METHODOLOGICAL NOTES

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On special relativity teaching using modern experimental data

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Contents

Abstract. A new methodological approach to the study of relativistic space-time properties is proposed based on the 20th century's vast experimental research on relativistic accelerator and cosmic ray particles. This approach vividly demonstrates that relativistic effects are the manifestation of fundamental space-time properties and refutes the false notion that relativity is only of relevance to light phenomena.

1. Introduction

The special theory of relativity (STR), created more than a hundred years ago, is one of the most difficult areas of physics to comprehend. Difficulties in perceiving the main ideas of spacetime, the kinematics and dynamics of motion with velocities close to the speed of light, seem to be related to the absence of any educational experimental base and of possibilities to observe motions with such velocities. Ordinary daily experience gives rise to doubt in the authenticity of relativistic effects and even causes aversion to them.

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Traditional STR exposition is based on the postulate of the speed of light being constant and on the relativity principle. Such an approach was the sole argument in favor of STR presented by its creators at the beginning of the 20th century. Therefore, the first encounter with STR often results in the false notion that relativistic effects are exclusively related to light signals, since the speed of light serves as a fundamental quantity in the Lorentz formulae. The situation is also aggravated by abstract clocks and rulers, traveling at near-light velocities, being 'used' in the exposition of STR fundamentals and by frequent reference to thought experiments that cannot be performed in real-life conditions.

In this initial familiarization with the fundamentals of relativity, it is not quite clear either how to imagine a clock traveling with a very high velocity, comparable to the velocity limit. Moreover, such a clock covers a typical laboratory distance $l \approx 1$ m in $t = l/c \approx 3$ ns, so the possibility of measuring time intervals on the order of 1 ns must be ensured.

As to the Lorentz contraction of 'ruler' lengths (by the way, what do such rulers look like?), it is better not to even mention it. Readers perceive with a considerable degree of suspicion the picture in the remarkable book [1] by E F Taylor and J A Wheeler showing a runner with a pole 20 meters long, which is kept in a 10-meter long shed.

From time to time, the discoveries are announced of objects traveling with velocities exceeding the conventionally adopted speed of light in a vacuum. As usual, 'scientific studies' are issued, the authors of which reveal 'errors' in the calculation of interferograms in Michelson–Morley experiments and provide 'proofs' of the existence of ether and, thus, of STR not being valid.

Even quite recently (only some 25–30 years ago), discussions were under way concerning the fundamental concepts of 'mass' and 'energy' in relativity theory. To our knowledge, the most consistent and logical interpretation of these concepts is given by L B Okun in Ref. [2].

Long-term teaching experience at the Moscow State University Department of Physics has profoundly convinced the author of this note that the exposition of STR fundamentals requires a novel approach, based on the large amount of experimental results accumulated in the 20th century and in no way related to measurement of the speed of light.

These are, first of all, experiments performed with relativistic particles from accelerators and with cosmic rays [3]. The evidential base for proving the validity of STR will also certainly be reinforced by experiments involving atomic clocks on board cosmic vehicles, in particular, satellites used in GPS (Global Positioning System) and GLONASS (Global Navigation Satellite System), including recent experiments on direct measurement of the propagation velocity of synchrotron radiation pulses [4] and for establishing the independence of the speed of light from the velocity of its source [5]. Within this approach, the speed of light serves as a fundamental physical constant defining the propagation velocity limit of an interaction.

The purpose of this article is to describe a new methodological approach to studying spacetime properties within STR. Since the issues of relativistic dynamics are expounded well in numerous textbooks and tutorials, including the aforementioned review by L B Okun [2], it makes sense to concentrate on the essentials of time and coordinate measurements and to deal only with kinematics in the case of motion with relativistic velocities.

In writing this note, no attempt was made to totally avoid repeating well-known facts and methodological approaches. Different methods complement each other and only facilitate a deeper understanding of the STR foundations. The material presented in the article is adapted for a very broad audience of readers.

2. Lorentz transformations

The history of STR creation deserves a brief digression.

In 1898, a philosophical journal published an article by Jules Henri Poincaré, "The measurement of time" [6], dealing with issues of simultaneity. This work contained practically all the main STR stipulations. Hermann Minkowski, a close friend of Poincaré and Professor at the Zurich Technological Institute, was familiar with this article, and he recommended his student, Albert Einstein, to study the work. Nearly 7 years later, in 1905, Einstein published the article "On the electrodynamics of moving bodies" [7], in which STR was presented in its modern form. However, no reference to the work of Poincaré was presented.

As is known, Hendrik Antoon Lorentz tried to find such transformations of coordinates and time, under which the Maxwell equations would be invariant in the case of transition from one inertial reference system (IRS) to another. He derived these transformations in 1904. While delivering lectures to his students, Minkowski found a mistake in them and corrected it, and in publishing a synopsis of his lectures he called them Lorentz transformations.

Lorentz had established that in transition to a frame of reference K′, moving with respect to a frame of reference K with a velocity V along the x-axis (the other axes are aligned

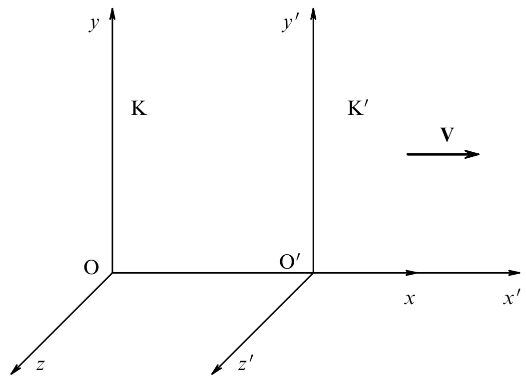

Figure 1. Motion of reference system K′ with velocity V with respect to reference system K along the x-axis.

similarly) (Fig. 1), the coordinates and time of an event change in accordance with the following relationships:

$$\begin{aligned} x' &= \gamma(x - Vt)\,, \\ y' &= y\,, \\ z' &= z\,, \\ t' &= \gamma\left(t - \frac{V}{c^2}\,x\right). \end{aligned} \tag{1}$$

Later on, in 1905, Poincaré called these relations Lorentz transformations and demonstrated their fundamental character. Then, in 1905, Einstein derived them from the constancy postulate of the speed of light and from the relativity principle.

The scale factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2}$$

was termed the Lorentz factor. Here, $\beta = V/c$. Evidently, at small velocities ($V \ll c$) and small coordinate values ($x \ll tc^2/V$) formulae (1) turn into Galilean transformations.

At relativistic velocities ($V \leqslant c$), Lorentz transformations differ essentially from Galilean transformations and result in absolutely new effects. Thus, for example, the factor γ being present in them 'gives rise' to a slowing down of time and contraction of length, while the term $(V/c^2)\,x$ is responsible for a violation of synchronism in the rates of clocks situated at different points in space.

If expressions (1) are resolved with respect to the variables x, y, z, t, the inverse Lorentz transformations are obtained:

$$\begin{aligned} x &= \gamma(x' + Vt')\,, \\ y &= y'\,, \\ z &= z'\,, \\ t &= \gamma\left(t' + \frac{V}{c^2}\,x'\right). \end{aligned} \tag{3}$$

The inverse transformations can also be obtained from the direct ones taking advantage of the relativity principle, according to which the velocity of the frame of reference K with respect to the K′ frame is $-V$. Therefore, if the sign of the velocity V in Eqn (1) is changed and the variables in the left-hand parts of all the equalities of the set are stripped of their primes, which are, then, assigned to the variables in the right-hand parts, one obtains relations (3). From expressions (1) it is seen that, if $V = 0$, then $\gamma = 1$, and that, when $V \to c$, the factor $\gamma \to \infty$. It is also clear that the velocity V cannot equal the speed of light c, or exceed it, since in this case either the denominator turns to zero or a negative number will appear under the root sign.

From a physical point of view, the condition $V < c$ signifies that no objects have been revealed (and a moving reference system can only be attached to such objects) with velocities equal to or exceeding the speed of light in a vacuum. A propagation speed limit of interactions also exists.

3. A limit speed of propagation of interactions

On the basis of the results of experiments performed in the 20th century, it is not at all difficult to arrive at the conclusion that there is a finite speed of propagation of interactions. Indeed, in all accelerators, when the energy of charged elementary particles increases, their velocity tends toward the limit $c \approx 3 \times 10^8$ m s^{-1}. Nobody has ever observed particles with velocities exceeding this limit. For example, electrons in SLAC (Stanford Linear Accelerator Center) (USA) move in an electric field across a region under a difference of potentials $\Delta U = 2.3 \times 10^{10}$ V. According to Newton's classical mechanics, the velocity of such electrons at the end of their acceleration path should amount to $v = 9.9 \times 10^{10}$ m s^{-1}, i.e., about 300 (!) times more than the aforementioned limit. However, the actual velocity of electrons at the end of their acceleration path is 0.075 m s^{-1} less than the limit c.

In cosmic rays, particles, presumably protons, are encountered that have energies $E \approx 5$ J. The velocity of such particles, estimated indirectly, is also inferior to the limit c, contrary to the fantastically large computed value of $v = \sqrt{2E/m} \approx 2.4 \times 10^{14}$ m s^{-1}(!).

Note that the limit velocity of particles, c, coincides with the speed of light (of a flux of photons) in a vacuum. The latter has been measured over and over again, and its present value is conventionally considered to be $c = 299792458$ m s^{-1}.

On the 24 February 1987, astronomers observed a very rare event: a flash of light due to the supernova outburst in the Large Magellanic Cloud, which is a small galaxy close to our Galaxy. Since the Kepler Star (1604), this happened to be the first supernova outburst visible to the naked eye. The distance from the Large Magellanic Cloud is about 1.6×10^{21} m. Evidently, the time that passed after the burst occurred till it was registered on Earth was $t \approx 5.4 \times 10^{12}$ s. In a time $\Delta t \approx 10^4$ s before the burst occurred, a flux was registered of neutrinos, the production of which, according to available theory, preceded the burst itself by approximately the same time. This observation signifies that the velocity c_{ν} of neutrinos equals the speed of light within a relative uncertainty of $|c_{\nu}/c - 1| < 10^{-8}$.

Thus, all modern measurements confirm the conclusion that there is a particle's velocity limit.

In modern physics, fundamental interaction is interpreted as an exchange of appropriate particles. For example, strong interaction between the quarks inside nucleons (protons and neutrons) is realized by an exchange of gluons, the carriers of such an interaction. So-called electroweak interactions are due to an exchange of vector bosons.

If interactions are interpreted as the exchange of particles, the existence of a particle's velocity limit c also imposes a restriction on the speed of propagation of interactions. Consequently, the following assertion is absolutely natural: *the speed of propagation of fundamental interaction* or, in conventional wording, *the signal velocity does not exceed the limit* $c \approx 3 \times 10^8$ m s^{-1}.

4. Measurement of time intervals

For the measurement of small time intervals, it is possible to make use of the decay phenomenon of nonstable particles (pions, kaons, muons, and others). Such particles traveling with high velocities are produced in accelerators as a result of proton interactions with the nuclei of target atoms.

The main stages in the generation of nonstable particles are the following. First, it is necessary to accelerate protons. To avoid their collision with nuclei of the atoms of air, the protons are accelerated in a vacuum chamber (with a residual pressure of about 10^{-9} Torr) — a pipe about 10 cm in diameter. Since, in a time of about several seconds of acceleration (an acceleration cycle) up to velocities practically indistinguishable from the speed of light, protons cover enormous distances, the pipe must obviously be made circular. The radius of such a ring amounts to several kilometers.

For protons to travel along a circle, they must be under the action of a magnetic field, the value of which during the acceleration process increases up to several tesla. The magnets (most recently made of current-carrying devices using superconducting alloys cooled to liquid-helium temperatures) creating the magnetic field are situated along the perimeter of the ring. Powerful high-frequency electric field sources are installed at certain segments of the ring for accelerating the protons up to energies on the order of 10^{12} eV owing to the protons repeatedly passing through these segments. For this to take place, the protons must happen to be within these particular segments in phase with the high-frequency electric field.

