

CONFERENCES AND SYMPOSIA PACS number: 01.10.Fv, **02.50.–r**, **05.40.–a**, 05.45.Df,, 89.65.Gh, **89.75.–k**

Econophysics and evolutionary economics
(Scientific session of the Physical Sciences Division of the Russian Academy of Sciences, 2 November 2010)

DOI: 10.3367/UFNe.0181.201107e.0753

The scientific session ''Econophysics and evolutionary economics'' of the Division of Physical Sciences of the Russian Academy of Sciences (RAS) took place on 2 November 2010 in the conference hall of the Lebedev Physical Institute, Russian Academy of Sciences.

The session agenda announced on the website www.gpad.ac.ru of the RAS Physical Sciences Division listed the following reports:

(1) **Maevsky V I** (Institute of Economics, RAS, Moscow) ''The transition from simple reproduction to economic growth'';

(2) **Yudanov A Yu** (Financial University of the Government of the Russian Federation, Moscow) ''Experimental data on the development of fast-growing innovative companies in Russia'';

(3) **Pospelov I G** (Dorodnitsyn Computation Center, RAS, Moscow) ''Why is it sometimes possible to successfully model an economy?''

(4) **Chernyavskii D S** (Lebedev Physical Institute, RAS, Moscow) ''Theoretical economics'';

(5) **Romanovskii M Yu** (Prokhorov Institute of General Physics, RAS, Moscow) ''Nonclassical random walks and the phenomenology of fluctuations of the yield of securities in the securities market'';

(6) **Dubovikov M M, Starchenko N V** (INTRAST Management Company, Moscow Engineering Physics Institute, Moscow) ''Fractal analysis of financial time series and the prediction problem.''

Papers written on the basis of these reports are published below.

Uspekhi Fizicheskikh Nauk **181** (7) 753 – 786 (2011)
DOI: 10.3367/UFNr.0181.201107e.0753
Translated by V I Kisin; edited by A M Semikhatov and A Radzig

PACS numbers: 89.65.Gh, **89.75.–k**
DOI: 10.3367/UFNe.0181.201107f.0753

The transition from simple reproduction to economic growth

V I Maevsky, S Yu Malkov

1. Introduction. Representation of the macro economy by a population of macroeconomic subsystems

At the current stage, the theory of economics offers a large number of models of the economy achieving static market equilibrium (see, e.g., [1]) as well as models describing how a macro system reaches the trajectory of stable, steady economic growth [2]. But there are no models showing how growth emerges at the macro level from an equilibrium situation.

It seems that the reason for this lacuna is of a fundamental, methodological nature: by virtue of a well-rooted tradition, the macro level is regarded as a complete entity in which the behavior of each element is identical to the behavior of any other part. Because any economy engages simultaneously in the production of consumer goods and investments in fixed capital and current assets, the tradition is to implicitly assume that every part of the macro economy is capable of conducting these two sorts of activities simultaneously (the *coproduction* mode). In our opinion, this well-established view on the macro level should not be treated as absolute, i.e., regarded as the only one acceptable. Another approach is possible, associated with the so-called *cycled* production–reproduction mode. To better understand the essential features of this approach, we consider some of the peculiarities of the machine-building industrial complex.

We assume that this complex includes a full set of subbranches of the machine-building industry capable of creating the active part of fixed capital (machine tools, machinery, equipment, instruments, and so on) both for itself and for the 'rest' of the economy. In terms of the tradition of coproduction, this complex is perceived as an aggregate unit in which all elements are operating simulta-

V I Maevsky Institute of Economics, Russian Academy of Sciences, Moscow, Russian Federation. E-mail: maev@orc.ru
S Yu Malkov Academy of Military Science, Moscow, Russian Federation

Uspekhi Fizicheskikh Nauk **181** (7) 753 – 758 (2011)
DOI: 10.3367/UFNr.0181.201107f.0753
Translated by V I Kisin; edited by A M Semikhatov

neously for the unit and for the 'rest' of the economy. But it is possible to argue differently.

All branches of the machine-building complex are composed of plants for which the ages of fixed capital in a year t are different. If we break down the set of plants in the year t into age groups, we obtain a set of subsystems; in the current year, the oldest among them needs to undertake the reproduction of its fixed capital, while the other subsystems are busy providing the growth in the rest of the economy. In the year $t+1$, the 'rejuvenated' subsystem of plants of the complex switches to providing the growth in the rest of the economy and another subsystem of plants, the oldest in year $t+1$, works on self-reproduction of its fixed capital. We can therefore say that acting within the machine-building complex is a population of machine-building subsystems distributed nonuniformly according to the age of fixed capital and hence also in efficiency, each of which is characterized by a cycled production–reproduction mode.

Because the operation of a machine-building complex predetermines the development of the economy as a whole, we decided to extend the production–reproduction mode to the macro level of the economy. For us, the macro level is not the traditional mono-unit but a population of macroeconomic subsystems (nonidentical in age and in the degree of efficiency); each subsystem operates in a year t either in the mode of self-reproduction of fixed capital or in the production of consumer goods, but not the two simultaneously.

This interpretation of the macro level already deserves attention because it helps pinpoint the competition between the older, less efficient, and the newer, more efficient, macroeconomic subsystems. Newer subsystems, like Glaz'ev's technological structures [3], are capable of forcing out older subsystems from the economic space. The processes activated in this case are those of merger and of absorption of capital, while the number of bankruptcies increases. A different scenario is possible, however: older subsystems succeed in modernizing themselves, without 'help' from new subsystems. Then the evolution unfolds in a quieter mode.

Before modeling the process of development, we must consider the behavior of a population of macroeconomic subsystems in an equilibrium situation in which the efficiency of fixed capital does not increase and the simple reproduction mode is established.

