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Similarity theory in neutron kinetics and its implications for RFNC–VNIIEF applied research

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Abstract. Exact similarity relations resulting from the neutron kinetic equation are presented. A number of eigenvalue problem solutions are obtained. Analytical solutions for the stationary and nonstationary formulations of the Milne problem are given.

1. Introduction

The 1938 discovery of uranium nuclei fission by neutrons (O Hahn and F Strassmann [1], L Meitner and O R Frisch [2]) has been followed by intense international research on the physics of atomic nucleus fission and the properties of neutrons interacting with matter.

The Manhattan Project in the USA and the Soviet Atomic project in the USSR made the neutron one of the most active subjects of investigation, both experimentally and theoretically.

The establishment in 1946 of Design Bureau No. 11 (KB-11 in *Russ. abbr.*), later renamed as the Russian Federal Nuclear Center — All-Russian Research Institute of Experimental Physics (RFNC–VNIIEF) for such studies was quite soon followed by setting up and launching experimental facilities for measuring neutron interaction cross sections with both fissionable and nonfissionable nuclei. The research team led by A N Protopopov at the Neutron Physics Measurement Laboratory pioneered the field.

In 1949, a facility named FKBN (*Russ. abbr.* for Fast Neutrons Physical Reactor) was designed and put into operation for determining the critical parameters of fissile material assemblies, including those with various types of neutron reflectors. To carry out this task on this test bench, a special laboratory was established with G N Flerov, a future academician, as its head (and with Yu S Zamyatnin, D P Shirshov, A A Berezin, and some others among its first staff).

Soon afterwards, valuable experimental data were obtained that allowed the validation of theoretical computational work on the choice of designs and on the justification of the characteristics of the first and subsequent USSR-built nuclear and thermonuclear weapons. Modified over time, these facilities and test benches are still in operation at RFNC–VNIIEF as a source of important experimental information necessary for maintaining the reliability and safety of the country's nuclear arsenal under the provisions of the 1996 Comprehensive Nuclear Test-Ban Treaty (CNTBT).

Theoretical and computational work on neutron transport in various, including layered, systems relied heavily on the contribution of prominent theoretical physicists and mathematicians assigned to the Soviet Atomic project, with special mention being due to N A Dmitriev, who started in the project as a staff member in Ya B Zel'dovich's department.

Of immense importance was the theoretical work on neutron kinetics by a group led by I E Tamm, Corresponding Member of the USSR Academy of Sciences at that time. The group was assigned to KB-11 in 1950 in the context of

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developing the first Soviet thermonuclear bomb (RDS-6S) successfully tested on 12 August 1953.

Comprising I E Tamm' group were the following: Andrey Sakharov, Candidate of Sciences in Physics and Mathematics (a degree equivalent to Ph.D. in many respects); S Z Belen'ky, Doctor of Sciences in Physics and Mathematics; Yu A Romanov, researcher; N N Bogoliubov, Academician of the Academy of Sciences of the Ukrainian SSR; I Ya Pomeranchuk, Doctor of Sciences in Physics and Mathematics, and V N Klimov and D V Shirkov, both researchers. The last two, later joined by D N Zubarev and still later, in 1951, by Yu A Tserkovnikov and V S Vladimirov, worked under the immediate direction of N N Bogoliubov, the founder of the RFNC–VNIIEF scientific school of mathematics.

The key research priorities of the N N Bogoliubov group were, among others: to develop maximum efficiency numerical methods for calculating the critical parameters of multilayered symmetric nuclear systems; to set up the calculation of energy release from such systems; to determine the probability of an incomplete explosion, and to settle the issue of neutrons slowing down. The then available methods of solving the Peierls integral equation proved unsuitable for multilayered systems. Listed below are the numerical methods that were developed and put into practice by the Bogoliubov group for solving (by hand) neutron transport equations:

— method of characteristics (V S Vladimirov, 1951),

— factorization method (V S Vladimirov, 1953),

— spherical harmonics method (V S Vladimirov, E V Malinovskaya),

— approaches to treating the problems of neutron propagation and slowing down (V N Klimov, D V Shirkov, V S Vladimirov, I A Zhernak, A A Bunatyan), and

— approaches to treating energy release problems for multilayered spherical systems (V S Vladimirov, E V Malinovskaya).

N N Bogoliubov was among those who initiated the use of computers for solving Soviet Atomic project-related problems, in particular, that of neutron transport in layered systems. Preceding the launch of the first KB-11 computer (named Strela, Russian for 'Arrow'), the Bogoliubov group performed the computer programming of those computational algorithms that had worked well when used manually.

Yu A Romanov in 1951 used the exact solutions of the single-velocity kinetic equation to develop an improved method for solving diffusion problems that allowed a sufficiently accurate theoretical estimation to be made of the critical masses of systems involving different fissile materials.

Following the first successful field tests was the stage of producing nuclear and thermonuclear weapons with improved (and added) physical characteristics. Paralleling this work was computational theoretical research, in particular on the neutron transport theory, which was greatly contributed to by V G Zagrafov, V G Morozov, and other theoretical physicists.

This paper presents some theoretical results from neutron kinetics studies overlapping with or relevant to the main activities of RFNC–VNIIEF.

The theory of neutron transport is fundamentally based on kinetic equations which describe how spatial and energy distributions of particles vary with time in various systems. Although a wide variety of numerical methods have been developed for numerically solving the kinetic equation for neutrons, their analytical counterparts still retain their topical interest. Of particular importance in this context are the exact analytical solutions of the kinetic equation. Although obtainable, of course, only in relatively rare cases and under certain simplifying assumptions, such solutions enable a detailed insight into the general behavior of neutron kinetics processes, as well as allowing the verification of the mathematical techniques used. Therefore, it is exact relations and analytic solutions that will be our main concern below.

2. The structure of the kinetic equation

Let us introduce the neutron distribution function $\psi(t, \mathbf{r}, \mathbf{V})$ in the phase space of vectors $\mathbf{r}$, $\mathbf{V}$ in such a way that $\psi(t, \mathbf{r}, \mathbf{V})\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{V}$ is the number of neutrons with velocity $\mathbf{V}$ (to within $\mathrm{d}\mathbf{V}$) in the neighborhood of point $\mathbf{r}$ inside volume $\mathrm{d}\mathbf{r}$ at time t.

In the linear transport theory — and this is the one we will consider below — the distribution function obeys an integro-differential equation which, neglecting the delayed neutrons (the reason being that we are concerned with fast pulsed systems in this paper), has the form

$$\frac{\partial\psi(t, \mathbf{r}, \mathbf{V})}{\partial t} + \left(\mathbf{V}\,\frac{\partial}{\partial \mathbf{r}}\right)\psi(t, \mathbf{r}, \mathbf{V}) + \zeta(t, \mathbf{r}, \mathbf{V})\,\psi(t, \mathbf{r}, \mathbf{V}) = \int_{(V')} \Gamma(t, \mathbf{r}, \mathbf{V}'\mathbf{V})\,\psi(t, \mathbf{r}, \mathbf{V}')\,\mathrm{d}\mathbf{V}'\,. \tag{1}$$

Here, $\zeta(t, \mathbf{r}, \mathbf{V})\,\mathrm{d}t$ is the probability of a neutron traveling with velocity $\mathbf{V}$ and interacting with a nucleus in time interval $\mathrm{d}t$, and $\Gamma(t, \mathbf{r}, \mathbf{V}', \mathbf{V})\,\mathrm{d}t\,\mathrm{d}\mathbf{V}$ is the probability of a neutron traveling with velocity $\mathbf{V}'$ and interacting with matter in time interval $\mathrm{d}t$, producing in the process a neutron with velocity $\mathbf{V}$ in the interval from $\mathbf{V}$ to $\mathbf{V} + \mathrm{d}\mathbf{V}$.

In the general case, the right-hand side of Eqn (1) should be supplemented by the source term $q(t, \mathbf{r}, \mathbf{V})$: $q(t, \mathbf{r}, \mathbf{V})\,\mathrm{d}t\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{V}$ is the number of neutrons with a velocity between $\mathbf{V}$ and $\mathrm{d}\mathbf{V}$ emitted by independent sources in time interval $\mathrm{d}t$ in volume element $\mathrm{d}\mathbf{r}$ around a point with radius vector $\mathbf{r}$.

