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On time as a quantum observable canonically conjugate to energy

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Contents

Abstract. A number of studies on time as a quantum observable canonically conjugate to energy, with reference to systems with continuous and discrete energy spectra, are briefly reviewed in this minimal-mathematics methodological note. The results of and prospects for the time analysis of quantum processes using time as a quantum observable are examined.

1. Introduction

It has been known for a long time — since the work of W Pauli some 90 years ago [1] — that, except in special abstract systems (as it later turned out), time cannot be represented by a self-adjoint operator.[1] This is clearly inconsistent with the common expectation that, depending on the case, time — similarly to space — can either play the role of a simple parameter or *be a physical observable* which *should be* represented by an operator. The available literature on the problem of time in quantum mechanics is quite extensive (see, for example, Refs [2–50] and references cited therein). The same situation exists in quantum electrodynamics and in relativistic quantum field theory (see, for example, Refs [11–13, 45, 50]).

The first series of studies (after Ref. [1]) on time in quantum mechanics included, among the most known and cited, Refs [2–20]. In the second series of work on the time as a quantum observable [21–50], published starting in the late 1980s but mostly in the 1990s and later, the primary concern was how the duration of a quantum collision and the tunneling time can be defined in a self-consistent manner. However, some studies in this second series ignored the mathematical theorem proved by Naimark [51], which had crucially underlain the results of Refs [14–20]. This theorem states that the nonorthogonal spectral decomposition $E(\lambda)$ of a Hermitian operator H is the Carleman type (and is unique for a maximal Hermitian operator), i.e., *can be approximated* by a succession of self-adjoint operators[2] whose spectral decompositions weakly converge to the spectral decomposition $E(\lambda)$ of the operator H.

It is precisely Naimark's theorem which was used in Refs [14–20] (more detailed treatments are available in Refs [22–25, 30–33, 45, 48–50]) to show that physical systems with a continuous energy spectrum allow time to be introduced as a quantum-mechanical observable canonically conjugate to energy. It was shown thereby that the operator for time in such systems is usually the maximal Hermitian operator. In Refs [23, 33] (see also Refs [48–50]), it was found that, for systems with discrete energy spectra, time is also a quantum-mechanical observable canonically conjugate to energy and that the time operator is a quasiself-adjoint operator (more precisely, for systems with more than one bound state the time operator can be chosen to be practically self-adjoint to an arbitrary degree of accuracy).

[1] That time cannot be so represented follows precisely from the semi-boundedness of continuous energy spectra (usually zero-bounded from below). It is only for an electrically charged particle in an infinitely extended uniform electric field or for certain special kinds of systems that a continuous energy spectrum is not bounded but rather extends over the entire energy axis, from $-\infty$ to ∞. For such systems, time can be represented by a self-adjoint operator, as it can for other special abstract systems with a continuous energy spectrum bounded from above or below.

[2] Reference [51] took from Ref. [52] a Carleman type expression for the spectral decomposition of unity for a Hermitian operator. Naimark referred to paper [52] as providing a method for proving that a maximal (and not only maximal) Hermitian operator H can be approximated by a succession of bounded self-adjoint operators which are such that the spectral decompositions of unity for them weakly converge to the spectral decomposition $E(\lambda)$ of unity for H. This fact was kindly pointed out to the author by A S Holevo.

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Another approach in the literature is to use the positive operator valued measure (POVM), a concept which can be traced back to the early versions of the quantum theory of measurements and is in wide utilization even today (see, in particular, Refs [26–29, 34–44, 46, 47]). This approach has been known by and large since about the 1960s in the quantum theory of measurements, and it was as early as in Ref. [53] that its simplest version was used as an illustrative tool to describe free motion. It was stated in Refs [26–29, 34–44, 46, 47], with reference to Naimark's extension theorem [54] and using sometimes simplified and abridged arguments, that a generalized decomposition of unity (or the POV measure) can be reproduced from any self-adjoint extension of the time operator into extended Hilbert space (with negative values of energy E on the left half-axis). In actual fact, however, this was not done for other than the simplest cases (for example, free motion of a particle).

Our approach relies on *another* Naimark's theorem [51] (see above) and does *not* involve any extension of the physical Hilbert space of usual wave functions (wave packets) followed by an inverse projection onto the preceding space of wave functions; moreover, it was published in Refs [11–17, 20] (and independently in Refs [18, 19]) prior to works [26–29, 34–44, 46, 47]. Based on Naimark's theorem from Ref. [51] already published earlier, this approach *is much more direct, simpler, and more general — while at the same time no less mathematically rigorous* — than the POVM approach.

