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LETTERS TO THE EDITORS

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Spatial structures can form in stochastic dynamic systems due to near-zero-probability events

(comment on "21st century: what is life from the perspective of physics?" (*Usp. Fiz. Nauk* **180** 337 (2010) [*Phys. Usp.* **53** 327 (2010)]) by G R Ivanitskii)

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Abstract. This letter is aimed to briefly highlight the fact that, along with many probabilistic models (including the random graphs and chains currently in wide use), there is a direct universal approach to describing the formation of stochastic structures with probability unity in random media (in 'chaos'), i.e., in almost all random field realizations. The mathematical level adopted is accessible to early undergraduate students. It can thus be concluded that clustering in chaos is not only a physical phenomenon but also a universal phenomenon inherent in nature — with some provisos, of course.

Such random nonstationary phenomena as mixing and *clustering* in phase and physical spaces can occur in *individual realizations of random fields* in stochastic parametrically excitable dynamical systems described by partial differential equations. Clustering of a field is the appearance of compact regions with high field intensities against the background of surrounding regions with relatively low fields.

The physical phenomenon of structure formation itself is well known in physics. Suffice it to mention *Anderson dynamic localization* of eigenfunctions of the one-dimensional stationary Schrödinger equation with random potential and, accordingly, *dynamic localization* of wave intensity in the boundary problem of waves propagating in random layered media (Helmholtz stochastic equation). This phenomenon is closely related to the question of intermittency of random processes.

Intermittence of stochastic processes is a more or less 'regular' alternation of excursions of a process toward higher or lower values relative to a certain determinate curve (regardless the magnitude of excursions) characterizing the dynamics of individual realizations as a whole over the entire interval of evolution. This curve, known as the typical realization curve $f^*(t)$, has the following properties for any stochastic process $f(t)$ (see, for instance, review [1] and monographs [2, 3]).

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On the one hand, the following expression holds for probabilities at any fixed instant t:

$$P\{f(t) > f^*(t)\} = P\{f(t) < f^*(t)\} = \frac{1}{2}\,.$$

On the other hand, a specific property of curve $f^*(t)$ consists in the fact that it is '*wrapped around*' with a random process $f(t)$ for any time interval (t_1, t_2), so that the mean time for which the inequality $f(t) > f^*(t)$ is fulfilled coincides with the mean time during which the opposite inequality $f(t) < f^*(t)$ is satisfied, namely

$$\langle T_{f(t)>f^*(t)}\rangle = \langle T_{f(t)<f^*(t)}\rangle = \frac{1}{2}(t_2 - t_1)\,.$$

In the case of a Gaussian random process $u(t)$, the typical realization curve coincides with the mean value of process: $u^*(t) = \langle u(t)\rangle$.

Inherent in parametrically excitable processes are large but rare excursions due to gently sloping 'tails' of the respective probability distributions. All the statistics for such processes is dominated by these large excursions. For the simplest model of a parametrically excited positive lognormal stochastic process $f(t)$, the typical realization curve also coincides with the statistical Lyapunov exponent $f^*(t) = \exp(\langle \ln f(t)\rangle) = f(0)\exp(-\alpha t)$, where the parameter $\alpha = -\lim_{t\to\infty} \partial\langle \ln f(t)\rangle/\partial t$ is the statistical Lyapunov characteristic index. This index can be either positive or negative, or vanish (a critical case). For $\alpha > 0$, the behavior of individual realizations of a random process corresponds to an exponential decrease, and for $\alpha < 0$ to an increase, with time. At $\alpha = 0$, the intermittency evolves with respect to the straight line $f^*(t) = f(0)$. Note that the positivity of index α for one-dimensional problems corresponds to the physical phenomenon of dynamic localization.