The conditions under which protons are accelerated were formulated by the Russian physicist V I Veksler. These conditions require the protons not to be distributed uniformly along the ring, but to be concentrated in bunches with longitudinal dimensions varying for different accelerators from several meters down to several centimeters, while their transversal dimensions amount to several micrometers. The number of protons in a bunch may amount to several billion, while the number of bunches in a ring varies between several dozen up to several thousand. Special magnets direct the accelerated protons toward the target (Fig. 2).

The interaction of a sole proton with any single nucleus of a target atom results in the production of a large number (10–100 per interaction) of nonstable secondary particles (mainly pions and kaons). These secondary particles are separated by a magnetic field according to their sort and velocity and are

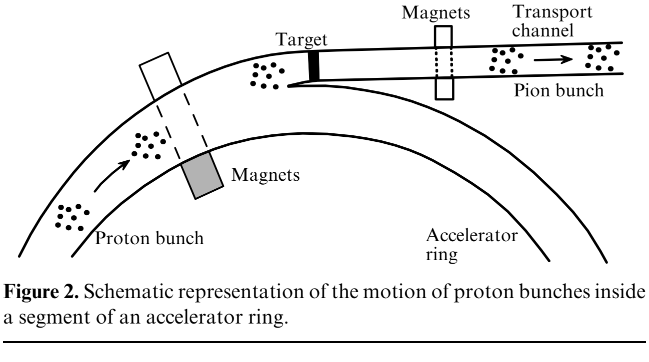

Figure 2. Schematic representation of the motion of proton bunches inside a segment of an accelerator ring.

directed toward different beam transport channels, which in the case of nonstable particles are sometimes called decay channels. Since protons hit the target in bunches, the nonstable particles are also produced in bunches. These bunches travel inside a transport channel with a constant velocity. In the most general terms, such is the formation scheme of a beam (a sequence of bunches) for various nonstable particles.

Thus, for example, in the Large Hadron Collider, bunches are formed, each of which contains a number on the order of 10^8 protons accelerated up to velocities $V = 0.999999991c$. The length of a bunch varies from 3 to 5 cm, and its transverse size is about 16 μm. About 3000 bunches are present in the ring simultaneously, and they undergo 6×10^8 collisions per s. If the channel is considered to be 27 km long, the average distance between bunches amounts to about 90 m.

The particle decay phenomenon can be used for measuring time. Let us associate the frame of reference K′ with the moving particles (particle bunch). We shall call this inertial reference system the proper reference system, and the time t' in it proper time. Time in the laboratory system K, in which the accelerator and the transport channels are at rest, will be denoted by t. The probability $\mathrm{d}P$ for a single particle to decay in time $\mathrm{d}t' \ll \tau$ is determined by the relationship

$$\mathrm{d}P = \frac{\mathrm{d}t'}{\tau}\,, \tag{4}$$

where the constant τ is the average proper lifetime of the particle (in the frame of reference, where it is at rest). This constant can be measured in various experiments, and it can also be calculated on the basis of radioactive decay laws. Its value for different nonstable particles varies within extremely broad limits (from 10^{-16} up to 10^3 s), If we consider a bunch of $N(t')$ nonstable particles, a change in this number by $\mathrm{d}N$ in a time $\mathrm{d}t'$ is determined by the product of the number of particles and the decay probability of a single particle:

$$\mathrm{d}N = -N(t')\,\mathrm{d}P = -N(t')\,\frac{\mathrm{d}t'}{\tau}\,. \tag{5}$$

The minus sign here signifies a decrease in the number of particles. It is easy to integrate equation (5) and to determine the average number $N(t')$ of particles in the bunch at any instant of time t':

$$N(t') = N(0)\exp\left(-\frac{t'}{\tau}\right), \tag{6}$$

where $N(0)$ is the number of particles in the bunch at $t' = 0$.

Measurements of the quantities $N(0)$ and $N(t')$ also permit resolving the inverse problem: to calculate the proper time t' from formula (6):

$$t' = \tau \ln \frac{N(0)}{N(t')}\,. \tag{7}$$

To avoid the significant influence of the fluctuations in random decay phenomena on the accuracy of time measurements, it is necessary to deal with bunches containing a large number N of particles.

Thus, if a known number of particles, $N(0)$, is formed and their number $N(t')$, which decreases owing to decays, is determined, relationship (7) permits measuring the proper time t' in inertial reference system K′, i.e. a bunch of nonstable particles can play the part of a clock moving with a velocity close to the speed of light.

5. Slowing of the rates of moving clocks

Evidently, it is possible to compare the readings of a moving clock at various instants of time with the readings of different clocks at rest. Therefore, the rate of a moving clock can be controlled in the following way.

Let there be, in the laboratory frame of reference K, a set of synchronized clocks at rest, which measure time t in this system (see Fig. 3 in Section 7) and that are situated on the x-axis along the transport channel of the accelerator at equal distances $l_0 = \text{const}$ from each other. We associate the inertial reference system K′ with the bunch of particles moving inside this channel with a constant velocity V. Let the coordinate axes of both reference systems be directed likewise, and let their origins coincide at the instant of time $t = t' = 0$. When the bunch passes the next-in-turn clock at rest number i (we shall call this instant the ith event), the clock will show the time

$$t_i = \frac{L_i}{V}\,, \tag{8}$$

where $L_i = il_0$, $i = 1, 2, 3, \ldots.$

In the reference system K′, associated with the bunch, the times of these very events are t'_i. It is obvious that to determine a concrete value of t'_i from formula (7) measurements are required of the numbers of particles, $N(0)$ and $N(t'_i)$. Thus, knowledge of the readings of laboratory clocks for certain events and measurements of the amounts of particles in a moving bunch at respective moments of time permit determining whether the rate of a moving clock differs from the rates of clocks at rest or not.

In one of the numerous experiments carried out at the European Organization for Nuclear Research (CERN) in Geneva, a beam of $N(0) = 1.5 \times 10^7$ (from the results of measurements) positively charged ultrarelativistic pions, the Lorentz factor of which amounted to $\gamma = 857$, was guided into a transport channel of length $L_0 = 100$ m. The velocity of these pions coincided with c to the sixth decimal place. The time the pions travelled through the transport channel was

$$t = \frac{L_0}{c} = \frac{10^2}{3 \times 10^8}\ \text{s} \approx 3.33 \times 10^{-7}\ \text{s}\,. \tag{9}$$

A pion can decay into a muon and neutrino inside the transport channel. Therefore, from measurements of the number of muons produced, N_μ, it is possible to determine the number of pions that decayed. Measurements revealed $N_\mu = 2.27 \times 10^5$ at the end of the transport channel. Consequently, the number of pions at the end of the channel equaled $N(t') = N(0) - N_\mu = 1.477 \times 10^7$. Since the pion proper lifetime $\tau = 2.56 \times 10^{-8}$ s, we obtain from formula (7) the following estimate for t':

$$t' = \tau \ln \frac{N(0)}{N(t')} = 3.90 \times 10^{-10}\ \text{s}\,. \tag{10}$$

From estimates (9) and (10) we find the following value for the ratio t/t' characterizing a slowing of the rates of moving clocks:

$$\frac{t}{t'} = 854\,, \tag{11}$$

which coincides with the above value of $\gamma = 857$ with an error of 0.35%. Thus, it is believed that

$$\frac{t}{t'} = \gamma\,, \tag{12}$$

or for time intervals

$$\frac{\Delta t}{\Delta t'} = \gamma \,. \tag{13}$$

The latter relationship is often represented in the form

$$\Delta t' = \sqrt{1-\beta^2}\,\Delta t \,. \tag{14}$$

In a great number of experiments performed at accelerators in many countries, for instance, at the largest accelerators in Russia (at the Institute for High Energy Physics in Protvino), in the USA [at the Fermi National Accelerator Laboratory (Fermilab) in Chicago], and at CERN, the length L_0 of the transport channel varied from several meters up to several kilometers, while the factor γ varied from several units up to several thousand. Experiments were performed with different nonstable particles, the proper lifetimes of which changed within very broad range (from 10^{-16} up to 2.2×10^{-6} s). None of the experiments revealed any deviations from dependence (13).

Formula (13) has actually become the basis of engineering calculations in designing transport channels for any nonstable particles. We note that in the high-precision experimental studies of muon decays occurring in a storage ring, performed by F Combley, F Farley, and E Picasso (CERN), relationship (13) was satisfied with a relative error amounting to $(0.9\pm0.4)\times10^{-3}$ [8].

Historically, the first experiments for testing (13) were performed by B Rossi in the USA (1939–1941) [9]. In these experiments, the decrease in the number of muons (at the time, they were called mesotrons) traveling through the atmosphere was studied. The interaction of cosmic rays with atomic nuclei at large altitudes in the atmosphere results in the production of pions. These pions decay quite rapidly into muons and neutrinos. The lifetime proper of muons is relatively small: $\tau\approx2.2\times10^{-6}$ s. If the muon lifetime t_μ measured by laboratory clocks were equal to their proper lifetime, then, in the direction toward Earth, muons would have covered, on the average, a distance

$$\lambda_0 = ct_\mu = c\tau \approx 660 \text{ m}. \tag{15}$$

Assume the number of muons at a certain point in the upper atmosphere to be equal to $N(0)$. Since muons practically travel at the speed of light, the distance l covered by them on the way towards Earth by the instant of time t will amount to

$$l = ct \,. \tag{16}$$

If there were no slowing down of time (i.e., if equality $t'=t$ were satisfied), the following dependence of the number of muons on the distance l would result from formula (6) with account of expressions (15) and (16):

$$N(l) = N(0)\exp\left(-\frac{tc}{\tau c}\right) = N(0)\exp\left(-\frac{l}{\lambda_0}\right), \tag{17}$$

i.e., the number of muons would decrease by a factor of e along a distance $\lambda_0 = 660$ m. If the slowing does take place [see formula (12), then, instead of muon descending dependence (17), we find

$$N(l) = N(0)\exp\left(-\frac{l}{\gamma\lambda_0}\right), \tag{18}$$

i.e., a decrease by e times takes place along a distance $\lambda=\gamma\lambda_0$, which is significantly larger (by a factor of γ) than λ_0. Measurements of the numbers of muons were performed in Chicago (at a height of 180 m above sea level) and in the mountains of Colorado at heights of 1600, 3240, and 4300 m. These measurements revealed a decrease in the number of muons by a factor of e along a distance on the order of several kilometers. From the performed estimation of the muon energy, it followed that the Lorentz factor was $\gamma\approx5$; hence, the distance $\lambda=\gamma\lambda_0\approx3$ km, which is consistent with the results of direct measurements, as presented above.

Modern studies of cosmic rays reveal that formula (13) also holds valid at very large values of the Lorentz factor, $\gamma\approx10^{10}-10^{11}$.

Besides charged pions and other particles, the interactions of primary particles of the cosmic radiation with atomic nuclei high up in the atmosphere result in the production of neutral pions, the proper lifetime of which is very small: $\tau\approx0.8\times10^{-16}$ s. Therefore, such neutral pions decay practically instantaneously. However, at very high cosmic ray energies, the Lorentz factor of these pions can reach the aforementioned values and the distance λ along which their number, owing to decays, decreases by a factor of e becomes larger:

$$\lambda = \gamma\tau c \approx 2.4\times10^3 \text{ m}. \tag{19}$$

In this case, neutral pions have time to undergo interaction with atomic nuclei in the atmosphere. Indirect measurements show that in the region of ultrahigh energies neutral pions, indeed, do not decay, but interact with atomic nuclei, i.e., formula (13) is satisfied also in the case of $\gamma\sim10^{10}$.

After the appearance of precise atomic clocks, the possibility also arose of testing formula (12) for the slowing down of time at a low velocity V, for example, in airplane flights, when $V\approx10^3$ km h^{-1}, $\beta=V/c\approx10^{-6}$, and, correspondingly, γ exceeds unity by a very small value on the order of 10^{-12}.