2. Simple reproduction model

Before we tackle the building of a model of simple reproduction in a population of macroeconomic subsystems, we note that the first economist who created a numerical macroeconomic model of simple reproduction was the French physiocrat François Quesnay [4]. Karl Marx [5] followed him with his model of simple reproduction. However, neither Quesnay nor Marx, nor their numerous followers, were interested in the phenomenon of a cycled production–reproduction mode, and they did not regard the macro level as a population of macroeconomic subsystems. We were the first to suggest a simple model of this type in 1980 [6]. We now consider this model.

Let $T_{\text{fixed cap}}$ be the average service life of capital assets; we assume that in the economy of a country, it is only three years ($T_{\text{fixed cap}} = 3$), and let T_{repr} be the average time of reproduction of fixed capital, equal to one year ($T_{\text{repr}} = 1$). We also assume that the distribution of the fixed capital in the economy over age is uniform. In this case, we can single out three specific macroeconomic subsystems of the economy,

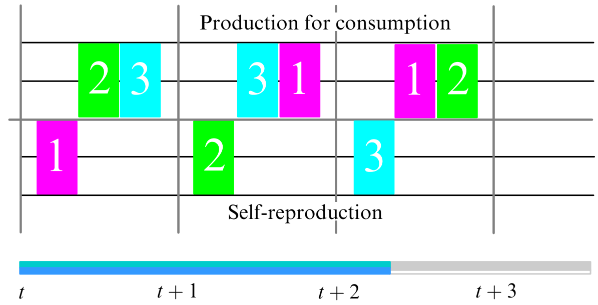

Figure 1. Subsystems 1, 2, 3 operating in the production–reproduction mode in years t, $t+1$, $t+2$.

each of which is capable of reproducing its fixed capital (Program A) and producing consumer goods (Program B) in the cycled production–reproduction mode. The execution of programs A and B by the subsystems of the macro level is accompanied by accumulation and expenditure of monetary funds, i.e., of 'depreciation' money. The subsystems differ from each other only in the age of fixed capital by the beginning of year t. The contractors of the subsystems are households (which supply the workforce to all three subsystems and are consumers of their products), and the bank plays an intermediary role. Finally, we note that all indicators of macroeconomic subsystems are measured in *current* prices, and hence the gross domestic product (GDP) produced by their combined effort is the *nominal* GDP.

The first subsystem is the oldest, and the age of its fixed capital at the beginning of year t is two years. By that time, it has accumulated the necessary depreciation savings and is to reproduce its fixed capital during year t (Program A). The age of the fixed capital of the second subsystem at the beginning of year t is 1 year; its tasks are to produce and sell consumer goods to households and accumulate savings (Program B). The third subsystem is the newest: its age is 0 years; it behaves during year t exactly as the second subsystem (Program B).

The following year, subsystems swap places in the process of operation: the first subsystem becomes the newest after the renewal of fixed capital, the third becomes a year older, and the second becomes the oldest and begins to renew its capital (Fig. 1).

The quarter-by-quarter sequence of events in year t unfolds as follows. At the beginning of the first quarter of year t, subsystems 1 and 2 have depreciation funds accumulated earlier and kept in the bank. One part of these funds is used up by subsystem 1 over the quarters of year t to pay wages to its employees who this year renew the fixed capital of subsystem 1. These workers take their earnings home. In this way, the money reach households (families) 1 that concentrate around subsystem 1. The other part of the depreciation funds 1 and 2 (kept in the bank) is used as credit serving to form the working capital of subsystems 2 and 3. It is assumed that by the beginning of year t, subsystems 1 and 2 have sold all their output produced by the end of year $t-1$ (warehouses are empty): the goods are bought up by households 1, 2, 3, which finance purchases with the money earned at the end of year $t-1$ (Fig. 2).

During the first quarter, the money from subsystems 1, 2, 3 flows to households 1, 2, 3 as wages, warehouses fill up with finished products, and households consume the products stored earlier. Having received their wages, households start buying consumer goods produced by subsystems 2 and 3,

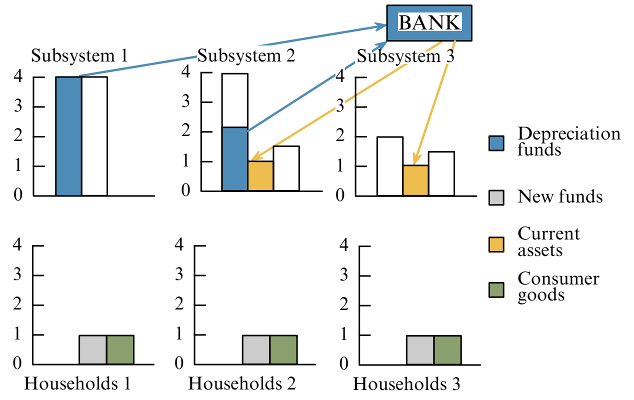

Figure 2. The status of the economic system at the beginning of the first quarter of year *t*.

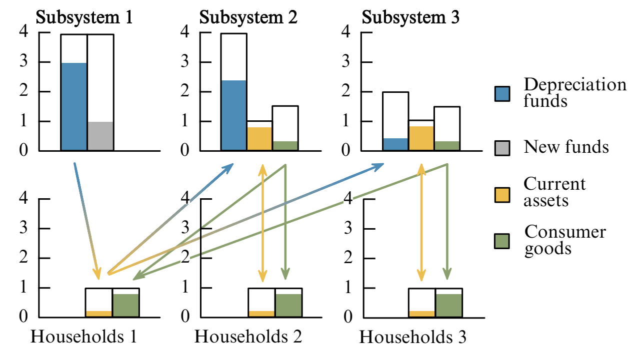

Figure 3. Functioning of the economic system in the first quarter of year *t*.