2. Time as a quantum observable and the general definition for the average time and duration of a quantum process

For systems with a continuous energy spectrum, we can, following Refs [14–20], introduce the following simple operator $\hat{t}$ for time:

$$\hat{t} = \begin{cases} t & \text{in the time representation,} \quad (1a) \\ -\mathrm{i}\hbar \dfrac{\partial}{\partial E} & \text{in the energy } (E) \text{ representation,} \quad (1b) \end{cases}$$

which is canonically conjugate to energy. The operator $\hat{t}$ is not self-adjoint but is *Hermitian* and acts on square-integrable space–time wave packets and their Fourier transforms in the respective representations (1a) and (1b), with the point $E = 0$ excluded (which limits consideration to only moving wave packets, when *nonmoving* back tails and zero-flow situations are excluded). This condition, incidentally, is quite sufficient for the operator (1) to be, in the terminology of Ref. [55], a '*maximal Hermitian operator*' [14 –20] (see also Refs [24, 25, 31–33, 48–50, 51, 55]). Notice that the exclusion of the point $E = 0$ involves no loss of physical generality because neither zero-velocity states of rest, nor wave packets with *non-moving* back tails, nor zero-flow wave packets are observable.

Operator (1b) is defined as acting on the space P of continuous, differentiable, square-integrable functions $f(E)$ that satisfy the conditions

$$\int_0^\infty \left|f(E)\right|^2 \mathrm{d}E < \infty\,, \qquad \int_0^\infty \left|\frac{\partial f(E)}{\partial E}\right|^2 \mathrm{d}E < \infty\,, \qquad \int_0^\infty \left|f(E)\right|^2 E^2\,\mathrm{d}E < \infty\,, \quad (2)$$

$$f(0) = 0\,. \quad (3)$$

The space P is *dense* in the Hilbert space of L^2 functions defined (only) for all nonnegative E. Most obviously, operator (1) is Hermitian — that is, the relation $(f_1, \hat{t} f_2) = \left((\hat{t} f_1), f_2\right)$ holds true if and only if all square-integrable $f(E)$ functions, whose space is the definition space of operator (1), vanish at the point $E = 0$. The operator $\hat{t}$ has no Hermitian extensions because, otherwise, at least one function $f_0(E)$ would be found to satisfy the condition $f_0(0) \neq 0$, which is inconsistent with Hermiticity. Therefore, in accordance with Refs [51, 55], $\hat{t}$ is a maximal Hermitian operator.

It was for these reasons that Pauli earlier dismissed the idea of introducing the operator for time (see, for example, Ref. [1]), thus virtually postponing for nearly fifty years the study of time as a quantum observable (Naimark's mathematical work [51] was long unknown to those involved, as was the still earlier (1923) work of Carleman [52]). Still, J von Neumann argued as long ago as 1932 [56] that it would be too restrictive to confine quantum mechanics to self-adjoint operators alone. He gave the following example [56] to illustrate his point. Consider a particle moving freely in the half-space x $(0 \leqslant x < \infty)$ bounded by a rigid wall along the $x = 0$ plane. The x-component of the momentum operator for such a particle, viz.

$$\hat{p}_x = -\mathrm{i}\hbar \frac{\partial}{\partial x}\,,$$

is defined as acting in the space of continuous, differentiable, square-integrable functions $f(x)$ satisfying the conditions

$$\int_0^\infty |f(x)|^2\,\mathrm{d}x < \infty\,, \qquad \int_0^\infty \left|\frac{\partial f(x)}{\partial x}\right|^2 \mathrm{d}x < \infty\,, \qquad \int_0^\infty |f(x)|^2 x^2\,\mathrm{d}x < \infty\,,$$

and $f(0) = 0$. This space Q of functions is *dense* in the Hilbert space of L^2 functions defined (only) on the semiaxis $0 \leqslant x < \infty$. Therefore, the operator $\hat{p}_x = -\mathrm{i}\hbar\,\partial/\partial x$ has the same mathematical properties as $\hat{t}$ (1) and, hence, is not self-adjoint but maximal Hermitian. Still, it remains observable and has an obvious physical meaning. The same properties apply to the radial momentum operator

$$\hat{p}_r = -\mathrm{i}\hbar \frac{\partial}{\partial r} + \frac{1}{r}\,, \qquad 0 < r < \infty\,.$$

Curiously, it was also noticed long ago (see, for example, Refs [3, 4]) that the time operator (1) for hypothetical quantum-mechanical systems with continuous energy spectra *bounded both from below and above* $(E_{\min} < E < E_{\max})$ will be a *real self-adjoint operator* with a discrete time spectrum quantized in units of $\tau = \hbar/d$, where $d = |E_{\max} - E_{\min}|$.

