity of plasma resonance [13–15].

polarization, and the direction of propagation of electromagnetic modes.

### 2.1 Representation in the basis of medium eigenpolarizations

We assume for definiteness that a medium is dissipation-free. Then the dielectric permittivity tensor is Hermitian and can be 'diagonalized' with the help of a unitary transition matrix:

$$\varepsilon_{ij}^{\mathrm{d}} = U_{im}^{-1}\varepsilon_{mn}U_{nj} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}. \tag{4}$$

Here, $U_{nj}$ is the transition matrix composed of the eigenvectors of the dielectric tensor, $U_{im}^{-1}$ is the inverse of the transition matrix, and $\varepsilon_i$ are the eigenvalues of the dielectric tensor. The columns of the transition matrix can generally be interpreted as the components of three complex eigenvectors defined in the initial Cartesian coordinate system as

$$\mathbf{e}_i = (U_{1i},\ U_{2i},\ U_{3i}),$$

where $i = 1, 2, 3$. Because the transition matrix is unitary, the complex scalar products of these vectors satisfy the orthonormalization conditions $\mathbf{e}_i\mathbf{e}_j \equiv U_{ki}U_{kj}^* = U_{ki}U_{jk}^{-1} = \delta_{ij}$. Thus, the transition matrix diagonalizing the dielectric permittivity tensor of a nondissipative medium defines the orthonormalized basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We note that the absence of dissipation is a sufficient but not necessary condition for the analysis that follows. Our calculations remain valid for a dissipative medium once its dielectric tensor can be diagonalized in a certain orthonormalized basis because we nowhere use the property of realness of $\varepsilon_{ij}^{\mathrm{d}}$ distinguishing Hermitian media. Such a dissipative medium is exemplified by a magnetoactive plasma with collisions (see the example at the end of Section 2.1).

The analysis of the set of wave equations significantly simplifies after transition to the field representation in terms of projections onto the new basis vectors, $\mathbf{E} = \mathcal{E}_j\,\mathbf{e}_j$ or

$$E_i = U_{ij}\mathcal{E}_j\,, \qquad \mathcal{E}_j = U_{ij}^*E_i\,.$$

In what follows, the eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the dielectric permittivity tensors are referred to as the *medium eigenpolarizations*, and the vector

$$\boldsymbol{\mathcal{E}} = (\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$$

as the electric field representation in the eigenpolarization basis. As is easy to see, multiplication of the complex eigenpolarization vector by a constant $\exp(\mathrm{i}\varphi)$ preserving the normalization reduces the vector to the form $\mathbf{e}_i = \mathbf{a} + \mathrm{i}\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are two orthogonal real vectors satisfying the condition $|\mathbf{a}|^2 + |\mathbf{b}|^2 = 1$. The vectors $\mathbf{a}$ and $\mathbf{b}$ determine elliptic polarization of an electric field with the amplitude $\mathcal{E}_i$. The real vector $\mathbf{e}_i$ [or a vector reduced to a real one after multiplication by a complex constant $\exp(\mathrm{i}\varphi)$] corresponds to linear field polarization, i.e., $\mathbf{b} = 0$.

Wave equation (3) in the medium eigenpolarization basis takes the form

$$D_{ij}\mathcal{E}_j = 0\,, \qquad D_{ij} = U_{im}^{-1}(k^2\delta_{mn} - k_mk_n - k_0^2\varepsilon_{mn})\,U_{nj}\,. \tag{5}$$

We stress that normal mode polarizations defined as eigenvectors of the matrix $D_{ij}$ generally do not coincide with medium eigenpolarizations defined as eigenvectors of the matrix $\varepsilon_{ij}$. The wave operator can be represented in a form invariant with respect to the choice of the initial Cartesian coordinate system:

$$D_{ij} = \begin{bmatrix} \left(1-|\kappa_1|^2\right)n^2-\varepsilon_1 & -n^2\kappa_1^*\kappa_2 & -n^2\kappa_1^*\kappa_3 \\ -n^2\kappa_2^*\kappa_1 & \left(1-|\kappa_2|^2\right)n^2-\varepsilon_2 & -n^2\kappa_2^*\kappa_3 \\ -n^2\kappa_3^*\kappa_1 & -n^2\kappa_3^*\kappa_2 & \left(1-|\kappa_3|^2\right)n^2-\varepsilon_3 \end{bmatrix}, \tag{6}$$

where $n = ck/\omega$ is the refractive index of the medium and $\kappa_j = \mathbf{e}_j\mathbf{k}/k = U_{ij}k_i/k$ are coefficients of the decomposition of the unit vector along the wave vector direction in terms of medium eigenpolarization vectors (i.e., coordinates of $\mathbf{k}/k$ in the new basis). The coefficients $\kappa_i$ can take complex values related by the normalization condition

$$|\kappa_1|^2 + |\kappa_2|^2 + |\kappa_3|^2 = 1\,. \tag{7}$$

It follows from (5) that the dispersion relation $\det D_{ij} = 0$ can be represented in the symmetrized invariant form

$$(\varepsilon_2 - n^2)(\varepsilon_3 - n^2)\,\varepsilon_1|\kappa_1|^2 + (\varepsilon_1 - n^2)(\varepsilon_3 - n^2)\,\varepsilon_2|\kappa_2|^2 + (\varepsilon_1 - n^2)(\varepsilon_2 - n^2)\,\varepsilon_3|\kappa_3|^2 = 0\,. \tag{8}$$

In what follows, it is assumed that the solution of this equation agrees with the finite refractive indices $n^2$. In this way, we exclude electrostatic waves corresponding to $n^2 \to \infty$ (as a rule, spatial dispersion of the medium becomes essential for such waves [4, 7, 17]).

As mentioned, the numerical triplet $(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ can formally be regarded as components of a vector $\boldsymbol{\mathcal{E}}$. In the case of polarization degeneracy, there are two linearly independent vectors $\boldsymbol{\mathcal{E}}^{(1)}$ and $\boldsymbol{\mathcal{E}}^{(2)}$ satisfying linear wave equations (5). Evidently, any linear combination of $\boldsymbol{\mathcal{E}}^{(1)}$ and $\boldsymbol{\mathcal{E}}^{(2)}$ is also a solution of the set of wave equations. Hence, there is a distinguished direction in the case of polarization degeneracy,

$$\boldsymbol{\tau} = \left[\boldsymbol{\mathcal{E}}^{(1)}\boldsymbol{\mathcal{E}}^{(2)}\right],$$

such that any vector lying in the orthogonal plane, $\boldsymbol{\mathcal{E}} \perp \boldsymbol{\tau}$, is a solution of the wave equation. From the standpoint of wave equations (5), the three eigenvectors of the medium are equivalent, which permits fixing one vector projection, e.g., $\tau_3 = -1$. It then follows from the orthogonality condition $\boldsymbol{\mathcal{E}}\boldsymbol{\tau} = 0$ that $\mathcal{E}_3 = \tau_1\mathcal{E}_1 + \tau_2\mathcal{E}_2$, where $\tau_1$ and $\tau_2$ are certain (in general, complex) constants. Substituting this expression in system (5), we obtain

$$\begin{cases} (D_{11} + \tau_1D_{13})\,\mathcal{E}_1 + (D_{12} + \tau_2D_{13})\,\mathcal{E}_2 = 0\,, \\ (D_{21} + \tau_1D_{23})\,\mathcal{E}_1 + (D_{22} + \tau_2D_{23})\,\mathcal{E}_2 = 0\,, \\ (D_{31} + \tau_1D_{33})\,\mathcal{E}_1 + (D_{32} + \tau_2D_{33})\,\mathcal{E}_2 = 0\,. \end{cases}$$

These relations must be satisfied for any $\mathcal{E}_1$ and $\mathcal{E}_2$. Obviously, this is possible if and only if all coefficients of $\mathcal{E}_1$ and $\mathcal{E}_2$ vanish:

$$D_{11} + \tau_1D_{13} = D_{12} + \tau_2D_{13} = D_{21} + \tau_1D_{23} = D_{22} + \tau_2D_{23} = D_{31} + \tau_1D_{33} = D_{32} + \tau_2D_{33} = 0\,. \tag{9}$$

It is worth noting that these equalities automatically imply dispersion equation (8). We consider all situations in which these equalities can be satisfied.

Let $n^2 = 0$. It then follows from definition (6) that all off-diagonal components of the tensor $D_{ij}$ vanish and from conditions (9) that the diagonal components also vanish; moreover, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$. In other words, this case corresponds to complete polarization degeneracy at which all components $D_{ij} = 0$; therefore, any field polarization satisfies the system of wave equations. We assume below that $n^2 > 0$.

Let neither $\tau_1$ nor $\tau_2$ be zero. It then follows from the hermiticity of $D_{ij}$ and conditions (9) that

$$|\tau_2|^2 D_{11} = |\tau_1|^2 D_{22} = |\tau_1\tau_2|^2 D_{33} = \tau_1\tau_2^* D_{12} = -\tau_1|\tau_2|^2 D_{13} = -\tau_2|\tau_1|^2 D_{23}\,.$$

The expressions for off-diagonal terms, $D_{ij} = -n^2\kappa_i^*\kappa_j$ $(i \neq j)$, can be used to deduce relations between polarization coefficients $\kappa_i$ at the polarization degeneracy point,

$$\kappa_1 = -\tau_1\kappa_3\,, \qquad \kappa_2 = -\tau_2\kappa_3\,.$$

With these relations, we find that the relation

$$D_{ii} = -|\kappa_i|^2 n^2$$

is satisfied for all diagonal elements, e.g., $D_{11} = -\tau_1 D_{13} = n^2\kappa_1^*(\tau_1\kappa_3) = -n^2\kappa_1^*\kappa_1$. Hence, $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = n^2$, i.e., the case being considered corresponds to complete isotropy of the dielectric response of the medium. The components of this field are related to the polarization degeneracy condition $\boldsymbol{\mathcal{E}} \perp \boldsymbol{\tau}$, where $\boldsymbol{\tau} = (\kappa_1, \kappa_2, \kappa_3)$. Evidently, $\boldsymbol{\tau}$ is merely the wave vector $\mathbf{k}$ in the eigenpolarization basis. Therefore, this condition in the original Cartesian framework is equivalent to the field transversality condition, $\mathbf{E} \perp \mathbf{k}$, natural for an isotropic medium.