Possible interaction channels between neutrons and nuclei in a material are characterized by the following elementary (microscopic) interaction cross sections: σ_c, σ_f, σ_s, and σ_in, respectively, for neutron capture, nuclear fission, and elastic and inelastic scatterings, all summing up to the total cross section $\sigma = \sigma_\mathrm{c} + \sigma_\mathrm{f} + \sigma_\mathrm{s} + \sigma_\mathrm{in}$. Therefore, one obtains

$$\Gamma = \Gamma_\mathrm{s} + \Gamma_\mathrm{f} + \Gamma_\mathrm{in}\,, \tag{2}$$

$$\zeta = \zeta_\mathrm{s} + \zeta_\mathrm{f} + \zeta_\mathrm{in} + \zeta_\mathrm{c}\,, \tag{3}$$

with subscripts having the same meaning as for the cross sections.

Expressions for the terms in Eqns (2) and (3) contain the relevant macroscopic cross sections, for example, $\alpha_{si} = n_\mathrm{nucl}\sigma_{si}$, where n_nucl is the number of particles per unit volume, and σ_{si} is the elementary cross section of neutron elastic scattering on the i-kind nucleus.

When considered for practical application, the complexity of equation (1) prevents it from being solved exactly for the necessary boundary and initial equations. Still, some general conclusions can be reached even without knowing the explicit form of the functions $\zeta(t, \mathbf{r}, \mathbf{V})$ and $\Gamma(t, \mathbf{r}, \mathbf{V}, \mathbf{V}')$.

3. Some consequences of the exact kinetic equation invariance under similarity transformations

Because the density of nuclei $n_{\mathrm{nucl}}(\mathbf{r})$ is proportional to the density of the substance $\rho(\mathbf{r})$, it follows that

$$\Gamma \sim \rho, \quad \zeta \sim \rho\,.$$

Let us first consider the case where both the density of nuclei and the composition of the substance are uniform. Let us make the following changes of variables in the kinetic equation (1):

$$t \to t' = \frac{\rho}{\rho'}\,t\,, \tag{4}$$

$$\mathbf{r} \to \mathbf{r}' = \frac{\rho}{\rho'}\,\mathbf{r}\,. \tag{5}$$

When transformed to the new variables, Eqn (1) retains its form in terms of the primed function $\psi'(t', \mathbf{r}', \mathbf{V})$. This means that kinetic equation (1) is invariant under transformations (4) and (5), which we will call similarity transformations.

In what follows, we explore critical active systems — that is, ones containing fissionable substances. In such systems, $\partial\psi/\partial t = 0$, and Eqn (1) becomes stationary.

Suppose we have an arbitrarily shaped homogeneous critical body of characteristic size $R = R_*$ (hereinafter all critical parameters are asterisked). The problem under consideration involves, along with Eqn (5), the partial similarity transformation

$$R'_* = \frac{\rho_* R_*}{\rho'_*}\,. \tag{6}$$

The volume of any body can be written out in the form $V_{\mathrm{T}} = C_{\mathrm{T}} R^3$, where the constant C_{T} is determined by the geometric form of the body, and by choosing R as a characteristic size. Then, the mass m of the body is $C_{\mathrm{T}}\rho R^3$, and relation (6) yields the constancy of the product of the critical mass m_* and the square of the density:

$$m_* \rho_*^2 = \mathrm{const}\,, \tag{7}$$

which holds for arbitrarily shaped similar bodies.

As long ago as 1943 or even earlier, American scientists derived relation (7) for the special case of a homogeneous active sphere using a solution of the neutron diffusion equation (see Ref. [3]).

Now let us consider the general case of density arbitrarily dependent on the coordinates, $\rho(\mathbf{r}) = \bar{\rho}\varphi(\xi)$, where $\bar{\rho} = \int \rho(\mathbf{r})\,\mathrm{d}\mathbf{r} / \int \mathrm{d}\mathbf{r}$ is the average density of an active system arbitrary in composition and geometry, $\xi = \mathbf{r}/R$ is a dimensionless coordinate, and $\varphi(\xi)$ is a profile function normalized as follows: $\int \varphi(\xi)\,\mathrm{d}\xi / \int \mathrm{d}\xi = 1$.

It is straightforward to show that the kinetic equation is now invariant under the transformations

$$t \to t' = \frac{\bar{\rho}}{\bar{\rho}'}\,t\,, \quad \mathbf{r} \to \mathbf{r}' = \frac{\bar{\rho}}{\bar{\rho}'}\,\mathbf{r}\,, \tag{8}$$

and that the similarity condition

$$R'_* \bar{\rho}'_* = R_* \bar{\rho}_*$$

holds true for profile critical systems with similar density profiles.

This exact relation between the critical sizes and the averaged densities is in correspondence with the following theorem which is cited but not proved in Ref. [4]: "If, in any critical system, all its components are uniformly decreased (increased) in density then, to make the system again critical, all its linear sizes should be increased (decreased) in the same proportion."

Notice that for the class of geometrically similar systems considered here, the following formula is valid:

$$m_* \bar{\rho}_*^2 = \mathrm{const}.$$

4. Similarity theory within the framework of single-velocity neutron kinetics and the general solution of the eigenvalue problem

The single–velocity approximation is fully justified for fast neutron systems, when a construction from fissile materials does not contain moderators for fission spectrum neutrons. Intermediate or thermal neutron systems are not amenable to the single-velocity theory.

In the single-velocity approximation all neutrons have the same velocity V. We will also simplify things by assuming that the indicatrix of neutron elastic scattering on nuclei is isotropic and that inelastic processes are absent.

Accordingly, the kinetic equation for the distribution function in the phase space of the vectors $\mathbf{r}$, $\mathbf{\Omega} = \mathbf{V}/V$ is written in the form

$$\frac{1}{V}\frac{\partial\psi(t,\mathbf{r},\mathbf{\Omega})}{\partial t} + \left(\mathbf{\Omega}\,\frac{\partial}{\partial\mathbf{r}}\right)\psi(t,\mathbf{r},\mathbf{\Omega}) + \alpha\psi(t,\mathbf{r},\mathbf{\Omega}) = \frac{\beta}{4\pi}\,n(t,\mathbf{r}). \tag{9}$$

The right-hand side of equation (9) contains the neutron density

$$n(t,\mathbf{r}) = \int \psi(t,\mathbf{r},\mathbf{\Omega}')\,\mathrm{d}\mathbf{\Omega}'\,.$$

In the general case of a multicomponent medium, when the spatial dependences of $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ parameters are determined not only by the function $\rho(\mathbf{r})$ but also by the way in which the concentration $\mu_i(\mathbf{r})$ of the i-kind nuclei depends on coordinates, we have

$$\beta(\mathbf{r}) = n_{\mathrm{nucl}}(\mathbf{r}) \sum_i \mu_i(\mathbf{r})(\sigma_{\mathrm{s}i} + \nu_i\,\sigma_{\mathrm{f}i})\,, \tag{10}$$

$$\alpha(\mathbf{r}) = n_{\mathrm{nucl}}(\mathbf{r}) \sum_i \mu_i(\mathbf{r})(\sigma_{\mathrm{s}i} + \sigma_{\mathrm{f}i} + \sigma_{\mathrm{c}i})\,, \tag{11}$$

where

$$n_{\mathrm{nucl}}(\mathbf{r}) = \frac{N_{\mathrm{A}}\rho(\mathbf{r})}{\sum_i \mu_i(\mathbf{r})\,A_i}$$

is the density of nuclei, A_i is the mass number of the i-kind nucleus, μ_i is the average number of secondary neutrons emitted in a single fission event of a nucleus of the i-kind, and N_{A} is the Avogadro constant.

The quantity $h = \beta/\alpha$ is called the activity of the medium. For inert, neutron-multiplying, and neutron-absorbing media, $h = 1$, $h > 1$, $h < 1$, respectively.

The kinetic equation should be supplemented by the initial and boundary conditions

$$\psi(t=0,\mathbf{r},\mathbf{\Omega}) = \psi_0(\mathbf{r},\mathbf{\Omega})\,,$$

$$\psi|_S = 0\,, \quad \text{if} \quad (\mathbf{\Omega}\mathbf{N}_S) < 0\,,$$

where $\mathbf{N}_S$ is the unit normal to the surface of the system, pointing to the vacuum. Our assumed boundary condition — that there is no flux of neutrons from the vacuum to the substance — restricts the systems to be considered to the class of simply connected objects with everywhere nonconcave outer surfaces.

