# On relativistic motion of a pair of particles having opposite signs of masses 

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Abstract. In this methodological note, we consider, in a weakfield limit, the relativistic linear motion of two particles with masses of opposite signs and a small difference between their absolute values: $\boldsymbol{m}_{1,2}= \pm(\mu \pm \Delta \mu), \boldsymbol{\mu}>\mathbf{0},|\Delta \mu| \ll \mu$. In 1957, H Bondi showed in the framework of both Newtonian analysis and General Relativity that, when the relative motion of particles is absent, such a pair can be accelerated indefinitely. We generalize the results of his paper to account for the small nonzero difference between the velocities of the particles. Assuming that the weak-field limit holds and the dynamical system is conservative, an elementary treatment of the problem based on the laws of energy and momentum conservation shows that the system can be accelerated indefinitely, or attain very large asymptotic values of the Lorentz factor $\gamma$. The system experiences indefinite acceleration when its energy-momentum vector is null and the mass difference $\Delta \mu \leqslant 0$. When the modulus of the square of the norm of the energy-momentum vector, $\left|N^{2}\right|$, is sufficiently small, the system can be accelerated to very large $\gamma \propto\left|N^{2}\right|^{-1}$. It is stressed that, when only leading terms in the ratio of a characteristic gravitational radius to the distance between the particles are retained, our elementary analysis leads to equations of motion equivalent to those derived from relativistic weak-field equations of motion by Havas and Goldberg in 1962. Thus, in the weak-field approximation it is possible to bring the system to the state with extremely high values of $\gamma$. The positive energy carried by the particle with positive mass may be conveyed to other physical bodies, say by intercepting this particle with a target. If we suppose that there is a process of production of such

[^0]pairs and the particles with positive mass are intercepted, while the negative mass particles are expelled from the region of space occupied by the physical bodies of interest, this scheme could provide a persistent transfer of positive energy to the bodies, which may be classified as 'perpetual motion of the third kind'. Additionally, we critically evaluate some recent claims regarding the problem.

## 1. Introduction

In 1957, Bondi [1] pointed out that in the Newtonian approximation two particles with opposite signs of masses at rest with respect to each other accelerate indefinitely in an inertial frame. This process is allowed by the conservation laws, since the kinetic energy and angular momentum of such a system are conserved, being exactly zero, while the potential energy only depends on the relative distance between the particles. In the same paper, he generalized this result by finding an appropriate static accelerated solution in General Relativity and discovered that a uniformly accelerated pair of particles with masses of opposite signs must have a mass difference determined by the fact that constant-in-time particle accelerations must be different to keep them static with respect to each other.

It is trivial to show that, in the Newtonian approximation (see Section 2), when the two particles with opposite signs of masses have a relative velocity, its value is approximately conserved. As a result, the acceleration period is finite and the pair as a whole, being initially at rest, attains a finite velocity. We also show that when the initial relative velocity of the particles is sufficiently small, the pair can be accelerated to a relativistic speed.

In Section 3 we consider the problem in the relativistic setting and generalize Bondi's analysis, considering pairs of particles with masses of opposite sings and a small difference between their absolute values: $m_{1,2}= \pm(\mu \pm \Delta \mu), \mu>0$, $|\Delta \mu| \ll \mu$ and having an initial relative velocity $v_{\text {in }}$ in a fixed lab frame where the pair as a whole is initially at rest. We assume that gravitational interaction is weak and, therefore, $G \mu /\left(c^{2} D_{\text {in }}\right) \ll 1$, where $D_{\text {in }}$ is the initial distance between the
particles. Also, for simplicity, it is assumed in the relativistic treatment that the orbital angular momentum of the system is equal to zero, and the motion is linear.

We analyze this situation by elementary means. The equations of motion are obtained from the laws of energy and momentum conservation. It is assumed that the energy and momentum of a system in a Lorentz frame instantaneously comoving with the motion of the pair are given by the Newtonian expressions, and that they form time and spacial components of a local four-vector. This energy-momentum vector is then projected onto the lab frame. Since energy and momentum in the lab frame are conserved under the assumption that gravitational radiation from the system is insignificant, we get two first-order equations in time fully describing the dynamics of the system. We also show how to derive an equivalent pair of second-order equations considering Newton's law of gravity in a frame accelerating with the particles.

It is shown that the pair as a whole always has a positive acceleration, with its asymptotic value being either zero or a nonzero constant, depending on initial conditions. The relative distance between the particles can either have a turning point or increase monotonically. The system accelerates indefinitely when the mass difference $\Delta \mu \leqslant 0$ and the norm of the energy-momentum vector

$$
N=\sqrt{\left(2 \Delta \mu c^{2}+\frac{G \mu^{2}}{D_{\text {in }}}\right)^{2}-\mu^{2} v_{\mathrm{in}}^{2}}=0,
$$

and, accordingly, the energy-momentum vector is null. In this case, the relative distance increases monotonically. When $N^{2}$ is sufficiently small, for the initial conditions corresponding to the monotonic behavior of the relative distance, the acceleration period is finite, but the asymptotic value of the Lorentz gamma factor is large, being proportional to $\left|N^{2}\right|^{-1}$.