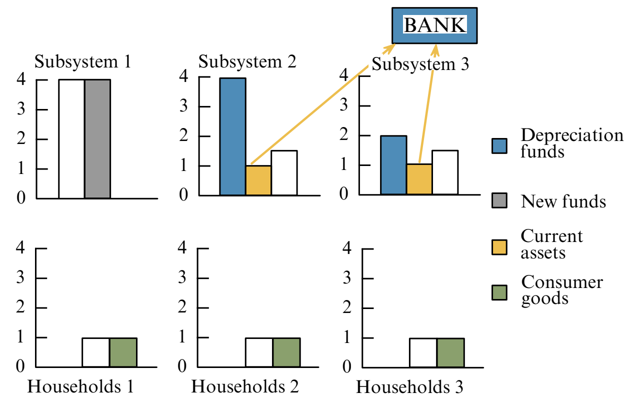

Figure 4. The status of the economic system at the end of year *t*.

company warehouses empty, and the money returns to the businesses of subsystems 2 and 3, which thus replenish their current assets and partly pour it into the depreciation fund (the corresponding flows of goods and money are shown in Fig. 3).

The circulation of goods and floating funds continues similarly in the second, third, and fourth quarters of year *t*. As a result, the first subsystem replenishes its fixed capital, and its depreciation funds are 'pumped out' to the depreciation funds of subsystems 2 and 3 (Fig. 4).

At the end of year *t*, subsystems 2 and 3 repay the bank for the loans received early in the year to support ongoing activities and the economic systems return to their original status (see Fig. 2), except that the subsystems have swapped places: the place of the first subsystem is now occupied by the second, that of the second by the third, and the place of the third by the first, with updated fixed capital (see Fig. 1).

With simple reproduction, cycles of this type follow one another indefinitely long and the population of macroeconomic subsystems persists in dynamic equilibrium. Incidentally, the presence of an intermediary bank makes the 'depreciation' fund sufficient for servicing all exchange operations in a given economy, while the depreciation fund itself completes the turnaround: this money transforms in its flow into 'consumer' money, with the consumer money again transforming into the depreciation fund.[1]

3. Transition to economic growth

We now assume that in year t, the macroeconomic subsystem 1 implemented the self-reproduction of fixed capital and introduced new technologies, thus creating a more efficient fixed capital. Then in year $t+1$ it can produce more consumer goods (at *current* prices) than the third subsystem, which also produces consumer goods in year $t+1$. Accordingly, the aggregate supply of consumer goods in year $t+1$ increases. Is this a sufficient condition for ensuring the resulting economic growth? Generally speaking, no: the additional product would not be bought if the amount of money at the disposal of households did not increase. Additional output leads to economic growth under the condition that the aggregate solvent demand increases simultaneously.

The aggregate solvent demand can only increase if the monetary supply and consumer preferences of households also increase. In turn, the availability of monetary supply depends on the monetary policy of the financial authorities. The following three scenarios of monetary policy are then possible.

First scenario. The amount of money issued supports an increment in aggregate demand from households equal to the increment in the aggregate supply of consumer goods: the result is noninflationary growth.

Second scenario. The amount of money issued generates demand that exceeds the growth in the aggregate supply of consumer goods: economic growth is accompanied by inflation.

Third scenario. Zero monetary emission: growth is impossible, and the crisis of overproduction of consumer goods sets in. Because the first subsystem succeeded in achieving higher productivity and became more competitive, it either economically strangles the third subsystem or absorbs its capital with time. The process of strangling inevitably leads to increasing unemployment and a decrease in the aggregate consumer demand, which is accompanied by economic recession and growing social tensions.[2]

To summarize, the innovations introduced in macroeconomic subsystem 1 in year *t* generated a *bifurcation state* in the

[1] By our estimate, by the end of 2007, the aggregate depreciation fund in the USA reached approximately \$17 trillion. This is nearly 2.5 times the USA M2 (the amount of cash in circulation, term deposits, checks, demand deposits), which in 2007 was \$7.4 trillion, and is considerably higher than the annual GDP of \$13.8 trillion.

[2] Historically, the third scenario has repeatedly manifested itself in the form of social explosions (e.g., the Luddite revolt at the beginning of the 19th century). Later, a practice was adopted of retraining redundant workers (e.g., for work in the services industry). It was the field of services which in the 20th century grew into the macroeconomic subsystem that absorbed the labor force released as a result of innovations.

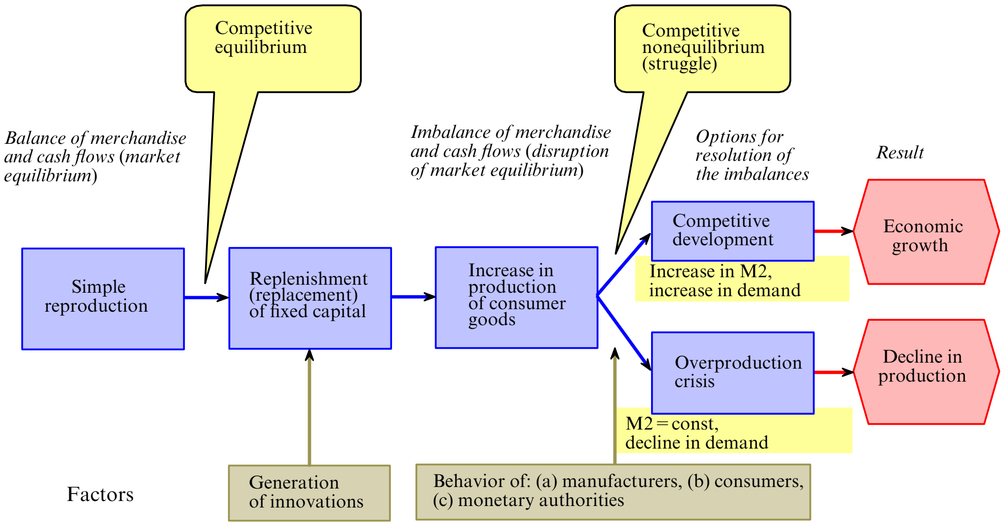

Figure 5. Diagram of the transition from simple reproduction to economic growth.

next year $(t+1)$; the output from the fork depends on the policies of the monetary authorities and on the evolution of consumer preferences. In this situation, the economy cannot be described as in the case of the simple reproduction model. The main difference is that the intermediary bank not engaged in issuing money needs to be replaced by a bank issuing new money or, to quote Schumpeter, a bank creating new purchasing power [7]. At the macro level, this function is fulfilled by the central bank, and the main method of delivering the new money emitted by the central bank and of placing it at the disposal of households is typically (at least in modern industrialized countries) the mechanism of raising public debt and, respectively, the budget deficit.