4.1 Elements of the similarity theory of nonstationary homogeneous systems

We will follow Ref. [5] by considering the parameters α and β to be constant and by introducing new dimensionless arguments

$$\mathbf{z} = \beta\mathbf{r} = h\alpha\mathbf{r}\,, \quad \tau = h\alpha V t\,.$$

Then, instead of equation (9), we obtain for the distribution function in the phase space of the vectors $\mathbf{z}$, $\mathbf{\Omega}$:

$$\left[\frac{\partial}{\partial\tau} + \left(\mathbf{\Omega}\,\frac{\partial}{\partial\mathbf{z}}\right)\right]\psi(\tau,\mathbf{z},\mathbf{\Omega}) + \frac{1}{h}\,\psi(\tau,\mathbf{z},\mathbf{\Omega}) = \frac{1}{4\pi}\int \mathrm{d}\mathbf{\Omega}'\psi(\tau,\mathbf{z},\mathbf{\Omega}')\,, \tag{12}$$

with the corresponding initial and boundary conditions

$$\psi(\tau=0,\mathbf{z},\mathbf{\Omega}) = \psi_0(\mathbf{z},\mathbf{\Omega})\,,$$

$$\psi\big|_{\Sigma} = 0 \quad \text{при} \quad (\mathbf{\Omega}\,\mathbf{N}_\Sigma) < 0\,,$$

where the vector $\mathbf{N}_\Sigma$ is normal to the surface of the system in $\mathbf{z}$ space and points to vacuum.

Similarity theorem. The product of the exponential $\exp(\tau/h)$ and the neutron distribution function in the phase space of the vectors $\mathbf{z}$, $\mathbf{\Omega}$ is independent of the nuclear physical properties of the material of the system and is determined only by the boundary and initial conditions.

Proof. We seek the solution of equation (12) in the form

$$\psi(\tau,\mathbf{z},\mathbf{\Omega}) = f(\tau,\mathbf{z},\mathbf{\Omega})\exp\left(-\frac{\tau}{h}\right). \tag{13}$$

This, when substituted into Eqn (12), yields the integro-differential equation

$$\left[\frac{\partial}{\partial\tau} + \left(\mathbf{\Omega}\,\frac{\partial}{\partial\mathbf{z}}\right)\right] f(\tau,\mathbf{z},\mathbf{\Omega}) = \frac{1}{4\pi}\int \mathrm{d}\mathbf{\Omega}' f(\tau,\mathbf{z},\mathbf{\Omega}') \tag{14}$$

with the initial and boundary conditions

$$f(\tau=0,\mathbf{z},\mathbf{\Omega}) = \psi_0(\mathbf{z},\mathbf{\Omega})\,, \tag{15}$$

$$f\big|_{\Sigma} = 0 \quad \text{at} \quad (\mathbf{\Omega}\,\mathbf{N}_\Sigma) < 0\,. \tag{16}$$

Because neither kinetic equation (14) nor the conditions (15), (16) contain the parameters h and α, the solution $f(\tau,\mathbf{z},\mathbf{\Omega})$ is independent of the nuclear physical properties of the medium. This had to be proven.

It should be noted that on going over to the usual coordinates t and $\mathbf{r}$, equation (14) yields the kinetic equation for the function $f(t,\mathbf{r},\mathbf{\Omega})$:

$$\left[\frac{1}{V}\frac{\partial}{\partial t} + \left(\mathbf{\Omega}\,\frac{\partial}{\partial\mathbf{r}}\right)\right] f(t,\mathbf{r},\mathbf{\Omega}) = \frac{\beta}{4\pi}\int \mathrm{d}\mathbf{\Omega}' f(t,\mathbf{r},\mathbf{\Omega}')\,,$$

which is invariant under the similarity transformations

$$t \to t' = \frac{\beta}{\beta'}\,t\,,$$

$$\mathbf{r} \to \mathbf{r}' = \frac{\beta}{\beta'}\,\mathbf{r}\,.$$

Based on the boundary condition-dictated similarity criterion for two systems of the same geometry type, viz.

$$\beta_2 R_2 = \beta_1 R_1\,, \tag{17}$$

an exact relation between the functions $\psi_2(t,\mathbf{r},\mathbf{\Omega})$ and $\psi_1(t,\mathbf{r},\mathbf{\Omega})$ was obtained [5].

4.2 General solution of the eigenvalue problem

It is well known (see, for example, monograph [6]) that, in the case of two finite-size systems, the general solution of the kinetic equation (9) for neutrons is the following superposition

$$\psi(t,\mathbf{r},\mathbf{\Omega}) = \sum_m a_m \exp(\lambda_m t)\,\psi_m(\mathbf{r},\mathbf{\Omega})\,, \tag{18}$$

which contains the eigenvalues $\lambda = \lambda_0 > \lambda_1 > \lambda_2 \ldots$ and their corresponding eigenfunctions ψ_m. For $t \gg t_0 = 1/\lambda_0$, the distribution function (18) takes the form

$$\psi(t,\mathbf{r},\mathbf{\Omega}) = \exp(\lambda t)\,\psi(\mathbf{r},\mathbf{\Omega})\,, \tag{19}$$

where $\lambda = \lambda_0$ is the principal eigenvalue (PEV); $\psi(\mathbf{r},\mathbf{\Omega}) = \psi_0(\mathbf{r},\mathbf{\Omega})$ is the principal eigenfunction (PEF).

The distribution function in the form (19), where the solution distinctly contains a temporal part and a spatial part, will be called equilibrium in what follows.

In the case of Eqn (19), the following stationary kinetic equation is valid:

$$\left(\mathbf{\Omega}\,\frac{\partial}{\partial\mathbf{r}}\right)\psi(\mathbf{r},\mathbf{\Omega}) + \left[\alpha + \frac{\lambda}{V}\right]\psi(\mathbf{r},\mathbf{\Omega}) = \frac{\beta}{4\pi}\int \mathrm{d}\mathbf{\Omega}'\psi(\mathbf{r},\mathbf{\Omega}')\,, \tag{20}$$

which in $\mathbf{z}$ space transforms to

$$\left(\mathbf{\Omega}\,\frac{\partial}{\partial\mathbf{z}}\right)\psi(\mathbf{z},\mathbf{\Omega}) + E\psi(\mathbf{z},\mathbf{\Omega}) = \frac{1}{4\pi}\int \mathrm{d}\mathbf{\Omega}'\psi(\mathbf{z},\mathbf{\Omega}')\,, \tag{21}$$

with $E = (\alpha + \lambda/V)/\beta$ being the new principal eigenvalue. Apart from the quantity E, equation (21) contains no other parameters, and the corresponding boundary condition involves the characteristic size of the system.

From systems with the characteristic size R and with the same type of geometry, let us single out those with the same product βR. In a $\mathbf{z}$ space, such systems all have not only the same geometric form but also the same characteristic size $Z = \beta R$. Upon going over to dimensionless variables, similarity condition (17) for systems in $\mathbf{r}$ space reduces to the equality of characteristic sizes:

$$Z_2 = Z_1\,.$$

In the case of a fixed-geometry object, for each Z there will be its own and besides single number E. Thus, a universal function $E = E(\beta R)$ exists, whose explicit form is determined by the geometric form of the object.

Using the relation between E and λ, we obtain the dependence of PEV on the characteristics of the system:

$$\lambda = \beta V\left[E(\beta R) - \frac{1}{h}\right] = \alpha V\left[hE(\beta R) - 1\right].$$

For the optical depth of the object, $p = \alpha R$, tending to zero, the neutron escape from the medium can be neglected, and $\lambda \to \lambda_\infty = (\beta - \alpha)\, V = (h-1)\,\alpha V$, where λ_∞ is a known quantity for the case of an infinite homogeneous medium. Thus, one has

$$\lim_{\alpha R\to\infty} E(h\alpha R) = \lim_{\beta R\to\infty} E(\beta R) = 1\,.$$

Also, for any critical system $\lambda = 0$ and

$$E(\beta_* R_*) = \frac{1}{h}\,.$$

At $E = 0$, kinetic equation (21) degenerates into the following:

$$\left(\mathbf{\Omega}\,\frac{\partial}{\partial \mathbf{z}}\right)\psi(\mathbf{z},\mathbf{\Omega}) = \frac{1}{4\pi}\int \mathrm{d}\mathbf{\Omega}'\psi(\mathbf{z},\mathbf{\Omega}')\,.$$

For systems with arbitrary h in a degenerate state, $\lambda = -\alpha V$. If the optical thickness of a system is less than for the degenerate case, then the function $E(\beta R)$ is negative.