In the experiments performed by Hafele and Keating [10] on board commercial airliners, there were four sets of cesium atomic clocks. The airplanes flew round the globe twice, first eastward, then westward, upon which the readings were compared of the clocks that had been 'flying' and of the similar clocks that remained at the U.S. Naval Observatory in Washington. The clocks on board the airplane were behind the terrestrial ones by (59 ± 10) ns when the airplane flew eastward, and were fast by (273 ± 70) ns if it flew westward. The error in testing formula (12) amounted to 10%. Theoretical estimates of these quantities were: (40 ± 23) ns and (275 ± 21) ns, respectively. In later experiments performed by Alley et al. [11] in 1980, the error was lowered to 1%.

Thus, the set of experimental data obtained for the interval of values of the Lorentz factor γ from 1 up to 10^{11} permits making the following conclusion: the rate of a sole moving clock compared with the readings of a set of clocks situated at different points in the frame of reference at rest slows by a factor of γ.

Such a slowing of the rate of moving clocks has been termed Lorentzian, since formula (13) follows from the Lorentz transformations. Indeed, a clock moving in the laboratory frame has corresponding to it a fixed value of the coordinate $x'_{\rm w}=$ const in the frame of reference, where it is at rest. Therefore, on the basis of expressions (3) we have the following for two instants of time, t_1 and t_2, in the laboratory frame of reference:

$$t_1 = \gamma\left(t_1' + \frac{V}{c^2}\,x_{\mathrm{w}}'\right),$$
$$t_2 = \gamma\left(t_2' + \frac{V}{c^2}\,x_{\mathrm{w}}'\right), \tag{20}$$

where t_1' and t_2' are the respective instants of proper time. Subtracting the first equality in time dependences (20) from the second one and introducing the notation $\Delta t = t_2 - t_1$ and $\Delta t' = t_2' - t_1'$, we arrive at relationship (13).

To conclude the discussion of this issue, we note that atomic clocks installed in navigational satellites traveling at altitudes of about 20,000 km with velocities $V = 4000\ \mathrm{m\ s^{-1}}$ will lag behind in 24 hours ($\Delta t = 8.64 \times 10^4$ s) by

$$\Delta t' = \left(1 - \sqrt{1-\beta^2}\right)\Delta t \approx \frac{1}{2}\left(\frac{V}{c}\right)^2 \Delta t$$
$$= \frac{1}{2}\left(\frac{4}{3}\right)^2 10^{-10} \times 8.64 \times 10^4\ \mathrm{s} = 7.7\ \mu\mathrm{s}\,.$$

6. Longitudinal length contraction of moving bodies

The frame of reference in which a body is at rest is termed proper for this body. The length of a body (a rod, a ruler) measured in such a frame of reference is termed proper length. To study the length of a body at rest is no problem. For example, let there be in the laboratory frame of reference K a motionless rod placed along the x-axis. The length L_0 of the rod is determined as the difference between the coordinates of its right (r) and left (l) ends x_{r} and x_{l}, respectively:

$$L_0 = x_{\mathrm{r}} - x_{\mathrm{l}}. \tag{21}$$

Here, it is irrelevant at which instants of time t, shown by clocks in the laboratory frame of reference, the coordinates are measured.

An alternative definition of length is also possible, and its advantages will become clear below. Imagine an accelerator transport channel of length L_0 to be a certain rod. Consider a bunch of particles moving at a constant velocity V from the left end of the channel to the right end. Then, if at the instant of time t_1, shown by the clock in the laboratory frame of reference, this bunch happens to be at the left end of the rod and at the instant of time t_{r} it happens to be at the right end, the proper length L_0 of the transport channel (which for reasons of brevity will further be called a rod) will be equal to the distance covered by the bunch:

$$L_0 = V(t_{\mathrm{r}} - t_1)\,. \tag{22}$$

In measuring the length L of a moving body (rod), it is necessary to have the coordinates of both its ends measured at one and the same instant of time t, shown by the clocks in that frame of reference, in which this length is determined:

$$L = x_{\mathrm{r}}(t) - x_{\mathrm{l}}(t)\,. \tag{23}$$

An accelerator transport channel moving with a velocity close to the speed of light with respect to the frame of reference associated with a bunch of particles can serve as an example of a rapidly moving rod. In this case, its length

$$L = x_{\mathrm{r}}'(t') - x_{\mathrm{l}}'(t') \tag{24}$$

is determined by the coordinates x' fixed at one and the same instant of time t'. It is easy to see, however, that the determination of length by formula (23) or (24) is extremely difficult, since it is necessary to establish beforehand clocks at both ends of the moving rod, the length of which is to be measured. A better way is to take advantage of an alternative method [formula (22)], based on application of the law of motion. If in the frame of reference K$'$ (associated with a bunch of particles) a bunch of particles is, first, overtaken by the left end of the rod at the instant of time t_{l}' and, then, at the instant t_{r}' by the right end, then the length of the moving rod will be

$$L = V(t_{\mathrm{r}}' - t_{\mathrm{l}}')\,. \tag{25}$$

In applying formula (25) for practical estimation of times t_{l}' and t_{r}', it is necessary to measure the number of particles in the bunch at those instants of time when the bunch enters the transport channel and when it leaves it.

From a comparison of formulae (25) and (22), it is immediately clear that the length L of a moving body differs from its proper length L_0. If the time differences are denoted as $\Delta t = t_{\mathrm{r}} - t_{\mathrm{l}}$ and $\Delta t' = t_{\mathrm{r}}' - t_{\mathrm{l}}'$, then, with account of the rate of moving clocks slowing down, we obtain the relationship between the length L_0 and the length L of a moving body:

$$\frac{L_0}{L} = \gamma\,. \tag{26}$$

Thus, we arrive at the following conclusion: the longitudinal dimensions of moving bodies undergo contraction by a factor of γ.

For example, in the aforementioned experiment at CERN, the proper time of motion of the pion bunch amounted to $t' = 3.90 \times 10^{-10}$ s [see estimate (10)], while for the length L of a moving rod (the entire transport channel) we obtain

$$L = Vt' \approx 11.7\ \mathrm{cm}\,, \tag{27}$$

which is 857 times smaller than L_0.

The contraction of longitudinal dimensions of moving bodies is termed Lorentzian, since it also follows from the Lorentz transformations. Indeed, let a body be at rest in the laboratory frame of reference; then, in accordance with the inverse Lorentz transformations, at one and the same instant of time t' in the frame of reference K$'$, with respect to which the body moves, we have

$$x_1 = \gamma(x_1' + Vt')\,, \tag{28}$$

$$x_2 = \gamma(x_2' + Vt')\,. \tag{29}$$

Let us denote the distances by $L_0 = x_2 - x_1$ and $L = x_2' - x_1'$. From formulae (28), (29) it follows that $L_0/L = \gamma$, which coincides with formula (26).

The transverse dimensions of moving bodies do not change. Such a conclusion can be made on the basis of experiments at accelerators with colliding beams. Imagine two proton bunches moving toward each other. We shall determine the total number of proton collisions. At high energies, one can neglect the Coulomb interaction of particles. As to nuclear forces, they are effective only at distances comparable to particle radii, so in the case dealt with protons can be considered spheres of finite radius.

It is obvious that two spheres will collide only when the distance between their centers does not exceed the sum of their radii, $r_1 + r_2$. This sum determines the so-called interaction cross section $\sigma = \pi(r_1 + r_2)^2$. If the spheres are travelling inside a pipe of radius R and they are distributed randomly across the pipe, the probability of a single collision occurring is $P = (r_1 + r_2)^2/R^2$. Clearly, if the bunches contain n protons each, the total number of collisions

$$N = n^2 \frac{(r_1 + r_2)^2}{R^2} . \tag{30}$$

If the transverse dimensions of particles (the radii r_1 and r_2) were to change during motion, the total number of collisions in the experiment would differ from the estimate (30). Since this does not occur, one can draw the conclusion that the transverse dimensions of moving bodies remain unchanged. This also follows from the Lorentz transformations for coordinates, the axes of which are perpendicular to the direction of motion.

Thus, the implication of the above analysis is that the transverse dimensions of moving bodies are invariant.

7. A violation of synchronism in the rates of sets of moving clocks

As follows from the Lorentz transformations, an observer in the frame of reference K will note nonsynchronicity of the rates of sets of clocks moving with respect to him in the frame of reference K$'$, and vice versa.

For a quantitative estimate of this nonsynchronicity, we turn to the example involving particle bunches moving at a velocity V in the transport channel of an accelerator (Fig. 3).

Let there be synchronized clocks located at equal distances l_0 from each other in this channel, and let them show the same time $t = 0$, when the first bunch-clock flies past the clock that is at the origin of the frame of reference. One can agree to set the proper time, shown at this instant by this clock, equal to zero, as well: $t' = 0$. Obviously, when the bunch-clock flies past the laboratory clock number i, the latter will show the time t_i [see formula (8)].

In the frame of reference K$'$ associated with the bunches, the distance between the laboratory clocks, owing to Lorentzian contraction, will be

$$l = l_0 \sqrt{1 - \beta^2} . \tag{31}$$

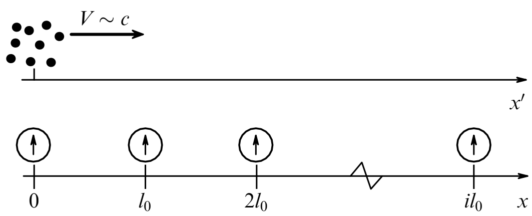

Figure 3. Schematic picture of a bunch of particles moving with respect to synchronized clocks located at equal distances along the transport channel of an accelerator.

Therefore, in the frame of reference K$'$ this event by the bunch-clock should take place at the instant of time

$$t_i' = \frac{il}{V} = \frac{il_0\sqrt{1 - \beta^2}}{V} = \frac{L_i\sqrt{1 - \beta^2}}{V} . \tag{32}$$

In the frame of reference K$'$, the ith clock moves toward the bunch with a velocity V. From the point of view of an observer in the frame of reference K$'$, the rate of the moving laboratory clocks should slow by the factor γ. Therefore, according to clocks in the frame of reference K, this observer can expect the event considered to take place at the instant

$$t_i^{\exp} = \frac{t_i'}{\gamma} = \frac{L_i(1 - \beta^2)}{V} . \tag{33}$$

The difference between the expected time $t_i^{\exp}$ (33) and the actual time t_i (8) can only be interpreted in a single way: the laboratory clocks numbers 0 and i moving in the frame of reference K$'$ are not synchronized. The difference δ_i in the readings of the clocks amounts to

$$\delta_i = t_i - t_i^{\exp} = \frac{L_i}{V} - \frac{L_i(1 - \beta^2)}{V} = \frac{L_i\beta^2}{V} . \tag{34}$$

From the point of view of an observer in the frame of reference K$'$, the clock number i shows a later time than the fixed clock number 0. In accordance with formula (34), this difference in the readings is larger, the farther the clocks are located from each other.

An illustration of the slowing down phenomena of the rates of clocks moving relative to other clocks, of a violation of synchronism of their rates, and of the contraction of distances between moving clocks can be represented by Fig. 4, in which the itemized (numbers in squares) laboratory clocks (indicated by circles) are installed at a distance $l_0 \approx 30$ m from each other along the transport channel (x-axis in the laboratory frame of reference K). The bunch-clocks (enclosed in oval lines) travel along this channel with a relativistic velocity, for which $\gamma = 10$. In the frame of reference K$'$, which is proper for the bunches, the distance between them is also equal approximately to 30 m. The laboratory clocks, as shown in the figure, move with respect to the bunch-clocks with a velocity V from right to left. At the instant $t' = 0$ (shown by the bunch-clocks), the origins of both coordinate systems coincide (Fig. 4a). The laboratory clock number 0, located at point $x = 0$, also shows time $t = 0$.