Such pairs can play a role in the realization of a hypothetical effect, which we call 'perpetual motion of the third kind' [2], hereafter PMT. In its most general formulation, this effect is the possibility of a persistent energy transfer from a subsystem having negative energy to a subsystem with positive energy in classical theories, where negative energy subsystems are possible. Indeed, the positive mass particle can, in principle, be used to transfer positive energy to other physical bodies after the pair has been accelerated to high values of the Lorentz factor. Iterating this process as many times as we need, we can extract as much positive energy as we wish. Note, however, that this is not the only 'working model' of PMT, and that, in principle, in order to make PMT, we need systems with neither negative rest mass nor gravitational interactions. As is shown in Ref. [2], it suffices to have a medium violating the weak energy condition with certain additional properties and mere hydrodynamical interaction 'to construct a PMT'.

Additionally, we comment on several statements in paper [3], where the Kepler problem for a binary with opposite signs of masses has been considered and which may, in our opinion, lead to a misunderstanding of the problem.

## 2. Newtonian treatment of the problem

First, let us consider the problem in the Newtonian approximation, where mutual gravitational accelerations acting on particles of masses $m_{1}$ and $m_{2}$ are given by conventional
expressions

$$
\begin{equation*}
\ddot{\mathbf{r}}_{1}=-\frac{G m_{2}}{|\mathbf{D}|^{3}} \mathbf{D}, \quad \ddot{\mathbf{r}}_{2}=\frac{G m_{1}}{|\mathbf{D}|^{3}} \mathbf{D} \tag{1}
\end{equation*}
$$

where $\mathbf{r}_{i}$ are position vectors of particles with subscripts $i=1,2$ and $\mathbf{D}=\mathbf{r}_{1}-\mathbf{r}_{2}$. Setting $\mu \equiv G m_{1}=-G m_{2}$, we obtain from equations (1):

$$
\begin{equation*}
\dot{\mathbf{V}}=\frac{\mu}{|\mathbf{D}|^{3}} \mathbf{D}, \quad \dot{\mathbf{v}}=0 \tag{2}
\end{equation*}
$$

where $\mathbf{V} \equiv\left(\dot{\mathbf{r}}_{1}+\dot{\mathbf{r}}_{2}\right) / 2$, and $\mathbf{v} \equiv \dot{\mathbf{D}}$. It follows from equation (2) that, when $\mathbf{v}=0$ at some moment of time, it remains zero in the course of evolution of the system. Thus, in this case, the interparticle distance $\mathbf{D}$ does not change during the evolution, and the system constantly accelerates as a whole, with the acceleration vector

$$
\begin{equation*}
\mathbf{a} \equiv \dot{\mathbf{V}}=\frac{\mu}{|\mathbf{D}|^{3}} \mathbf{D} \tag{3}
\end{equation*}
$$

being constant. The conservation laws are nonetheless respected, since the kinetic energy and momentum of the system are precisely zero, while the potential energy depends only on the relative separation distance. ${ }^{1}$

When $\mathbf{v}(t=0) \equiv \mathbf{v}_{\text {in }} \neq 0$, the absolute value of the relative distance changes with time. Accordingly, the absolute value of the acceleration changes, as well, and eventually decays provided that $\mathbf{D}_{\text {in }} \mathbf{v}_{\text {in }} \neq-\left|\mathbf{v}_{\text {in }}\right|\left|\mathbf{D}_{\text {in }}\right|,{ }^{2}$ where $\mathbf{D}_{\text {in }} \equiv \mathbf{D}(t=0)$. We have

$$
\begin{equation*}
\mathbf{D}=\mathbf{v}_{\text {in }} t+\mathbf{D}_{\text {in }} \tag{4}
\end{equation*}
$$

and, thus, integrating equation (3) we obtain

$$
\begin{equation*}
|\mathbf{V}(t)|=\frac{1}{\sqrt{1-\alpha^{2}}} \frac{\mu}{D_{\text {in }} v_{\text {in }}} \sqrt{2\left(1-\frac{\epsilon^{1 / 2}+\alpha \tau}{\Delta}\right)} \tag{5}
\end{equation*}
$$

where we introduced the dimensionless time

$$
\begin{aligned}
& \tau=\sqrt{\frac{\mu}{D_{\mathrm{in}}^{3}}} t, \quad D_{\mathrm{in}}=\left|\mathbf{D}_{\mathrm{in}}\right|, \quad v_{\mathrm{in}}=\left|\mathbf{v}_{\mathrm{in}}\right| \\
& \epsilon=\frac{\mu}{D_{\mathrm{in}} v_{\mathrm{in}}^{2}}, \quad \alpha=\frac{\mathbf{v}_{\mathrm{in}} \mathbf{D}_{\mathrm{in}}}{v_{\mathrm{in}} D_{\mathrm{in}}}, \quad \Delta=\sqrt{\epsilon+\tau^{2}+2 \alpha \epsilon^{1 / 2} \tau}
\end{aligned}
$$

Note that, when the system moves along a straight line with an increasing value of $|\mathbf{D}|$ and, accordingly, $\alpha=1$ equation (5) yields

$$
\begin{equation*}
|\mathbf{V}(t)|=\frac{\mu}{D_{\text {in }} v_{\text {in }}} \frac{\tau}{\epsilon^{1 / 2}+\tau} . \tag{6}
\end{equation*}
$$

In the limit $\tau \rightarrow \infty$, we get from equations (5) and (6) the asymptotic velocity

$$
\begin{equation*}
V_{\infty} \equiv|\mathbf{V}(\tau \rightarrow \infty)|=\sqrt{\frac{2}{1+\alpha}} \frac{\mu}{D_{\text {in }} v_{\text {in }}} \tag{7}
\end{equation*}
$$

[^1]It follows from formula (7) that, when

$$
\begin{equation*}
v_{\text {in }}<v_{\text {crit }}=\sqrt{\frac{2}{1+\alpha}} \frac{\mu}{D_{\text {in }} c}, \tag{8}
\end{equation*}
$$

the asymptotic value of the velocity of the system, $V_{\infty}$, formally exceeds the speed of light, $c$. Clearly, a relativistic approach to the problem is to be used in this situation.