Let either $\tau_1$ or $\tau_2$ be nonzero, e.g., for definiteness, $\tau_1 = 0$ and $\tau_2 \neq 0$. It then follows from (9) that

$$D_{21} = D_{31} = D_{12} + \tau_2 D_{13} = 0\,, \tag{10}$$

$$D_{11} = D_{22} + \tau_2 D_{23} = D_{32} + \tau_2 D_{33} = 0\,. \tag{11}$$

Relations (10) are equivalent to the conditions $\kappa_2^*\kappa_1 = \kappa_3^*\kappa_1 = 0$ and can be satisfied in two cases, $\kappa_2 = \kappa_3 = 0$ or $\kappa_1 = 0$. We consider the first case. As follows from normalization condition (7), $|\kappa_1| = 1$, i.e., the eigenpolarization vector must be parallel to the wave vector $\mathbf{k}$. Relations (11) give $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon_3 = n^2$, meaning that $D_{ij} = 0$, i.e., the case of interest corresponds to the *complete* polarization degeneracy. In the second case, $\kappa_1 = 0$, expressions (11) can be rewritten as

$$n^2 - \varepsilon_1 = 0\,, \qquad |\kappa_3|^2 n^2 - \varepsilon_2 = \tau_2 n^2 \kappa_2^*\kappa_3\,,$$

$$\tau_2\left(|\kappa_2|^2 n^2 - \varepsilon_3\right) = n^2\kappa_3^*\kappa_2\,.$$

In the last two expressions, we use the relation $|\kappa_2|^2 + |\kappa_3|^2 = 1$. By eliminating $\tau_2$, we finally obtain the conditions

$$\varepsilon_1 = \frac{\varepsilon_2\varepsilon_3}{\varepsilon_2|\kappa_2|^2 + \varepsilon_3|\kappa_3|^2} = n^2 \quad \text{at} \quad \kappa_1 = 0\,.$$

We note that in this case, the eigenpolarization vector $\mathbf{e}_1$ is orthogonal to the wave vector $\mathbf{k}$. By determining $\tau_2$, we find that the polarization of degenerate normal modes is orthogonal to the vector $\boldsymbol{\tau} = (0, \varepsilon_2\kappa_2, \varepsilon_3\kappa_3)$, $\boldsymbol{\mathcal{E}} \perp \boldsymbol{\tau}$. Obviously, this condition in the original Cartesian coordinate system is equivalent to the relation $\varepsilon_{ij}k_i E_j = 0$ that follows directly from the Maxwell equation $\operatorname{div}\mathbf{D} = 0$.

Finally, let $\tau_1 = 0$ and $\tau_2 = 0$. Then $D_{33}$ can take any value and all the other components $D_{ij}$ vanish. This is equivalent to the condition $\kappa_2^*\kappa_1 = \kappa_3^*\kappa_1 = \kappa_3^*\kappa_2 = 0$ satisfied when any two of the coefficients vanish. In view of normalization (7), this means that one of the eigenpolarization vectors $\mathbf{e}_i$ is parallel to the wave vector, the corresponding coefficient being $|\kappa_i| = 1$. It can be shown that

$$\varepsilon_1 = 0\,, \quad \varepsilon_2 = n^2 \quad \text{if} \quad |\kappa_1| = 1\,,$$

$$\varepsilon_1 = n^2\,, \quad \varepsilon_2 = 0 \quad \text{if} \quad |\kappa_2| = 1\,,$$

$$\varepsilon_1 = n^2\,, \quad \varepsilon_2 = n^2 \quad \text{if} \quad |\kappa_3| = 1\,.$$

We further note that $\varepsilon_3$ can take any value, thus eliminating the complete polarization degeneracy considered above. Polarization of degenerate modes is orthogonal to the eigenpolarization vector $\mathbf{e}_3$.

Up to permutations of indices, we have listed all possible combinations of parameters at which polarization degeneracy can occur. To summarize, there are five characteristic cases in which polarization degeneracy occurs.

**2.1.1 Polarization degeneracy under conditions of partial anisotropic degeneracy.** This case is realized when one of the medium eigenpolarization vectors is codirectional with the wave vector and the corresponding eigenvalue of the dielectric permittivity tensor differs from zero, while the other two eigenvalues coincide:

$$\mathbf{e}_i \parallel \mathbf{k}\,, \quad \varepsilon_i \neq 0\,, \quad \varepsilon_j = \varepsilon_k = n^2 \quad (j \neq k \neq i)\,. \tag{12}$$

That the eigenvalue of the dielectric permittivity tensor is equal to the squared refractive index automatically follows from the above conditions. Degenerate mode polarizations are orthogonal to the eigenpolarization vector $\mathbf{e}_i$. Strictly speaking, the condition $\varepsilon_j = \varepsilon_k$ requires not only the isotropy of the dielectric response in the plane orthogonal to $\mathbf{e}_i$ but also the absence of gyrotropy along $\mathbf{e}_i$. For brevity, we use the notion of partial anisotropic degeneracy to also mean the simultaneous degeneracy of gyrotropy.

**2.1.2 Polarization degeneracy in the vicinity of a medium resonance.** Polarization degeneracy without the anisotropy degeneracy is realized in the vicinity of a medium resonance when one of the eigenpolarization vectors is codirectional with the wave vector and the corresponding eigenvalue of the dielectric permittivity tensor is zero while the other two eigenvalues are not equal to each other (with one of them being automatically equal to the squared refractive index):

$$\mathbf{e}_i \parallel \mathbf{k}\,, \quad \varepsilon_i = 0\,, \quad \varepsilon_j = n^2\,, \quad \varepsilon_k \neq n^2 \quad (j \neq k \neq i)\,. \tag{13}$$

Polarizations of degenerate modes are orthogonal to the eigenpolarization vector $\mathbf{e}_k$.

**2.1.3 Complete polarization degeneracy.** If the wave vector is codirectional with one of the eigenpolarization vectors, the corresponding eigenvalue of the dielectric permittivity tensor is zero (medium resonance), and the other two eigenvalues are equal,

$$\mathbf{e}_i \parallel \mathbf{k}\,, \quad \varepsilon_i = 0\,, \quad \varepsilon_j = \varepsilon_k = n^2 \quad (j \neq k \neq i)\,, \tag{14}$$

then the *complete* polarization degeneracy is realized, with any field polarization satisfying the system of wave equations. This case corresponds to only a partial degeneracy of anisotropy because the medium reserves the distinguished direction $\mathbf{e}_i$.

**2.1.4 Complete degeneracy of medium anisotropy.** When all three eigenvalues of the dielectric tensor are equal,

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = n^2\,, \tag{15}$$

polarization degeneracy occurs under conditions of the complete degeneracy of medium anisotropy. Evidently, this condition is satisfied in the absence of spatial dispersion for all wave propagation directions. Polarizations of degenerate modes are orthogonal to $\mathbf{k}$. In the special case of the medium resonance at $n^2 = 0$, complete degeneracy is realized for both wave polarization and medium anisotropy.

It follows from the above conditions that the wave vector at the polarization degeneracy point is parallel to a certain eigenpolarization vector. In the case of propagating waves with real $\mathbf{k}$, the parallel eigenpolarization vector must be either *real* or reducible to a real one. In other words, one of the eigenpolarizations of the medium must be either linear or oriented along the degenerate mode propagation direction. However, this is not the only situation possible.

**2.1.5 Wave vector orthogonal to the medium eigenpolarization vector.** Polarization degeneracy can occur without both the anisotropy degeneracy and medium resonance. In this case, unlike all preceding ones, the wave vector is *orthogonal* to one of the eigenpolarization vectors, while the conditions

$$\mathbf{e}_i \perp \mathbf{k}\,, \quad \varepsilon_i = \frac{\varepsilon_j \varepsilon_k}{\varepsilon_j |\kappa_j|^2 + \varepsilon_k |\kappa_k|^2} = n^2 \quad (j \neq k \neq i) \tag{16}$$

are imposed on the eigenvalues of the dielectric permittivity tensor. For a real eigenpolarization vector $\mathbf{e}_i$, these conditions define the vector $\mathbf{k}$ (the first condition specifies the plane while the second sets the direction in this plane and the modulus of $\mathbf{k}$) but do not impose any additional constraints on the dielectric tensor components and normal mode polarization, besides the solvability conditions for Eqns (16). In the general case, the vector $\mathbf{e}_i$ may not be reducible to a real one by a complex vector. In that case, the first condition uniquely defines the direction of the wave vector $\mathbf{k} \parallel [\mathrm{Re}\,\mathbf{e}_i \times \mathrm{Im}\,\mathbf{e}_i]$ and the second establishes an additional relation between the dielectric tensor eigenvalues. This case is more convenient to study in the representation of principal optical axes considered in the next section.

**2.1.6 Examples of polarization degeneracy in a nondissipative medium.** We consider the most typical cases that can occur in a nondissipative medium. The medium can have a gyrotropy axis and be isotropic in the plane across this axis. The most common type of such a medium is plasma in a magnetic field, where one of the eigenpolarization vectors is real and is directed along the gyrotropy axis, while the other two eigenpolarization vectors are complex. It follows from the above conditions that the coupling between waves with real $\mathbf{k}$ is possible only when the wave vector is parallel or orthogonal to the gyrotropy axis.[2] If the medium is a uniaxial or biaxial crystal, all its three eigenpolarization vectors are real and are directed along the principal optical axes of the crystal. The linear interaction of propagating waves is possible if the wave vector is either parallel or orthogonal to one of the principal optical axes. Finally, if an anisotropic crystal has a gyrotropy direction that does not coincide with the principal optical axes (e.g., the one induced by an arbitrarily directed external magnetic field), then all three eigenpolarization vectors are complex (not reducible to real ones). Then the only possibility for propagating waves to interact linearly is realized as in case 2.1.5 with complex $\mathbf{e}_i$. The description of this rather exotic case requires a somewhat different formalism, discussed below.

To illustrate applications of the above classification, we consider the propagation of high-frequency waves in a cold magnetoactive plasma. If the wave frequency is much higher than all ion frequencies and the effective collision rate, then the medium is described by the gyrotropic dielectric permittivity tensor [2, 5, 30]

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_\perp & \mathrm{i}g & 0 \\ -\mathrm{i}g & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{bmatrix},$$

where

$$\varepsilon_\perp = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2 - \omega_{\mathrm{ce}}^2}\,, \quad \varepsilon_\parallel = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\,, \quad g = \frac{\omega_{\mathrm{ce}}\omega_{\mathrm{pe}}^2}{\omega(\omega^2 - \omega_{\mathrm{ce}}^2)}\,,$$

with $\omega_{\mathrm{ce}}$ and $\omega_{\mathrm{pe}}$ being electron cyclotron and plasma frequencies. This tensor is written in the Stix coordinate system where the $x_3$ and $x_2$ axes are directed along the external magnetic field and across $\mathbf{k}$. The eigenpolarization vectors of the medium and the corresponding diagonal elements of the dielectric tensor can be defined as

$$\mathbf{e}_1 = \left(\frac{1}{\sqrt{2}}\,, \frac{\mathrm{i}}{\sqrt{2}}\,, 0\right), \qquad \varepsilon_1 = \varepsilon_\perp - g = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega(\omega - \omega_{\mathrm{ce}})}\,,$$

$$\mathbf{e}_2 = \left(\frac{1}{\sqrt{2}}\,, -\frac{\mathrm{i}}{\sqrt{2}}\,, 0\right), \quad \varepsilon_2 = \varepsilon_\perp + g = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega(\omega + \omega_{\mathrm{ce}})}\,,$$

$$\mathbf{e}_3 = (0,\, 0,\, 1)\,, \qquad \varepsilon_3 = \varepsilon_\parallel\,.$$