Two facts are of interest here. First, in accordance with formula (34), the readings of the laboratory clocks differ (the time is indicated inside the circles and ovals in nanoseconds; for convenience, the readings of all the clocks are rounded off): clock number 2 is $\delta = 99$ ns fast with respect to the neighboring clock, and it, in turn, shows a time 99 ns superior to the reading of clock number 0, and so on. Second, in the frame of reference K$'$, the distance between the laboratory clocks is $\Delta x' \approx 3$ m, i.e., 10 times less than in the frame of reference K. After a time $\Delta t' \approx 10$ ns passes (Fig. 4b), clock number 1, showing a time of 100 ns, will be opposite the bunch at point $x' = 0$. The difference in the readings of neighboring laboratory clocks is still 99 ns, as before, and the readings of all the laboratory clocks have increased by $\Delta t = 1$ ns. In a time $\Delta t' \approx 20$ ns, laboratory clock number 2, showing a time 200 ns (Fig. 4c), will be opposite the bunch, while $\Delta t = 2$ ns. The situation illustrated in Fig. 4d is realized after the passage of $\Delta t' \approx 100$ ns, when the laboratory clock

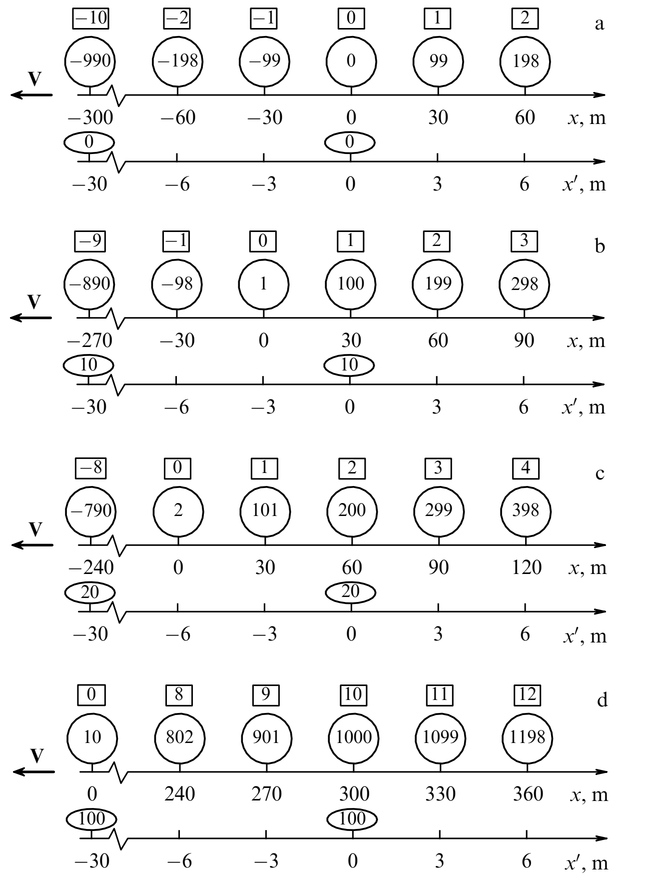

Figure 4. Illustration of the slowing down of clock rates, a violation of synchronism in their rates, and of contraction of the distance between moving clocks.

number 0 happens to be near the next bunch, and for it $\Delta t = 10$ ns.

We shall now discuss the issue of the slowing of the rate of a clock. Since the laboratory clocks and the bunch-clocks are in relative motion, this slowing down is peculiar to clocks of both types. If one follows the readings of one and the same laboratory clock, for instance, clock number 1, when comparing the situations presented in Figs 4a and 4b, then one can see that during the time $\Delta t' = 10$ ns (by the bunch-clock) its reading has increased by $\Delta t = 100-99 = 1$ ns, which signifies a 10-fold slowing down of their rate as compared to the rate of the bunch-clocks. On the other hand, by comparing the readings of one and the same bunch-clock (which is at point $x' = 0$) with the readings of different laboratory clocks that happened to be near the bunch, it is readily seen (Fig. 4b) that the time according to the bunch-clock is 10 ns, while clock number 1 shows 100 ns, in Fig. 4c the bunch-clock shows 20 ns, while clock number 2 shows 200 ns, and the bunch-clock in Fig. 4d shows 100 ns, while laboratory clock number 10 shows 1000 ns. This signifies a 10-fold slowing of the rate of bunch-clocks.

Similar reasoning can be performed making use of the next bunch-clock located at the point $x' \approx -30$ m. Comparing, for instance, Figs 4a and 4d, we arrive at the conclusion that, when 100 ns passes (according to the bunch-clock), the time interval measured with the aid of different laboratory clocks but which happen to be close to the bunch, amounts to $10 - (-990) = 1000$ ns.

Finally, we shall also point out the relativity of simultaneity: events that take place simultaneously in one frame of reference will not be simultaneous in another coordinate system. Indeed, consider that bunches with coordinates $x_1' \approx -30$ m and $x_2' = 0$ m at the instant of time $t' = 20$ ns (Fig. 4c) 'caused damage' to the laboratory clocks that were nearby. Their hands 'stopped' respectively at the readings $t_1 = -790$ ns and $t_2 = 200$ ns. Consequently, for an experimentalist working at the accelerator, the damage to the clocks was caused in sequence with a time interval of 990 ns.

8. Nonsynchronicity and the equivalence principle

Albert Einstein, having taken into account that both forces due to inertia and forces due to gravity are proportional to the masses of the bodies involved, proposed a simple thought experiment. Assume there is an observer shut inside a tightly closed elevator cabin. If the elevator cabin is at rest on Earth, the observer will see usual manifestations of the forces of gravity: all bodies will fall with the same acceleration.

In an elevator moving with a constant acceleration g directed toward the ceiling in a space without the forces of gravity, the observer will also find the accelerations of all the bodies falling onto the floor of the cabin to be the same. Therefore, these two cases can be distinguished from each other by phenomena occurring within the cabin, only if the cabin is sufficiently large. Indeed, owing to the field of gravity being inhomogeneous, it is possible to observe an insignificant change in the distance between two falling bodies, while this change will not be noticed in the field of inertial forces. However, it is always possible to choose such a local frame of reference in which the action of forces of inertia and of gravity will be indistinguishable. This assertion is known as Einstein's famous equivalence principle. On the basis of this principle, Einstein developed the general theory of relativity (GTR).

One of the main effects of GTR, namely, the red shift in a gravitational field, is explained with the aid of the equivalence principle. Consider a facility containing a source of photons with frequency ω and a receiver that can absorb these photons and is situated at a height H above the receiver (Fig. 5).

Consider photons moving upward in Earth's gravitational field, which we shall consider to be homogeneous. The effect consists in the fact that the frequency registered by the receiver is smaller than ω by $\Delta\omega$. The following equality holds valid in the first approximation:

$$\frac{\Delta\omega}{\omega} = -\frac{gH}{c^2}\,, \tag{35}$$

where c is the speed of light.

Let us analyze this effect in the frame of reference attached to the elevator (see Fig. 5) when it falls freely with respect to a facility with an acceleration g. Gravity is absent in the elevator, so there is no reason for the photon frequency to change. If at the instant of time $t = 0$, the velocity of the elevator is $v = 0$, and at the same instant of time a photon of frequency ω is emitted, then at the instant $t = H/c$ this photon will arrive at the receiver, which in the frame of reference associated with the elevator will have a velocity

$$v = gt = \frac{gH}{c}\,, \tag{36}$$

directed upward. Thus, the photon will be catching up with the receiver. But, in this case, there must be a manifestation of

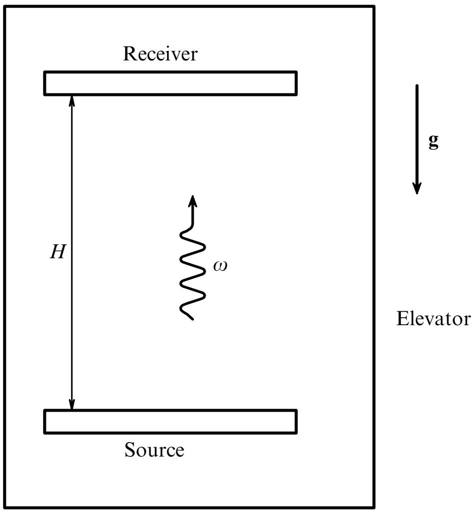

Figure 5. Illustration of the equivalence principle for explaining the gravitational red shift.

the Doppler effect. In accordance with this effect, the frequency registered by the receiver will be smaller than ω by $\Delta\omega$:

$$\frac{\Delta\omega}{\omega} = -\frac{v}{c}\,. \tag{37}$$

If the velocity v (36) is substituted into formula (37), we obtain formula (35), predicted in GTR. From the aforementioned, it is clear that the frequency of the photon ω remains unchanged, while the effect is due to motion of the receiver.

We shall now consider what causes $\Delta\omega$ to appear in the gravitational field. According to GTR, the frequency ω of the photon does not change with the height H in a static gravitational field. Consequently, the only possible explanation of the appearance of $\Delta\omega$ can be the difference between the rates of clocks located near the photon source and near the receiver.

The rate of clocks indeed increases as they rise in the gravitational field. If, for example, clocks on Earth 'measure' a time t, clocks lifted to a height H will be fast by

$$\Delta t = t\,\frac{gH}{c^2}\,. \tag{38}$$

Straightforward experiments with precise clocks, one of which was in an airplane at a certain altitude, while the other one remained on Earth, confirmed this effect, first, with an error of $\approx 10\%$ (experiments performed by J C Hafele and R E Keating) [10], and then with an error of $\approx 1\%$ (experiments performed by C O Alley et al.) [11].

In 1976, R Vessot and M Levine from the Smithsonian Astrophysical Observatory of Harward University, together with their colleagues from NASA, installed an atomic clock with a precision of one trillionth of a second per hour [12] on the Scout D rocket launched from Wallops Island in Virginia. The readings of two atomic clocks were compared with the aid of a reciprocal exchange of microwave signals. It was established that by the time the rocket reached the maximum altitude of 10,000 km, the atomic clock in it was four billionths of a second faster than the clock that remained on Earth. The difference between the experimental data and the results of theoretical calculations amounted to less than 0.01%.

When an atomic clock rises in the gravitational field, the spacing between the energy levels of electrons in the atom increases, while in the case of nuclear clocks it is the spacing between nuclear levels that increases. When the difference in energies between levels in an atom (nucleus) increases, the emission frequency also increases, and the oscillation period decreases. This means that the rate of such atomic (nuclear) clocks will increase.

In one case, the photon frequency is measured by a 'fast' clock that is high up, and in the other case by a 'slow' clock that is at the bottom. In the first approximation, it is precisely formula (35) for the red shift $\Delta\omega$ that is obtained. Evidently, $\Delta\omega$ is just the difference between frequencies that arises when measurements are performed with clocks exhibiting different rates.

Owing to the presence of gravity, in 24 hours the readings of atomic clocks on board navigational satellites will become fast by the quantity

$$\Delta t = t\,\frac{gH}{c^2} = t\,\frac{\Delta U}{c^2}\,, \tag{39}$$

where ΔU is the change in the gravitational potential:

$$\Delta U = -GM\left(\frac{1}{R+H} - \frac{1}{R}\right).$$

For $H = 3R = 19{,}200$ km and $t = 8.6 \times 10^4$ s, the quantity $\Delta t = 46$ μs.

Thus, with account of the relativistic lag, the difference between the readings of clocks on board a satellite and of clocks on Earth reached in 24 hours will amount to $\Delta t_0 = \Delta t - \Delta t' = (46 - 7.7)$ μs $= 38.3$ μs. If this difference were not taken into account, the error in determining the distance between a satellite and an object on Earth would be $\Delta r = c\Delta t_0 = 11.5$ km.