From the general solution of a PEV problem, it is an easy matter to obtain similarity formulas. For example, using the condition $E(\beta' R') = E(\beta'' R'')$, we have for similar systems with sizes R', $R'' = (\beta'/\beta'')\,R'$:

$$\lambda'' = \left[\frac{h''}{h'}\left(1 + \frac{\lambda'}{\alpha' V'}\right) - 1\right]\alpha'' V''\,.$$

The above formulas were obtained for the PEV $\lambda = \lambda_0$. In Ref. [7] it is shown that the m-th eigenvalue is given by

$$\lambda_m = \beta V\left[E_m(\beta R) - \frac{1}{h}\right].$$

It should be emphasized that the eigenvalue (EV) formulas presented above have the same accuracy as their underlying single-velocity kinetic equation. The formulas give visual insight into what quantities determine EVs and how.

General formulas for λ allow testing both the analytical solutions of the EV problem and their numerical counterparts obtained by various mathematical methods. Expressions for EV may also prove useful in constructing interpolation relations and in extrapolating known data to the as-yet-unexplored range of variability of physical parameters.

4.3 Similarity formulas for systems with coordinate-dependent parameters α and β

Let us consider the general case [see formulas (10), (11)] in which the spatial variation of α and β parameters is determined not only by the function $\rho(\mathbf{r})$ but also by the concentrations of various nuclear species, $\mu_i = \mu_i(\mathbf{r})$.

We introduce the dimensionless variable $\xi = \mathbf{r}/R$ and write down formulas (10), (11) in the form $\alpha = \bar{\alpha}A(\xi)$, $\beta = \bar{\beta}B(\xi)$, where $\bar{\alpha}$ and $\bar{\beta}$ are the volume-averaged quantities, and the profile functions are normalized as follows:

$$\frac{\int \mathrm{d}\xi\, A(\xi)}{\int \mathrm{d}\xi} = \frac{\int \mathrm{d}\xi\, B(\xi)}{\int \mathrm{d}\xi} = 1\,.$$

Kinetic equation (20) then becomes

$$\left(\mathbf{\Omega}\,\frac{\partial}{\partial \xi}\right)\psi(\xi,\mathbf{\Omega}) + \left[A(\xi) + \frac{\lambda}{\bar{\alpha}V}\right]\bar{\alpha}R\psi(\xi,\mathbf{\Omega}) = \frac{\bar{\beta}RB(\xi)}{4\pi}\int \mathrm{d}\mathbf{\Omega}'\psi(\xi,\mathbf{\Omega}')\,.$$

Once this equation is solved for an arbitrary initial system '1', passage to any similar system '2' does not change the PEF, namely

$$\psi_2\left(\frac{\mathbf{r}}{R_2},\mathbf{\Omega}\right) = \psi_1\left(\frac{\mathbf{r}}{R_1},\mathbf{\Omega}\right),$$

provided the conditions

$$\left[A_2(\xi) + \frac{\lambda_2}{\bar{\alpha}_2 V}\right]\bar{\alpha}_2 R_2 = \left[A_1(\xi) + \frac{\lambda_1}{\bar{\alpha}_1 V}\right]\bar{\alpha}_1 R_1\,, \tag{22}$$

$$\bar{\beta}_2 R_2 B_2(\xi) = \bar{\beta}_1 R_1 B_1(\xi) \tag{23}$$

are satisfied.

Integrating expressions (22) and (23) over the dimensionless volume, we arrive at

$$\left[1 + \frac{\lambda_2}{\bar{\alpha}_2 V}\right]\bar{\alpha}_2 R_2 = \left[1 + \frac{\lambda_1}{\bar{\alpha}_1 V}\right]\bar{\alpha}_1 R_1\,, \tag{24}$$

$$\bar{\beta}_2 R_2 = \bar{\beta}_1 R_1\,, \tag{25}$$

leading to the following expression for the PEV:

$$\lambda_2 = \bar{\alpha}_2 V\left[\frac{\bar{\alpha}_1\bar{\beta}_2}{\bar{\alpha}_2\bar{\beta}_1}\left(1 + \frac{\lambda_1}{\bar{\alpha}_1 V}\right) - 1\right].$$

The profile functions $A_2(\xi)$ and $B_2(\xi)$ are found, using expressions (24), (25), to be

$$A_2(\xi) = 1 + \frac{\bar{\alpha}_1\bar{\beta}_2}{\bar{\alpha}_2\bar{\beta}_1}\left[A_1(\xi) - 1\right], \tag{26}$$

$$B_2(\xi) = B_1(\xi)\,. \tag{27}$$

The functions $A_1(\xi)$ and $B_1(\xi)$ corresponding to the initial system can be piecewise continuous and have any number of points of discontinuity (or jumps).

It should be emphasized that, in the class of new similar systems being considered, the parameter $\beta_2 = \bar{\beta}_2 B_2$ at a fixed characteristic size R_2 is fully determined by formulas (25) and (27). If we choose $\alpha_2 = \bar{\alpha}_2 A_2$, only the profile function A_2 should satisfy condition (26), whereas $\bar{\alpha}_2$ can be chosen arbitrarily.

In the special case of critical systems ($\lambda_2 = \lambda_1 = 0$), the following equalities are valid:

$$\bar{\alpha}_{*2} R_{*2} = \bar{\alpha}_{*1} R_{*1}\,,$$

$$\bar{\beta}_{*2} R_{*2} = \bar{\beta}_{*1} R_{*1}\,.$$

The formulas above can be applied to layered systems, as illustrated in Ref. [8] by specific examples.

4.4 Particular solutions of the principal eigenvalue problem as applied to profile systems

We will abandon the single-velocity approximation for a moment and will rely on the results of Section 2 instead.

Based on the exact equation (1), let us consider finite-size inhomogeneous systems in which only the density of substance $\rho(\mathbf{r}) = \bar{\rho} u(\mathbf{r})$ (where $\bar{\rho}$ is the average density of substance, and $u(\mathbf{r})$ is the profile function, with $\int \mathrm{d}\mathbf{r}\, u(\mathbf{r}) = 1$) depends on coordinates (importantly, in the same manner in all the systems).

Let us solve the problem on the PEF $\psi(\mathbf{r}, \mathbf{V})$ and the PEV λ. Setting $\psi(t, \mathbf{r}, \mathbf{V}) = \exp(\lambda t)\, \psi(\mathbf{r}, \mathbf{V})$ in the kinetic equation (1) yields

$$\lambda \psi(\mathbf{r}, \mathbf{V}) + \left(\mathbf{V} \frac{\partial}{\partial \mathbf{r}}\right) \psi(\mathbf{r}, \mathbf{V}) + \zeta(\mathbf{r}, \mathbf{V})\, \psi(\mathbf{r}, \mathbf{V}) = \int_{(V')} \Gamma(\mathbf{r}, \mathbf{V}', \mathbf{V})\, \psi(\mathbf{r}, \mathbf{V}')\, \mathrm{d}\mathbf{V}' . \tag{28}$$

Because Eqn (28) must be invariant under the coordinate transformation $\mathbf{r} \to \mathbf{r}' = \bar{\rho}\mathbf{r}/\bar{\rho}'$, the following equality holds:

$$\lambda' = \frac{\bar{\rho}'}{\bar{\rho}}\, \lambda . \tag{29}$$

The invariant of transformation (8) is the product of the average density $\bar{\rho}$ of substance and the characteristic size R of the system. Therefore, the solution of the PEV problem in the case discussed has the form

$$\lambda = \bar{\rho} F(\bar{\rho} R) . \tag{30}$$

The dimensional function $F(\bar{\rho} R)$ is determined by the geometry of the object.