We note first of all that the plasma is isotropic in the plane orthogonal to the magnetic field, and the condition $\varepsilon_1 = \varepsilon_2$ of partial degeneracy of the anisotropy actually reduces to the gyrotropy degeneracy condition $g \to 0$. This condition is realized in three cases: when radiation escapes into the vacuum ($\omega_{\mathrm{pe}} \ll \omega, |\omega \pm \omega_{\mathrm{ce}}|$), at weak magnetization ($\omega_{\mathrm{ce}} \ll \omega$), and in a strong magnetic field ($\omega_{\mathrm{ce}} \gg \omega, \omega_{\mathrm{pe}}$). In the first two cases, linear wave coupling occurs under conditions of complete anisotropy degeneracy (case 2.1.4)

[2] For definiteness, we assume that $\mathbf{e}_1 \in \mathrm{R}$ is directed along the gyrotropy axis and $\mathbf{e}_2$ and $\mathbf{e}_3$ are complex. In then follows from the orthonormality condition for these vectors that $\mathbf{e}_2 = \cos\gamma\,\mathbf{a} + \mathrm{i}\sin\gamma\,\mathbf{b}$ and $\mathbf{e}_3 = \sin\gamma\,\mathbf{a} - \mathrm{i}\cos\gamma\,\mathbf{b}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{e}_1$ make an orthonormalized triplet of *real* vectors. Conditions (12)–(14) can be satisfied only at $\mathbf{k} \parallel \mathbf{e}_1$, i.e., for propagation along the gyrotropy axis. We assume that condition (16) is satisfied at $\mathbf{k} \perp \mathbf{e}_2$, i.e., at $\mathbf{k} \perp \mathbf{a}$ and $\mathbf{k} \perp \mathbf{b}$. Then necessarily $\mathbf{k} \parallel \mathbf{e}_1$, i.e., case 2.1.5 reduces to the case of anisotropy degeneracy 2.1.1. Therefore, without loss of generality, it suffices to consider case 2.1.5 with $\mathbf{k} \perp \mathbf{e}_1 \in \mathrm{R}$, i.e., propagation across the gyrotropy axis.

and in the last case, under partial anisotropy degeneracy (case 2.1.1); radiation must be outside the plasma cut-off region. Case 2.1.2 is realized in the vicinity of the plasma cut-off frequency, $\omega \to \omega_{\rm pe}$, when the transition to complete polarization degeneracy (case 2.1.3) occurs in a strong magnetic field ($\omega_{\rm ce} \gg \omega_{\rm pe}$). It is easy to show by direct substitution that case 2.1.5 can be realized in a magnetized plasma only if $\omega_{\rm pe} = 0$ or $\omega_{\rm ce} = 0$; in other words, it does not exist as a separate case. Physically, this situation can be interpreted as resulting from the impossibility of the intersection of dispersion branches of the ordinary and extraordinary waves propagating strictly across the magnetic field. Thus, the wave vector of interacting modes is always aligned with the magnetic field except in the case of the complete anisotropy degeneracy. Taking collisional wave dissipation [2] into account does not change the picture because the vectors $\mathbf{e}_i$ remain unaltered (the effective collision frequency enters only the diagonal elements $\varepsilon_i$, thereby making them complex); the emerging imaginary part of the wave vector is codirected with the real one.

### 2.2 Representation in the basis of principal optical axes

In this section, we consider an alternative approach to the description of polarization degeneracy conditions based on the field representation in terms of linear polarizations, i.e., in a basis of real vectors. Compared with the method in Section 2.1, this approach looks somewhat more 'physically sound' because it explicitly discriminates between gyrotropic and anisotropic effects, even if in most cases this leads to more complicated linear interaction equations. Nevertheless, this approach proves to be more convenient and informative for the analysis of one special case (namely, 2.1.5) in which the wave vector is *orthogonal* to one of the eigenpolarization vectors.

We divide the Hermitian dielectric permittivity tensor into symmetric and antisymmetric parts,

$$\varepsilon_{ij} = \varepsilon'_{ij} + \mathrm{i}\varepsilon''_{ij}\,, \qquad \varepsilon''_{ij} = e_{ijk}\, g_k\,,$$

where $\varepsilon'_{ij}$ and $\varepsilon''_{ij}$ are respectively symmetric and antisymmetric matrices, with real coefficients, $e_{ijk}$ is the antisymmetric unit tensor, and summation over the index $k$ is implied. In the second relation, we used the known fact that in a three-dimensional space, an antisymmetric tensor can be determined by a three-dimensional vector. In the present case, this is the real vector $\mathbf{g}$ with Cartesian components $g_k$, known as the gyration vector [16]. The symmetric part $\varepsilon'_{ij}$ can be diagonalized by passing to a new real orthogonal basis $\mathbf{e}'_i$ that determines directions of the principal optical axes of the medium [16]. The full set of basis vectors forms the transition matrix $U'_{ij}$. We emphasize that the principal optical axes basis, unlike the eigenpolarization basis considered in Section 2.1, is always real by construction, i.e., corresponds to linear polarizations.

We consider wave equations in the basis of principal optical axes. We represent the electric field in the form of an orthogonal linear polarization expansion $\mathbf{E} = \mathcal{E}'_j \mathbf{e}'_j$. In the new basis, the antisymmetric part of the dielectric tensor is naturally defined by the same gyration vector $\mathbf{g}$ whose components are $G_j = U'_{ij} g_i$. As a result, the complete dielectric tensor in the new basis has the form

$$U'^{-1}_{im}\varepsilon_{mn}U'_{nj} = \begin{bmatrix} \varepsilon'_1 & \mathrm{i}G_3 & -\mathrm{i}G_2 \\ -\mathrm{i}G_3 & \varepsilon'_2 & \mathrm{i}G_1 \\ \mathrm{i}G_2 & -\mathrm{i}G_1 & \varepsilon'_3 \end{bmatrix}.$$

The corresponding wave operator can be constructed similarly to (6):

$$D'_{ij} = \begin{bmatrix} (1-\kappa_1'^2)n^2 - \varepsilon'_1 & -n^2\kappa'_1\kappa'_2 + \mathrm{i}G_3 & -n^2\kappa'_1\kappa'_3 - \mathrm{i}G_2 \\ -n^2\kappa'_1\kappa'_2 - \mathrm{i}G_3 & (1-\kappa_2'^2)n^2 - \varepsilon'_2 & -n^2\kappa'_2\kappa'_3 + \mathrm{i}G_1 \\ -n^2\kappa'_1\kappa'_3 + \mathrm{i}G_2 & -n^2\kappa'_2\kappa'_3 - \mathrm{i}G_1 & (1-\kappa_3'^2)n^2 - \varepsilon'_3 \end{bmatrix};$$

$\kappa'_j = \mathbf{e}'_j\,\mathbf{k}/k$ are real coordinates of the unit vector along the direction of the wave vector in the new basis.

Next, we introduce the vector $\boldsymbol{\tau} = (\tau_1, \tau_2, -1)$ determining the plane of polarization of degenerate modes and repeat the logic of the preceding section. In the system of principal optical axes, conditions (9) then become

$$n_2^2 + n_3^2 - \varepsilon'_1 = \tau_1(\mathrm{i}G_2 + n_1 n_3)\,, \tag{17}$$

$$n_1^2 + n_3^2 - \varepsilon'_2 = -\tau_2(\mathrm{i}G_1 - n_2 n_3)\,, \tag{18}$$

$$\mathrm{i}G_2 - n_1 n_3 = -\tau_1(n_1^2 + n_2^2 - \varepsilon'_3)\,, \tag{19}$$

$$\mathrm{i}G_1 + n_2 n_3 = \tau_2(n_1^2 + n_2^2 - \varepsilon'_3)\,, \tag{20}$$

$$\mathrm{i}G_3 - n_1 n_2 = \tau_2(\mathrm{i}G_2 + n_1 n_3)\,, \tag{21}$$

$$\mathrm{i}G_3 + n_1 n_2 = \tau_1(\mathrm{i}G_1 - n_2 n_3)\,, \tag{22}$$

where $n_i = n\kappa_i$ are coordinates of the vector $\mathbf{k}/k_0$ in the new basis. The consistency conditions for these equations impose additional constraints on the polarization, the wave vector, and the components of $\varepsilon_{ij}$ at the polarization degeneracy point. For example, Eqns (21) and (22) define the vector $\boldsymbol{\tau}$. The substitution of this vector in (19) and (20) and elimination of the factor $(n_1^2 + n_2^2 - \varepsilon'_3)$ yield the consistency condition for (19)–(22):

$$n_1 n_2 n_3 (G_1 n_1 + G_2 n_2 + G_3 n_3) = 0\,.$$

Similarly, substituting the vector $\boldsymbol{\tau}$ in (17) and (18) and setting the imaginary part equal to zero gives

$$G_1 G_2 G_3 = n_1 n_2 n_3 (G_1 n_1 + G_2 n_2 + G_3 n_3) = 0\,.$$

Hence, the wave vector must be *orthogonal* either to one of the principal optical axes or to the gyration vector if the consistency between the above equations is to be achieved. Moreover, the gyration vector must be orthogonal to one of the principal optical axes.

Let both $\tau_1$ and $\tau_2$ be nonzero. Two cases are possible, $\mathbf{k} \perp \mathbf{e}'_i$ and $\mathbf{k} \perp \mathbf{g}$. For definiteness, we examine the case $\mathbf{k} \perp \mathbf{e}'_1$, i.e., $n_1 = 0$. As follows from Eqn (21), $\tau_2 = G_3/G_2$ is a real quantity. Setting the imaginary part in (18) equal to zero, we find that $G_1 = 0$, i.e., the gyration vector must be orthogonal to the same optical axis $\mathbf{e}'_1$ as the wave vector. Finding $\tau_1 = -\mathrm{i}G_3/n_2 n_3$ from Eqn (22) and eliminating $\boldsymbol{\tau}$ from the equations, we arrive at the polarization degeneracy condition in the form

$$\begin{cases} (n^2 - \varepsilon'_1)\, n_3 n_2 = G_2 G_3\,, \\ (n^2 - \varepsilon'_1)(n_3^2 - \varepsilon'_2) = G_3^2\,, \\ (n^2 - \varepsilon'_1)(n_2^2 - \varepsilon'_3) = G_2^2\,, \end{cases}$$

where $n^2 = n_2^2 + n_3^2$. Solving this set of equations for $n_{2,3}$ and $\varepsilon_1$ yields the modulus and direction of the wave vector and an additional condition for the dielectric permittivity tensor:

$$n^2 = \frac{\varepsilon_2'^2 G_2^2 + \varepsilon_3'^2 G_3^2}{\varepsilon'_2 G_2^2 + \varepsilon'_3 G_3^2}\,, \qquad \kappa'_2\varepsilon'_2 G_2 + \kappa'_3\varepsilon'_3 G_3 = 0\,,$$

$$\varepsilon'_1 - n^2 = \frac{G_2^2}{\varepsilon'_3} + \frac{G_3^2}{\varepsilon'_2}\,.$$