In 1960, R Pound and G Rebka confirmed formula (35) with an error of $\sim 10\%$ (later in 1964, Pound and J Sneider lowered the uncertainty to $\sim 1\%$) [13] in experiments with gamma rays emitted by nuclei of the iron isotope ^{57}Fe. The photons were emitted upward (in another series of experiments they travelled downward) in a tower 22 m high. The observed shift in frequency amounted to a very small magnitude, $\Delta\omega/\omega \approx 10^{-15}$. The high measurement precision was achieved owing to the Mössbauer effect: when a photon is emitted, the recoil is not felt by the sole emitting atom, but by the whole crystal. Since the mass of the crystal is large, its recoil velocity is small. Therefore, the photons emitted (absorbed) are extremely monochromatic, which is precisely what permitted achieving a high measurement precision.

We shall now turn to a violation of synchronism in the rates of clocks moving at a velocity V with respect to an observer in the frame of reference K. Before attaining the velocity V, the frame of reference K$'$ must move with acceleration. Consider the initial stage of accelerated motion of the frame of reference K$'$ ($\gamma = 1$) and assume the acceleration a of the frame of reference to remain constant during a small time interval t. Then, it will attain the velocity $V = at$ at the end of this time interval t. In agreement with the equivalence principle, in the frame of reference gathering speed, the readings of clocks located at a distance L from each other in the direction of motion will, from the point of view of an observer at rest in this coordinate system, differ, according to formula (39), by

$$\Delta t = t\,\frac{aL}{c^2} = \frac{VL}{c^2}\,. \tag{40}$$

To eliminate this difference, and thus synchronize the clocks in the frame of reference K′, the observer in this coordinate system must 'turn backward the hands' of the clocks that became fast. An observer in the frame of reference K will register the above event as the appearance of nonsynchronicity: a successive decrease in the readings of clocks located one after another in the direction of their motion.

9. Minkowski space

The manifold of variables x, y, z, ct forms the Minkowski space. The cause–effect relationships between events in the simplest case of their occurring at different points on the x-axis are conveniently analyzed, taking advantage of the so-called Minkowski plane (Fig. 6). The variables x and ct serve as the Cartesian coordinates in this plane. An event taking place on the x-axis in the frame of reference K is mapped as a point onto the Minkowski plane. If a flash of light occurs at the origin at the instant $t = 0$, the coordinates x of the wavefront are determined by the relation

$$x = \pm ct\,. \tag{41}$$

In the Minkowski plane, the law of motion (41) is mapped by the two bisectors of the angles between the coordinate axes (dashed straight lines). In the case of two spatial coordinates (x and y), instead of lines there will be a cone, called the light cone. For three spatial coordinates, the notion of a 'light cone' is also introduced, even though it cannot be represented graphically. For an observer at the origin at the instant $t = 0$ (note that any point on the x-axis can be chosen as the origin), the light cone partitions Minkowski space into three parts. The region inside the upper cone is called the 'future', and the region inside the lower cone is the 'past'. The remaining part is the 'neutral' region. Obviously, any events mapped by points onto the surface of the light cone are related to each other by timelike intervals:

$$\Delta S^2 = (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = 0\,. \tag{42}$$

Events from the regions future, P_2, or past, P_1, are related to the 'zeroth' event at point O by timelike intervals $\Delta S^2 > 0$.

An arbitrary event P_1 from the past region took place earlier than the zero event and may be the cause of the latter, while any event P_2 in the future occurs later than the zero event and can, therefore, be a consequence of it. Thus, for an observer at point $x = 0$ all the events from the lower cone took place in the past, while the events from the upper cone will happen in the future.

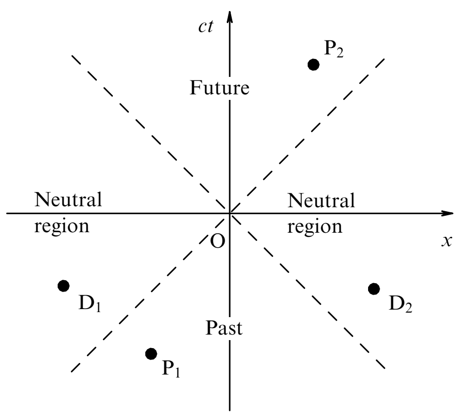

Figure 6. Minkowski plane.

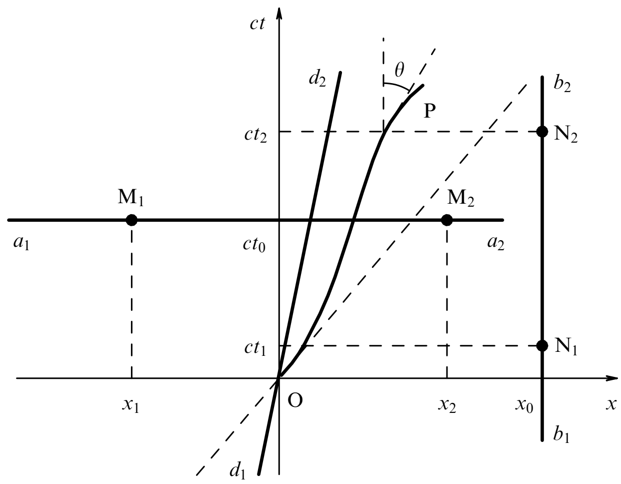

Figure 7. Mapping of the motion of a particle onto the Minkowski plane.

Any events from the neutral region (points D_1 and D_2 in Fig. 6) are related to the event at point O by spacelike intervals ($\Delta S^2 < 0$). Clearly, no cause–effect relationships are possible between these and the zero events.

The x coordinate axis and all possible straight lines parallel to it (for instance, the line a_1a_2) are so-called lines of simultaneity in the frame of reference K (Fig. 7). Any two events belonging to these lines (for example, events M_1 and M_2 on the straight line a_1a_2) are simultaneous, i.e., they take place at the same instant of time t_0 by the clock in the frame of reference K, but at different points of space x_1 and x_2. These events are clearly related by a spacelike interval and, therefore, no cause–effect relationships between them are established.

All events occurring at one and the same point in space, but at different instants of time, are mapped either onto the ordinate axis ($x = 0$) or onto all other possible straight lines $x = \text{const}$ parallel to it (for example, the line b_1b_2). Any two events mapped onto points belonging to these straight lines are characterized by one and the same coordinate, but by different instants of time (for example, events N_1 and N_2 on the line b_1b_2 occurred at the same point with the coordinate x_0, but at different instants of time, t_1 and t_2). It is clear that these events are related by a timelike interval, and that cause–effect relationships between them are possible. Obviously, the event that took place earlier could be the cause, while the events that occurred later are the effect.

In the Minkowski plane, it is possible to map not only events, but also the motions of particles. For example, particles at rest residing at different points x are mapped by straight lines $x = \text{const}$. These straight lines are called 'world' lines of particles at rest. The motion of a particle traveling with a velocity V is mapped onto this plane by a straight line, the angle of which to the ct-axis does not exceed 45°. Such lines are called world lines of moving particles. For example, if a particle at the instant $t = 0$ is at the origin of the frame of reference, its world line d_1d_2 passes through the past and future regions. The equation describing such a line is determined by the following expression

$$ct = \frac{x}{\beta}\,, \tag{43}$$

where $\beta = V/c$.

If the velocity V of a particle is variable, its world line is mapped by a certain curve OP, the equation of which is as follows:

$$ct = f(x) \,. \tag{44}$$

The angle θ between the tangent to this curve at any point and the ordinate axis is such that

$$\tan\theta = \beta \,. \tag{45}$$

Since $V \leqslant c$, then $\theta \leqslant 45°$.

10. Velocity transformations

According to STR, the velocity addition law following from Galilean transformations is not obeyed for motion at relativistic velocities. We shall consider one of the experiments carried out at the CERN accelerator by T Alväger, F Farley, J Kjellman, and L Wallin [14] in 1964.

Protons were accelerated using a variable electric field of frequency $\nu = (9.53220 \pm 0.00005)$ MHz. For the protons to undergo acceleration, they must happen to be within the acceleration segment in phase with the electric field. Therefore, the proton bunches in the accelerator ring must follow each other with the same frequency. For a fixed length of the ring, the neighboring proton bunches were at a distance $s = 31.45$ m from each other (Fig. 8). The length l of each bunch was approximately 1 m.

Interactions of the accelerated protons with nuclei of the target atoms resulted in the production of various secondary particles, including neutral pions π^0. Since the protons impinged upon the target in bunches with a frequency of ν, the neutral pions were also produced in bunches with the same frequency. Upon production, the neutral pions moved with a velocity $V \sim c$ along the tangent to the circular ring of the accelerator toward the detectors D_1 and D_2, which were also located at a distance $s = 31.45$ m from each other, while all the charged particles were deflected with the aid of magnets. Neutral pions are unstable particles with an average lifetime $\tau = 0.83 \times 10^{-16}$ s that decay into two gamma quanta at a distance $\lambda = \gamma V \tau \approx 1.1 \times 10^{-6}$ m ($\gamma \approx 45$), i.e., close to their production point. In the experiment, such gamma quanta were selected that travelled in the same direction as the pions having velocities $V > 0.99975c$ (the Lorentz factor $\gamma > 45$). Thus, the bunches of neutral pions that moved with a velocity V along the x-axis and followed each other in succession with a frequency ν gave rise to bunches of gamma quanta, which followed each other in succession with the same frequency and the velocities of which were equal to the speed of light with respect to the pions.

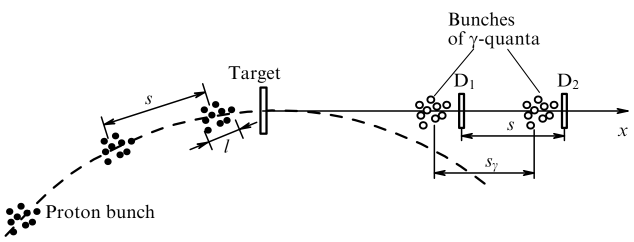

Figure 8. Layout of an experiment performed at the CERN accelerator.

If the Galilean transformations are valid, the velocity of the gamma quanta with respect to the detectors should be expressed as

$$v_\gamma = V + c \,. \tag{46}$$

In this case, two neighboring bunches of gamma quanta that are at a distance $s_\gamma = v_\gamma/\nu$ from each other will not arrive at the detectors at the same time, but will be separated by the time interval

$$\Delta t = \frac{s_\gamma - s}{v_\gamma} = \frac{1}{\nu} - \frac{c}{\nu v_\gamma} = \frac{1}{\nu}\left(1 - \frac{c}{v_\gamma}\right) = \frac{1}{\nu}\,\delta \,, \tag{47}$$

where $\delta = 1 - c/v_\gamma$. The experiment revealed that $\Delta t = -(0.005 \pm 0.013)$ ns; therefore, within the limits of experimental error, $\delta = -4.7 \times 10^{-5}$, and

$$v_\gamma = (2.9977 \pm 0.0004) \times 10^8 \text{ m s}^{-1}, \tag{48}$$

which coincides with the speed of light.

Thus, new laws for velocity transformation are necessary, instead of formula (46). They can be readily obtained on the basis of the Lorentz transformations (1). The differentials of the left-hand and right-hand parts of these transformations have the form

$$\begin{aligned} dx' &= \gamma(dx - V\,dt) \,, \\ dy' &= dy \,, \\ dz' &= dz \,, \\ dt' &= \gamma\left(dt - \frac{V}{c^2}\,dx\right) . \end{aligned} \tag{49}$$

We divide the left-hand and right-hand parts of the first three equalities (49) by the respective left-hand and right-hand parts of the last equality and take into account the kinematic definitions of velocity projections in the frames of reference K and K$'$. As a result, we obtain the sought-after transformations of velocity projections onto the coordinate axes:

$$\begin{aligned} v'_x &= \frac{v_x - V}{1 - v_x V/c^2} \,, \quad v'_y = \frac{\sqrt{1 - V^2/c^2}}{1 - v_x V/c^2}\, v_y \,, \\ v'_z &= \frac{\sqrt{1 - V^2/c^2}}{1 - v_x V/c^2}\, v_z \,. \end{aligned} \tag{50}$$

Relations (50) are known as the Lorentz transformations of velocity projections. In the case of small velocities ($v \ll c$, $V \ll c$), they transform into the Galilean velocity transformations.