Furthermore, *clustering of both a passive scalar admixture* (density fields) *and a vector admixture* (magnetic field energy) sometimes occurs in *turbulent transfer* problems described by partial differential equations. Stochastic structure formation may also have the form of a *caustic structure* of wave field intensity in the problems of waves propagating in randomly inhomogeneous media in the framework of the stochastic parabolic Leontovich equation (Schrödinger nonstationary equation with a random potential). All these stochastic structures form (under proper conditions) with probability unity, i.e., almost in its each realization (see Refs [1–3]).

Certainly, any stochastic field $f(\mathbf{r}, t)$ possesses the property of intermittency as well. First and foremost, time evolution $f(\mathbf{r},t)$ for any fixed spatial point $\mathbf{r}$ is a random process for which all the aforesaid holds true. For a statistically homogeneous problem, all one-point statistical characteristics of field $f(\mathbf{r},t)$ are unrelated to point $\mathbf{r}$, and positive index $\alpha = -\lim_{t\to\infty} \partial \langle \ln f(\mathbf{r},t)\rangle/\partial t$ for a log-normal field $f(\mathbf{r},t)$ means that its realizations diminish with time at any point in space, even if large rare excursions occur in a log-normal process. In this case, the characteristic time of field diminution is $t \sim 1/\alpha$. However, if this field decreases almost everywhere, it must be somewhere concentrated, i.e., clusterization must take place.

The phenomenon of spatial structure formation (clustering) in individual realizations of stochastic fields can be detected and described only by the analysis of simultaneous single-point probability densities of solutions of the respective equations, based on the ideas of statistical topography.

The main subject of study in statistical topography of random fields, just as in the traditional topography of massifs, is a set of contours, i.e., level lines (in the two-dimensional case) or surfaces (in the three-dimensional case) of constant value given by the equality $f(\mathbf{R},t) = f = \text{const}$.

A system of contours (for simplicity and clearness, we confine ourselves to considering the two-dimensional case, $\mathbf{R} = \{x,\ y\}$) is convenient to analyze by introducing the Dirac delta-function $\varphi(\mathbf{R},t;f) = \delta(f(\mathbf{R},t) - f)$, termed the *indicator function*, concentrated on these contours.

This function serves to express, for example, such quantities as the total area enclosed by level lines of the regions in which the random field $f(\mathbf{R},t)$ surpasses the specified f level, i.e., $f(\mathbf{R},t) > f$:

$$S(t;f) = \int \theta\big(f(\mathbf{R},t) - f\big)\,\mathrm{d}\mathbf{R} = \int_f^\infty \mathrm{d}f' \int \mathrm{d}\mathbf{R}\,\varphi(\mathbf{R},t;f')\,.$$

The total field 'mass' in these regions is given by

$$M(t;f) = \int f(\mathbf{R},t)\,\theta\big(f(\mathbf{R},t) - f\big)\,\mathrm{d}\mathbf{R} = \int_f^\infty f'\,\mathrm{d}f' \int \mathrm{d}\mathbf{R}\,\varphi(\mathbf{R},t;f')\,,$$

where $\theta\big(f(\mathbf{R},t) - f\big)$ is the Heaviside function. The value of the indicator function averaged over the ensemble of random $f(\mathbf{R},t)$-field realizations determines the probability density $P(\mathbf{R},t;f) = \langle \delta(f(\mathbf{R},t) - f)\rangle$ simultaneous in time and one-point in space. This means that values of $S(t;f)$ and $M(t;f)$ averaged over the realization ensemble are directly specified by this probability density:

$$\big\langle S(t;f)\big\rangle = \int_f^\infty \mathrm{d}f' \int \mathrm{d}\mathbf{R}\,P(\mathbf{R},t;f')\,,$$

$$\big\langle M(t;f)\big\rangle = \int_f^\infty f'\,\mathrm{d}f' \int \mathrm{d}\mathbf{R}\,P(\mathbf{R},t;f')\,.$$