**Table 1.** Polarization degeneracy conditions in the representations of eigenpolarizations and principal optical axes. Some (unprimed) quantities in the right column correspond to the medium eigenpolarization representation. This means that they coincide in both representations. This is possible either at $\mathbf{g}=0$ (when the two representations are identical) or at $\mathbf{g}\parallel\mathbf{e}_i'$ (when the unit vectors coincide, $\mathbf{e}_i=\mathbf{e}_i'$).

| Case | Representation in the basis of medium eigenpolarizations | Representation in the basis of principal optical axes |
|---|---|---|
| 2.1.1 | $\mathbf{k}\parallel\mathbf{e}_i,\ \varepsilon_i\neq 0,\ \varepsilon_j=\varepsilon_k$ | $\mathbf{g}\to 0,\ \mathbf{k}\parallel\mathbf{e}_i,\ \varepsilon_i\neq 0,\ \varepsilon_j=\varepsilon_k$ |
| 2.1.2 | $\mathbf{k}\parallel\mathbf{e}_i,\ \varepsilon_i=0,\ \varepsilon_j\neq\varepsilon_k$ | $\mathbf{k}\parallel\mathbf{e}_i'\parallel\mathbf{g},\ \varepsilon_i=0,\ (n^2-\varepsilon_j')(n^2-\varepsilon_k')=g^2$ |
| 2.1.3 | $\mathbf{k}\parallel\mathbf{e}_i,\ \varepsilon_i=0,\ \varepsilon_j=\varepsilon_k$ | $\mathbf{g}\to 0,\ \mathbf{k}\parallel\mathbf{e}_i,\ \varepsilon_i=0,\ \varepsilon_j=\varepsilon_k$ |
| 2.1.4 | $\varepsilon_1=\varepsilon_2=\varepsilon_3$ | $\mathbf{g}\to 0,\ \varepsilon_1=\varepsilon_2=\varepsilon_3$ |
| 2.1.5 | $\mathbf{k}\perp\mathbf{e}_i,\ \varepsilon_i=\dfrac{\varepsilon_j\varepsilon_k}{\varepsilon_j\lvert\kappa_j\rvert^2+\varepsilon_k\lvert\kappa_k\rvert^2}$ | $\mathbf{k}\perp\mathbf{e}_i',\ \mathbf{k}\perp\hat{\varepsilon}'\mathbf{g},\ \mathbf{g}\perp\mathbf{e}_i',\ \varepsilon_i'=\dfrac{G_j^2}{\varepsilon_k'}+\dfrac{G_k^2}{\varepsilon_j'}+\dfrac{\varepsilon_j'^2G_j^2+\varepsilon_k'^2G_k^2}{\varepsilon_j'G_j^2+\varepsilon_k'G_k^2}$ |
| | | $\mathbf{k}\perp\mathbf{g},\ \mathbf{g}\perp\mathbf{e}_i',\ \varepsilon_j'=\varepsilon_k',\ \varepsilon_i'-\varepsilon_j'=\dfrac{G_j^2}{\varepsilon_j'\kappa_j'^2}$ |
| | | $\mathbf{k}\perp\mathbf{e}_i\in\mathrm{R},\ \mathbf{g}\parallel\mathbf{e}_i',\ \varepsilon_i=\dfrac{\varepsilon_j'\varepsilon_k'-g^2}{\varepsilon_j'\kappa_j'^2+\varepsilon_k'\kappa_k'^2}$ |

The first two relations give a condition of the same form as (16):

$$n^2=\frac{\varepsilon_2'\varepsilon_3'}{\varepsilon_2'\kappa_2'^2+\varepsilon_3'\kappa_3'^2}\,.$$

The condition for the direction can also be interpreted in the invariant form as $\mathbf{k}\perp\hat{\varepsilon}'\mathbf{g}$. We consider the second case, $\mathbf{k}\perp\mathbf{g}$. One of the $G_i$ components must be zero, e.g., $G_1=0$. Eliminating $\boldsymbol{\tau}$ from Eqns (17)–(22) and using the orthogonality condition $\mathbf{k}\perp\mathbf{g}$ (i.e., $G_2n_2+G_3n_3=0$), we obtain

$$\begin{cases}(G_2^2+n_1^2n_3^2)(n^2-\varepsilon_2')=G_2n_2(G_2n_2+G_3n_3)=0\,,\\(G_3^2+n_1^2n_2^2)(n^2-\varepsilon_3')=G_3n_3(G_2n_2+G_3n_3)=0\,,\\(n^2-\varepsilon_1')n_2n_3=G_2G_3\,,\end{cases}$$

where $n^2=n_1^2+n_2^2+n_3^2$. Hence, $\varepsilon_2'=\varepsilon_3'=n^2$ and $\varepsilon_1'-n^2=G_2^2/n_2^2=G_3^2/n_3^2$. In nongyrotropic media, this case corresponds to complete anisotropic degeneracy (15). It follows that the two cases considered correspond to (16) with a complex eigenpolarization vector ($\mathbf{k}\perp\mathbf{e}_i\in\mathrm{C}$) because they completely determine the wave vector and specify one additional condition for the components of the dielectric tensor.

Let only $\tau_1$ or $\tau_2$ be nonzero, e.g., for definiteness, $\tau_1=0$ and $\tau_2\neq 0$. It then follows from Eqns (19) and (22) that $G_2=G_3=n_1n_2=n_1n_3=0$, i.e., $\mathbf{e}_1'\parallel\mathbf{g}$ and either $\mathbf{k}\perp\mathbf{g}$ or $\mathbf{k}\parallel\mathbf{g}$. As follows from the condition $\mathbf{e}_1'\parallel\mathbf{g}$, one of the eigenpolarization vectors of the medium is real and coincides with the principal optical axis $\mathbf{e}_1'$ and corresponds to the eigenvalue $\varepsilon_1'$; moreover, $G_1=\pm|\mathbf{g}|$. In the case $\mathbf{k}\perp\mathbf{g}$, the consistency conditions reduce to

$$\varepsilon_1'=\frac{\varepsilon_2'\varepsilon_3'-g^2}{\varepsilon_2'\kappa_2'^2+\varepsilon_3'\kappa_3'^2}=n^2\,.$$

This case corresponds to case (16) with a real eigenpolarization vector ($\mathbf{k}\perp\mathbf{e}_i\in\mathrm{R}$) because it determines only the wave vector (its modulus and direction). In the case $\mathbf{k}\parallel\mathbf{g}$, the consistency conditions reduce to

$$\varepsilon_1'=0\,,\quad (n^2-\varepsilon_2')\,(n^2-\varepsilon_3')=g^2\,.$$

This case corresponds to (13) or, for $\mathbf{g}\to 0$, to (14).

Finally, let $\tau_1=0$ and $\tau_2=0$. It then follows from Eqns (19)–(21) that the medium becomes nongyrotropic, $\mathbf{g}=0$; this case reduces either to (12) or to special case (13).

### 2.3 Comparison of the two representations

We have found some constraints on the dielectric tensor of the medium at the polarization degeneracy point. The results of this analysis are summarized in Table 1. It can be seen that the representation in the basis of principal optical axes allows obtaining somewhat more detailed information about the dielectric tensor structure at the polarization degeneracy point because it is based on a more detailed model of dielectric response where gyrotropic effects are described by a separate vector $\mathbf{g}$. The next step is to construct truncated wave equations in the vicinity of the polarization degeneracy point in a weakly inhomogeneous medium. Such an analysis is possible for both field representations considered above; however, the use of principal optical axes involves much more cumbersome calculations than the crude description in terms of medium eigenpolarizations. The methods of analysis are identical in both cases. Therefore, we confine ourselves in the next section to a more demonstrative analysis of wave equations in the eigenpolarization representation.

## 3. Reference equations describing linear wave coupling in the vicinity of polarization degeneracy points

It is known that polarization degeneracy is removed in spatially inhomogeneous media. We consider a weakly inhomogeneous medium without spatial dispersion with the Hermitian dielectric permittivity tensor $\varepsilon_{ij}(\omega,\mathbf{r})$ varying in space more slowly than the wavelength far from the interaction region,

$$k_0L\gg 1\,,$$

where $L$ is the characteristic scale of variation in medium dielectric properties. This condition permits distinguishing normal waves propagating independently in the geometric optics approach [2, 5, 6]. But this approach tends to be violated in the vicinity of the polarization degeneracy points found in Section 2, resulting in the coupling between normal waves (linear interaction) in a weakly inhomogeneous medium. In this case, the electromagnetic field distribution can be described by reference wave equations derived from the truncated Maxwell equations in the vicinity of polarization degeneracy points. In this section, we find and classify such equations.

For this, we diagonalize the tensor $\varepsilon_{ij}(\mathbf{r})$ at each point in space as described in Section 2.1. This yields a diagonal matrix $\varepsilon_{ij}^{\rm d}(\mathbf{r})$ with elements $\varepsilon_i(\mathbf{r})$ and a transition matrix $U_{ij}(\mathbf{r})$ composed of medium eigenpolarization vectors $\mathbf{e}_i(\mathbf{r})$. All these quantities have a smooth coordinate dependence with a characteristic variation scale $L$ that may be different for $\varepsilon_i(\mathbf{r})$ and $\mathbf{e}_i(\mathbf{r})$. For example, a rotation of vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ under anisotropy degeneracy conditions ($\varepsilon_i \approx \varepsilon_j$) typically occurs much faster than $\varepsilon_i$ and $\varepsilon_j$ change. In such cases, $L$ is set equal to the smallest of the scales. Let there be a point in space at which one of the polarization degeneracy conditions found in Section 2.1 for a plane electromagnetic wave with a certain fixed wave vector $\mathbf{k}^0$ is satisfied. We consider a neighborhood of this point under the assumption that components of the dielectric permittivity tensor and of the eigenpolarization vector of the medium are continuous and do not substantially differ from the corresponding quantities at the polarization degeneracy point,

$$U_{ij}(\mathbf{r}) = U_{ij}^0 + \delta U_{ij}(\mathbf{r}) \,, \qquad |\delta U_{ij}| \ll |U_{ij}^0|$$

or

$$\mathbf{e}_i(\mathbf{r}) = \mathbf{e}_i^0 + \delta\mathbf{e}_i(\mathbf{r}) \,, \qquad |\delta\mathbf{e}_i| \ll |\mathbf{e}_i^0| \,.$$

The polarization degeneracy point may be infinitely distant or even nonexistent, as in the case where the medium smoothly passes into the vacuum [2, 5, 8]. But this does not preclude considering the neighborhood of this point defined as a region in space where the medium and field parameters almost satisfy the polarization degeneracy conditions. We stress that the requirement of continuity of eigenpolarization vectors makes the choice of these vectors unique in the cases corresponding to a degeneracy of the medium anisotropy.