Resolving expressions (50) with respect to v_x, v_y, and v_z, one can obtain the inverse transformations of velocity projections:

$$\begin{aligned} v_x &= \frac{v'_x + V}{1 + v'_x V/c^2} \,, \quad v_y = \frac{\sqrt{1 - V^2/c^2}}{1 + v'_x V/c^2}\, v'_y \,, \\ v_z &= \frac{\sqrt{1 - V^2/c^2}}{1 + v'_x V/c^2}\, v'_z \,. \end{aligned} \tag{51}$$

Let us apply the transformations found for determining the velocities of gamma quanta in the experiment considered above. Since neutral pions move with a velocity $V = 0.99975c$, and the gamma quanta with the speed of light, $v'_x = c$, then the first of formulae (51) in this case gives

$$v_\gamma = \frac{c + 0.99975c}{1 + c\,0.99975c/c^2} = c\,, \tag{52}$$

which is in total agreement with the result of measurements (48).

The relativistic formulae (50) and (51) for the velocity transformation have been confirmed by all relevant experiments, without any exceptions. In Sections 11–13, we shall examine a few important examples.

11. Accelerators with colliding particles

The physical result of collisions of elementary particles depends on their energy in their center-of-mass system, in which the sum of the relativistic momenta is equal to zero. The higher this energy, the richer is the spectrum of new phenomena.

Let us now consider the collision of two identical particles moving toward each other with a velocity v_0, to which a Lorentz factor γ_0 corresponds. In this case, the center-of-mass system is at rest, and the energy of each of the particles is proportional to γ_0. The collision of these particles can also be considered in a frame of reference attached to one of them. In this case, the entire spectrum of phenomena remains the same; however, the Lorentz factor γ corresponding to the relative velocity v of particles approaching each other will be significantly larger than γ_0. Indeed, setting $v_x = v_0$, $V = -v_0$ in formulae (50), we obtain the following for the relative velocity $v = v'_x$:

$$v = \frac{2v_0}{1 + v_0^2/c^2}\,. \tag{53}$$

The relativistic factor γ is expressed as

$$\gamma = 2\gamma_0^2 - 1\,. \tag{54}$$

For example, when $\gamma_0 = 10$, we obtain $\gamma \approx 200$, and when $\gamma_0 \approx 10^3$, the factor γ becomes 2000 (!) times greater. Hence follows an extremely important conclusion: in order to obtain the same effect as in a collision with a particle at rest (a target), it is much more advantageous to deal with collisions of particles moving toward each other with velocities to which comparatively low values of the Lorentz factor γ_0 correspond.

Such an idea is realized in accelerators, in which two beams of particles moving toward each other with relativistic velocities collide. For example, to achieve values of $\gamma \approx 10^6$, the dimensions of an ordinary accelerator (with a fixed target) may have to exceed Earth's radius, which makes its construction impossible. As to accelerators with colliding beams, their dimensions (for a given magnetic field induction) and, consequently, cost will be $\gamma/\gamma_0 = 2\gamma_0$ times less. Still another advantage of such accelerators consists in the symmetry of the spatial recession of secondary particles. The idea of such an accelerator was first put forward by G A Budker, and one of the first accelerators was constructed near Novosibirsk.

12. Jets of particles

The collision of a relativistic particle with a target at rest results in the production of new (secondary) particles moving inside a narrow cone, the axis of which is oriented along the velocity vector of the primary particle, while its opening angle depends on the Lorentz factor γ (the larger γ, the smaller the angle). Secondary particles form so-called jets and, when the opening angle of the cone is small, they are barely detectable, since they are very close to each other. In this case, it becomes very difficult to distinguish one particle from another.

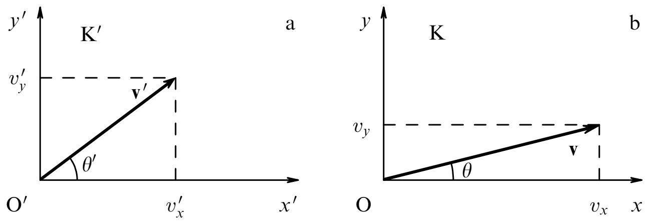

Figure 9. Directions of motion of particles forming jets in different frames of reference.

Particle collisions are conveniently examined in the center-of-mass system (CMS), in which the secondary particles lie within cones with large opening angles, since $\gamma_0 < \gamma$. Therefore, the problem of detection and identification of secondary particles is significantly facilitated at accelerators with colliding beams.

The directions of motion of particles forming jets can be calculated using the velocity addition law. Consider primary particles moving in the frame of reference K′ (CMS) toward each other along the x'-axis, and after the collision let one of the secondary particles have a velocity $\mathbf{v}'$ directed at a zenith angle θ' to the x'-axis (Fig. 9a).

The velocity projections of this secondary particle are

$$v'_x = v'\cos\theta'\,, \quad v'_y = v'\sin\theta'\,. \tag{55}$$

The frame of reference K′ (CMS) moves with respect to the frame of reference K, in which one of the particles is at rest, with a certain velocity V. The velocity projections of the secondary particle in the frame of reference K, in accordance with the velocity addition law, are as follows:

$$v_x = \frac{v'\cos\theta' + V}{1 + v'V\cos\theta'/c^2}\,, \quad v_y = \frac{\sqrt{1 - V^2/c^2}}{1 + v'V\cos\theta'/c^2}\,v'\sin\theta'\,. \tag{56}$$

If these projections are written down as

$$v_x = v\cos\theta\,, \quad v_y = v\sin\theta\,, \tag{57}$$

we obtain the following for the zenith angle θ (Fig. 9b):

$$\tan\theta = \frac{v_y}{v_x} = \frac{1}{\gamma}\,\frac{v'\sin\theta'}{v'\cos\theta' + V}\,, \tag{58}$$

where $\gamma = (1 - V^2/c^2)^{-1/2}$. Hence, it is seen that in the ultrarelativistic case, when $\gamma \to \infty$, the zenith angle $\theta \to 0$, whatever the value of θ'. Thus, at large values of γ nearly all secondary particles move within a very narrow cone.

13. Stellar aberration

If a star is seen at an angle θ to the x-axis of the frame of reference K, then, in another frame of reference K′ moving along this axis with a velocity V, the angle will be different (Fig. 10). The projections of the velocity vector of a light wave

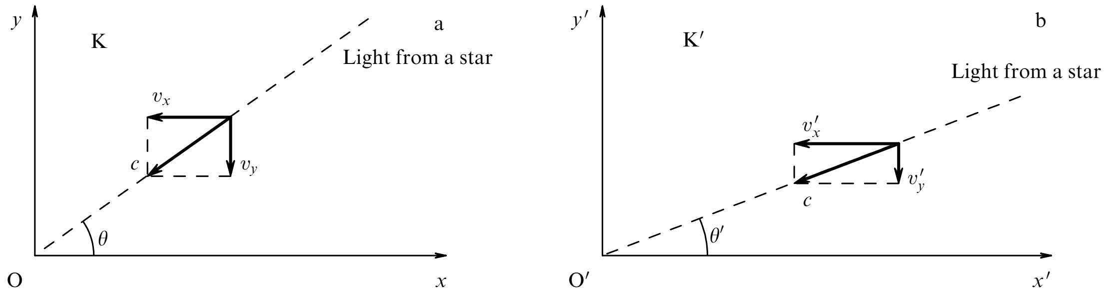

Figure 10. Directions of propagation of a light beam from a star at the zenith in different frames of reference.

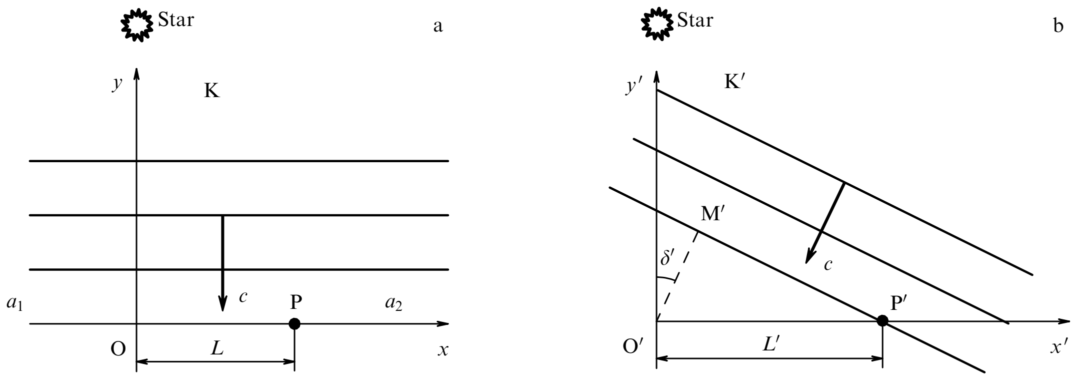

Figure 11. Position of the flat wavefront of a light wave arriving from a star at the zenith in different frames of reference.

in the frame of reference K (Fig. 10a) are

$$v_x = -c\cos\theta\,, \quad v_y = -c\sin\theta\,. \tag{59}$$

According to the transformation formulae of velocity projections, in the frame of reference K$'$ we have

$$v'_x = \frac{-c\cos\theta - V}{1 + V\cos\theta/c}\,, \quad v'_y = -\frac{\sqrt{1 - V^2/c^2}}{1 + V\cos\theta/c}\,c\sin\theta\,. \tag{60}$$

At the same time, we note that $(v'^2_x + v'^2_y)^{1/2} = v' = c$. Since, according to Fig. 10b, these velocity projections are also given by

$$v'_x = -c\cos\theta'\,, \quad v'_y = -c\sin\theta'\,, \tag{61}$$

we can write the following for the sought-after angle θ':

$$\tan\theta' = \frac{v'_y}{v'_x} = \frac{\sin\theta}{\gamma(V/c + \cos\theta)}\,. \tag{62}$$

In the case of ultrarelativistic velocities, when $\gamma \to \infty$, a star, whatever its position, will seem to be situated at a very small angle to the direction of motion of the frame of reference K$'$.

If the angle $\delta' = \pi/2 - \theta'$ is introduced, which is used in astronomy for characterizing the aberration of starlight, we obtain from formula (62) for a star at the zenith ($\theta = \pi/2$) the relationship

$$\sin\delta' = \frac{V}{c}\,, \tag{63}$$

which, in the case of relativistic velocities, differs essentially from the classical formula

$$\tan\delta' = \frac{V}{c}\,. \tag{64}$$

Note that the phenomenon considered is related to the change of direction from which a light wave propagating from a distant star arrives. Let us examine how the wavefront is oriented in different coordinate systems.

Consider, for reasons of simplicity, a star that is at the zenith ($\theta = \pi/2$) in the frame of reference K. Then, the flat front $a_1 a_2$ will reach points O and P on the x-axis simultaneously (Fig. 11a). An observer will register (visually or with the aid of an optical device) the star's position in the direction of motion of the wave, which is perpendicular to its front.

In the frame of reference K$'$ (Fig. 11b), light will arrive at point O$'$ later than at point P$'$. If the length of the segment OP $= L$, then, in accordance with the formula for time transformation (1), such retardation will be equal to

$$\Delta t' = \gamma\,\frac{V}{c^2}\,L\,. \tag{65}$$

An observer in the frame of reference K$'$ will find the normal to the wavefront to be inclined at an angle δ' to the y'-axis. In a time $\Delta t'$, light will cover the distance M$'$O$' = c\Delta t'$. The distance between points O$'$ and P$'$ in the frame of reference K$'$, in accordance with the Lorentz transformations for coordinates, is $L' = \gamma L$. Hence, one obtains

$$\sin\delta' = \frac{\mathrm{O'M'}}{\mathrm{O'P'}} = \frac{c\Delta t'}{L'} = \frac{V}{c}\,, \tag{66}$$

which coincides with formula (63).