The study of time as a quantum-mechanical observable cannot be complete without properly defining two observed average characteristics of particle motions and collisions, namely the time and duration of their occurrence. This, in turn, raises the question of determining the measure (or weight function) for averaging over time in accordance with the averaging procedures and known principles of quantum mechanics. In the case of one-dimensional unidirectional motion of a particle, such a *measure* (or weight function) can be obtained using the following simple quantity

$$W(x,t)\,\mathrm{d}t = \frac{j(x,t)\,\mathrm{d}t}{\int_{-\infty}^{\infty} j(x,t)\,\mathrm{d}t}\,, \quad (4)$$

where the probabilistic interpretation of $j(x,t)$ (precisely in terms of *time*) corresponds to the probability flux density of a particle passing point x at the instant of time t (more precisely, passing point x within a unit time interval centered at t), assuming the particle travels in the positive x direction. This measure is not postulated but rather directly follows from the well-known probabilistic (*spatial*) interpretation of the probability density $\rho(x,t)$ and the continuity equation

$$\frac{\partial\rho(x,t)}{\partial t}+\operatorname{div} j(x,t)=0 \tag{5}$$

for a particle moving in the field of any Hamiltonian described by a one-dimensional Schrödinger equation. Recall that $\rho(x,t)$ here is the probability of finding a moving particle in a unit spatial interval centered at x at the instant of time t. The probability density $\rho(x,t)$ and the probability flux density $j(x,t)$ are, as usual, related to the particle's wave function $\Psi(x,t)$ by the respective relations $\rho(x,t)=|\Psi(x,t)|^2$ and $j(x,t)=\operatorname{Re}[\Psi^*(x,t)(\hbar/\mathrm{i}\mu)\,\partial\Psi(x,t)/\partial x]$. Measure (4) was first introduced and investigated in Refs [20, 22–25, 30–33].

If the flux density $j(x,t)$ changes sign, the quantity $W(x,t)\,\mathrm{d}t$ is no longer positively defined and acquires the physical meaning of the probability density *only* within those bounded time intervals where the flux density $j(x,t)$ maintains its sign. In this case, it is natural to introduce two measures, distinguishing between the positive and negative probability fluxes by their signs:

$$W_{\pm}(x,t)\,\mathrm{d}t=\frac{j_{\pm}(x,t)\,\mathrm{d}t}{\int_{-\infty}^{\infty}j_{\pm}(x,t)\,\mathrm{d}t} \tag{6}$$

with $j_{\pm}(x,t)=j(x,t)\,\Theta(\pm j)$, where $\Theta(z)$ is the Heaviside step function, which is zero for $z<0$, and unity for $z>0$ (see Appendix).

The average value of $\langle t(x)\rangle$ at the instant of time t, when a particle passes through the point x (assuming it travels unidirectionally along the x-axis), and the average value of $\langle t_{\pm}(x)\rangle$ at the instant of time t, when the particle passes through the point x (assuming it travels in the positive or negative x direction), can be defined as

$$\begin{aligned}\langle t(x)\rangle&=\frac{\int_{-\infty}^{\infty}t\,j(x,t)\,\mathrm{d}t}{\int_{-\infty}^{\infty}j(x,t)\,\mathrm{d}t}\\&=\frac{\int_0^{\infty}\mathrm{d}E\,[G^*(x,E)\,\hat{t}vG(x,E)+vG^*(x,E)\,\hat{t}G(x,E)]/2}{\int_0^{\infty}\mathrm{d}E\,v|G(x,E)|^2},\end{aligned} \tag{7}$$

where v is the velocity of the particle passage through the point x, $G(x,E)$ is the Fourier transform of the moving one-dimensional (1D) wave packet:

$$\begin{aligned}\Psi(x,t)&=\int_0^{\infty}G(x,E)\exp\left(-\frac{\mathrm{i}Et}{\hbar}\right)\mathrm{d}E\\&=\int_0^{\infty}g(E)\,\varphi(x,E)\exp\left(-\frac{\mathrm{i}Et}{\hbar}\right)\mathrm{d}E,\end{aligned} \tag{8}$$

on changing from the time to the energy representation:

$$\langle t_{\pm}(x)\rangle=\frac{\int_{-\infty}^{\infty}t\,j_{\pm}(x,t)\,\mathrm{d}t}{\int_{-\infty}^{\infty}j_{\pm}(x,t)\,\mathrm{d}t}\,. \tag{9}$$

For the 1D passage of a particle from the point x_i to $x_\mathrm{f}>x_\mathrm{i}$, and for 1D reflection of a particle from region (x_i,∞) to region $x_\mathrm{f}\leqslant x_\mathrm{i}$, the average durations are given respectively by

$$\begin{aligned}\langle\tau_\mathrm{T}(x_\mathrm{i},x_\mathrm{f})\rangle&=\langle t_+(x_\mathrm{f})\rangle-\langle t_+(x_\mathrm{i})\rangle\,,\\\langle\tau_\mathrm{R}(x_\mathrm{i},x_\mathrm{f})\rangle&=\langle t_-(x_\mathrm{f})\rangle-\langle t_+(x_\mathrm{i})\rangle\,.\end{aligned} \tag{10}$$

3. Energy-time uncertainty relation for continuous energy spectrum systems

It is easy to see that the operator for time (1) and its canonically conjugate energy operator

$$\hat{E}=\begin{cases}E & \text{in the energy } (E) \text{ representation},\\ \mathrm{i}\hbar\,\dfrac{\partial}{\partial t} & \text{in the time } (t) \text{ representation},\end{cases} \tag{11}$$

satisfy the standard commutation relation

$$[\hat{E},\hat{t}\,]=\mathrm{i}\hbar\,. \tag{12}$$

It should be noted that relation (12), in accordance with the Stone–von Neumann theorem [57], is interpreted as holding *only* for a pair of self-adjoint canonically conjugate to each other operators in both representations, and to date has not been directly generalized to *maximal Hermitian operators*. Nevertheless, the difficulty of directly proving this generalization was in fact indirectly surmounted by introducing $\hat{t}$ using a *unique* Fourier (Laplace) transform from the t-axis $(-\infty<t<\infty)$ to the E-semiaxis $(0<E<\infty)$, and in doing so, exploiting, as done in Refs [18–20, 22, 23, 31–33, 48–50], the special mathematical properties of maximal symmetric operators described in detail in, for example, Ref. [55].

From Eqn (12) and using a simple generalization of standard transforms for self-adjoint canonically conjugate quantities (see Refs [16–20, 22, 23, 31–33, 48–50]), one directly derives the uncertainty relation

$$\Delta E\,\Delta t\geqslant\frac{\hbar}{2}\,, \tag{13}$$

where uncertainties are presented as root-mean-square deviations $\Delta a=\sqrt{\mathrm{D}a}$, with $\mathrm{D}a=\langle a^2\rangle-\langle a\rangle^2$ $(a=E,t)$ and $\langle\ldots\rangle$ standing for averaging over t with a measure $W(x,t)\,\mathrm{d}t$ or $W_{\pm}(x,t)\,\mathrm{d}t$ in the t representation or averaging over E [using the uniqueness of the right-hand sides of Eqns (7) and (9)] in the E representation. Moreover, relation (13) satisfies the *Dirac correspondence principle* because the classical Poisson brackets $\{q_0,p_0\}$ with $q_0=t$ and $p_0=-E$ are unity [58]. Reference [20] also reported, as did Refs [22, 23], that the *differences* between the average passage times of wave packets through a successive *pair* of points satisfies the Ehrenfest correspondence principle, which is in fact equivalent to a generalization of Ehrenfest's theorem [20, 22, 23].