Evidently, contour clustering condition with probability unity for the *positive field* $f(\mathbf{R},t)$ in the general case, i.e., for almost all its realizations, is a tendency toward simultaneous fulfillment of asymptotic equalities as $t \to \infty$:

$$\big\langle S(t;f)\big\rangle \to 0\,, \qquad \big\langle M(t;f)\big\rangle \to \int \mathrm{d}\mathbf{R}\,\big\langle f(\mathbf{R},t)\big\rangle\,.$$

The absence of structure formation corresponds to the tendency toward simultaneous fulfillment of asymptotic equalities as $t \to \infty$:

$$\big\langle S(t;f)\big\rangle \to \infty\,, \quad \big\langle M(t;f)\big\rangle \to \int \mathrm{d}\mathbf{R}\,\big\langle f(\mathbf{R},t)\big\rangle\,.$$

For a spatially homogeneous field $f(\mathbf{R},t)$ for which the one-point probability density $P(\mathbf{R},t;f)$ is $\mathbf{R}$-independent, the statistical averages of all expressions (without integration over $\mathbf{R}$) describe the respective specific (i.e., per unit area) values of these quantities. By way of example, specific mean area $\big\langle S_{\text{hom}}(t;f)\big\rangle$ on which the random field $f(\mathbf{R},t)$ exceeds the specified level f coincides with the probability of an event $f(\mathbf{R},t) > f$ at any point in space, namely

$$\big\langle S_{\text{hom}}(t;f)\big\rangle = \big\langle \theta(f(\mathbf{R},t) - f)\big\rangle = P\big\{f(\mathbf{R},t) > f\big\}\,;$$

thus, mean specific area is a geometric interpretation of the probability of an event $f(\mathbf{R},t) > f$, which is naturally independent of point $\mathbf{R}$. Consequently, conditions of structure formation (clustering) for a *homogeneous* case reduce to the tendency to satisfy asymptotic equalities as $t \to \infty$:

$$\big\langle S_{\text{hom}}(t;f)\big\rangle = P\big\{f(\mathbf{r},t) > f\big\} \to 0\,,$$

$$\big\langle M_{\text{hom}}(t;f)\big\rangle \to \big\langle f(t)\big\rangle\,.$$

The absence of clustering corresponds to the tendency toward fulfilling asymptotic equalities as $t \to \infty$:

$$\big\langle S_{\text{hom}}(t;f)\big\rangle = P\big\{f(\mathbf{r},t) > f\big\} \to 1\,,$$

$$\big\langle M_{\text{hom}}(t;f)\big\rangle \to \big\langle f(t)\big\rangle\,.$$

To sum up, *clustering in a spatially homogeneous problem is a physical phenomenon that occurs with probability unity, i.e., for almost all random positive field realizations, and is generated by a rare event, the probability of which tends to vanish*.

In the present case, the very existence of rare events provides a *triggering mechanism* that initiates structure formation, while the latter itself is a property of a stochastic medium.

The characteristic time of cluster formation in space therewith depends on the character of the above asymptotic expressions at large times. This time is determined not only by the statistical Lyapunov characteristic index α but also by diffusion coefficient D_f in the phase space of the positive field $f(\mathbf{r},t)$. Certainly, this time is longer than the characteristic time of realization decrease at each point in space.

Thus, the problem of physical field clustering for concrete physical dynamical systems reduces to the calculation of the stochastic Lyapunov index α and diffusion coefficient D_f; generally speaking, it is a rather cumbersome problem for concrete partial differential equations.

Many authors argue that if an event is destined to happen, it must be most probable. In a recent (very interesting) work [4], *a hypothesis of the origin of life from the perspective of physics* was suggested. "*Life can be briefly described as a result of a game process, an interplay between part of the system and its environment. During the game, this part acquired an ability to remember the probabilities of gains and losses in previous rounds, which gave it a chance to exist in the following ones*."

I cannot share the opinion that the origin of life is a game process. I believe it is rather an event that occurred with the probability unity.

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clustering phenomenon in random media in the framework of the idea of statistical topography.

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