14. Acceleration transformations

In the case of Galilean transformations, acceleration is an invariant quantity under transition from one inertial reference system to another. When moving with a relativistic velocity, the acceleration projections a_x, a_y, a_z cannot evidently remain constant in different inertial reference systems, since neither the differentials of velocity projections nor the time differential $\mathrm{d}t$ is an invariant of the Lorentz transformations. From formulae (51) one can find the differentials of velocity projections:

$$\mathrm{d}v_x = \frac{\mathrm{d}v'_x}{\gamma^2(1+v'_x V/c^2)^2}\,,$$
$$\mathrm{d}v_y = \frac{1}{\gamma}\,\frac{\mathrm{d}v'_y}{1+v'_x V/c^2} - \frac{1}{\gamma}\,\frac{v'_y V/c^2}{(1+v'_x V/c^2)^2}\,\mathrm{d}v'_x\,, \tag{67}$$
$$\mathrm{d}v_z = \frac{1}{\gamma}\,\frac{\mathrm{d}v'_z}{1+v'_x V/c^2} - \frac{1}{\gamma}\,\frac{v'_z V/c^2}{(1+v'_x V/c^2)^2}\,\mathrm{d}v'_x\,,$$

and the time differential:

$$\mathrm{d}t = \gamma\left(\mathrm{d}t' + \frac{V}{c^2}\,\mathrm{d}x'\right). \tag{68}$$

If the left-hand and right-hand parts of the three equalities (67) are divided by the respective left-hand and right-hand parts of expression (68), we obtain the following transformations of the acceleration projections:

$$a_x = \frac{a'_x}{\gamma^3(1+v'_x V/c^2)^3}\,,$$
$$a_y = \frac{1}{\gamma^2}\,\frac{a'_y}{(1+v'_x V/c^2)^2} - \frac{1}{\gamma^2}\,\frac{v'_y (V/c^2)\,a'_x}{(1+v'_x V/c^2)^3}\,, \tag{69}$$
$$a_z = \frac{1}{\gamma^2}\,\frac{a'_z}{(1+v'_x V/c^2)^2} - \frac{1}{\gamma^2}\,\frac{v'_z (V/c^2)\,a'_x}{(1+v'_x V/c^2)^3}\,.$$

The noninvariance of acceleration is readily illustrated with the aid of formulae (69) by taking advantage of a simple example. If a particle moves uniformly in the frame of reference K′ along a straight line perpendicular to the x'-axis ($v'_y \neq 0$, $v'_z \neq 0$, $a'_y = 0$, $a'_z = 0$), and with an acceleration a'_x along the x'-axis, in the frame of reference K it attains accelerations a_y and a_z differing from zero.

15. Co-moving frame of reference

Let us introduce the important concept of a co-moving inertial frame of reference K'_c traveling along the x-axis of the laboratory frame of reference K. Its velocity V is determined by the condition that the velocity projection v'_x be zero ($v'_x = 0$).

In the case of a particle motion with acceleration along the x-axis, the velocity V of the co-moving frame of reference K'_c at different instants of time should be various:

$$V = v_x\,, \tag{70}$$

since the velocity v_x is variable. In particular, we stress that the system K'_c is an inertial reference system and must move with a constant velocity, $V = \mathrm{const}$. Condition (70) can only be satisfied by a set of reference systems K'_c moving with different velocities, and at each instant of time t it is necessary to make use of one of the systems for which at this instant $V = v_x$.

With account of $v'_x = 0$, the acceleration projections in the reference system K and in the system K'_c, moving at the given instant with velocity V, are related by simple relationships:

$$a_x = \frac{a'_x}{\gamma^3}\,, \quad a_y = \frac{a'_y}{\gamma^2} - v'_y\,\frac{V}{c^2}\,\frac{a'_x}{\gamma^2}\,, \quad a_z = \frac{a'_z}{\gamma^2} - v'_z\,\frac{V}{c^2}\,\frac{a'_x}{\gamma^2}\,. \tag{71}$$

We recall that the velocity V of the frame of reference K'_c is determined in formulae (71) from condition (70).

If the velocity and acceleration of a particle in the co-moving frame of reference K'_c are directed along the x'-axis ($v'_y = v'_z = 0$, $a'_y = a'_z = 0$), only the projection a_x differs from zero in the laboratory frame of reference:

$$a_x = \frac{a'_x}{\gamma^3}\,. \tag{72}$$

For an example, we shall consider the motion of an elementary particle in a constant electric field E directed along the x-axis. The velocity of the particle in the frame of reference K increases under the action of the field; however, the motion will not be uniformly accelerated. Indeed, a particle of charge q in the co-moving frame of reference is under the action of a force $f = qE$, since the field strength along the x-axis is the same in all IRSs. In this frame ($v'_x = 0$), the particle's acceleration can be found from the classical equation of motion

$$a'_x = a_0 = \frac{qE}{m} = \mathrm{const}\,. \tag{73}$$

Returning to formula (72), we note that, as γ increases, the acceleration $a_x \to 0$, although the force $f = \mathrm{const}$. From formula (72) with account of formula (73), one readily obtains the time dependence of the velocity v_x and the law of motion $x(t)$ of the particle. Indeed, we rewrite formula (72) as

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = a_0\left[1 - \left(\frac{v_x}{c}\right)^2\right]^{3/2}, \tag{74}$$

where equality (70) was taken into account. Separating variables v_x and t, we obtain

$$\frac{\mathrm{d}v_x}{\left[1-(v_x/c)^2\right]^{3/2}} = a_0\,\mathrm{d}t\,. \tag{75}$$

If we assume that at the instant $t = 0$ the velocity $v_x = 0$, integration of equality (75) results in the relationship

$$\frac{v_x}{\sqrt{1-(v_x/c)^2}} = a_0 t\,. \tag{76}$$

Further expressing v_x, we obtain the velocity increase law for v_x:

$$v_x = \frac{a_0 t}{\sqrt{1+(a_0 t/c)^2}} = \frac{ct/t_0}{\sqrt{1+(t/t_0)^2}}\,, \tag{77}$$

where $t_0 = c/a_0$ is a characteristic time scale. Upon substitution of formula (77) into (74), we have

$$a_x = a_0\,\frac{1}{\left[1+(t/t_0)^2\right]^{3/2}}\,. \tag{78}$$

The time dependences (77) and (78) are shown in Fig. 12. The acceleration of the particle is seen to decrease steadily, while its velocity tends asymptotically to its limit value equal to c. We note that $a_x = a_0$ in Newton's mechanics, and the particle

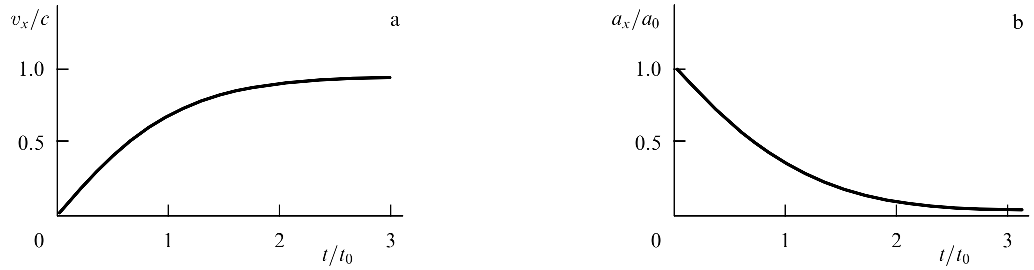

Figure 12. Time dependences of velocity and acceleration of a relativistic particle under the action of a constant force.

velocity would increase without limits, so already in time t it would have reached the value equal to the speed of light.

Of practical interest is the time t_β which it takes a particle to be accelerated up to a given relative velocity $\beta = v_x/c$. From expression (77) it follows that

$$t_\beta = \beta\gamma t_0 \,. \tag{79}$$

Since $\beta \approx 1$, the time t_β is γ times greater than the characteristic time t_0. For example, the international project TESLA (TeV-Energy Superconducting Linear Accelerator) foresees the acceleration of electrons up to the velocity at which the Lorentz factor reaches the value $\gamma = 10^6$ at $a_0 = 3 \times 10^{18}$ m s^{-2}. Because under these conditions $t_0 \approx 10^{-10}$ s, the time t_β may comprise $\approx 10^{-4}$ s.

Integrating expression (77), we obtain the law of motion

$$x = \int_0^t v_x \,\mathrm{d}t = ct_0\left(\sqrt{1 + \left(\frac{t}{t_0}\right)^2} - 1\right), \tag{80}$$

where it was taken into account that $x = 0$ at $t = 0$. It is not difficult to see that, for $t \ll t_0$, one finds

$$x = \frac{ct^2}{2t_0} = \frac{a_0 t^2}{2} \,, \tag{81}$$

which coincides with the classical law of uniformly accelerated motion. However, for $t \gg t_0$, the increase in the coordinate x with time is approximately linear. The asymptote of the law of motion is a straight line (Fig. 13), and its equation has the form

$$x = c(t - t_0) \,. \tag{82}$$

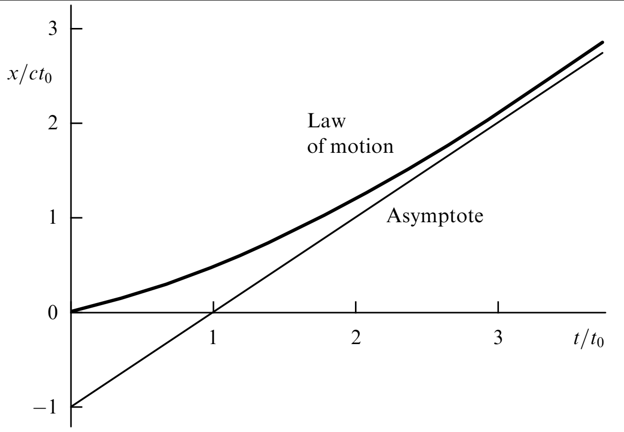

Figure 13. The law of motion of a relativistic particle under the action of a constant force.

As follows from Eqns (79) and (80), the particle will cover in time t_β the distance

$$x_\beta = ct_0(\gamma - 1) \,. \tag{83}$$

Thus, for instance, at $t \approx 10^{-4}$ s, required for achieving the value of $\gamma = 10^6$, an electron will cover the distance $x_\beta \approx 30$ km. Naturally, the cost of constructing such a long linear accelerator is enormous, and therefore TESLA is an international project.

Let us find the dependence of the proper time τ (determined by the clock associated with the particle) on time t in the laboratory frame of reference. With account of expression (77), this dependence takes the form

$$\tau = \int_0^t \sqrt{1 - \beta^2}\,\mathrm{d}t = \int_0^t \frac{\mathrm{d}t}{\sqrt{1 + (t/t_0)^2}} = t_0 \operatorname{arsinh} \frac{t}{t_0} \,. \tag{84}$$

At large times of motion, $t \gg t_0$, one obtains

$$\tau = t_0 \ln \frac{2t}{t_0} \,. \tag{85}$$

Here, use has been made of the well-known representation of the inverse hyperbolic sine, namely

$$\operatorname{arsinh} \zeta = \ln\left(\zeta + \sqrt{1 + \zeta^2}\right).$$

For electrons participating in the process of their acceleration considered above, when $t \approx 10^{-4}$ s, the proper time $\tau \approx 1.45 \times 10^{-9}$ s, i.e., five orders of magnitude less.

The enormous effect of time slowing down essentially permits accelerating nonstable particles, as well. For example, in the case of muons, which are about 200 times heavier than electrons, the characteristic time t_0 will also be 200 times longer: $t_0 \approx 2 \times 10^{-8}$ s. To reach the value of $\gamma \approx 10^6$ during the time of acceleration $t_\beta \approx 2 \times 10^{-2}$ s, muons will cover a path $x_\beta \approx 6000$ km. In accordance with formula (85), the proper time will amount to

$$\tau_\beta = t_0 \ln\left(\frac{2t_\beta}{t_0}\right) \approx 3 \times 10^{-7}\ \mathrm{s}, \tag{86}$$

i.e., will happen to be an order of magnitude smaller than the muon proper lifetime. In this case, the accelerator must be circular.