Thus, for quantum-mechanical systems with continuous energy spectra the mathematical properties (described, in particular, in Refs [48–50]) of maximal Hermitian operators, including the operator $\hat{t}$ in relations (1), are quite *sufficient* for these operators to be considered quantum observables: the *unique* expansion of unity (also referred to as the spectral function) for operators $\hat{t}$ and, incidentally, for $\hat{t}^n$ $(n>1)$ ensures — despite the nonorthogonality of such an expansion and for any analytical functions of time — the *equivalence* of whether the average value of the function is calculated in the t or E representation. In other words, the existence of such an

expansion is equivalent to the *completeness relation* for (formal) eigenfunctions of $\hat{t}^n$ $(n \geqslant 1)$ corresponding with any degree of accuracy to *real eigenvalues* within the continuous spectrum; such eigenfunctions belong to the space of the square-integrable functions of E (2) with type (3) boundary conditions (see Refs [48–50] for details).

From this point of view, *there is no practical* (*physical*) *difference between self-adjoint and maximal Hermitian operators* for systems with continuous energy spectra. It should be therefore emphasized that the *mathematical properties of* $\hat{t}^n$ $(n \geqslant 1)$ are quite sufficient for *time* to be considered *a quantum-mechanical observable* (like energy, momentum, and spatial coordinates) *without postulating new physics*.

4. Time as an observable and the energy-time uncertainty principle for discrete spectrum quantum-mechanical systems

For systems with discrete energy spectra, one naturally introduces, following Refs [22, 23, 48–50], wave packets in the form

$$\psi(x,t) = \sum_{n=0}^{\infty} g_n \varphi_n(x) \exp\left[-\mathrm{i}(\varepsilon_n - \varepsilon_0)\,\frac{t}{\hbar}\right], \tag{14}$$

where $\varphi_n(x)$ are the orthonormalized bound state wave functions of the system, satisfying the equation $\hat{H}\varphi_n(x) = \varepsilon_n \varphi_n(x)$, with $\hat{H}$ being the system's Hamiltonian, and where $\sum_n^{\infty} |g_n|^2 = 1$; we have here dropped the unimportant phase factor $\exp(-\mathrm{i}\varepsilon_0 t/\hbar)$ because it is the same for all terms in the sum over n for describing the evolution of systems in the purely discrete spectrum range. Without loss of generality, let the initial instant of time be at $t = 0$.

Let us first consider systems in which *there really exists* a greatest common divisor for the values of energy level spacings. For such idealized systems, which are exemplified by the *harmonic oscillator, a particle in a rigid box*, and *the spherical rotator*, wave packet (14) is a periodic function of time with period (*Poincaré cycle time*) $T = 2\pi\hbar/D$, where D is the greatest common divisor for the values of the energy level spacings in the system.

In the t representation, the energy operator and Hamiltonian of the system are self-adjoint operators acting in the space of *periodic* functions of time, to which, importantly, the function $t\psi(t)$ does not belong. But in the space of periodic functions of time, the operator $\hat{t}$ for time should also be a periodic function of time, even in its own representation. This situation is totally analogous to that with the azimuthal angle φ canonically conjugate to the angular moment $\hat{L}_z$ (see, for example, Refs [59, 60]). Using then an example and a result from Ref. [61], we choose, instead of t, the *periodic* saw-tooth (Fig. 1) function of t:

$$\hat{t} = t - T\sum_{n=0}^{\infty} \Theta\left(t - [2n+1]\,\frac{T}{2}\right) + T\sum_{n=0}^{\infty} \Theta\left(-t - [2n+1]\,\frac{T}{2}\right). \tag{15}$$

This choice is quite justifiable physically, because the periodic function of the operator (15) for time is linear within each Poincaré interval, i.e., time maintains its course and its usual meaning of the '*ordering*' parameter of the evolution of the system.

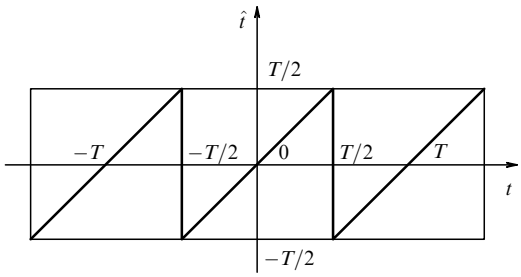

Figure 1. Periodic saw-tooth function approximating operator (15) for time.