## 3. Relativistic treatment

### 3.1 Derivation of dynamical equations

In order to keep our study as simple as possible, let us consider in the relativistic case only motion along a straight line with increasing $\mathbf{D}(\alpha=1)$. Additionally, in this section we use natural units, setting the speed of light and the gravitational constant to unity. However, unlike the Newtonian case, here we would like to consider particles having a small mass difference: $m_{1,2}= \pm(\mu \pm \Delta \mu)$, where it is assumed that $\mu>0$ and $|\Delta \mu| \ll \mu$.

It is useful to introduce two local frames of reference and the associated coordinate systems: (1) a fixed lab frame with global Lorentzian coordinates ( $x, t$ ), and (2) a local Lorentzian frame instantaneously comoving with the motion of the point $R(t)=\left(x_{1}(t)+x_{2}(t)\right) / 2$, where $x_{1}(t)$ and $x_{2}(t)$ are positions of the particles in the lab frame, with associated Lorentzian coordinates $\left(x^{\mathrm{com}}, t^{\mathrm{com}}\right)$. It is assumed that at some particular moment of time $t=t_{*}$, the coordinates of the event $\left(t_{*}, R\left(t=t_{*}\right)\right)$ in the comoving coordinate system are equal to $(\tau, 0)$, where $\tau$ is the proper time associated with the world line $(t, R(t))$. Hereafter, the world line $(t, R(t))$ is referred to as the 'reference world line'.

When $t^{\text {com }}=\tau$, the positions of particles are given by $x_{1,2}^{\mathrm{com}}\left(t^{\mathrm{com}}\right)$, and their velocities are $v_{1,2}^{\mathrm{com}}=\mathrm{d} x_{1,2}^{\mathrm{com}} / \mathrm{d} t^{\mathrm{com}}$. Let us also introduce the relative position and velocity in the comoving coordinate system: $D=x_{1}^{\text {com }}-x_{2}^{\text {com }}, v^{\mathrm{com}}=$ $\mathrm{d} D / \mathrm{d} t^{\mathrm{com}}$. Without loss of generality, we assume hereafter that $D^{\text {com }}>0$. When the interparticle distance remains sufficiently small along the reference world line, we have $x_{2}^{\text {com }} \approx-x_{1}^{\text {com }}$.

In the global coordinates at the time slice $t=t_{*}$, the velocity of motion of the system as a whole is given by $V=(1 / 2)\left(\mathrm{d} x_{1} / \mathrm{d} t+\mathrm{d} x_{2} / \mathrm{d} t\right)\left(t=t_{*}\right)$, while the relative position and velocity of the relative motion are $D_{\text {lab }}=$ $x_{1}\left(t_{*}\right)-x_{2}\left(t_{*}\right)$ and $v=\mathrm{d} D / \mathrm{d} t$.

Introducing the Lorentz gamma factor $\gamma=1 / \sqrt{1-V^{2}}$ associated with the reference world line, we may write in the limit of small separations:

$$
\begin{equation*}
D\left(t^{\mathrm{com}}\right)=\gamma D_{\mathrm{lab}}, \quad \frac{\mathrm{~d} t}{\mathrm{~d} t^{\mathrm{com}}}=\gamma \tag{9}
\end{equation*}
$$

and, accordingly, for the velocity

$$
\begin{equation*}
v^{\mathrm{com}}=\gamma \frac{\mathrm{d}}{\mathrm{~d} t}\left(\gamma D_{\mathrm{lab}}\right)=\gamma^{2} v+\gamma \frac{\mathrm{d} \gamma}{\mathrm{~d} t} D_{\mathrm{lab}} \tag{10}
\end{equation*}
$$

Supposing below that, on the one hand, the relative distance $D \gg \mu$, and, therefore, a weak-field approximation holds and, on the other hand, it is not too large for the local Lorentzian coordinates to be adequate and, respectively, for equations (9), (10) to be valid, we can use the Newtonian expression for the energy, $E_{\mathrm{c}}$, and momentum, $P_{\mathrm{c}}$, of the
system in the comoving frame at the instant of time $t^{\mathrm{com}}=\tau$ :

$$
\begin{equation*}
E_{\mathrm{c}}=2 \Delta \mu+\frac{\mu^{2}}{D}, \quad P_{\mathrm{c}}=\mu \dot{D} \tag{11}
\end{equation*}
$$

where the dot stands for differentiation with respect to the proper time $\tau$.