4.5 Features of underlying kinetic processes in homogeneous systems with ultimately high optical thicknesses

We start by asking ourselves what types of physically meaningful solutions can be obtained based on the two-constant (α and β) inhomogeneous kinetic equation written in $\mathbf{r}$ space:

$$\frac{\partial \psi(t, \mathbf{r}, \mathbf{\Omega})}{\partial t} + V\left(\mathbf{\Omega} \frac{\partial}{\partial \mathbf{r}}\right) \psi(t, \mathbf{r}, \mathbf{\Omega}) + \alpha V \psi(t, \mathbf{r}, \mathbf{\Omega}) = \frac{\beta V}{4\pi}\, n(t, \mathbf{r}) + q(t, \mathbf{r}, \mathbf{\Omega}) . \tag{31}$$

Integrating equation (31) over the angle $\mathbf{\Omega}$ and then over the volume of the system yields

$$\frac{\partial n(t, \mathbf{r})}{\partial t} + \nabla \mathbf{j}(t, \mathbf{r}) + \alpha V n(t, \mathbf{r}) = \beta V n(t, \mathbf{r}) + \int \mathrm{d}\mathbf{\Omega}\, q(t, \mathbf{r}, \mathbf{\Omega}) ,$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda(t)\, N(t) + Q(t) , \tag{32}$$

$$\lambda(t) = (h - 1)\, \alpha V - W(t) ,$$

$$W(t) = \frac{\Pi(t)}{N(t)} = \frac{\int \mathbf{j}(t, \mathbf{r})\, \mathrm{d}\mathbf{S}}{\int n(t, \mathbf{r})\, \mathrm{d}\mathbf{r}} ,$$

where $\mathbf{j}(t, \mathbf{r}) = V \int \mathbf{\Omega} \psi(t, \mathbf{r}, \mathbf{\Omega})\, \mathrm{d}\mathbf{\Omega}$ is the vector neutron flux, $Q(t)$ is the number of neutrons emitted by the outer source per unit time, $N(t)$ is the total number of neutrons in the system, $\Pi(t) = \int \mathbf{j}(t, \mathbf{r})\, \mathrm{d}\mathbf{S}$ is the neutron escape rate to vacuum, and $W(t)$ is the probability per unit time for a neutron to escape from the system.

Let the characteristic size of the object under consideration be R. The escape probability is inversely proportional to the surface-to-volume ratio of the system, i.e., $W \sim 1/R$ ($W \sim 1/\beta R = 1/hp$ in a $\mathbf{z}$ space).

Let us consider equation (32). If we suppose that the outer source of neutrons is constant, $Q(t) = Q_0$, and pass to the limit $p = \alpha R \to \infty$, then the neutron escape from the system can be neglected, and

$$\lambda(t) \to \lambda_\infty = (h - 1)\, \alpha V .$$

Thus, the equation for the function $N(t)$ reduces to

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \lambda_\infty N(t) + Q_0 \tag{33}$$

with the initial condition $N(t = 0) = N_0$.

Equation (33) has different types of solutions depending on the sign of λ_∞:

(1) $\lambda_\infty = 0$ (inert medium with $h = 1$). In this case, the total number of neutrons increases linearly with time:

$$N(t) = N_0 + Q_0 t ,$$

in the presence of a source ($Q_0 \neq 0$) or remains constant, $N(t) = N_0$, otherwise.

(2) $\lambda_\infty > 0$ (above-critical system with $h > 1$). The total number of neutrons in such a system increases exponentially with time:

$$N(t) = \left[N_0 + \frac{Q_0}{\lambda_\infty}\right] \exp(\lambda_\infty t) - \frac{Q_0}{\lambda_\infty} = \left[N_0 + \frac{Q_0}{\lambda_\infty}\left(1 - \exp(-\lambda_\infty t)\right)\right] \exp(\lambda_\infty t) . \tag{34}$$

For $t \gg 1/\lambda_\infty$, solution (34) changes over to exponential behavior:

$$N(t) = \left(N_0 + \frac{Q_0}{\lambda_\infty}\right) \exp(\lambda_\infty t) ,$$

independent of whether a source of neutrons is present or not.

(3) $\lambda_\infty < 0$ (subcritical system with $h < 1$). In this case, the total number of neutrons is given by expression (34) with $\lambda_\infty = -|\lambda_\infty|$. If a source Q_0 operates, then after a lapse of time $t \gg 1/|\lambda_\infty|$ the total number of neutrons ceases to change further, and we arrive at the stationary solution

$$N = \frac{Q_0}{|\lambda_\infty|} .$$

If equation (33) is homogeneous ($Q_0 = 0$), the total number of neutrons $N(t)$ decreases exponentially with time:

$$N(t) = N_0 \exp(-|\lambda_\infty| t) .$$

Similarity limit theorem. If the condition $p = \alpha R \to \infty$ holds, then the equilibrium spatial distribution of neutrons $\psi(\mathbf{z}, \mathbf{\Omega})$ inside a homogeneous system in the phase space of the vectors $\mathbf{z} = h\alpha\mathbf{r}$, $\mathbf{\Omega}$ is independent of the nuclear physical properties of the materials the system is made of.

Reference [9] offers two proofs of this theorem, one of which follows.

Let us omit the source term q in equation (31) (which does not lose the generality of the argument). After the solution has acquired the equilibrium form, the distribution function and the neutron density become

$$\psi(t, \mathbf{z}, \mathbf{\Omega}) = C \exp\left[(h - 1)\, \alpha V t\right] \psi(\mathbf{z}, \mathbf{\Omega}) , \tag{35}$$

$$n(t, \mathbf{z}) = C \exp\left[(h - 1)\, \alpha V t\right] n(\mathbf{z}) . \tag{36}$$

Substituting Eqns (35) and (36) into Eqn (31) yields the equation for the function $\psi(\mathbf{z}, \mathbf{\Omega})$:

$$\left(1 + \mathbf{\Omega}\,\frac{\partial}{\partial \mathbf{z}}\right)\psi(\mathbf{z}, \mathbf{\Omega}) = \frac{n(\mathbf{z})}{4\pi}\,. \tag{37}$$

Neither kinetic equation (37) nor the boundary condition (no neutron flux from outside, see above) contain the characteristics of the medium, which proves the theorem.

Some remarks are in order here.

The theorem just proved is valid for any class of arbitrary simply connected systems bounded everywhere by nonconcave surfaces.

In vacuum outside a system with arbitrary optical thickness, the following kinetic equation holds:

$$\left(\mathbf{\Omega}\,\frac{\partial}{\partial \mathbf{r}}\right)\psi_{\text{ex}}(\mathbf{r}, \mathbf{\Omega}) + \frac{\lambda}{V}\,\psi_{\text{ex}}(\mathbf{r}, \mathbf{\Omega}) = 0\,.$$

Here, the second term on the left-hand side is, for $\lambda > 0$, responsible for λ-absorption of neutrons, so that the neutron density in vacuum is a decreasing function. But, if $\lambda < 0$, a λ source of neutrons emerges.

When $p \to \infty$, in a vacuum we have

$$\left(\mathbf{\Omega}\,\frac{\partial}{\partial \mathbf{z}}\right)\psi_{\text{ex}}(\mathbf{z}, \mathbf{\Omega}) + \frac{h-1}{h}\,\psi_{\text{ex}}(\mathbf{z}, \mathbf{\Omega}) = 0\,.$$

5. The Milne problem in neutron transport theory

Milne's neutron-kinetics problem, so named after the scientist who had earlier addressed a similar problem in astrophysics, has gained recognition in the context of atomic bomb creation efforts in the middle of the last century. At the time, the researchers assigned to the U.S. and USSR atomic projects were faced with a number of totally new and complex science and technology problems, one of which was Milne's problem concerning the exact solution of the neutron transport equation for various types of semiinfinite media with a plane boundary between them. What made this problem pressing at the time was a lack of knowledge about the neutron kinetics effects present in the atomic devices under development. Computer technology was then unavailable, and the existing elementary diffusion theory could not provide reliable estimates for the critical masses of the fissile materials, let alone the spatial distributions of neutrons in even the simplest systems. The use of the exact analytical solutions to the transport equation for such systems could add greatly to the reliability of computational and theoretical studies into the choice of atomic weapon design. Therefore, the Milne problem was the focus of much theoretical research globally during the creation of early nuclear weapons — with the result that the sought-for solutions of the stationary kinetic equations were obtained and the issue was closed, even though in the case of neutron-multiplying media the formal stationary solutions proved to be physically inconsistent. A multiplying medium is an above-critical system which requires the solution of a nonstationary kinetic equation in order to describe neutron kinetics.