The general diagram illustrating the transition from simple reproduction to economic growth is shown in Fig. 5.

The transition from competitive equilibrium (in the case of simple reproduction) to intensified competitive wars (in the case of imbalance in the economic system), which may lead to economic growth but may also result in an economic crisis (see Fig. 5), can be illustrated by the growth model of two competing macroeconomic subsystems. We assume that the following logic diagram reflects the dynamics of the production of goods by each of these subsystems: *the change in production output equals the increase in the output under conditions of no resource constraints*[3] *minus a correction taking the resource constraints into account, minus a correction taking the effect of the competitor subsystem into account*.

Mathematically, this logic reduces to the basic model of competition that is widely used in studies of social systems [8, 9]:

$$\frac{dx_1}{dt} = a_1 x_1 - b_1 x_1^2 - c_1 x_1 x_2 \,, \tag{1}$$

$$\frac{dx_2}{dt} = a_2 x_2 - b_2 x_2^2 - c_2 x_1 x_2 \,, \tag{2}$$

where x_i is the total output of the ith subsystem $(i = 1, 2)$.

[3] Here, we interpret the 'resources' in a broad sense: they include raw materials, manpower, monetary resources, solvent demand of the production output, etc.

The first two terms in the right-hand sides of Eqns (1) and (2) characterize the process of autonomous development of the subsystems under resource constraints, but without taking competition into account. The third terms in the right-hand sides of (1) and (2) take competition into account. They enter with the minus sign, which indicates that the emergence of competitors obviously worsens the economic situation of the subsystem in question and may even threaten its existence. A threat to their existence pressurizes the competing subsystems into intensification of their activities (into increasing a_i, in terms of the model), and the higher the level of threat from the competitors is, the more active the efforts need to be to build up the subsystem capabilities. In view of this, we can write

$$\frac{dx_1}{dt} = a_1(1 + h_1 x_2)x_1 - b_1 x_1^2 - c_1 x_1 x_2 = a_1 x_1 - b_1 x_1^2 + (h_1 a_1 - c_1) x_1 x_2 \,, \tag{3}$$

$$\frac{dx_2}{dt} = a_2(1 + h_2 x_1)x_2 - b_2 x_2^2 - c_2 x_1 x_2 = a_2 x_2 - b_2 x_2^2 + (h_2 a_2 - c_2) x_1 x_2 \,. \tag{4}$$

We see that in contrast to Eqns (1) and (2), Eqns (3) and (4) can describe both economy in recession (if $h_1 a_1 - c_1 < 0$) and economic growth (if $h_i a_i - c_i > 0$). The quantity $h_i a_i - c_i$ is the bifurcation parameter that determines the characteristics of system dynamics. The quantity $h_i a_i - c_i$, in turn, is a function of the parameter h_i, whose value is affected by a number of factors: availability of credit, cheap raw materials, skilled labor, modern technologies, and market demand for manufactured products. The bifurcation parameter takes different values depending on specific combinations of the above factors, and these also determine the type of dynamics of the economic system (growth, decline, or stagnation).

We need to remember that the parameters a_i, b_i, c_i, and h_i of the set of equations (3), (4) are not constant but are in fact functions of time. First, their values are affected by external circumstances, e.g., changes in resource costs in the markets of labor, raw materials, and capital. Second, they depend on the institutional features of the economic system under consideration and on the previous history of the processes occurring in it. Third, the competing subsystems may influence, to a certain extent, the values of the current parameters (e.g., by enhancing the innovative activity or by increasing pressure on the competitor). The situation is therefore shifting, and each imbalance in the economic system can generate a variety of diverging outcomes.

It is important that dynamic models of competition such as (3), (4) allow taking this diversity into account and can be the basis for a mathematical description of nonequilibrium situations that arise as a result of the presence of cycled production–reproduction modes in the economy.

4. Conclusion

As a rule, *mainstream* mathematical models analyze either 'pointlike' states of market equilibrium or the resulting trends of sustained economic growth. These models *do not solve* the problem as we have formulated it, of simulating the transition from simple reproduction to growth. They are difficult to use as a tool for supporting decision making on economic policies. We believe that one cause of this state of affairs is that economic theorists still perceive the macro economy as a system exclusively implementing the reproduction of itself in the mode of *coproduction* and the production of consumer goods. Economic theorists do not consider the alternative approach to the macro economy as a population of macroeconomic subsystems performing the same functions but in the *cycled production–reproduction* mode.

In our opinion, it is precisely this approach that offers good prospects for creating fundamentally new economic models, describing:

— competitive interaction at the macro level;

— macroeconomic bifurcation states;

— states of dynamic inequilibrium of merchandise and cash flows in the implementation of innovations and subsequent changes in the behavior of producers, consumers, and monetary authorities.

The important feature of the proposed approach is that it does not focus on seeking a trend of sustained growth. On the contrary, it shows how the economy now enters the trajectory of economic growth, now falls into recession, now stagnates, now resumes growth again, all of it as a result of systematic transitions from one bifurcation state to another (Fig. 6).