In the same limit, $E_{\mathrm{c}}$ and $P_{\mathrm{c}}$ represent the time and spacial components of a local four-vector, and, therefore, their values in the lab frame, $E$ and $P$, respectively, can be obtained from formula (11) by the standard Lorentz transformation. We have

$$
\begin{align*}
& E=\gamma\left(2 \Delta \mu+\frac{\mu^{2}}{D}+V \mu \dot{D}\right),  \tag{12}\\
& P=\gamma\left[\mu \dot{D}+V\left(2 \Delta \mu+\frac{\mu^{2}}{D}\right)\right],
\end{align*}
$$

where it is assumed that the velocity of the systems as a whole, $V$, is a function of the proper time $\tau$. Since energy and momentum in the lab frame are obviously conserved, equations (12) fully describe the dynamics of our system. They should be solved subject to the condition that the system is initially at rest with respect to the lab frame: when $\tau=0$, we have $V=0$, and

$$
\begin{equation*}
E=E_{\mathrm{in}}=2 \Delta \mu+\frac{\mu^{2}}{D_{\mathrm{in}}}, \quad P=P_{\mathrm{in}}=\mu v_{\mathrm{in}} \tag{13}
\end{equation*}
$$

where $D_{\text {in }}$ and $v_{\text {in }}$ are the initial interparticle distance and the relative velocity, respectively. It is assumed below that $v_{\text {in }}>0$.

Although our derivation of dynamical equations (12) may look somewhat heuristic, it is worth mentioning that when terms next to the leading order in $\mu$ are discarded they can be derived from the precise weak-field equations of reference [4] in the limit of small separations, and $|\Delta \mu| \ll \mu$.

It is convenient to transform equations (12) into another form using their linear combination $E-V P$ and calculating the square of the norm of the energy-momentum vector, $N^{2}=E^{2}-P^{2}$. Then, we get

$$
\begin{align*}
& E-V P=\gamma^{-1}\left(2 \Delta \mu+\frac{\mu^{2}}{D}\right)  \tag{14}\\
& N^{2}=\left(2 \Delta \mu+\frac{\mu^{2}}{D}\right)^{2}-\mu^{2}(\dot{D})^{2} \tag{15}
\end{align*}
$$

We also obviously have $N^{2}=\left(2 \Delta \mu+\mu^{2} / D_{\text {in }}\right)^{2}-\mu^{2} v_{\text {in }}^{2}$. Note that, contrary to the usual situation, the energy-momentum vector can be null, time-like, or space-like, depending on initial conditions.

Equations (12) are first-order integrals of two dynamical equations of second order in time. One of these equations can be obtained from formula (15) by differentiating it over $\tau$ with the result

$$
\begin{equation*}
\ddot{D}=-\frac{2 \Delta \mu}{D^{2}}-\frac{\mu^{2}}{D^{3}} \tag{16}
\end{equation*}
$$

and the second one by differentiating either of equations (12) and using equation (16):

$$
\begin{equation*}
\gamma^{2} \dot{V}=\frac{\mu}{D^{2}} \tag{17}
\end{equation*}
$$

Equations (16) and (17) can be obtained from other independent qualitative arguments. The derivation of the second-order dynamical equations, which relate dynamical variables with different values of time coordinates, is not, however, convenient in the local Lorentzian coordinates introduced above, since these coordinates are defined with respect to some particular event on the reference world line and, therefore, the definition is different for different events along this world line. It is much more convenient to use a coordinate system where the proper time $\tau$ plays the role of coordinate time. To do so, let us consider another one, the socalled local Fermi-Walker coordinate system $(\tau, y)$ (see, e.g., monograph [5]), where the proper time $\tau$ is the coordinate time, and the unit vector in the spatial direction $y$ is always perpendicular to the four-velocity along the reference world line. The coordinates of the reference world line in this coordinate system are simply $(\tau, 0)$.

At the time slice $t^{\text {com }}=\tau$, the local Lorentz coordinates and the Fermi-Walker coordinates coincide: $x_{1,2}^{\text {com }}=y_{1,2}$, but the Fermi-Walker coordinate system accelerates with respect to the local Lorentz coordinate system with an acceleration $g(y)$. Clearly, $g(y=0)$ must coincide with the modulus of four-acceleration of the reference world line with respect to the lab frame. The equations of motion in the Fermi-Walker coordinates are assumed to be determined by the Newton's law (1) with an added acceleration term $-g$, which accounts for the fact that this coordinate system is not inertial:

$$
\begin{equation*}
\ddot{y}_{1,2}=\frac{\mu \mp \Delta \mu}{D_{\mathrm{FW}}^{2}}-g, \tag{18}
\end{equation*}
$$

where $D_{\mathrm{FW}}=y_{1}-y_{2}$, and we took into account that the acceleration term depends on the coordinate $y$ (see, e.g., Ref. [5]): $g=a+a^{2} y$. For the average distance $Y=$ $\left(y_{1}+y_{2}\right) / 2$ to be at rest, $Y(\tau)=0$, the acceleration term $a$ must be balanced by the gravity term $\mu / D_{\mathrm{FW}}^{2}$ :

$$
\begin{equation*}
a=\frac{\mu}{D_{\mathrm{FW}}^{2}} . \tag{19}
\end{equation*}
$$

Taking into account that in the lab frame the spatial coordinate of four-acceleration is related to $a$ as $a^{x}=\gamma a$, we arrive at

$$
\begin{equation*}
\dot{U}^{x}=\gamma^{3} \dot{V}=\gamma a, \quad \dot{V}=\frac{\mu}{\gamma^{2} D_{\mathrm{FW}}^{2}} . \tag{20}
\end{equation*}
$$

It is clear that the last relationship coincides with equation (17).