As in a homogeneous medium, we seek the wave field in the eigenpolarization representation,

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}_i(\mathbf{r})\, \mathbf{e}_i(\mathbf{r}) \exp\left(\mathrm{i}\mathbf{k}\mathbf{r} - \mathrm{i}\omega t\right) ,$$

where $\mathcal{E}_i(\mathbf{r})$ are slow functions on the wavelength scale. The equations for slow wave field amplitudes can be derived by formally substituting a differential operator for the wave vector in Eqns (5):

$$\mathbf{k} \to \hat{\mathbf{k}} = \mathbf{k}^0 - \mathrm{i}\,\frac{\partial}{\partial \mathbf{r}} \,.$$

In the vicinity of the polarization degeneracy point, the spatial derivative is small compared with the 'carrier' wave vector $\mathbf{k}_0$, which allows restricting to first-order terms when calculating bilinear operators:

$$\hat{k}_1\hat{k}_1 = \hat{k}_1\hat{k}_2 = \hat{k}_2\hat{k}_2 = 0 \,,$$

$$\hat{k}_1\hat{k}_3 = -\mathrm{i}k_0 n\,\frac{\partial}{\partial x_1} \,, \qquad \hat{k}_2\hat{k}_3 = -\mathrm{i}k_0 n\,\frac{\partial}{\partial x_2} \,,$$

$$\hat{k}_3\hat{k}_3 = k_0^2 n^2 - 2\mathrm{i}k_0 n\,\frac{\partial}{\partial x_3} \,.$$

Here, we have chosen a Cartesian coordinate system such that the $x_3$ axis is parallel to the wave vector $\mathbf{k}^0$. The reflective index $n = |\mathbf{k}^0|/k_0$ is calculated at the polarization degeneracy point. As a result, the system of truncated wave equations takes the form

$$\hat{D}_{ij}\mathcal{E}_j = 0 \,, \qquad \hat{D}_{ij} = U_{mi}^*\hat{K}_{mn}U_{nj} - k_0^2\varepsilon_{ij}^{\rm d} \,, \tag{23}$$

where

$$\hat{K}_{mn} = k_0 n \begin{bmatrix} k_0 n - 2\mathrm{i}\,\dfrac{\partial}{\partial x_3} & 0 & \mathrm{i}\,\dfrac{\partial}{\partial x_1} \\ 0 & k_0 n - 2\mathrm{i}\,\dfrac{\partial}{\partial x_3} & \mathrm{i}\,\dfrac{\partial}{\partial x_2} \\ \mathrm{i}\,\dfrac{\partial}{\partial x_1} & \mathrm{i}\,\dfrac{\partial}{\partial x_2} & 0 \end{bmatrix} .$$

Evidently, the coordinate derivatives entering the operator $\hat{D}_{ij}$ act not only on the wave field but also on the components of medium eigenpolarization vectors defined by the matrix $U_{nj}$. However, if we confine ourselves to the terms of the zeroth and first order in $\delta U_{ij}$, then wave operator (23) can be readily rewritten in the form with the derivatives with respect to coordinates acting only on the wave field:

$$\begin{aligned} \hat{D}_{ij} &= U_{mi}^{0*}U_{nj}^0\hat{K}_{mn} - k_0^2\varepsilon_{ij}^{\rm d} + \delta D_{ij} \,, \\ \delta D_{ij} &= k_0^2 n^2 (\delta U_{mi}^*\sigma_{mn}U_{nj}^+ U_{mi}^{0*}\sigma_{mn}\delta U_{nj}) \,, \end{aligned} \tag{24}$$

where $\sigma_{mn} = \delta_{mn} - \delta_{3m}\delta_{3n}$. The term $\delta D_{ij}$ describes the 'shear' part of the wave operator associated with the rotation of the distinguished directions in the medium, i.e., optical axes or the gyrotropy axis. This term can be somewhat simplified, using the orthonormality of the vectors $\mathbf{e}_i(\mathbf{r})$ at each point in space. Indeed, it follows from the equality $\mathbf{e}_i(\mathbf{r})\,\mathbf{e}_j(\mathbf{r}) = \delta_{ij}$ up to the first-order terms that $U_{ni}^0\delta U_{nj}^* + \delta U_{mi}U_{mj}^{0*} = 0$, which means that the matrix $\sigma_{mn}$ in (24) can be replaced by the simpler matrix $\sigma_{mn} = -\delta_{3m}\delta_{3n}$, in which all elements except the last element on the diagonal vanish.

There is a clear physical meaning in the fact that derivatives of medium eigenpolarization vectors (23), unlike derivatives of electric field components, can be neglected except in come special cases. Indeed, the terms taken into account in (23) have the form $(x_i/L)\mathcal{E}_j$ and $(k_0^{-1}\,\partial/\partial x_i)\mathcal{E}_j$, where $L$ is the inhomogeneity scale of the dielectric response of the medium. It is easy to see when comparing these terms that the wave field changes on the characteristic scale $L/\sqrt{k_0 L} \ll L$, i.e., faster than the dispersive properties of the medium. The eigenpolarization vectors of the medium vary on the scale $L$, and hence their derivatives can be dropped, in contrast to the first derivatives of the field amplitude, $\mathcal{E}_j\,\partial U_{ij}/\partial x_k \ll U_{ij}\,\partial\mathcal{E}_j/\partial x_k$.

We analyze the cases of polarization degeneracy identified in Section 2.1. In cases 2.1.1–2.1.3, the condition $\mathbf{e}_i = (0, 0, 1)$ is imposed on the medium eigenvector basis because the $x_3$ axis is directed along the carrier wave vector $\mathbf{k}^0$. It is easy to see that the unitary matrix $U_{ij}^0$, up to a permutation of indices, has the general form

$$U_{ij}^0 = \begin{bmatrix} \cos\gamma\cos\varphi - \mathrm{i}\sin\gamma\sin\varphi & \sin\gamma\cos\varphi + \mathrm{i}\cos\gamma\sin\varphi & 0 \\ \cos\gamma\sin\varphi + \mathrm{i}\sin\gamma\cos\varphi & \sin\gamma\sin\varphi - \mathrm{i}\cos\gamma\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{25}$$

The coordinates axes $x_1$ and $x_2$ can always be rotated with respect to the $x_3$ axis such that $\varphi = 0$; in this case, the coordinate axes are directed along the axes of the ellipses of medium eigenpolarizations in the plane $(x_1, x_2)$. The remaining parameter $\gamma$ characterizes the axial ratio of the ellipses of medium eigenpolarizations. In what follows, we deal with exactly such a coordinate system, denoted by $(x, y, z)$, where the $z = x_3$ azis is directed along the wave vector and is simultaneously parallel to the real eigenvector of the medium. After the substitution of the above matrix $U_{ij}^0$ at $\varphi = 0$ in (24), the system of wave equations $\hat{D}_{ij}\mathcal{E}_j = 0$ reduces to the truncated form

$$\begin{cases} \hat{D}_{11}\mathcal{E}_1 + \hat{D}_{13}\mathcal{E}_3 = 0\,, \\ \hat{D}_{22}\mathcal{E}_2 + \hat{D}_{23}\mathcal{E}_3 = 0\,, \\ \hat{D}_{13}^*\mathcal{E}_1 + \hat{D}_{23}^*\mathcal{E}_2 = \varepsilon_3\mathcal{E}_3\,, \end{cases} \tag{26}$$

where

$$\hat{D}_{11} = n^2 - \varepsilon_1 - 2\mathrm{i}nk_0^{-1}\,\frac{\partial}{\partial z}\,,$$

$$\hat{D}_{22} = n^2 - \varepsilon_2 - 2\mathrm{i}nk_0^{-1}\,\frac{\partial}{\partial z}\,,$$

$$\hat{D}_{13} = n^2(\delta e_{3x}\cos\gamma - \mathrm{i}\delta e_{3y}\sin\gamma) + \mathrm{i}nk_0^{-1}\left(\cos\gamma\,\frac{\partial}{\partial x} - \mathrm{i}\sin\gamma\,\frac{\partial}{\partial y}\right),$$

$$\hat{D}_{13}^* = n^2(\delta e_{3x}\cos\gamma + \mathrm{i}\delta e_{3y}\sin\gamma) + \mathrm{i}nk_0^{-1}\left(\cos\gamma\,\frac{\partial}{\partial x} + \mathrm{i}\sin\gamma\,\frac{\partial}{\partial y}\right),$$

$$\hat{D}_{23} = n^2(\delta e_{3x}\sin\gamma + \mathrm{i}\delta e_{3y}\cos\gamma) + \mathrm{i}nk_0^{-1}\left(\sin\gamma\,\frac{\partial}{\partial x} + \mathrm{i}\cos\gamma\,\frac{\partial}{\partial y}\right),$$

$$\hat{D}_{23}^* = n^2(\delta e_{3x}\sin\gamma - \mathrm{i}\delta e_{3y}\cos\gamma) + \mathrm{i}nk_0^{-1}\left(\sin\gamma\,\frac{\partial}{\partial x} - \mathrm{i}\cos\gamma\,\frac{\partial}{\partial y}\right).$$

In deriving these expressions, we used the relations

$$\hat{D}_{11} = -\varepsilon_3\,, \quad \hat{D}_{12} = \hat{D}_{21} = 0\,, \quad \hat{D}_{31} = \hat{D}_{13}^*\,, \quad \hat{D}_{32} = \hat{D}_{23}^*\,,$$

where the asterisk denotes Hermitian conjugation of an operator. We note that the longitudinal derivative (along the wave vector) enters only the diagonal operators $\hat{D}_{11}$ and $\hat{D}_{22}$, while all off-diagonal operators contain only transverse derivatives. We further stress that rotation of eigenpolarization vectors of the medium contributes to the linear wave coupling; however, this contribution is determined solely by transverse perturbations $\delta\mathbf{e}_3$ of the distinguished eigenvector $\mathbf{e}_3$ along which the wave vectors of the interacting waves align. We recall that the component $\mathcal{E}_3$ in the above equations defines the longitudinal electric field $E_z = \mathcal{E}_3$, with $\mathbf{e}_3^0 \,||\, \mathbf{k}^0$.

### 3.1 Polarization degeneracy under conditions of partial anisotropy degeneracy (case 2.1.1)

In the case of polarization degeneracy under conditions of partial anisotropic degeneracy, the following relations are satisfied in accordance with (12):

$$|n^2 - \varepsilon_1| \ll 1\,, \quad |n^2 - \varepsilon_2| \ll 1\,.$$

It follows that all $\hat{D}_{ij}\mathcal{E}_k$-like terms in Eqns (26) are small quantities of the same order. On the other hand, the medium must be far from the longitudinal resonance; therefore, we can assume that $\varepsilon_3 \sim 1$ or, to be precise, $|\hat{D}_{13}^*|, |\hat{D}_{23}^*| \ll \varepsilon_3$. Hence, by virtue of the last equation in (26), the longitudinal electric field is small, $\mathcal{E}_3 \ll \mathcal{E}_1, \mathcal{E}_2$. Disregarding the small component of the electric field yields a set of two decoupled equations:

$$\hat{D}_{11}\mathcal{E}_1 = 0\,, \quad \hat{D}_{22}\mathcal{E}_2 = 0\,.$$

In this approximation, two transverse modes with polarizations matching two eigenpolarizations of the medium propagate independently, in accordance with geometric optics equations. Physically, this means that the coupling is due to the disregarded higher-order terms and occurs on geometric optical scales of the order of $L$ rather than $L/\sqrt{k_0 L}$, as in the derivation of the truncated equations. With those terms included, the equations for interacting geometric optical modes can be written in form (1),

$$\begin{cases} \left(\dfrac{\mathrm{d}}{\mathrm{d}l} - \mathrm{i}k_0 n_1\right)E_1 = \eta E_2\,, \\ \left(\dfrac{\mathrm{d}}{\mathrm{d}l} - \mathrm{i}k_0 n_2\right)E_2 = \eta^* E_1\,, \end{cases}$$

where the derivative is taken along the geometric optical path $l$, $E_{1,2}$ and $n_{1,2}$ are the amplitudes and refractive indices of normal modes, and $\eta$ is the coupling coefficient between the modes. Such equations for coupled modes in weakly anisotropic media were systematically investigated in [6, 8]; we do not dwell on this well-known issue here. As noted in the Introduction, mode coupling in the case under consideration is effectively one-dimensional because the effect 'accumulates' as the waves propagate along the geometric optical path.