The commutation relation for *self-adjoint* energy and time operators in this case (discrete energies and periodic functions of time) has the form

$$[\hat{E}, \hat{t}\,] = \mathrm{i}\hbar\left\{1 - T\sum_{n=0}^{\infty} \delta\big(t - [2n+1]T\big)\right\}. \tag{16}$$

Now recall (see, for example, Ref. [61]) the generalized uncertainty relation

$$(\Delta A)^2 (\Delta B)^2 \geqslant \hbar^2 \big[\langle N \rangle\big]^2 \tag{17}$$

for two self-adjoint operators $\hat{A}$ and $\hat{B}$ canonically conjugate to each other through the commutator

$$[\hat{A}, \hat{B}\,] = \mathrm{i}\hbar\hat{N}, \tag{18}$$

where $\hat{N}$ is a third self-adjoint operator. From this it straightforwardly follows that

$$(\Delta E)^2 (\Delta t)^2 \geqslant \hbar^2 \left[1 - \frac{T\big|\psi(T/2+\gamma)\big|^2}{\int_{-T/2}^{+T/2} \big|\psi(t)\big|^2\,\mathrm{d}t}\right], \tag{19}$$

where the parameter γ, arbitrary between $-T/2$ and $+T/2$, is introduced to ensure that the integral over $\mathrm{d}t$ from $-T/2$ to $+T/2$ on the right-hand side of Eqn (19) is calculated uniquely using the procedure analogous to that used in Ref. [59] (see also Ref. [61]).

From relation (19) it follows that when $\Delta E \to 0$ (i.e., when $|g_n| \to \delta_{nn'}$), the right-hand side of Eqn (19) also tends to zero because $|\psi(t)|^2$ tends to a constant. In this case, wave packet times of passage through point x within a single Poincaré cycle are distributed uniformly. When $\Delta E \gg D$ and

$$T\big|\psi(T/2+\gamma)\big|^2 \ll \int_{-T/2}^{T/2} \big|\psi(t)\big|^2\,\mathrm{d}t,$$

the periodicity condition for $\Delta t \ll T$ becomes unimportant, i.e., relation (19) *changes to uncertainty relation* (13) typical of continuous energy spectrum systems.

In principle, the operator (15) for time can also be expressed in the energy representation.

In the more general case of excited states — nuclear, atomic, or molecular — *the values of energy level spacings in discrete spectra have no strictly defined greatest common divisor and hence no strictly defined Poincaré cycle time*. Moreover, in such systems there is no clearly defined transition from the discrete to the continuous part of the spectrum. Still, even for these systems it proves possible to introduce an approximate description of the quasiperiodic

evolution with Poincaré quasicycles (with, importantly, any degree of accuracy within a chosen maximum limit of the energy level widths, for example, γ_{lim}). For sufficiently large time intervals of motion within such systems (say, greater than $\hbar/\gamma_{\text{lim}}$), neglecting the decay of states conforming to individual levels, motion within such systems can be considered *periodic to any degree of accuracy*. For these systems, the Poincaré cycle time can be chosen (defined) with any degree of accuracy, with one cycle containing as many quasicycles as needed to achieve the desired accuracy. Then, with the same accuracy, *quasiself-adjoint operator* (15) *for time* can be introduced, followed by the determination of all temporal characteristics.

In the degenerate case, when the sum $\sum_{n=0}^{\infty}$ in state (14) contains only one term ($g_n \to \delta_{nn'}$), there is no evolution and the entire Poincaré cycle time is formally equal to infinity.

For systems with mixed continuous and discrete energy spectra, forms (1) and (15) are naturally applied to the former and the latter, respectively.

5. Temporal analysis of quantum processes using time as a quantum observable

It should be reiterated that Refs [20, 22–25, 45, 48–50] have, in fact, settled with all mathematical rigor (Naimark's theorem [51]) the question of time as a quantum-mechanical observable (canonically conjugate to energy) for systems with a continuous energy spectrum. As a matter of fact this work also received support from an alternative approach [39–41, 43, 44, 46, 47], which is not inconsistent with that of Refs [20, 22–25, 45, 48–50] but relies on a different mathematical framework (another Naimark's theorem [54] which goes beyond the physical Hilbert space). For systems with discrete energy spectra, the question of time as a quantum-mechanical observable canonically conjugate to energy was settled in Refs [22, 23, 48–50].

In what follows, only a few important conclusions and results obtained in the temporal analysis of quantum processes are cited from the literature.

(1) There is every reason to believe [48–50] that the energy–time uncertainty relations (13) and (19), derived from the properties of the operator for time as a quantum observable, will have a strong attenuating effect on the long-standing debate (triggered by the discussion in Refs [62–65]) over the interpretation of such relations.