The dynamical equation for the relative distance $D_{\text {FW }}$ directly follows from expressions (18) and (19):

$$
\begin{equation*}
\ddot{D}_{\mathrm{FW}}=-\frac{2 \Delta \mu}{D_{\mathrm{FW}}^{2}}-a^{2} D_{\mathrm{FW}}=-\frac{2 \Delta \mu}{D_{\mathrm{FW}}^{2}}-\frac{\mu^{2}}{D_{\mathrm{FW}}^{3}} . \tag{21}
\end{equation*}
$$

It coincides with equation (16).
The last term on the right-hand side of Eqn (21) is due to the nonuniform acceleration force appearing in the FermiWalker coordinates. Because this term is $\propto \mu^{2}$, technically, it is a post-Newtonian term. Since we are considering the gravitational force in the Newtonian approximation in Eqn (21), it is important to check whether or not postNewtonian corrections to the gravitational force are comparable with the acceleration term in Eqn (21). In fact, as is described in standard textbooks (see, e.g., Ref. [6]), the postNewtonian corrections are either proportional to $\Delta \mu$ or $\dot{y}_{1,2}$.

The mass difference and velocities are assumed to be small and, therefore, the terms in Eqn (21) arising from the postNewtonian corrections appear to be small compared to the terms taken into account.

From Eqn (21) it follows that when the mass difference is negative and $D_{\mathrm{FW}}=2|\Delta \mu|$ the particles are at rest with respect to each other. In this case, the Fermi-Walker coordinate system locally coincides with the Rindler one, and the particles accelerate indefinitely. Thus, unlike the Newtonian case considered in Section 2, particles accelerating indefinitely and at rest with respect to each other must have a small mass difference. This effect was first noted by Bondi [1] in 1957. It is obviously due to the nonuniform character of the acceleration term.

### 3.2 Solution of dynamical equations

Since equation (14) contains only $V$ and $D$, it can be employed to express $V$ in terms of $D$ :

$$
\begin{equation*}
V=\frac{E P}{E_{\mathrm{c}}^{2}+P^{2}}\left(1 \mp \frac{E_{\mathrm{c}}}{E P} \sqrt{E_{\mathrm{c}}^{2}-N^{2}}\right) \tag{22}
\end{equation*}
$$

where $E_{\mathrm{c}}$ is expressed through $D$ in equation (11), and $E$ and $P$ are given in equation (13). As discussed above, we assume that at the initial moment of time $t=\tau=0$ we have $V=0$. This means that initially we have to choose the $(-)$ sign in formula (22). However, under certain conditions discussed below the direction of motion of the particles relative to each other and, accordingly, $\dot{D}$, changes sign. At the turning point $\dot{D}=0$, we have $N^{2}=E_{\mathrm{c}}^{2}$. Since velocity $V$ must grow monotonically according to equation (17), we must take the $(+)$ sign in formula (22) beyond the turning point.

On the other hand, equation (15) contains only $D$ and its derivative with respect to time $\tau$, and, therefore, it can be integrated to yield the dependence of $D$ on time. Explicitly, we have

$$
\begin{equation*}
\int_{D_{\min }}^{D} \frac{x \mathrm{~d} x}{\sqrt{R(x)}}=\frac{\tau}{\mu} \tag{23}
\end{equation*}
$$

where

$$
\begin{equation*}
R(x)=\left(\mu^{2}+2 \Delta \mu x\right)^{2}-N^{2} x^{2} \tag{24}
\end{equation*}
$$

The integral in formula (23) can be evaluated by a standard substitution to give an explicit relation between $\tau$ and $D$. However, the final expressions are rather cumbersome, and we do not present them here. Instead, in general, we analyze qualitatively solutions to equation (15) based on an analogy between this equation and one describing the motion of a particle in a potential well.

For that, we put equation (15) into a standard form:

$$
\begin{equation*}
\frac{\dot{D}^{2}}{2}+U(D)=\mathcal{E}, \quad U(D)=-\frac{2 \Delta \mu}{D}-\frac{\mu^{2}}{2 D^{2}} \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{E}=\frac{4 \Delta \mu^{2}-N^{2}}{2 \mu^{2}}=\frac{v_{\mathrm{in}}^{2}}{2}-\frac{2 \Delta \mu}{D_{\mathrm{in}}}-\frac{\mu^{2}}{2 D_{\mathrm{in}}^{2}} \tag{26}
\end{equation*}
$$

Introducing natural units $\tilde{U}=\left(\mu^{2} / \Delta \mu^{2}\right) U$ and $\tilde{D}=$ $\left(|\Delta \mu| / \mu_{\tilde{\nu}}^{2}\right) D$, we can express $\tilde{U}$ in terms of $\tilde{D}$ in a very simple form: $\tilde{U}=\mp 2 / \tilde{D}-1 /\left(2 \tilde{D}^{2}\right)$, where the sign $-(+)$ corre-


Figure 1. The dependence of the potential $U$ on the spatial coordinate $D$. The solid curve corresponds to the case of $\Delta \mu>0$, while the dashed one is for the case of $\Delta \mu<0$.
sponds to $\Delta \mu>0(\Delta \mu<0)$. The dependence $\tilde{U}(\tilde{D})$ is shown in Fig. 1.

First, let us consider in detail an important case of the zero norm of the energy-momentum vector, $N^{2}=0$, and set, accordingly, $P=E$. A simple analysis of equation (22) shows that, in this case, there are no turning points, the relative separation $D$ grows with time, and the value of $V=1$ can be achieved in the asymptotic limit, $\tau \rightarrow \infty$. Therefore, the system may accelerate in this case indefinitely.