### 3.2 Polarization degeneracy in the vicinity of the medium resonance (case 2.1.2)

Wave equations have a quite different character in the case of polarization degeneracy in the vicinity of the medium resonance. By virtue of conditions (13), one of the diagonal operators, either $\hat{D}_{11}$ or $\hat{D}_{22}$, is not small. For definiteness, let the following conditions be satisfied in the vicinity of the polarization degeneracy point:

$$|n^2 - \varepsilon_1| \ll 1\,, \quad |n^2 - \varepsilon_2| \gtrsim 1\,, \quad |\varepsilon_3| \ll 1\,. \tag{27}$$

This means that $\hat{D}_{22}$ in (25) is not small, while all the remaining parameters are of the same order and small compared with $\hat{D}_{22}$. The component $\mathcal{E}_2 \ll \mathcal{E}_1, \mathcal{E}_3$ can therefore be omitted and Eqns (26) take the form

$$\begin{cases} \hat{D}_{11}\mathcal{E}_1 + \hat{D}_{13}\mathcal{E}_3 = 0\,, \\ \hat{D}_{13}^*\mathcal{E}_1 - \varepsilon_3\mathcal{E}_3 = 0\,. \end{cases} \tag{28}$$

To the best of our knowledge, equations of this type were first introduced in [18] in application to gyroropic plasma electrodynamics; an understanding of the physical meaning of these equations and their solution were obtained in [19] and independently in [20]. These papers were followed by a large number of publications devoted to the analysis of wave processes described by these equations. A detailed discussion of interaction between ordinary and extraordinary waves can be found in [19, 20, 31–36] for a two-dimensionally inhomogeneous magnetoactive plasma and in [37] for a three-dimensionally inhomogeneous plasma. In Ref. [21], the problem was generalized to a case unrelated to magnetoactive plasma electrodynamics.

The above equations describe a totally new type of wave coupling that does not comply with the standard picture. This can be illustrated in the framework of the geometric optics approach. Formally, this approximation is inapplicable here, but it gives a clear qualitative insight into the main properties of the linear interaction in the case of interest. The dispersion relation corresponding to conditions (28) for propagation angles $k_x, k_y \ll k_z \approx nk_0$ can be written as

$$k_x^2 \cos^2\gamma + k_y^2 \sin^2\gamma = k_0^2 \varepsilon_3 \left(\frac{\varepsilon_1}{n^2} - 1\right). \tag{29}$$

This condition is easy to obtain by setting $\partial_x = \mathrm{i}k_x$, $\partial_y = \mathrm{i}k_y$, $\partial_z = 0$, and $\delta e_{3x,y} = 0$ in the initial equations. Obviously, the waves described by this dispersion relation can propagate in the regions where the right-hand side of (29) is positive. The opposite condition, $\varepsilon_3(\varepsilon_1 - n^2) < 0$, defines the nontransparency region for the modes being considered. The third mode with a nonzero $\mathcal{E}_2$ 'decouples' and cannot propagate within this region in principle.

Figure 1 shows the structure of the linear interaction region in (a) one-dimensional and (b) two-dimensional cases. The regions of propagation of waves with a fixed linear wave number are separated by a nontransparent (evanescent) region with the 'radiation cut-off surfaces' $\varepsilon_1 = n^2$ and $\varepsilon_3 = 0$ serving as its boundaries. The mode nomenclature is introduced such that a mode of some type propagates on the one side and the mode of a different type on the other side.

In the one-dimensional case, the cut-off surfaces can be considered parallel in the first approximation, with the width of the evanescent region depending on the wave number. At a certain optimal longitudinal wave number $n$, the cut-off planes may coincide; then the evanescent region is absent and the wave propagates perfectly freely, which corresponds to the case of complete transformation of one mode into another. In the presence of the evanescent region in the geometric optics approximation, radiation is reflected back and no interaction with another mode occurs. However, tunneling of electromagnetic radiation through the nontransparency region is possible beyond the framework of geometric optics; for a sufficiently thin layer, it may lead to an effective transformation [13–15].

The situation in the two- and three-dimensional cases is topologically different: the cut-off surfaces intersect in space along a certain line perpendicular to the plane of the figure. There is no evanescent region for a geometric optical beam crossing this line, which formally corresponds to complete transformation. In this case, in contrast to the one-dimensional one, such a beam exist not for a single optimal $n$ value but for a certain continuous range of values at which the line intersecting the cut-off surfaces exists. A change of $n$ within this range results only in a shift of the transformation region following the intersection line between the cut-off surfaces.

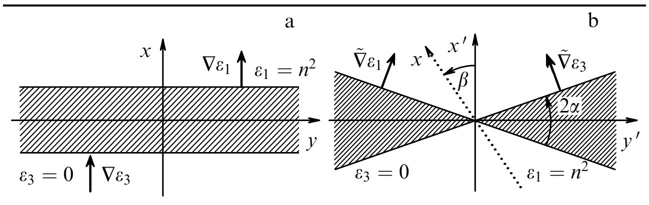


**Figure 1.** Cut-off surfaces $\varepsilon_1 = n^2$ and $\varepsilon_3 = 0$ in one- and two-dimensional cases. The evanescent region is hatched.

Clearly, the above geometric optical description becomes incorrect in the mode-coupling region, but a comprehensive wave analysis of the transformation effect yields a similar result. Once the cut-off surfaces intersect in a given geometry, there is an optimal field distribution at which the incident wave beam passes through the interaction region without being reflected.

Equations (28) can be conveniently analyzed in the approximation of planar cut-off surfaces. As is shown below, these equations describe wave coupling on smaller scales than the medium inhomogeneity scales. Therefore, in a localized linear coupling region, we can confine ourselves to variations of coefficients in Eqns (28) that are linear in coordinates:

$$k_0(\varepsilon_1 - n^2) = a_1 x + a_2 y + a_3 z\,, \quad k_0 \varepsilon_3 = b_1 x + b_2 y + b_3 z\,,$$

where $a_1 = k_0\, \partial\varepsilon_1/\partial x$, $b_1 = k_0\, \partial\varepsilon_3/\partial x$, etc. are assumed to be constants. Here, we took into account that rigorous conditions of polarization degeneracy are satisfied at the origin: $\varepsilon_1 = n^2$ and $\varepsilon_3 = 0$. To simplify calculations, we consider the simplest case of a fixed gyrotropy axis in the medium when the direction of the eigenpolarization vector $\mathbf{e}_3$ in the vicinity of the interaction region can be considered constant (a variable gyrotropy axis is considered in [37]). As a result, Eqns (28) become

$$\begin{cases} \mathrm{i}n\left(\cos\gamma\,\dfrac{\partial}{\partial x} - \mathrm{i}\sin\gamma\,\dfrac{\partial}{\partial y}\right)\mathcal{E}_3 = \left(a_1 x + a_2 y + a_3 z + 2\mathrm{i}n\,\dfrac{\partial}{\partial z}\right)\mathcal{E}_1\,, \\ \mathrm{i}n\left(\cos\gamma\,\dfrac{\partial}{\partial x} + \mathrm{i}\sin\gamma\,\dfrac{\partial}{\partial y}\right)\mathcal{E}_1 = (b_1 x + b_2 y + b_3 z)\,\mathcal{E}_3\,. \end{cases} \tag{30}$$

An important special case is given by a two-dimensionally inhomogeneous medium in which all parameters vary across a distinguished axis parallel to the eigenpolarization vector $\mathbf{e}_3$, i.e., the gyrotropy axis. In such a medium, $a_3 = 0$, $b_3 = 0$, and $\partial/\partial z = k_z = \mathrm{const}$. The corresponding wave equations describe the propagation of independent Fourier harmonics along $z$, and the general solution of these equations is constructed as their superposition. As can be readily seen, the term with the wave number $k_z$ in the above equations can be taken into account by shifting the origin along the $x$ and $y$ axes, which results in the same set of equations for each Fourier harmonic. Rotating coordinate axes and rescaling the fields, we reduce (30) to the reference equations

$$\begin{cases} -\left(\mathrm{i}\,\dfrac{\partial}{\partial x'} - \dfrac{\partial}{\partial y'}\right)\mathcal{E}_3' = (x'\cos\alpha + y'\sin\alpha)\mathcal{E}_1'\,, \\ -\left(\mathrm{i}\,\dfrac{\partial}{\partial x'} + \dfrac{\partial}{\partial y'}\right)\mathcal{E}_1' = (x'\cos\alpha - y'\sin\alpha)\mathcal{E}_3'\,. \end{cases} \tag{31}$$

The new coordinates are located in the linear interaction region as shown in Fig. 1b, where $x'$ and $y'$ axes are directed along bisectors of the angles formed by the intersecting vectors $\tilde{\nabla}\varepsilon_1 = (a_1\cos\gamma, a_2\sin\gamma, 0)$ and $\tilde{\nabla}\varepsilon_3 = (b_1\cos\gamma, b_2\sin\gamma, 0)$. The coordinates are normalized to the characteristic interaction scale

$$L_\nabla = \sqrt{n}\,\left(k_0^2|\tilde{\nabla}\varepsilon_1||\tilde{\nabla}\varepsilon_3|\right)^{-1/4}.$$

The degree of the problem 'non-one-dimensionality' is determined by the single dimensionless parameter $\alpha$ (half-angle between $\tilde{\nabla}\varepsilon_1$ and $\tilde{\nabla}\varepsilon_3$); as $\alpha \to 0$, transition to a one-dimensional problem occurs.

The above equations admit a complete analytic solution [19, 21, 31]. They describe linear coupling on the scale $L_\nabla \sim L/\sqrt{k_0 L} \ll L$ that is smaller than the medium inhomogeneity scale. This justifies the use of the approximation of linear variations of wave equation coefficients with respect to coordinates. We note, however, that in transparent media, the scale $L_\nabla \sim \lambda\sqrt{k_0 L} \gg \lambda$ is large compared with the radiation wavelength. In the two-dimensional geometry, new effects absent in the standard one-dimensional model arise, such as the existence of aperture-limited wave beams undergoing complete (reflectionless) transformation and the absence of symmetry between the processes of direct and reverse wave transformation in gyrotropic media.