16. Twin paradox

Consider twin brothers, one of whom has taken part in a space flight from Earth to a distant planet and then returned. From the point of view of the brother who stayed at home, owing to the Lorentzian contraction of time due to the motion of the rocket, when the cosmonaut returned, he was younger than his brother.

However, such reasoning should also mean that the brother-cosmonaut would be justified in expecting the ageing of his brother who had stayed at home to slow down, since the latter first 'flew away' together with Earth and then returned to the rocket. Precisely this is the twin paradox, which is due to incorrect application of the relativity principle.

Indeed, the brother who stays at home is always in the same inertial reference system (IRS) associated with Earth. Contrariwise, the reference system associated with the rocket is noninertial (NIRS) during the stages of speeding up and rocket drag, while along the path of uniform movement toward the distant planet, and then back to Earth, one should speak of two different IRSs traveling in opposite directions with the velocity of the rocket.

In Fig. 14a, the world line of a rocket (WLR) traveling with a velocity $V \sim c$ to a planet at a distance L_0 from Earth is presented in the Minkowski plane. Within the sections of speeding up and slowing down, which may occupy a small time as compared with the time of uniform motion, the world line is distorted. The world line of Earth (WLE) coincides with the ordinate axis. The dashed straight line indicates the light line coinciding with the bisectrix of the right angle between the coordinate axes.

We shall perform a most simple calculation of the flight duration, taking advantage of the Lorentz formula for time transformation:

$$t' = \gamma\left(t - \frac{V}{c^2}\,x\right). \tag{87}$$

We shall neglect the time (read by a clock on Earth) spent for speeding up and slowing down the rocket, and we shall represent the world line of the rocket as two straight lines, OA and AP (Fig. 14b). Obviously, the flight duration t_0 measured by a clock on Earth will amount to

$$t_0 = \frac{2L_0}{V}. \tag{88}$$

The clock on board the rocket that returned will show less time

$$t_0' = \frac{t_0}{\gamma}. \tag{89}$$

Thus, the brother-cosmonaut will age less. To him, there is nothing to be surprized at in this shorter time, since, owing to the Lorentzian contraction of length, Earth, in a time t_0', will have 'flown' with a velocity V through a distance $2L' = 2L_0/\gamma$. He may, however, make an incorrect conclusion concerning the time t_0 if he considers the time $\tilde{t}_0$ on Earth, which had slowed down compared to t_0' by the factor γ, owing to Earth's motion, to represent the total time:

$$\tilde{t}_0 = \frac{t_0'}{\gamma} = \frac{t_0}{\gamma^2}. \tag{90}$$

The error in the cosmonaut's conclusion consists in $\tilde{t}_0$ being only a small part of the time that passed on Earth during his flight mission. When the cosmonaut approached the planet and was in the frame of reference K_1', moving away from Earth, AB_1 was the simultaneity line, while, when he started returning to Earth, AB_2 became the simultaneity line. The actual time $\tilde{t}_0 = (OB_1 + B_2P)/c$. Thus, the cosmonaut had not taken into account the time interval B_1B_2/c.

For estimating this time interval, we make use of formula (87), in which we set $t'_{A_1} = t_0'/2$. At this instant, the rocket is at point A_1, as close as possible to the turning point A. In our reasoning, point A_1 is used to reflect the fact that the rocket was still in the frame of reference K_1' moving away from Earth. Then, from formula (87) follows the equation for the straight line B_1A_1:

$$ct = \frac{ct_0'}{2\gamma} + \frac{Vx}{c}. \tag{91}$$

All the events on this straight line take place simultaneously at the instant $t' = t_0'/2$ (from the cosmomaut's point of view). Obviously, the readings of clocks in the IRS associated with Earth will be different for these events. For example, clocks on Earth will show for event B_1 the time

$$t_{B_1} = \frac{OB_1}{c} = \frac{t_0'}{2\gamma} = \frac{1}{2}\,\tilde{t}_0 = \frac{1}{2}\,\frac{t_0}{\gamma^2}, \tag{92}$$

which follows from formula (91) at $x = 0$, while for event A_1 ($x \approx L_0$), clocks on Earth will show the time $t_{A_1} = t_0/2$.

Upon transferring to another frame of reference K_2', the clock of the cosmonaut shows $t'_{A_2} \approx t'_{A_1}$. At the same time as event A_2, events will occur in this frame that lie on the dashed

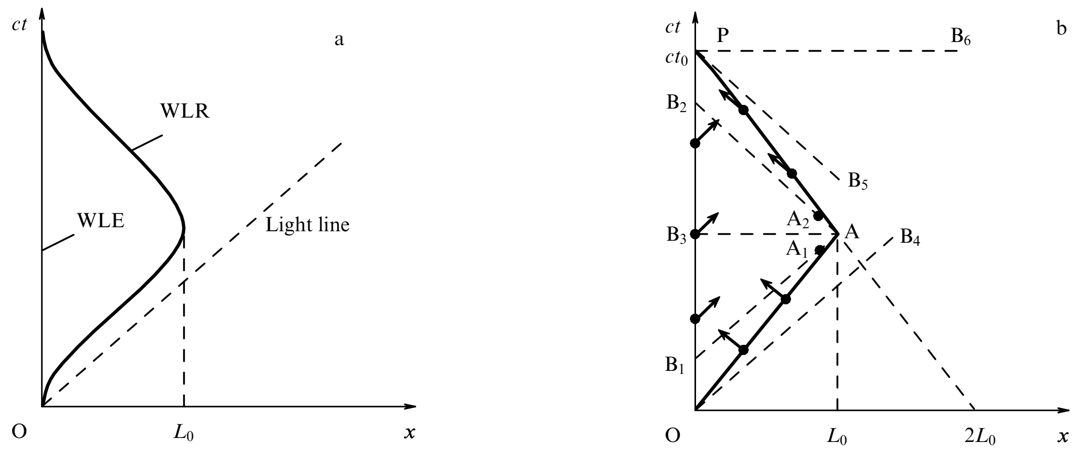

Figure 14. Illustration of the twin paradox in the Minkowski plane.

straight line B_2A_2. In the IRS associated with Earth, clocks will show differing times for these events. Thus, for example, in the case of happening the event B_2, time on Earth is

$$t_{B_2} = \frac{OB_2}{c}\,. \qquad (93)$$

In order to calculate this time, we shall obtain the equation for the simultaneity line B_2A_2.

From reasons of symmetry, it is obvious that at instant $t = t' = 0$ the origin of the frame of reference K_2' was at a distance $x = 2L_0$ from point O (Earth). Therefore, the coordinate x must be replaced in formula (87) by the quantity $x - 2L_0$. We also recall that the velocity V of the frame of reference K_2' is directed toward Earth. Then, one has

$$t' = \gamma\left[t + \frac{V}{c^2}\,(x - 2L_0)\right]. \qquad (94)$$

From the last formula we obtain for the simultaneity line B_2A_2 (again we assume $t' = t_0'/2 = t_0/2\gamma$):

$$ct = \frac{ct_0}{2\gamma^2} - \frac{V}{c}\,(x - 2L_0)\,. \qquad (95)$$

Thus, for example, for event A_2 infinitely close to A ($x \approx L_0$), from formula (95) we find $t_{A_2} = t_0/2$, and for event B_2 ($x = 0$), the time on Earth is

$$t_{B_2} = \frac{t_0}{2\gamma^2} + 2\,\frac{V}{c^2}\,L_0 = t_0 - \frac{t_0}{2\gamma^2}\,. \qquad (96)$$

Thus, for the aforementioned time B_1B_2/c, we obtain

$$\frac{B_1B_2}{c} = t_{B_2} - t_{B_1} = t_0 - \frac{t_0}{\gamma^2}\,. \qquad (97)$$

If this time is added to time (90), we will have

$$t = \frac{t_0}{\gamma^2} + t_0 - \frac{t_0}{\gamma^2} = t_0\,, \qquad (98)$$

which will give rise to no objections from the inhabitants of Earth.

Let us note the peculiarities of the flow of time during starts from Earth and from the planet and two landings. During the start from Earth, the rocket speeds up in a short time interval $\delta t \ll t_0$ and, by the end of acceleration of the rocket, the cosmonaut will perceive clocks on Earth and on the planet in the IRS of Earth to be no longer synchronized (clocks on the planet will show greater time) by the quantity

$$\Delta t = \frac{VL_0}{c^2} = \beta^2\,\frac{t_0}{2}\,. \qquad (99)$$

Therefore, during the approach to the planet at instant $t_0/2$ (readings by clocks on the planet), clocks on Earth will show the time

$$\frac{t_0}{2} - \Delta t = \frac{t_0}{2\gamma^2} = t_{B_1}\,. \qquad (100)$$

During the launch of a rocket from the surface of the Earth, the simultaneity line, which at the beginning coincided with the x-axis, will turn and occupy the position OB_4 (Fig. 14b), while during the space flight it will move in parallel to itself to position B_1A. When landing on the planet, the end of the simultaneity line AB_1 (point B_1) will start, during a time $\delta t \ll t_0$, to displace upwards (the nonsynchronicity will decrease), and, at the instant of time the rocket stops, the simultaneity line AB_3 will be parallel to the x-axis. This means that clocks on Earth and on the planet will show the same time, equal to $t_0/2$ (while the cosmonaut's clock will show a time $t_0/2\gamma$).

During the start of the rocket from the planet, nonsynchronicity will again arise, but the clocks on Earth will already be fast by the quantity (99), and during the start time δt the simultaneity line will move from position AB_3 to position AB_2. During the space flight toward Earth, this line will move in parallel to itself to position PB_5. In landing on Earth, which takes time δt, the simultaneity line will turn, having occupied position PB_6, and once again it will become parallel to the abscissa axis. The clocks on Earth and on the planet register the same flight time, equal to t_0, while the cosmonaut's clock will show the time t_0/γ. Thus, from the point of view of the cosmonaut, it was acceleration of the rocket's motion (two starts and two landings) that occupied most of the time that passed on Earth during the flight.

We shall once more note that both twins (both on Earth and on board the rocket) will feel the natural (uniform) course of time.

It is interesting to see how the traveller will perceive radio signals sent from Earth in identical time intervals, according to clocks on Earth, and how similar signals sent from the rocket will be perceived on Earth. From Fig. 14b, in which the radio signals sent out are indicated by arrows, it is seen that on the way to the planet the traveller will register only a small part of all the signals sent to him, while on the return trip he will register a much larger part. On Earth, the signals from the rocket will be registered in very long time intervals practically during the whole trip, and only during the period of the very short final stage in short intervals.

Naturally, even an inexperienced reader will understand that in reality it would be impossible to carry out such an experiment. First, for the rocket to achieve such velocities it would be necessary to spend an enormous amount of energy, which could not be 'stored' on board the rocket. Second, to achieve such velocities in a time comparable to the life span of human beings would require such large accelerations that the overloads would be essentially superior to the limit level compatible with life. And, finally, a flight at sublight velocities is incompatible with the possibilities of radiation safety owing to the presence of interstellar gas (hydrogen and much heavier ions).

17. Conclusion

The approach presented in this methodical note is certainly not to be considered universal and comprehensive: it should be perceived as a possible version of the exposition of STR fundamentals. If one abides by the main didactic principle of achieving maximum accessibility and clarity, this seems to have been sufficiently provided for by the description of real experiments, the results of which exhibit clear physical content and unquestionable reliability.

In this article we have deliberately dealt as little as possible with 'light' experiments confirming the validity of STR. This has permitted us to avoid speculation concerning the existence of ether and etheric wind, as well as 'sensational' announcements of measured values of the speed of light exceeding the conventional one.

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