The following sections present, together with stationary solutions, explicit solutions to the nonstationary Milne problem. The authors of appropriate studies showed that, in the case of an above-critical system, the general physical solution they found for the Milne nonstationary problem leads, in the region asymptotically distant from the boundary, to the neutron density varying linearly with coordinate, $n(x) \sim x + x_0$, in the active and the most active media in the one-region and two-region problems, respectively. Our literature survey failed to uncover this result.

It should be noted that neutron transport is far from being the only physical phenomenon that can be described by the linearized Boltzmann equation. Among the others are the propagation of radiation in stellar atmospheres, the transport of particles in gas discharges, and the passage of X-rays through scattering media, and so forth. Interestingly, some of the fundamental problems that arose in neutron transport theory had been considered and partly solved by astrophysicists (Schwarzschild, Milne, Hopf, and some others) even before the discovery of the neutron (see monographs [6, 10] and references cited therein).

5.1 Stationary Milne problem

Let us first consider the stationary one-region Milne problem, for which we specify that the semiinfinite space to the right of the plane boundary is filled by a homogeneous substance and that the semispace to the left is empty (a vacuum).

The task at hand is, in the case of a plane case, to solve the stationary kinetic equation

$$\mu\,\frac{\partial\psi(y,\mu)}{\partial y} + \psi(y,\mu) = \frac{h}{2}\,n(y) \tag{38}$$

with the boundary condition

$$\psi(0,\mu) = 0 \quad \text{for} \quad \mu > 0\,, \tag{39}$$

where μ is the cosine of the angle between the abscissa and the neutron flight direction toward the point of observation, $y = \alpha x$ is a dimensionless coordinate, and $n(y) = \int_{-1}^{1}\psi(y,\mu)\,\mathrm{d}\mu$ is the neutron density. For an absorbing ($h < 1$) and a multiplying ($h > 1$) medium, the constant-rate flux of neutrons is assumed to come from infinity.

We first reproduce several known results on inert media ($h = \beta/\alpha = 1$).

In their joint paper [11], Placzek and Seidel applied the Wiener–Hopf method to obtain the angular distribution of neutrons at the boundary, $\psi(0,\mu)$, and the Laplace image $\Phi_0(s)$ of the function $n(y)$:

$$\psi(0,\mu) = \frac{\sqrt{3}}{2}\,(1-\mu)\,\tau_-\!\left(-\frac{1}{\mu}\right) = \frac{\sqrt{3}}{2}\,(1-\mu)$$
$$\times\exp\left\{-\frac{\mu}{\pi}\int_0^{\pi/2}\frac{\ln\left[\sin^2 u/(1-u\cot u)\right]}{1-(1-\mu^2)\,\sin^2 u}\,\mathrm{d}u\right\},\ \ \mu < 0\,. \tag{40}$$

$$\Phi_0(s) = \frac{\sqrt{3}(s+1)\,\tau_-(s)}{s^2}\,. \tag{41}$$

For $s > 0$, one obtains

$$\ln\tau_-(s) = \frac{s}{\pi}\int_0^{\pi/2}\frac{\ln\left[\sin^2 u/(1-u\cot u)\right]}{\sin^2 u + s^2\cos^2 u}\,\mathrm{d}u\,.$$

The solution of the Milne problem for the neutron density can be presented as the superposition $n(y) = n_{\text{as}}(y) + \varepsilon(y)$, where $n_{\text{as}}(y)$ is the asymptotic part of the solution, and $\varepsilon(y)$ is a relatively small correction which rapidly decreases with increasing y.

From the Laurent series expansion of $\Phi_0(s)$ at $s = 0$, viz.

$$\Phi_0(s) = \frac{\sqrt{3}\tau_-(0)}{s^2} + \frac{\sqrt{3}\left[\tau_-(0) + \tau'_-(0)\right]}{s} + \dots ,$$

the authors of Ref. [11] concluded that the asymptotic part of the function $n(y)$ is a linear function of y, so that

$$n_{\rm as}(y) = 3(y + y_0) , \tag{42}$$

where y_0 is a constant equal to

$$y_0 = 1 + \frac{\tau'_-(0)}{\tau_-(0)} = \frac{1}{\pi}\int_0^{\pi/2}\left[\frac{3}{\sin^2 x} - \frac{1}{1 - x\cot x}\right] {\rm d}x \approx 0.7104 .$$

In his work [12], Placzek reduced distribution (40) to a more computation-friendly form and tabulated the function $\varphi(\mu) = \psi(0, -\mu)/\sqrt{3}$ in the interval $0 \leqslant \mu \leqslant 1$ with a step $\Delta\mu = 0.01$.

An exact formula for the neutron density in an inert semi-infinite medium, one which includes an asymptotic part and a correction, was given in C Mark's work. His paper [13] reports on how to choose a contour for the inverse Laplace transform of the function $\Phi_0(s)$ [41] and derives the desired expression:

$$n_{\rm M}(y) = 3\left\{y + y_0 - \frac{1}{4}\int_0^1 \frac{\exp(-y/\mu)\,{\rm d}\mu}{\left(\psi(0,-\mu)\left[(1-\mu\,{\rm artanh}\,\mu)^2 + \pi^2\mu^2/4\right]\right)}\right\} . \tag{43}$$

It is known from the work of LeCaine [14] that the stationary Milne problem for absorbing ($h < 1$) and multiplying ($h > 1$) media was first solved by Mark and Adler. These results were not published, however.

Yu A Romanov was another who was able to solve the Milne problem for arbitrary h. Some of his results are as follows [15].

For $h < 1$, the asymptotic neutron density varies exponentially with the dimensionless coordinate y:

$$n_{\rm as}(y) = \sqrt{\frac{2(1-k^2)}{k^2 - 1 + h}}\sinh k(y + y_0) ,$$

where the coefficient k is obtained from the transcendental equation

$$h\,\frac{{\rm artanh}\,k}{k} = 1 , \tag{44}$$

and where $y_0 = y_0(h)$ is a constant which can be approximated to good accuracy by the formula

$$y_0(h) \approx \frac{0.71}{h} .$$

For a neutron-multiplying medium ($h > 1$), the transcendental equation (44) has two purely imaginary roots, and the asymptotic solution takes the form

$$n_{\rm as}(y) = A\sin k(y + y_0) , \tag{45}$$

with A being the normalization constant, and k the absolute value of the root of equation (44).

Solution (45) produces regions with a negative neutron density and is therefore unphysical.

Along with the one-region Milne problem, Yu A Romanov [15] succeeded in solving the stationary problem of two media, i.e., determining the neutron field in two semiinfinite media which are uniformly filled with different substances and have a plane interface. Mathematically, the problem is set up as follows: solve in the $y = \alpha x$ space the system of equations

$$\begin{aligned} &\mu\,\frac{\partial\psi_1(y,\mu)}{\partial y} + \psi_1(y,\mu) = \frac{h_1}{2}\,n_1(y) \quad \text{for} \quad y < 0 , \\ &\mu\,\frac{\partial\psi_2(y,\mu)}{\partial y} + \psi_2(y,\mu) = \frac{h_2}{2}\,n_2(y) \quad \text{for} \quad y > 0 , \end{aligned} \tag{46}$$

subject to the boundary condition $\psi_2(0+,\mu) = \psi_1(0-,\mu)$.

If we assume for definiteness that the neutron flux is directed from medium 1 to medium 2, then the particle density $n(y)$ in the case of absorbing media ($h_1 < 1$, $h_2 < 1$) is given by

$$\begin{aligned} n_1(y) &= \exp(-k_1 y) + A\exp(k_1 y) + \varepsilon_1(y) \\ &= n_{1\rm as}(y) + \varepsilon_1(y) , \quad y < 0 , \\ n_2(y) &= B\exp(-k_2 y) + \varepsilon_2(y) = n_{2\rm as}(y) + \varepsilon_2(y) , \quad y > 0 , \end{aligned} \tag{47}$$

where $n_{1\rm as}(y)$ and $n_{2\rm as}(y)$ are the asymptotic densities, and the functions $\varepsilon_1(y)$ and $\varepsilon_2(y)$ are nonzero only near the interface.