In a three-dimensionally inhomogeneous medium, Eqns (30) can almost always be reduced, with a certain degree of accuracy, to Eqn (31), which admits a complete analytic solution. This permits regarding Eqns (31) as a new standard problem in the wave propagation theory, which implies a wide range of applicability. Specifically, it can typically be assumed in solving problems pertinent to magnetic plasma confinement that the equilibrium plasma density is rapidly redistributed along magnetic field lines; therefore, the condition $\nabla\varepsilon_3 \perp \mathbf{B}$ is a good approximation. The results presented in Section 2.1.6 give reason to suppose that the $z$ coordinate is directed along the magnetic field, $b_3 = 0$ and $\gamma = \pi/4$. Then the substitution

$$\mathcal{E}'_{1,3} = \mathcal{E}_{1,3} \exp\left(\mathrm{i}k_z z + \frac{\mathrm{i}a_3 z^2}{4n}\right)$$

eliminates the coordinate along the magnetic field from the equations, and the problem becomes two-dimensional:

$$\begin{cases} \mathrm{i}n\left(\dfrac{\partial}{\partial x} - \mathrm{i}\,\dfrac{\partial}{\partial y}\right)\mathcal{E}'_3 = \sqrt{2}\,(a_1 x + a_2 y - 2nk_z)\,\mathcal{E}'_1\,, \\ \mathrm{i}n\left(\dfrac{\partial}{\partial x} + \mathrm{i}\,\dfrac{\partial}{\partial y}\right)\mathcal{E}'_1 = \sqrt{2}\,(b_1 x + b_2 y)\,\mathcal{E}'_3\,. \end{cases}$$

It is easy to see that shifting and rotating coordinates reduces the problem to reference equations (31). We also note that the plasma density in strong external fields varies along the magnetic field; in this context, $\nabla\varepsilon_3 \,\|\, \mathbf{B}$ is of interest, i.e., $b_1 = b_2 = 0$. The substitution

$$\mathcal{E}'_{1,3} = \mathcal{E}_{1,3} \exp\left[-\frac{\mathrm{i}z(a_1 x + a_2 y)}{2n}\right]$$

eliminates the explicit dependence on the transverse coordinates $x, y$ from Eqn (30) and thereby reduces the problem to a one-dimensional one:

$$\begin{cases} \mathrm{i}n\left(\dfrac{\partial}{\partial x} - \mathrm{i}\,\dfrac{\partial}{\partial y}\right)\mathcal{E}'_3 = \dfrac{1}{2}(a_1 - \mathrm{i}a_2)\,z\mathcal{E}'_3 + 2\sqrt{2}\,\mathrm{i}n\,\dfrac{\partial \mathcal{E}'_1}{\partial z}\,, \\ \mathrm{i}n\left(\dfrac{\partial}{\partial x} + \mathrm{i}\,\dfrac{\partial}{\partial y}\right)\mathcal{E}'_1 = \left(\dfrac{1}{2}\,a_1 + \dfrac{1}{2}\,\mathrm{i}a_2 + \sqrt{2}b_3\right)z\mathcal{E}'_3\,. \end{cases}$$

The resultant set of wave equations describes the independent propagation of Fourier harmonics along $x$ and $y$ in a medium that is inhomogeneous along $z$.

The case of an anisotropic nongyrotropic medium deserves special attention. Such a medium is characterized by real eigenpolarization vectors and therefore linear polarization of all normal waves. In our notation, without loss of generality, $\gamma = 0$. Wave equations (28) corresponding to this case can be written as

$$\begin{cases} \left(k_0 n^2 \delta e_{3x} + \mathrm{i}n\,\dfrac{\partial}{\partial x}\right)\mathcal{E}'_3 = \left(a_1 x + a_2 y + a_3 z + 2\mathrm{i}n\,\dfrac{\partial}{\partial z}\right)\mathcal{E}'_1\,, \\ \left(k_0 n^2 \delta e_{3x} + \mathrm{i}n\,\dfrac{\partial}{\partial x}\right)\mathcal{E}'_1 = (b_1 x + b_2 y + b_3 z)\,\mathcal{E}'_3\,. \end{cases}$$

For generality, we included additional terms responsible for the change in the direction of the optical axis $\mathbf{e}_3$ in the vicinity of the interaction region; the absence of gyrotropy implies that $\delta e_{3x} = \delta e^*_{3x}$. As in the preceding cases, we assume that this direction in the vicinity of the polarization degeneracy point is subject to linear variation in space, $k_0 \delta e_{3x} = c_1 x + c_2 y + c_3 z$. Then the substitution

$$\mathcal{E}'_{1,3} = \mathcal{E}_{1,3} \exp\left[-\mathrm{i}n\left(\frac{c_1 x^2}{2} + c_2 xy + c_3 xz\right) + \frac{\mathrm{i}\gamma}{4nb_1}\right],$$

$$x = x' - \frac{zb_3}{b_1}\,,$$

where $\gamma = a_1 b_3 - a_3 b_1 - 2n^2 b_3 c_3$, removes the explicit dependence on the longitudinal coordinate $z$ from the equations:

$$\begin{cases} \mathrm{i}n\,\dfrac{\partial \mathcal{E}'_3}{\partial x'} = \left[(a_1 - 2n^2 c_3)\,x' + a_2 y + 2\mathrm{i}n\,\dfrac{\partial}{\partial z} + 2\mathrm{i}n\,\dfrac{b_3}{b_1}\,\dfrac{\partial}{\partial x'}\right]\mathcal{E}'_1\,, \\ \mathrm{i}n\,\dfrac{\partial \mathcal{E}'_1}{\partial x'} = (b_1 x' + b_2 y)\,\mathcal{E}'_3\,. \end{cases}$$

Formally, the system of wave equations thus obtained describes the independent propagation of Fourier harmonics over $z$ in a medium inhomogeneous along the $x'$ axis. The $y$ coordinate enters these equations as a parameter without affecting their dimensionality. Hence, wave coupling in anisotropic media without gyrotropy can always be regarded as a one-dimensional process in an effective plane-layered medium. Despite the one-dimensionality of this process, it cannot be described in terms of geometric optics due to its small scale, $L_\nabla \ll L$.

### 3.3 Complete polarization degeneracy (case 2.1.3)

We recall that in the case of complete polarization degeneracy in a homogeneous medium, any field polarization at the degeneracy point satisfies the Maxwell equations. In an inhomogeneous medium, in view of (14), the following conditions are satisfied in the vicinity of the complete polarization degeneracy point:

$$|n^2 - \varepsilon_1| \ll 1\,, \quad |n^2 - \varepsilon_2| \ll 1\,, \quad |\varepsilon_3| \ll 1\,.$$

It then follows that all terms in Eqns (26) are small quantities of the same order. Therefore, these conditions describe three coupled modes, with the field components $\mathcal{E}_1$ and $\mathcal{E}_2$ transverse to the propagation direction related through the longitudinal field $\mathcal{E}_3$. We do not consider this exotic case.

In the foregoing, we represented wave equations in form (26) corresponding to eigenpolarization matrix (25). This representation is inconvenient for characterizing the remaining case 2.1.4, and is inapplicable to the case in 2.1.5.

### 3.4 Complete anisotropy degeneracy (case 2.1.4)

Complete anisotropy degeneracy, in which all three $\varepsilon_i$ components become nearly the same in a certain neighborhood, can be analyzed with the use of equations resulting from diagonalization of the dielectric permittivity tensor; however, this implies rather cumbersome calculations.

We consider a simpler approach. By virtue of conditions (15), the dielectric permittivity tensor at the polarization degeneracy point has the form $\varepsilon_{ij} = n^2\delta_{ij}$ in the basis of eigenpolarization vectors; exactly the same form is preserved in any other orthogonal basis. We choose the Cartesian coordinate system $(x, y, z)$ with the $z$ axis directed along the carrier wave vector. It then follows from the wave equation in the vicinity of the zeroth-order degeneracy point that $E_z = 0$; therefore, this wave field component can be neglected in the first-order equations for transverse field components. Hence, it suffices to take into account only the part of the dielectric permittivity tensor that describes the dielectric properties of the medium in the plane orthogonal to the wave vector, i.e.,

$$\varepsilon_{ij}^{2\mathrm{D}}(\mathbf{r}) = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_{yy} \end{bmatrix}.$$

In a nondissipative medium, this tensor is Hermitian and as such cannot be diagonalized in a unitary basis $\mathbf{e}'_1(\mathbf{r}), \mathbf{e}'_2(\mathbf{r})$ that varies continuously in a neighborhood of the degeneracy point. The transverse electric field can be expressed through the new basis as

$$\mathbf{E}_\perp = \mathcal{E}'_i(\mathbf{r})\,\mathbf{e}'_i(\mathbf{r}) \exp(\mathrm{i}\mathbf{k}\mathbf{r} - \mathrm{i}\omega t).$$

The use of a procedure analogous to that described at the beginning of this section allows finding truncated wave equations for the new electric field amplitudes $\mathcal{E}'_i(\mathbf{r})$ near the complete anisotropic degeneracy point. These equations split into two independent ones:

$$\begin{cases} \left(n^2 - \varepsilon'_1 - 2\mathrm{i}nk_0^{-1}\dfrac{\partial}{\partial z}\right)\mathcal{E}'_1 = 0, \\ \left(n^2 - \varepsilon'_2 - 2\mathrm{i}nk_0^{-1}\dfrac{\partial}{\partial z}\right)\mathcal{E}'_2 = 0. \end{cases}$$

Hence, the case of complete anisotropy degeneracy is analogous to case 2.1.1, in which polarization degeneracy occurs under conditions of partial anisotropy degeneracy. It differs from 2.1.1 only in that the eigenvalues $\varepsilon'_1$ and $\varepsilon'_2$ and eigenpolarizations $\mathbf{e}'_1$ and $\mathbf{e}'_2$ are sought not for the full dielectric permittivity tensor but only for its 'transverse' part.