When $N^{2}=0$, equation (22) simplifies to

$$
\begin{equation*}
V=\frac{E^{2}-E_{\mathrm{c}}\left|E_{\mathrm{c}}\right|}{E^{2}+E_{\mathrm{c}}^{2}}, \quad \gamma=\frac{E^{2}+E_{\mathrm{c}}\left|E_{\mathrm{c}}\right|}{2 E E_{\mathrm{c}}}, \tag{27}
\end{equation*}
$$

and from equation (23) we get

$$
\begin{equation*}
\tau=\frac{1}{4 \Delta \mu^{2}}\left(2 \Delta \mu\left(D-D_{\min }\right)-\mu^{2} \log \frac{\mu^{2}+2 \Delta \mu D}{\mu^{2}+2 \Delta \mu D_{\min }}\right) . \tag{28}
\end{equation*}
$$

From equation (27) it follows that when $E_{\mathrm{c}}>0$, indefinite acceleration is possible only if $E_{\mathrm{c}} \rightarrow 0$ when $\tau \rightarrow \infty$, and from the expression (11) for $E_{\mathrm{c}}$ it is seen that the mass difference $\Delta \mu$ must be then negative. We consider below only this case in detail. When $|\Delta \mu| \neq 0, E_{\mathrm{c}} \rightarrow 0$ provided that $D \rightarrow D_{\text {crit }}=\mu^{2} /(2|\Delta \mu|)$. Equation (28) tells us that the logarithm on the right-hand side diverges, when $D \rightarrow D_{\text {crit }}$. That means that this limit does correspond to the limit $\tau \rightarrow \infty$. Let us estimate the dependence of the Lorentz factor $\gamma$ on time in this case.

To do so, let us introduce a new variable, $\Delta=D_{\text {crit }}-D$, and substitute it into expression (28) assuming that this variable is small. Then, we get

$$
\begin{equation*}
\tau \approx \frac{\mu^{2}}{4 \Delta \mu^{2}} \log \frac{\mu^{2}-2|\Delta \mu| D_{\mathrm{in}}}{2|\Delta \mu| \Delta} \tag{29}
\end{equation*}
$$

and, substituting this result into equation (27), we obtain

$$
\begin{equation*}
\gamma \approx \frac{\mu^{2}}{4|\Delta \mu| D_{\mathrm{in}}} \exp \frac{4 \Delta \mu^{2}}{\mu^{2}} \tau \tag{30}
\end{equation*}
$$

Equation (30) tells us that, when $D \approx D_{\text {crit }}$, acceleration is exponentially fast.

The degenerate case $\Delta \mu=0$ must be analyzed separately. In this case, from expression (28) we have

$$
\begin{equation*}
\tau=\frac{1}{2 \mu^{2}}\left(D^{2}-D_{\min }^{2}\right) \tag{31}
\end{equation*}
$$

and the distance $D$ increases indefinitely with time. From equation (27) it follows that

$$
\begin{equation*}
\gamma \approx \frac{\mu}{2 D_{\min }} \sqrt{2 \tau} \tag{32}
\end{equation*}
$$

Now let us turn to the general case of $N^{2} \neq 0$. Setting $\dot{D}=0$ in the first equation (25), we get a general equation for the turning points:
$D_{1,2}=\frac{\Delta \mu}{\mathcal{E}}\left(-1 \pm \sqrt{1-\frac{\mu^{2} \mathcal{E}}{2 \Delta \mu^{2}}}\right)=\frac{\Delta \mu}{\mathcal{E}}\left(-1 \pm \frac{\sqrt{N^{2}}}{2 \Delta \mu}\right)$.
Equation (33) shows that the turning points exist only when $N^{2}>0$. Their number depends on the signs of $\mathcal{E}$ and $\Delta \mu$. When $\Delta \mu>0$, the potential $U(D)$ is negative (see Fig. 1), and, therefore, the relative motion is finite for $\mathcal{E}<0$ with one turning point: ${ }^{3}$

$$
\begin{equation*}
D_{1}=\frac{\Delta \mu}{|\mathcal{E}|}\left(1+\frac{N}{2 \Delta \mu}\right) \tag{34}
\end{equation*}
$$

In the opposite case, $\mathcal{E}>0$, and, accordingly, $N<2 \Delta \mu$, the motion is unbound, and the relative distance $D$ grows indefinitely with time.

When $\Delta \mu<0$, the potential $U(D)$ acquires positive values for $D>\mu^{2} /(4|\Delta \mu|)$ (see Fig. 1). It tends to zero when $D \rightarrow \infty$ and has a maximum at $D=D_{\text {crit }}$. Note that, from the condition $U\left(D_{\text {crit }}\right)=\mathcal{E}=2 \Delta \mu^{2} / \mu^{2}$, we get there $N^{2}=0$. The character of the relative motion depends on whether $\mathcal{E}$ is negative, belongs to the interval $0<\mathcal{E}<2 \Delta \mu^{2} / \mu^{2}$ corresponding to $0<N<2|\Delta \mu|$, or $\mathcal{E}>2 \Delta \mu^{2} / \mu^{2}$ and, accordingly, $N^{2}<0$. When the energy $\mathcal{E}$ is negative, the motion is bound with one turning point:

$$
\begin{equation*}
D_{1}=\frac{|\Delta \mu|}{|\mathcal{E}|}\left(-1+\frac{N}{2 \Delta \mu}\right) . \tag{35}
\end{equation*}
$$

In the intermediate region, $0<\mathcal{E}<2 \Delta \mu^{2} / \mu^{2}$, there are two turning points

$$
\begin{equation*}
D_{ \pm}=\frac{|\Delta \mu|}{\mathcal{E}}\left(1 \pm \frac{N}{2 \Delta \mu}\right) \tag{36}
\end{equation*}
$$

When $D_{\text {in }}<D_{-}$the motion is bound, while for $D_{\text {in }}>D_{+}, D$ grows indefinitely. Finally, when $N^{2}<0$ the motion is always unbound.