With this interpretation of the macro economy, the center of gravity of research in economic theory shifts toward the analysis of the conflict of interest, which becomes acute every time radical innovations are introduced. As regards research in mathematical simulation, the following fields for advancing mathematical methods are pressing and important in this case:

— simulation of nonstationary and nonsynchronous modes of the functioning of economic systems;

— simulation of the interaction between merchandise and cash flow under nonstationary conditions;

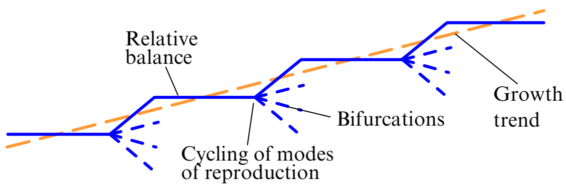

Figure 6. Sequence of bifurcations used to simulate economic growth.

— modeling the effects of positive feedback (effects of positive returns) on economic systems;

— simulation of bifurcation in economic systems, and determination of critical values of economic parameters that define the transition from one mode of operation to another.

The work was supported by RGNF funding (project No. 09-02-00747a).

References

1. Lancaster K *Mathematical Economics* (New York: Macmillan, 1968) [Translated into Russian (Moscow: Sovetskoe Radio, 1972)]
2. Sharaev Yu V *Teoriya Ekonomicheskogo Rosta* (Theory of Economic Growth) (Moscow: Izd. Dom GU VShE, 2006)
3. Glaz'ev S Yu *Teoriya Dolgosrochnogo Tekhniko-Ekonomicheskogo Razvitiya* (Theory of Long-Term Economic and Technological Development) (Moscow: VlaDar, 1993)
4. Quesnay F *Physiocratie* (Leyde et Paris: Chez Merlin, 1768) [Translated into Russian (Moscow: Eksmo, 2008) p. 279]
5. Marx K, Engels F *Izbrannye Trudy* (Selected Works) Vol. 8 (Moscow: Politizdat, 1987)
6. Maevsky V I *Izv. Akad. Nauk SSSR. Ser. Ekonom.* (6) 71 (1980)
7. Schumpeter J *Theorie der wirtschaftlichen Entwicklung* (The Theory of Economic Development) (Leipzig: Duncker & Humbolot, 1912) [Translated into English (New Brunswick, NJ: Transaction Books, 1983); Translated into Russian (Moscow: Progress, 1982)]
8. Chernavskii D S *Sinergetika i Informatsiya: Dinamicheskaya Teoriya Informatsii* (Synergetics and Information: Dynamic Information Theory) (Moscow: URSS, 2004)
9. Malkov S Yu *Sotsial'naya Samoorganizatsiya i Istoricheskii Protsess: Vozmozhnosti Matematicheskogo Modelirovaniya* (Social Self-Organization and the Historical Process: Potential Power of Mathematical Modeling) (Moscow: LIBROKOM, 2009)

PACS numbers: **02.50.–r**, 89.65.Gh
DOI: 10.3367/UFNe.0181.201107g.0758

High-growth firms in Russia: experimental data and prospects for the econophysical simulation of economic modernization

A Yu Yudanov

1. Introduction

The concept of the 'high-growth firm' or 'gazelle' was introduced in the 1980s by David Birch. It was established that the majority of both large and small companies grow slowly and contribute minimally to increasing employment and the gross domestic product (GDP) [1, 2]. But a small proportion of firms combine high dynamic stability and growth. Birch gave them the name gazelles to emphasize the similarity of these companies to the animal that is capable not only of reaching high speed but of sustaining it for a long time. In 1988–1992, by Birch's estimate, gazelles making up only 4% of the total number of firms created approximately 70% (!) of all new jobs in the U.S.

A Yu Yudanov Financial University of the Government of the Russian Federation, Moscow, Russian Federation
E-mail: yudanov@yandex.ru

Uspekhi Fizicheskikh Nauk **181** (7) 758 – 762 (2011)
DOI: 10.3367/UFNr.0181.201107g.0758
Translated by V I Kisin; edited by A M Semikhatov

This identification of the type of enterprise playing such an exceptional role in economic growth could not help attracting scrutiny. Despite some criticism (see, e.g., [3]), Birch's findings received representative confirmation [4–8]. The current assessment of the impact of gazelles on the economy has removed any doubt about the significance of this phenomenon. According to a review by a research group of the European Union (EU), "Often-cited studies suggest that anything from between 3% and 10% of any new cohort of firms will end up delivering from 50% to up to 80% of the aggregate economic impact of the cohort over its lifetime" [7, p. 6].

The amazingly large proportion contributed by a handful of gazelles indicates that the mechanism of growth of a mature market economy is subject to the so-called Pareto–Zipf principle, according to which the key role in the behavior of nonlinear multivariable systems is usually played by only a small number of high-performance factors. In this case, we witness an unevenly distributed 'growth energy' of the economy. Far from being 'spread' among all active businesses, it is concentrated in a small number of gazelles.

Do gazelles exist in Russia and do they play a significant role here? (See [9] for more general issues surrounding entrepreneurial activities.) We began studying gazelles in Russia in 2003 at the Financial University created under the auspices of the Government of the Russian Federation (Finuniversitet) [10–13].[1] The journal *Ekspert* (*Expert*) conducted an independent research project [14].[2] In 2007, the two groups joined forces. We studied three official, albeit not fully comparable, databases for 1999–2007 (and partly for 2008) of all Russian enterprises with revenues in excess of 300 million rubles. The information about each company included data on sales, fixed assets, noncurrent assets, receivables and payables, expenditures on research and development, and net profits. In fact, this project covered almost all large and medium businesses in the country, and the timeframe chosen for the study (1999–2007) fully covered the first strong economic expansion in Russia. In accordance with standard procedures, the set of permanent firms, i.e., companies that existed during the entire sample length (6.5 thousand companies during 1999–2007 and about 10 thousand companies for shorter five-year time intervals) were considered as approximations of the general set of firms of the country. Companies were identified as gazelles using the Birch algorithm, which requires that the revenue of each company grow by 20% or more every year over at least five consecutive years. The algorithm was adjusted to Russia by deciding to use the data after factoring out inflation. A substantive interpretation of statistical data relied on the array of data covering the most notable gazelles (about 500 data files), questionnaires, and in-depth interviews.