Notice that, for $h_1 > 1$ and $h_2 > 1$, the parameters k_1 and k_2 are purely imaginary, and solutions (47) are periodic.

The constants A and B are found from the boundary conditions for the asymptotic densities:

(1) the continuity of logarithmic derivatives at points y_1 and y_2 yields

$$\frac{n'_{1\rm as}(y_1)}{n_{1\rm as}(y_1)} = \frac{n'_{2\rm as}(y_2)}{n_{2\rm as}(y_2)} , \quad \text{and}$$

(2) density jumps at points y_1 and y_2 give

$$\frac{n_{1\rm as}(y_1)}{\gamma(h_1)} = \frac{n_{2\rm as}(y_2)}{\gamma(h_2)} ,$$

where

$$\gamma(h) = \sqrt{\frac{2k^2(1-k^2)}{3h(k^2 - 1 + h)}}, \quad k = k(h)$$

[see equation (44)].

The positions of points y_1 and y_2 are determined numerically from the formulas

$$\begin{aligned} y_1 &= f(k_1, k_1) - f(k_2, k_1) , \\ y_2 &= f(k_1, k_2) - f(k_2, k_2) , \\ \cosh s f(k,s) &= \sqrt{\frac{h(s)(k^2 - s^2)}{h(s) - h(k)}}\int_0^1 \frac{\mu\varphi(k,\mu)}{1 - s^2\mu^2}\,{\rm d}\mu . \end{aligned} \tag{48}$$

The function $\varphi(k,\mu)$ entering into equality (48) satisfies the integral equation

$$\int_0^1 \frac{\mu_0\varphi(k,\mu_0)}{\mu_0 + \mu}\,{\rm d}\mu_0 = \frac{h}{2\varphi(k,\mu)(1 - k^2\mu^2)} .$$

In the two-media problem, the asymptotic densities undergo a discontinuity at the interface, with neutron density being always greater in the medium where h is larger. The function $n(y)$ is continuous and has an infinite derivative at

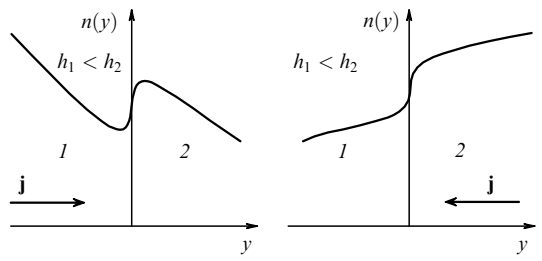

Figure 1. Schematic behavior of the function $n(y)$ near the interface for a stationary problem of two media with two absorbers (left: neutron flux is directed from medium *1* to medium *2*; right: from *2* to *1*).

the interface. Figure 1 illustrates qualitatively the variation of neutron density with the coordinate for the case $h_1 < 1$, $h_2 < 1$.

Alternatively, as shown by Case and Zweifel [10], the one- and two-region problems can be solved by expanding the characteristic transport equation in terms of the singular eigenfunctions. These authors have determined neutron density as a function of both the coordinate and the angular variable μ, and have considered the nonstationary Milne problem.

5.2 Nonstationary Milne problem

Let us consider first the nonstationary one-region Milne problem whose solution can be obtained using the kinetic equation

$$\frac{1}{V}\frac{\partial\psi(t,x,\mu)}{\partial t}+\mu\frac{\partial\psi(t,x,\mu)}{\partial x}+\alpha\psi(t,x,\mu)=\frac{h\alpha}{2}\,n(t,x)\,.$$

We will make use of the similarity limit theorem which was proven in Section 3 for the general case of arbitrary homogeneous systems with infinite optical thicknesses, the systems whose boundary surfaces nowhere concave. According to this theorem, the solution of the nonstationary one-region problem in $z = h\alpha x$ space has, in our case, the form

$$n(t,z)=C_1\exp\left[(h-1)\,\alpha Vt\right]n(z)\,, \tag{49}$$

and its spatial part $n(z)$ is independent of the nuclear physical properties of the medium, i.e., is independent of h.

The stationary solution of Refs [11, 13] for an inert ($h = 1$) medium is a special case of solution (49). Therefore, the required function $n(z)$ is obtained by simply replacing $y = \alpha x$ with $z = h\alpha x$ in expression (43) for $n_{\mathrm{M}}(y)$. The function $n_{\mathrm{M}}(z)$ represents the dependence of the neutron density on coordinate z in a medium with an arbitrary value of parameter h.

Section 3 demonstrated the validity of the stationary solutions of the $h = 1$ source-free problem and of the $h < 1$ problem with a source (a constant-rate neutron flux at an infinitely distant point). The $h < 1$ source-free case admits the nonstationary solution (49). For an above-critical ($h > 1$) system, the unique physically meaningful solution of the Milne problem is

$$n(t,z)=C\exp\left[(h-1)\,\alpha Vt\right]n_{\mathrm{M}}(z)\,, \tag{50}$$

with the asymptotic part increasing linearly with z: $n_{\mathrm{as}}(z) \sim z + z_0$, where $z_0 = 0.7104$.

The distance x_0 from the surface of the medium at which the asymptotic neutron flux vanishes is referred to as the extrapolation length.

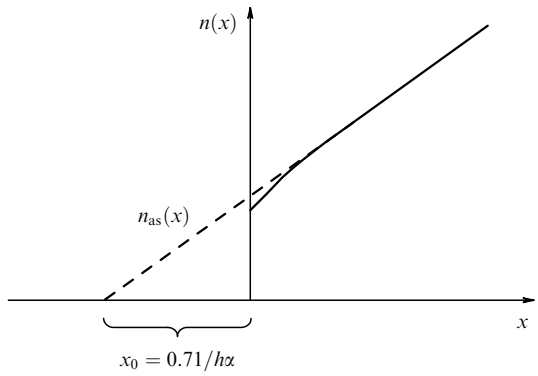

Figure 2. Behavior of $n(x)$ and $n_{\mathrm{as}}(x)$ functions near the interface.

In his fascinating theoretical paper [16] of 1946, R Feynman addressed the question of how the extrapolation length x_0 depends on the properties of the medium. Based on the solutions of the Milne problem known at the time, he arrived at the following result

$$x_0=\frac{0.7104g(h)}{h\alpha}\,.$$

The function $g(h)$ in this formula becomes unity at $h = 1$. With the then available solution of the corresponding stationary Milne problem at his disposal, it was Feynman's belief that the dependence $g(h)$ exists for $h > 1$ — the reason why it was computed and tabulated.

Today, with the availability of the physically meaningful solution (50), it can be argued that $g(h \geqslant 1) = 1$. The qualitative dependences $n(x)$ and $n_{\mathrm{as}}(x)$ for substances with $h \geqslant 1$ are illustrated in Fig. 2.

It should be noted that the above-described solutions of the stationary and nonstationary Milne problems and the results presented in Section 3 from the similarity theory of nonstationary uniform systems were used to verify one of the mathematical techniques developed for the numerical solution of the transport equation (see Refs [17, 18]).

The problem in which at least one of the semiinfinite media is neutron-multiplying is worth discussing here in some detail as one of the most interesting two-region nonstationary Milne problems.

The formulation of this version of the Milne problem is given in Ref. [19], where it is also shown that for the highest-activity $h > 1$ medium the asymptotic part of the solution is a linear function of z increasing away from the boundary. The complete solution of this problem was found by P S Bondarev in a work currently due for publication. Given below are some intermediate calculations and the final result of Bondarev's analysis.