When the motion is bound, the velocity $\dot{D}$ changes sign after passing the turning point and, in this case, we should use the $(+)$ sign in formula (22). Taking into account that $D$ decreases beyond the turning point and that $E_{\mathrm{c}} \propto D^{-1}$, we see from formula (22) that the velocity $V \rightarrow 1$. Then, the particles tend to collide. However, our assumption that $D \gg \mu$ breaks down in this case, and we cannot describe the motion on scales $D \sim \mu$ within the framework of our formalism. Note that we consider in this study only pairs of particles with

[^2]strictly zero angular momentum. In the situation where the particles have a small but nonzero angular momentum, they would miss each other, and at a certain moment of time the distance $D$ would become negative. In this case, the analysis given in this paper can be repeated without any major change for negative values of $D$, and one would conclude that for such parameters of motion there is another symmetric turning point at negative values of $D$. Thus, the relative motion of a pair of particles with small but nonzero orbital momentum would be periodic, very similarly to the case of ordinary particles with positive masses.

Now let us consider the case of unbound motion and estimate the maximal value of the Lorentz gamma factor the system can reach. As follows from our previous discussion, when $N^{2} \neq 0$, the distance $D$ grows indefinitely. This means that the energy in the comoving frame of reference, $E_{\mathrm{c}}$, must tend asymptotically to $2 \Delta \mu$. Note that when $\Delta \mu<0$, the asymptotic value of $E_{\mathrm{c}}$ is also negative. We have from formula (22), setting there $E_{\mathrm{c}}=2 \Delta \mu$, the following relationship

$$
\begin{equation*}
V=\frac{1}{4 \Delta \mu^{2}+E^{2}-N^{2}}\left(E \sqrt{E^{2}-N^{2}}-2 \Delta \mu \sqrt{4 \Delta \mu^{2}-N^{2}}\right) . \tag{37}
\end{equation*}
$$

Equation (37) shows that when $\Delta \mu>0$, the last term in the parentheses is negative, and the asymptotic value of velocity is smaller than 1. Large values of $V$ can be achieved in the opposite case, $\Delta \mu<0$, assuming $\left|N^{2}\right|<\Delta \mu^{2}$. In this case, we expand expressions in Eqn (37) in the Taylor series in $\left|N^{2}\right| / \Delta \mu^{2}$ to obtain

$$
\begin{equation*}
V=1-\frac{N^{4}}{32 \Delta \mu^{2} E^{2}}, \tag{38}
\end{equation*}
$$

and, accordingly, one finds

$$
\begin{equation*}
\gamma \approx \frac{1}{\sqrt{2(1-V)}}=\frac{4|\Delta \mu E|}{\left|N^{2}\right|} . \tag{39}
\end{equation*}
$$

Equation (39) tells us that, for fixed values of $E$ and $\Delta \mu<0$, the gamma factor can be made arbitrarily large by choosing arbitrarily small values of $\left|N^{2}\right|$. This conclusion is in agreement with our previous finding that the system accelerates indefinitely, when $N^{2}=0$.

## 4. Methodological comments

Here, I would like to comment on several methodological issues related to the problem.
(1) At first glance, the fact that the 'average' position of the pair $\left(x_{1}+x_{2}\right) / 2$ always grows with time may seem to contradict the law of conservation of the center of mass of the system. This contradiction is resolved by the observation that, for a system containing particles of opposite masses, the position of the center of mass, $R$, is determined by a difference of positions of particular particles. For example, in the Newtonian approximation, we have $R=$ $m_{1} x_{1}+m_{2} x_{2}=(\mu+\Delta \mu) x_{1}-(\mu-\Delta \mu) x_{2}$. In the relativistic case, the situation is analogous for systems with $N^{2}>0$. In the opposite case, the notion of center of masses is ill defined. Indeed, introducing the velocity of a coordinate system, where the center of mass is at rest, in a standard way as $V_{\mathrm{cm}}=P / E[6]$, we see that, when $N^{2}=0, V_{\mathrm{cm}}=1$, and when
$N^{2}<0, V_{\mathrm{cm}}$ formally exceeds the speed of light. It is obvious that the notion of the center of mass is redundant in both cases.
(2) In the Introduction to their paper, the authors of review [3] claim that the conception of PMT put forward by the author of this paper is related to the problem of indefinite acceleration of two gravitationally interacting particles. This statement needs, in my opinion, clarification. First, let me note that, as discussed above, even when only a finite acceleration of particles is attained, PMT is still possible in a situation where production of such pairs is provided by some physical mechanism. Second, the conception of PMT, in general, does not rely on gravitation interactions at all. In particular, I considered in paper [2] a model where there is a continuous flow of positive energy from some spatial regions having negative energy to other regions with positive energy provided by hydrodynamical effects. In this model, spacetime is assumed to be flat and gravitational interactions are absent. Moreover, in order to construct a PMT, it is not necessary to invoke objects having negative rest masses; it is enough to consider a medium with positive comoving energy density violating the weak energy condition [2]. Additionally, there are ways of constructing a PMT, where gravitational interaction plays a totally different role, say, transferring the energy from a nonstationary system having negative mass to gravitational waves, as, for example, in the model of a rotating relativistic string connected by two negative mass monopoles [2, 7]. The effects related to the dynamics of free negative mass particles are clearly irrelevant to such systems.
(3) The authors of Ref. [3] claim that it is impossible to obtain, in principle, an indefinite acceleration of a system containing two particles with masses of opposite signs. One may think that this clearly contradicts Bondi's result [1] and the conclusions of this paper. The conundrum is resolved by the observation that the authors of Ref. [3] consider only relative motions, while Bondi's analysis, as well as that presented in this paper, also deal with the motion of the pair of particles as a whole with respect to an inertial frame of reference.