2. Description of the population of gazelles in Russia

According to direct estimates (Table 1), gazelles make up 7 – 8% of the number of permanent firms, i.e., they are about twice as numerous as in the West. The peculiarities of Russian accounting (see [15] for the details) makes us believe that even these high figures dramatically underestimate the number of gazelles and the correct assessment should be 12–13% of the population of companies. Contrary to the common joke, Russia should be called not the 'motherland of elephants,' but rather the 'country of gazelles,' offering the possibility of rapid and sustainable growth for a strikingly large percentage of firms.

Table 1. Number of gazelles in Russia.

Interval, years	Number of permanent firms	Number of gazelles*	Percentage of gazelles among permanent firms, %
1999 – 2003	6524	484	7.4
2000 – 2004	7348	527	7.2
2001 – 2005	8244	587	7.1
2002 – 2006	9381	744	7.9
2003 – 2007	10,174	830	8.2

* Including subsidiaries of large corporations. Source: database of medium-size businesses, Mediaholding *Ekspert*–Finuniversitet.

Table 2. Revenue dynamics for groups of companies (constant prices).

Groups of companies	Revenue, billion rubles		Average annual growth, %	Reveneue increment, billion rubles	Increment in revenue in % of total
	2003	2007			
Gazelles	285	2900	78	2615	23.1
Top-10* (Rosstat version)	1969	2985	11	1016	9.0
Top-10* (Ekspert-400 version)	2413	4560	18	2147	19.0
Permanent firms	12,393	23,707	18	11,314	100

* Gazprom, Lukoil, Surgutneftegaz, Nornikel, Transneft, Tatneft, Severstal, Magnitogorsk metallurgical works, Novolipetsk metallurgical works, AvtoVAZ. The Rosstat (Federal State Statistics Service) version takes only parent companies into account, while the Expert-400 version also includes consolidated data for subsidiaries. Source: database of medium-size businesses, Mediaholding *Ekspert*–Finuniversitet.

Growth rates of Russian gazelles were impressive (Table 2). They increased revenue annually by 78% on average, while the average growth rate of all permanent firms did not exceed 18%.

An estimate of the contribution of gazelles to national economic growth is somewhat ambiguous. This contribution can be regarded as quite large in view of the tiny start-up size of gazelles. Indeed, in 2003, gazelles generated a mere 2% of the total revenue of permanent firms. But this did not stop them from generating almost a quarter (23.1%) of the increment in this index over the period 2003–2007. At the same time, however, the contribution of Russian gazelles was much lower than that typical of the West (50–80% of increment in GDP).

Is it possible to give a rational explanation for the strange combination of properties of the Russian population of gazelles, namely, their large number against a lower impact on the economy? In our opinion, both these effects stem from the excellent possibilities of rapid growth in a young, emerging market economy. Indeed, this situation increases the number of gazelles: Russia offers a multitude of promising unoccupied niches, making long-term dynamic development possible. But Russian 'non-gazelles' also grow fast, and for the same reason of youthfulness of the economy. In other words, the group of gazelles in the West constitutes almost the entire engine of growth of the economy. In contrast, Russian gazelles play a relatively lesser role, not because they are more passive or weaker than their foreign counterparts, but

[1] The main participants in the project, in addition to the author of this article, were N N Dumnaya, G V Kolodnaya, V V Razumov, O E Pyrkina, V A Uspenskii, and E V Medina.

[2] Yu A Polunin, T I Gurova, A V Vin'kov, and O L Ruban.

because although they are important, they are nevertheless not the only engines of economic growth.

At the same time, it would be wrong to downplay the role of gazelles in the economy of Russia. Table 2 provides information on revenue growth of the 10 largest corporations (top-10) in Russia. Public opinion clearly links the revival of 1999–2007 with their enrichment using the global growth in prices of raw materials and fuel. However, the increase in revenue of gazelles exceeds the growth rate of the revenue of the top-10 corporations in Russia, even if the latter includes the revenues of their subsidiaries.

Obviously, direct comparison of the revenue of gazelles and resource corporations is not completely correct. Firms operate in different industries, create a different added value, etc. It is also appropriate to ask what the origins of growth of gazelles were: it is very doubtful that it could reach the scale we have witnessed in the economy were it not propped up by resource revenues. It is nevertheless worthy of note that the largest growth in revenue came not from resource giants — helped by the unprecedented situation with world prices — but from gazelles, although 'resource gazelles' are a rarity.

It appears logical to interpret Russian gazelles as real agents of the process of transformation of oil revenues into general economic growth, i.e., precisely those agents of economic diversification that became an obsession with our politicians. It is not impossible for our gazelles, once they mature and grow the muscle, to become the future non-resource-based engine for the Russian economy.

3. Exponential growth and entrepreneurial nature of gazelles

The exponential growth of many Russian gazelles was an unexpected phenomenon identified in the study but not yet described in the literature. It was first considered accidental, because revenues of companies typically demonstrate high levels of fluctuation due to variability in demand for their products. But it turned out that the exponential growth is typical for gazelles.

Figure 1 shows the degree of correspondence of revenue dynamics and the exponential law of growth for gazelles and for nongazelles. We chose long-life gazelles as representative gazelles. Specifically, we considered all 74 firms (out of 484 gazelles of the generation of 1999–2003; see Table 1) that succeeded in sustaining high growth rate until 2006. An equal number in a random sample of permanent firms played the role of the control group.