(2) Time domain analysis of various types of motion of nonrelativistic particles and photons not only revealed an analogy between the motion of particles and photons [66–68, see also 32, 33, 45, 48–50], but allowed the introduction of the maximal Hermitian operator for time in quantum electrodynamics as well (at least for the one-dimensional motion of photons [45, 48–50]).

(3) In principle, two time averaging measures are known in quantum mechanics. Section 2 introduced an averaging measure related to the passage of a particle or a photon. The second measure relates to the *dwelling* (or *sojourning*) of particles or photons in a closed spatial volume as they pass through it. The latter and its relation to the former are detailed, in particular, in the reviews [45, 48–50].

(4). *The temporal analysis of tunneling processes* has provided the following results:

(a) it is established in Ref. [45] that virtually all of the previously known special definitions or definition proposals prove, within Olkhovsky–Recami's (OR) general definition framework, to be simple special cases [24, 25, 45] of either average tunneling time or of the root-mean-square fluctuation in the tunneling time distribution (or prove to take the form of these for special idealized boundary conditions). The OR approach was considered by some (see, for example, Ref. [69]) as the most self-consistent approach in the framework of standard quantum mechanics in its Copenhagen interpretation;

(b) it is established that there is an infinite succession of *multiple internal reflections of evanescent and antievanescent waves from the entrance and exit walls of the potential barrier* [70];

(c) the Hartman (or Hartman–Fletcher) 'saturation' effect was discovered and analyzed (see, for example, the reviews [24, 25, 45]), which comprises the fact that for sufficiently wide barriers tunneling times are independent of the barrier width both for complete emerged (tunneled) wave packet peaks and for the peak of each emerged packet in the process of all multiple internal reflections. For complete peaks emerged outward through the barrier rear wall, this effect was confirmed in tunneling experiments with microwave photons [71–79]. These phenomena triggered considerable discussion on relativistic causality, although as has long been clear the velocity of the wave front, which is well-defined for short leading edge pulses, cannot exceed the speed of light in vacuum. No consensus emerged, however, with some pointing to the nonlocality of barriers as the reason for the phenomena in question. This and the later discovery of superluminal phenomena (in the tunneling of microwaves and optical waves through optical fiber devices and in the tunneling of X-shaped optical solitons) will be discussed elsewhere;

(d) a large momentum spread in a wave packet leads theoretically to the strong deterioration of the Hartman effect, which shows up in that the tunneling time additionally decreases (and occasionally becomes negative) upon increasing the barrier width [80];

(e) if wave packets consist only of evanescent or only of antievanescent waves, then it is shown in Refs [70, 81, 82] that, both for nonrelativistic particles and for photons [50], nonstationary wave packet flows describe oscillations *that arise simultaneously all over the inner space of the barrier*. It is in this way that two factors — the influence of the *barrier as a whole* on particle and photon tunneling, and the *nonlocal* behavior of tunneling particles and photons within the barrier (see, for example, Refs [50, 70, 81, 82]) — manifest themselves. Curiously, mathematical tools exist [such as changing to virtual (or imaginary) pulses or to Fourier expansion, or an approach of instantons on the imaginary time axis] with which such wave packets can be described as *traveling* in an unusual metric space [50, 81, 82];

(f) for two identical *nonresonating* barriers a distance L apart, an enhanced Hartman effect was reported [83], in which not only the total tunneling time through the two barriers turned out to be equal to that through a single barrier but, furthermore, the total tunneling time was found to be absolutely independent of the barrier separation L. However, the presence of resonance between the barriers greatly complicates the situation [84];

(g) although tunneling under a potential barrier does not allow for a direct passage to the classical mechanics limit, it is established that there exist a nonzero probability density and a nonzero probability flow density for the motion of wave packets within such barriers and that, furthermore, a direct

classical limit explicitly exists for the motion under barrier of the wave packets of superposed evanescent and antievanescent waves: recall a similar motion, in time, of wave packets described by *the superposition of evanescent and antievanescent waves in classical optics and classical acoustics* (see, for example, Refs [48–50, 71–78, 85, 86]).