### 3.5 Wave vector orthogonal to the eigenpolarization vector of the medium (case 2.1.5)

In conclusion, we consider the last case, in which the wave vector is orthogonal to one of the eigenpolarizations of the medium. The general analysis of this problem requires bulky computations, and we therefore confine ourselves to a particular case reflecting all of basic physics. In many applications, at least one of the eigenpolarizations can be regarded as linear, i.e., corresponding to a real vector $\mathbf{e}_i$. As mentioned, the nontrivial case 2.1.5 is realized only if the wave vector is orthogonal to a real eigenpolarization vector. We assume for definiteness that

$$\mathbf{k}^0 \perp \mathbf{e}_1^0, \quad \mathbf{e}_1^0 \in \mathrm{R},$$

and direct the $x$ axis along $\mathbf{e}_1^0$ and the $z$ axis along the carrier wave vector $\mathbf{k}^0$. Then the unitary matrix $U_{ij}^0$ in the most general form is defined by two free parameters and can be represented as

$$U_{ij}^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \exp(\mathrm{i}\varphi)\cos\chi & \exp(\mathrm{i}\varphi)\sin\chi \\ 0 & -\sin\chi & \cos\chi \end{bmatrix}.$$

Substituting this matrix in (24) yields the operator $\hat{D}_{ij}$ and the corresponding set of wave equations $\hat{D}_{ij}\mathcal{E}_j = 0$. For brevity, we do not write this complicated system containing all three components of the vector $\boldsymbol{\mathcal{E}}$. However, the system can be simplified by singling out coupled modes. We recall that $\boldsymbol{\mathcal{E}} \perp \boldsymbol{\tau}$ in the vicinity of the polarization degeneracy point. In the case under consideration, $\boldsymbol{\tau} \perp \mathbf{e}_1^0$ as well; therefore, the wave field can be decomposed in terms of three orthogonal vectors $\boldsymbol{\tau}$, $\mathbf{e}_1^0$, and $\mathbf{u} = [\boldsymbol{\tau}\,\mathbf{e}_1^0]$:

$$\boldsymbol{\mathcal{E}} = \mathcal{E}_1\mathbf{e}_1^0 + \mathcal{E}_u\mathbf{u} + \mathcal{E}_\tau\boldsymbol{\tau},$$

where

$$\mathbf{e}_1^0 = (1, 0, 0), \quad \mathbf{u} = (0, \varepsilon_3\cos\chi, \varepsilon_2\sin\chi),$$
$$\boldsymbol{\tau} = (0, -\varepsilon_2\sin\chi, \varepsilon_3\cos\chi).$$

When the field in this form is substituted in the truncated equations, the $\mathcal{E}_\tau$ component is small in the vicinity of the degeneracy point and can be neglected. The following set of equations is valid for the remaining two components:

$$\begin{cases} \left(\Delta_1 - \dfrac{2\mathrm{i}n}{k_0}\dfrac{\partial}{\partial z}\right)\mathcal{E}_1 + \eta\left(\delta e_{1z}^* - \dfrac{\mathrm{i}n}{k_0}\dfrac{\partial}{\partial x}\right)\mathcal{E}_u = 0, \\ \eta\left(\delta e_{1z} - \dfrac{\mathrm{i}n}{k_0}\dfrac{\partial}{\partial x}\right)\mathcal{E}_1 \\ \qquad + \left[\Delta_u + \delta_u - \dfrac{2\mathrm{i}n}{k_0}\left(\dfrac{\partial}{\partial z} - \eta\cos\varphi\,\dfrac{\varepsilon_2\varepsilon_3}{\varepsilon_1}\dfrac{\partial}{\partial y}\right)\right]\mathcal{E}_u = 0, \end{cases} \tag{32}$$

where

$$\eta = (\varepsilon_3 - \varepsilon_2)\left(\frac{\varepsilon_1}{\varepsilon_2\varepsilon_3}\right)^2 \sin\chi\cos\chi, \quad \Delta_1 = n^2 - \varepsilon_1,$$
$$\Delta_u = n^2 - \frac{\varepsilon_2\varepsilon_3}{\varepsilon_2\sin^2\chi + \varepsilon_3\cos^2\chi},$$
$$\delta_u = 2n^2\eta\big(\varepsilon_2\sin\chi\,\mathrm{Re}(\delta e_{3z}) + \varepsilon_3\cos\chi\,\mathrm{Re}(\delta e_{2z})\big).$$

These equations describe a linear coupling of two modes that pass into the modes with dispersion relations $\Delta_1 = 0$ and $\Delta_u = 0$ outside the interaction region. The linear coupling results from the medium anisotropy across the $x$ axis and/or gyrotropy along this axis. Indeed, at $\varepsilon_2 = \varepsilon_3$, the coupling coefficient $\eta = 0$, and the modes in this approximation propagate independently; it can be verified that taking the next orders into account leads to a one-dimensional coupling described by equations of type (1). Moreover, the mode coupling disappears at $\chi = 0$ or $\chi = \pi/2$, which corresponds to the case of partial anisotropy degeneracy (case 2.1.1). We recall that in the interaction region, all coefficients in Eqns (32) are small quantities smoothly varying in space on the scale $L$. For $\eta \neq 0$, the characteristic scale of the linear interaction region is $L_\nabla \sim L/\sqrt{k_0 L} \ll L$, as in the case of the interaction in the vicinity of medium resonance (case 2.1.2), and the linear coupling can be essentially non-one-dimensional.

As an example, we consider a biaxial crystal with $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon_3$. It is easy to verify that the sole possibility for polarization degeneracy in this case is realized under conditions (16), which can be satisfied only for $\mathbf{k} \perp \mathbf{e}_1$. In a crystal with a constant direction of optical axes, Eqns (32) take the form

$$\begin{cases} \mathrm{i}n\eta \dfrac{\partial \mathcal{E}_u}{\partial x} = \left(k_0 \Delta_1 - 2\mathrm{i}n \dfrac{\partial}{\partial z}\right)\mathcal{E}_1 \,, \\ \mathrm{i}n\eta \dfrac{\partial \mathcal{E}_1}{\partial x} = \left(k_0 \Delta_u - 2\mathrm{i}n \left(\dfrac{\partial}{\partial z} - \eta' \dfrac{\partial}{\partial y}\right)\right)\mathcal{E}_u \,, \end{cases} \tag{33}$$

where $\eta' = \eta \varepsilon_2 \varepsilon_3 / \varepsilon_1$. Expanding in the vicinity of linear coupling,

$$k_0 \Delta_1 = a_1 x + a_2 y + a_3 z \,, \quad k_0 \Delta_u = b_1 x + b_2 y + b_3 z \,,$$

and substituting

$$\mathcal{E}'_{1,u} = \mathcal{E}_{1,u} \exp \frac{\mathrm{i}x\xi}{2n} \,, \quad \xi = (a_1 - b_1)\,\frac{y}{\eta'} + a_1 z \,,$$

we eliminate the $x$ coordinate from the equations:

$$\begin{cases} \eta\left(\dfrac{\xi}{2} + \mathrm{i}n \dfrac{\partial}{\partial x}\right)\mathcal{E}'_u = \left(a_2 y + a_3 z - 2\mathrm{i}n \dfrac{\partial}{\partial z}\right)\mathcal{E}'_1 \,, \\ \eta\left(\dfrac{\xi}{2} + \mathrm{i}n \dfrac{\partial}{\partial x}\right)\mathcal{E}'_1 = \left(b_2 y + b_3 z - 2\mathrm{i}n \left(\dfrac{\partial}{\partial z} - \eta' \dfrac{\partial}{\partial y}\right)\right)\mathcal{E}'_u \,. \end{cases}$$

Considering these equations for an individual Fourier harmonic with respect to $x$, we transform them to the form

$$\begin{cases} z'\mathcal{E}'_u = \left(c_2 y' + c_3 z' - \mathrm{i}c'_2 \dfrac{\partial}{\partial y'} - \mathrm{i}c'_3 \dfrac{\partial}{\partial z'}\right)\mathcal{E}'_1 \,, \\ z'\mathcal{E}'_1 = \left(d_2 y' + d_3 z' - \mathrm{i}d'_2 \dfrac{\partial}{\partial y'} - \mathrm{i}d'_3 \dfrac{\partial}{\partial z'}\right)\mathcal{E}'_u \,. \end{cases}$$

Here, we rotated the coordinate axes such that $z' = \xi / a_1$. Importantly, the differential operators in these equations are linearly independent at $\eta \neq 0$, i.e., $w = c'_2 d'_3 - c'_3 d'_2 \neq 0$. This allows the substitution

$$\mathcal{E}''_{1,u} = \mathcal{E}'_{1,u} \exp\left[\mathrm{i}(c'_2 d_2 - c_2 d'_2)\,\frac{y'z'}{w} + \mathrm{i}(c_2 d'_3 - c'_3 d_2)\,\frac{y'^2}{2w}\right]$$

that eliminates the $y'$ coordinate from the equations:

$$\begin{cases} z'\mathcal{E}''_u = \left(c''_3 z' - \mathrm{i}c'_2 \dfrac{\partial}{\partial y'} - \mathrm{i}c'_3 \dfrac{\partial}{\partial z'}\right)\mathcal{E}''_1 \,, \\ z'\mathcal{E}''_1 = \left(d''_3 z' - \mathrm{i}d'_2 \dfrac{\partial}{\partial y'} - \mathrm{i}d'_3 \dfrac{\partial}{\partial z'}\right)\mathcal{E}''_u \,. \end{cases}$$

The resulting system of equations describes the independent propagation of Fourier harmonics along $y'$ in a one-dimensionally homogeneous medium. Therefore, linear wave coupling in a three-dimensionally inhomogeneous crystal can be represented as a one-dimensional process in a certain effective plane-layered medium. Despite the one-dimensional character, this interaction is not of a 'geometric optical' nature because it occurs on a fast scale $L_\nabla \ll L$.

It can be readily seen that a change in the direction of the optical axes in the crystal does not alter the above conclusion as long as we confine ourselves to linear variations. Indeed, it follows from (32) that variations $\delta e_{3z}$ are taken into account additively together with variations of $\Delta_u$ and variations of $k_0 \delta e_{1z} = f_1 x + f_2 y + f_3 z$ at $\delta e_{1z} = \delta e^*_{1z}$ by the quadratic phase substitution

$$\mathcal{E}'_{1,u} = \mathcal{E}_{1,u} \exp\left[-\mathrm{i}n\left(\frac{f_1 x^2}{2} + f_2 xy + f_3 xz\right)\right],$$

after which the equations can be reduced to (33). This brings us back to the conclusion that the non-one-dimensional character of linear wave coupling can manifest itself only in gyrotropic media. In the case of interest, 'non-one-dimensionality' occurs because $\delta e_{1z} \neq \delta e^*_{1z}$.

## 4. Conclusion

We have demonstrated that linear coupling of electromagnetic waves in unbounded weakly inhomogeneous media follows two distinct scenarios. In the case of simultaneous anisotropy and gyrotropy degeneracy (cases 2.1.2 and 2.1.4), the linear wave coupling is realized on a slow scale $L$ determined by the scale of parameter variations. This situation can be interpreted as a scalar coupling of two geometric optical modes described by equations of type (1). Because the coupling occurs along the rays, the wave conversion is a one-dimensional process. In the vicinity of medium resonances (cases 2.1.2 and 2.1.3) and in the special case of transverse propagation (case 2.1.5), the fast wave coupling scenario is realized; in this case, the conversion occurs on a small scale $L_\nabla \ll L$, at which the geometric optics approximation is no longer valid. Wave coupling in gyrotropic media is typically essentially non-one-dimensional. In anisotropic media without gyrotropy, it can always be described as a one-dimensional (small-scale) process in an effective plane-layered medium.

To summarize, the conclusions in Section 2 concerning the classification of polarization degeneracy points, obtained in the approximation of a homogeneous medium, hold for spatially dispersive media. They are equally applicable to magnetic media because any linear medium characterized by dielectric and magnetic susceptibility can be equivalently described as a 'nonmagnetic' medium with spatial dispersion [5]. These results are of interest for the rapidly developing electrodynamics of metamaterials. At the same time, the conclusions in Section 3, where wave equations in weakly inhomogeneous media were considered, hold only for spatially dispersionless media and are therefore inapplicable to magnetic media.

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