The growth dynamics of about 50% of long-life gazelles followed the exponential curve practically perfectly ($R^2 > 0.98$), while ordinary companies with this behavior were exceptions (4% of firms).[3] Almost all long-life gazelles (89%) were described satisfactorily ($R^2 > 0.95$) by exponential growth.

The enigmatic ability of many gazelles to grow by a constant percentage from year to year irrespective, as it were, of external factors was unmistakable. An unavoidable question is: why do the factors that constantly perturb the progress of other firms fail to force a gazelle to stray away from the exponential?

In nature, the typical case where exponential dynamics is found is the development of reproductive processes in the

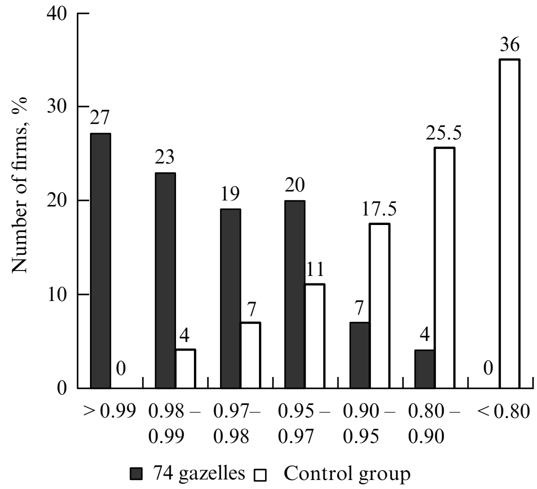

Figure 1. Degree of compliance to the exponential law for the revenue dynamics of the full set of identified gazelles that each year had the annual revenue growth rate greater than 20% in constant prices over the interval 1999–2006, and the data for nongazelles (equal number of 74 firms in a random sample from among the permanent firms for 1999–2006) to the exponential curve (percentage of firms with the corresponding determination coefficient R^2). Source: database of medium-size businesses, Mediaholding *Ekspert*–Finuniversitet.

absence of resource constraints. For example, the population of rabbits brought to Australia grew exponentially until it inhabited every green valley of the continent.

Could it be that this absence of constraints (constraints on demand in this case) is the key to the nature of the stable rapid growth of gazelles? Indeed, most of the factors that cause the instability of the dynamics of a firm evolution actually reduce to fluctuations in the demand for its products. It appears logical to link gazelles' escape from demand constraints to their entrepreneurial or innovative (after Schumpeter) nature. The products brought to the market by innovators are so much in demand that every item supplied by their companies is certainly purchased. The growth rate in such circumstances is limited not by demand but by the ability to increase the supply of the product or, in other words, by the rate of expansion of specific internal assets: personnel trained in advance for reaching the targets set by the company, managerial staff, and dedicated equipment.

It is significant that the growth of these factors tends to be exponential. An entrepreneur capable of yearly training two leaders with initiative, who then train two assistant managers each, and so on, is no different in the sense of process dynamics from a doe rabbit. If we find that Russkii Standart bank had a stuff of 221 employees in 2000 and 36617 in 2006, it is clear that the rate of growth of this particular gazelle was constrained by nothing else but the rate of training high-class managers. Otherwise, the tiny bank that has recruited more than 36 thousand rookies would simply lose control over its operations.

Our interviews showed that the above line of reasoning is not mere speculation. Leaders of gazelles (Fig. 2) indeed interpret the situation with their companies as conditions of quasi-unlimited demand. Only 10–14% of gazelles[4] point to insufficient demand as a factor that hampered the fast growth of their company! At the same time, resource constraints,

[3] Additional research has shown that these companies are also gazelles, but have a shorter longevity.

[4] We deal with the pre-crisis period here.

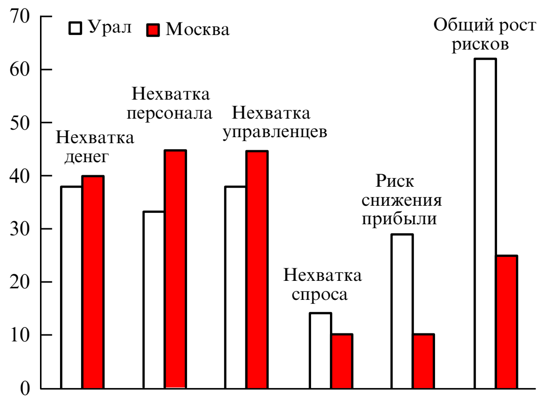

Figure 2. Answers of top managers of Russia's gazelles to the question about factors limiting the growth of their firms (by percentage of respondents). (Based on a survey of 20 gazelles in Moscow, November 2008, and 22 gazelles in the Ural region, March 2009.)

such as a shortage of money, problems with recruitment of personnel, and shortages of managerial staff, are regarded by gazelles as serious factors constraining growth (answers of 38–40%, 33–45%, and 38–45% of gazelles, respectively).

This pattern does not mean that gazelles are insensitive to the volume of demand. Rather the opposite: in all likelihood, a company can reach the rank of gazelle only after the problem of demand has been successfully resolved and the barrier of limited demand has been breached at least for some time. M Kershtein (Ramfood company, Meat products) formulated this idea almost word-for-word: "We produce as much output — of a quality that suits us and in a variety required by the market — as we can. It seems that if we had an opportunity to double our output within two or three months, the market would swallow this up."

The heretical idea that demand may not be the limitation of a company growth rate was theoretically developed by Penrose already in the 1950s [16, 17]: "So long as there are profitable production opportunities open anywhere in the economy, a firm can take advantage of them if its resources are versatile, in particular, if its management is imaginative, flexible and ambitious." The thing to do is to adjust in the right manner to the demand that always lurks somewhere. "Failure to grow is often incorrectly attributed to demand conditions rather than to the limited nature of entrepreneurial resources [of the company]" [16, p. 540].