The system of equations to be solved consists of two integro-differential equations:

$$\begin{aligned}
&\frac{1}{V}\frac{\partial\psi_1(t,x,\mu)}{\partial t}+\mu\frac{\partial\psi_1(t,x,\mu)}{\partial x}+\alpha_1\psi_1(t,x,\mu)\\
&\qquad=\frac{\beta_1}{2}\,n_1(t,x)\,,\quad x<0\,,\\
&\frac{1}{V}\frac{\partial\psi_2(t,x,\mu)}{\partial t}+\mu\frac{\partial\psi_2(t,x,\mu)}{\partial x}+\alpha_2\psi_2(t,x,\mu)\\
&\qquad=\frac{\beta_2}{2}\,n_2(t,x)\,,\quad x>0\,.
\end{aligned} \tag{51}$$

If at least one of the two semiinfinite media being considered behaves as neutron-multiplying, then the distribution function varies exponentially in time:

$$\psi(t,x,\mu) = \exp(\lambda t)\,\psi(x,\mu)\,. \tag{52}$$

The parameter λ is determined by that material for which the difference of the Peierls parameters $\beta - \alpha$ is largest. We will specify that $\beta_1 - \alpha_1$ is larger than $\beta_2 - \alpha_2$.

Substituting distribution function (52) into the system of equations (51) and making the replacements

$$x = \frac{z_1}{\alpha_1 + (\lambda/V)} = \frac{z_1}{\beta_1} \tag{53}$$

and

$$x = \frac{z_2}{\alpha_2 + (\lambda/V)} = \frac{z_2}{\alpha_2 - \alpha_1 + \beta_1} \tag{54}$$

in, respectively, the first and the second equations of the system, we obtain

$$\begin{aligned} &\mu\,\frac{\partial\psi_1(z_1,\mu)}{\partial z_1} + \psi_1(z_1,\mu) = \frac{H_1}{2}\,n_1(z_1)\,,\\ &\mu\,\frac{\partial\psi_2(z_2,\mu)}{\partial z_2} + \psi_2(z_2,\mu) = \frac{H_2}{2}\,n_2(z_2)\,, \end{aligned} \tag{55}$$

where

$$\begin{aligned} &H_1 = 1,\\ &H_2 = \frac{\beta_2}{\alpha_2 - \alpha_1 + \beta_1}\,, \end{aligned} \tag{56}$$

and z_1 and z_2 in system (55) are new dimensionless variables, which we will hereafter denote simply as z. For any parameters α_1, β_1, α_2, β_2, the value of H_2 is always less than unity provided at least one of the two media is active ($\lambda > 0$).

Thus, the nonstationary problem for two media is reduced to a stationary one with the parameters $H_1 = 1$, $H_2 < 1$, which can be solved, for example, by the Wiener–Hopf method.

The Milne problem under consideration has the following explicit form in a z space:

$$n_1(z) = -z + z_0 - \varepsilon_1(z)\,, \quad z < 0\,, \tag{57}$$

$$n_2(z) = A\exp(-K_2 z) + \varepsilon_2(z)\,, \quad z > 0\,, \tag{58}$$

where

$$\varepsilon_1(z) = \frac{\sqrt{1-H_2}}{2\sqrt{3}}\int_0^1 \frac{\exp(z/\mu)}{\xi(\mu)}\,\frac{\mathrm{d}\mu}{\left[(1-\mu\,\mathrm{artanh}\,\mu)^2 + \pi^2\mu^2/4\right]}\,,$$

$$\varepsilon_2(z) = \frac{\sqrt{1-H_2}}{2\sqrt{3}}\int_0^1 \frac{\xi(\mu)\,\exp(-z/\mu)\,\mathrm{d}\mu}{\left[(1-h_2\,\mu\,\mathrm{artanh}\,\mu)^2 + h_2^2\pi^2\mu^2/4\right]}\,,$$

$$A = \sqrt{\frac{1-H_2}{3}}\,\frac{2(1-K_2^2)}{H_2(1+K_2^2)(H_2-1+K_2^2)}\,\xi(K_2)\,,$$

$$\xi(\mu) = (1+K_2\mu)\,\exp\left[\frac{\mu}{\pi}\int_0^{\pi/2} \times \frac{\ln\left(1/(1+K_2^2\cot^2 x)\,(1-H_2 x\cot x)/(1-x\cot x)\right)}{\mu^2\sin^2 x + \cos^2 x}\,\mathrm{d}x\right],$$

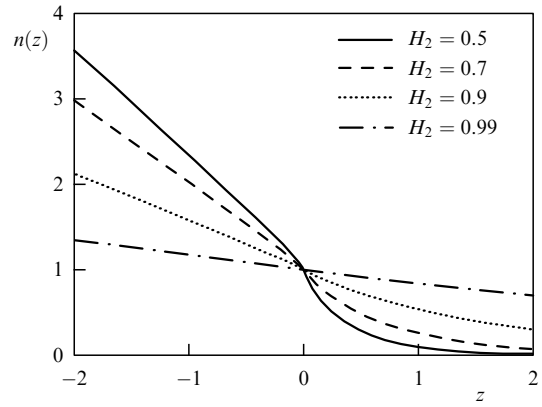

Figure 3. Plots of the neutron density $n(z)$ for several values of H_2.

$$z_0 = \frac{1}{K_2} + z_{01} - z_{02}\,,$$

$$z_{01} = \frac{1}{\pi}\int_0^{\pi/2}\left[\frac{3}{\sin^2 x} - \frac{1}{x\cot x}\right]\mathrm{d}x \approx 0.7104\,,$$

$$z_{02} = \frac{1}{\pi}\int_0^{\pi/2}\left[\frac{3+K_2^2\cot^2 x}{\sin^2 x + K_2^2\cos^2 x} - \frac{1+(1-H_2)\cot^2 x}{1-H_2 x\cot x}\right]\mathrm{d}x\,,$$

and $K_2 = K_2(H_2)$ [see Eqn (44)]. Reverting the replacements made earlier in Eqns (53) and (54) readily yields the function $n(x)$.

Notice that the solutions obtained in z space for any fixed value of the parameter H_2 holds for any infinite set of functions for which

$$\frac{\beta_2}{\alpha_2 - \alpha_1 + \beta_1} = \frac{h_2}{1 + (h_1 - 1)\,\alpha_1/\alpha_2} = H_2 < 1\,.$$

Figure 3 shows the dependence of the neutron density n (57), (58) on the dimensionless coordinate z for some values of the parameter H_2. As $H_2 \to 1$ (infinite homogeneous medium, $h_1 \to h_2$, $\alpha_1 \to \alpha_2$), the solution eventually becomes constant.

6. Conclusions

The invariance property of the kinetic equation under similarity transformations provides general insight into neutron kinetics processes in various media and systems and has been used to confirm some old results and to obtain new ones. For the sake of brevity, only those results valid in the one-velocity approximation are listed below.

(1) A similarity theory was developed for nonstationary homogeneous systems of arbitrary geometric shape. It was shown that, for systems consisting of different substances, a nonstationary problem reduces to a problem with one substance, which allows the verification of mathematical methods developed for the numerical solution of the nonstationary kinetic equation. Also, similarity formulas expressing exact relations between neutron distribution functions in geometrically similar nonstationary systems were obtained.

(2) For homogeneous systems of arbitrary geometry and composition, the general solution of the eigenvalue problem was found, which shows vividly how eigenvalues depend on the properties of the medium.

(3) Similarity formulas were obtained for systems in which macroscopic cross sections for the neutron–nucleus interactions are coordinate dependent.

(4) The similarity limit theorem valid for homogeneous systems was proven, which implies that, in the case of an exponential time dependence of the distribution function $\psi(t, \mathbf{z}, \mathbf{\Omega}) = \exp(\lambda t)\,\psi(\mathbf{z}, \mathbf{\Omega})$ of neutrons in the phase space of the vectors $\mathbf{z} = h\alpha\mathbf{r}$, $\mathbf{\Omega} = \mathbf{V}/V$, this function ceases to depend on the nuclear physical properties of the substance as the optical depth $p = \alpha R$ tends to infinity. This theorem is directly relevant to the Milne problem in the neutron transport theory.

(5) It was shown that, along with the known solutions of the stationary one-region Milne problem, there also exist solutions to its nonstationary versions. Moreover, instead of the formal mathematical solution to the stationary Milne problem which existed earlier — and which was unphysical for neutron-multiplying systems — an exact nonstationary solution was obtained. Based on the body of exact analytical solutions, a number of mathematical techniques were verified.

(6) The complete physical solution was found for the nonstationary two-region Milne problem, in which at least one of the semiinfinite media multiplies neutrons.

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