## 5. Conclusions

In this paper, we show by elementary means that in the weak limit approximation a pair of particles having opposite values of masses can be accelerated indefinitely provided that the energy-momentum vector characterizing the system is null. The system can also be accelerated to arbitrarily large Lorentz factors when the mass difference $\Delta \mu<0$ and the norm of the energy-momentum vector is sufficiently small.

Assuming that there is a process of production of such pairs and that the positive mass particles are intercepted by a target, while the negative mass particles are flying away, it is possible to transfer any desired amount of energy to the target. In a more natural situation, one can also consider a theory where the positive and negative mass particles interact differently with conventional matter. A general situation of this kind where there is a persistent transfer of energy from a subsystem with negative or almost zero energy (like this pair of particles) to a subsystem with positive energy was dubbed by us 'perpetual motion of the third kind' (PMT) [2]. Note, however, that it is just a classical analog of the well-known instability of a quantum system with a number of negative energy states unbound from below.

The question of whether the existence of PMT or constantly accelerating pairs of particles is a paradox depends, in our opinion, on the definition of what a paradox is. On the one hand, for example, W B Bonnor stated in 1989: "I regard runaway (or self-accelerating ) motion... as so preposterous that I prefer to rule it out by supposing that inertial mass is all positive or all negative" [8]. Clearly, the existence of PMT can also be classified as a kind of runaway. On the other hand, no laws of physics are broken in such systems. We believe that the existence of runaways of these kinds in theories is dangerous for them. To exemplify, an indefinite concentration of energy of different signs in spatially separated regions could lead to a highly inhomogeneous space-time hardly compatible with the presence of any life. Therefore, such theories should be ruled out, though some additional study of them in General Relativity may be of a certain interest.

Since in our approximation only linear metric perturbations and one next-to-the-leading order term determined by the acceleration of the pair as a whole are taken into account, it is interesting to estimate what kind of corrections can be obtained by considering other higher-order terms quadratic in metric perturbations. For a nonrelativistic motion with $V \ll c$, one can use for this purpose the well-known Einstein-Infeld-Hoffmann equations of motion (see, e.g., book [6]). In this way, it is convenient to consider particles with a large mass difference, as well as systems with nonzero angular momentum. There are, however, many corrections, which are absent in such a treatment, notably the emission of gravitational waves. Therefore, a self-consistent relativistic treatment of the problem in the next-to-the-weak-field approximation must be based on the second-order formalism of Havas and Goldberg from 1962. Such an approach is left for possible future work.

Although we consider in this paper only particles with no internal structure, our analysis may also be valid for a pair of extended objects with total energies of opposite signs, provided that they have a sufficiently large interparticle distance and that their relative velocities are sufficiently small. For example, Deser and Pirani [9] considered the behavior of systems with all possible inertial/gravitational mass signs and noted that a pair of geons having opposite signs of their total energies would behave as a pair of point particles in the appropriate limit.

It is also interesting to point out that the notion of 'perpetual motion of the third kind' was introduced in the context of thermodynamical systems having negative temperatures, where one can withdraw heat from a negative temperature reservoir and convert it completely to work (see, e.g., book [10], p. 176). Since thermodynamical systems with negative masses of their components should have negative temperatures (see, e.g., paper [11]), there is a link between the thermodynamical properties of such systems and the ones discussed in this paper. In particular, a runaway process occurring in a thermodynamical system having two subsystems involving particles with masses of opposite signs has been discussed in paper [12]. It has been mentioned that this process is analogous to the self-acceleration of a pair of particles with opposites signs of their masses.

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[^1]:    ${ }^{1}$ Note that it is easy to show that the same motion can be realized in a system containing $N$ particles provided that the total mass of the system $M=\sum_{i=1}^{i=N} m_{i}=0$ and the positions of the particles are chosen in a special way. For instance, for a system containing three particles, their relative positions must form an equilateral triangle.
    ${ }^{2}$ Clearly, the particles collide when $\mathbf{D}_{\text {in }} \mathbf{v}_{\text {in }}=-\left|\mathbf{v}_{\text {in }}\right|\left|\mathbf{D}_{\text {in }}\right|$.

[^2]:    ${ }^{3}$ Let us remember that we are considering only positive values of $D$.