(5) The following are some results from the *time analysis of nuclear reactions and decay processes*:

(a) in the region of isolated resonances strongly distorted by the nonresonant background, a paradoxical effect was exposed [87–92], in which the delay time transforms to an advance time near a resonance, or more precisely, near the cross section minimum in the region of the distorted resonance. A more careful analysis [93, 94] later showed, however, that this paradox results in the center-of-mass system from false interference which appears in the scattering amplitude due to the standard transformations from the laboratory to the center-of-mass frame being used formally, without taking into account the fact that the center of inertia moves differently for the mechanism involving direct (fast) processes (when it remains virtually unshifted by a collision) and for the mechanism involving processes that pass through the stage of a long-lived compound nucleus (when the motion of the decaying compound nucleus itself in precisely the laboratory frame should be taken into account). Because transformations of collision amplitudes for different mechanisms should be considered separately, taking into account the spacetime shift due to the motion of the compound nucleus, no advance time effects occurred for the total collision amplitude;

(b) there is a research [95] devoted to high-energy nuclear reactions involving the appearance of *temporal resonances* (*explosions*) that correspond not to the exponential, but to the resonant-like decay of compound nuclei and feature a certain delay in the onset of their decay — an effect by which a number of experimental data were explained;

(c) estimates of the age of planets and stars started to be revised [96, 97] as nuclear chronometry methods became modified by including the physical processes occurring both in the interior of the large masses of star material (in particular, the excitation of the alpha-radioactive nuclei-chronometers in nucleosynthesis processes and the formation of the successive pairs of gamma deexcitations and gamma excitations of the excited states of nuclei-chronometers, with the inclusion of recoil effects) and in the surface layers of planets (with the inclusion of space radiation).

6. Conclusion

(1) The possibility has arisen, in principle, of extending the temporal analysis of quantum processes proceeding in more complex motions of particles and photons, in particular, in spiral motions and in motions through two- and three-dimensional (plane, spherical, and nonspherical) potentials and barriers, and so forth.

(2) There also exists a fundamentally simple way to generalize the operator for time to relativistic quantum mechanics (Klein–Gordon and Dirac equations) by using appropriate averaging measures. It is also possible, although not that easy, to construct four-dimensional coordinate–time operators for other relativistic problems, particularly for localization problems. The existing work on this issue is reviewed in Ref. [98].

(3) Of interest would be a temporal analysis of oscillating wave packets together with the photon fluorescence of excited states within any quasidiscrete spectrum system (crystal) whose energy level separations are *multiples* of a certain 'greatest common divisor' D. A temporal analysis of possible laser-like phenomena migrating inside crystals would also be worthwhile.

(4) A promising area seems to be the temporal analysis of the tunneling of an early inflating universe through a barrier determined by the quantum gravitation curve in the quasilinear approximation, treating the operator for time with the aid of a Hamiltonian approach (described, in particular, in Refs [10, 48–50]) and using the operator equation

$$[H, T] = \mathrm{i}\hbar$$

and the dual equations

$$H\Psi = \mathrm{i}\hbar\,\frac{\partial\Psi}{\partial t}\,, \qquad T\Phi = -\mathrm{i}\hbar\,\frac{\partial\Phi}{\partial E}\,,$$

where Φ is the Fourier component of the wave function

$$\Psi = \int_{-\infty}^{\infty} \mathrm{d}E\,\exp\left(-\frac{\mathrm{i}Et}{\hbar}\right)\Phi(E)\,.$$

7. Appendix

Formula (9) was first obtained in Refs [24, 25, 30–33], where it was also justified by representing continuity equation (5) for time intervals with $j = j_+$ and $j = j_-$ in the respective forms

$$\frac{\partial\rho_>(x,t)}{\partial t} = -\frac{\partial j_+(x,t)}{\partial x} \quad \text{and} \quad \frac{\partial\rho_<(x,t)}{\partial t} = -\frac{\partial j_-(x,t)}{\partial x} \qquad \text{(A.1)}$$

(in the same papers relations (A.1) are also considered as the formal representations of $\partial\rho_>/\partial t$ and $\partial\rho_</\partial t$).

Clearly, in Eqn (9) we can, similarly to transformation (7), change to averaging over energy E, i.e., to integrals $\int_0^\infty \mathrm{d}E\ldots$ by using the uniqueness of the Fourier (Laplace) transformations and expanding the function $j_\pm(x,t) = j(x,t)\,\Theta(\pm j)$ in terms of the integral over energy. This requires a lot of tedious algebra, though.

A similar analysis can be applied to the motion of a particle in three dimensions (see, for example, Refs [20, 22, 23, 50]).

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