It is known that Penrose's concept failed to reach the status of generally accepted truth. The study of Russia's gazelles led us to understanding why this attractive idea was unsuccessful. We came to the conclusion that this sort of logic should be applied not to every company (as Penrose did) but only to entrepreneurial, innovative firms. Only such companies, gazelles among them, have sufficient entrepreneurial resources for cutting through demand constraints and reaching the stage of an ultrafast exponential growth trend. What Penrose regarded as a general rule is in fact by no means characteristic of the majority of firms as they slowly evolve in the conservative environment of a steady market. But the study of gazelles allows concluding that just the exponential growth — unconstrained by demand — of this small part of the population of firms constitutes the central mechanism driving the rapid changes in a market economy (see [18] for information on the model splitting of firms into conservative and innovative).

4. Gazelles and another approach to the problem of the modernization of Russia

Today, after decades of neglect, the question of the modernization of the economy has moved to the center of public attention and policy initiatives. The main emphasis is on stimulating the growth of hi-tech companies (like the creation of the Skolkovo center). This is undoubtedly an important task, because entrepreneurs building an innovative company in Russia without help and support face huge difficulties trying to overcome various obstacles. Professionals know, however, that in real life, the problem is not solved once the company has been created; far from it. The main hurdle is that an innovative enterprise in Russia, having reached a certain size, ceases to grow and never matures into a world-class player. Companies simply lack sufficient demand. In the simplified Russian economy, which shifted the focus towards primary products, sophisticated hi-tech products wilt in the limbo even despite unquestionably outstanding product characteristics.

"Russian innovative companies attempt to skip several institutional barriers in their evolution. They need access to capital at the early stage and later some help in conquering markets" [19, p. 24]. Direct stimulation of innovators by the state at this later stage is almost pointless: what use is there in multiplying their number if the market offers no demand for their products?

This is where Russian gazelles may come to the fore. In terms of the structure of the industries, they and their counterparts in other countries are typically not hi-tech firms (Table 3). On the contrary, most of them are concentrated in conservative industries (trade, construction, food) or industries at a moderate technological level (machine building, chemistry).

It has been fully understood, however, that a strong hi-tech component does not grow where it feeds itself, i.e., satisfies the demand within hi-tech branches, but it grows where its product is demanded by the entire economy, including the saturation of all, including low-tech, industries with hi-tech products [20, 21]. Gazelles reign supreme in identifying and building demand for their products at the consumer end and are therefore natural consumers/implementers of new technologies. The fact that gazelles have this ability to stimulate consumer frenzy in any (including traditional) industries additionally expands the scope of use of innovations. Owing to gazelles, high-tech firms gain potential access to massive sales of their products.

We have collected documented examples of borrowing, adapting, and implementing technological advances in the files of a number of gazelles. Nevertheless, this has not yet become a large-scale phenomenon in Russia because manufacturers of hi-tech products in this country do not yet know which specific innovations are in demand by gazelles, while gazelles have no information on what Russian hi-tech can supply to them. Obviously, the current predominant policy of stimulation to create hi-tech companies requires supplements aimed at facilitation of their contacts with the agents generating demand for innovations, such as gazelles.

5. Conclusion: prospects for the econophysics approach to simulation

The study of the phenomenon of Russian gazelles is still in its infancy. But even the initial results indicate its considerable significance. Gazelles have proved to be brilliant representatives of healthy, nonoligarchic Russian business resulting in

Table 3. The sectoral structure of gazelle population.*

Industry	Gazelles, %	All permanent firms, %	Number of gazelles per 100 permanent firms
All industries	100	100	7.9
Wholesale and retail trade	42.3	39.3	5.5
Construction and building materials	20.7	13.0	8.1
Machine building	7.8	7.6	5.2
Food industry	5.2	7.1	3.7
Engineering (including oilfield services)	4.1	1.2	17.6
Chemistry, pharmaceuticals, perfumery	3.6	2.8	6.5
Transportation, logistics, communications	3.3	4.1	4.1
Consumer goods	2.5	1.4	9.2
IT and Internet	2.3	0.6	19.7
Business services	1.9	0.5	19.6
Agro-industrial complex	1.7	1.7	5.2
Hotels, tourism, entertainment, public catering	1.4	1.0	7.4
Other	3.2	19.7	0.8
* Source: database of medium-size businesses, Mediaholding *Ekspert*–Finuniversitet.			

an appreciable impact, both quantitative (increase in GDP and employment) and qualitative (initiation of proprietary innovations and transfer of innovations from other sources), on the economy of the country.

From the standpoint of research, it appears promising to simulate the effect of gazelles on the development of high technologies in Russia in the framework of econophysical analogy to the phenomenon of a particle overcoming the potential barrier.[5] In this analogy, the innovative activity of a high-tech firm is regarded as the particle energy, which may be sufficient but is more often too small for the particle to overcome the 'work function for entering the market.'

Two potential barriers of different degrees of transparency have to be overcome for the successful development of a company. During the first stage of a company development, i.e., while the market tests the innovative product (annual revenue up to \$10 million, the first 'potential barrier'), the state support for innovative companies as such plays an important role; this factor is introduced into the model as an increased energy of particles.

At the second stage, when the firm reaches an annual turnover of about \$100 million, the main success factor is the access to the mass market of hi-tech products. Simulation of overcoming this second 'potential barrier' uses the concept of the tunneling effect. The 'transparency coefficient' of the barrier is then a function of the degree of development of the gazelle population, i.e., of the carriers of the external factor — the demand for innovative solutions — not constrained by the limits of hi-tech demand.

Next, based on the known quantum mechanical equations for the tunneling effect, the possibility of a mass emergence of hi-tech firms in some industries can be considered (tunneling through a potential barrier and the subsequent clusterization). The conditions necessary for the emergence and development of such a process can be estimated in the semiclassical approximation.

[5] This idea was suggested by O E Pyrkina (Financial University), who is currently working on the development of this model.

References

1. Birch D L *Job Creation in America. How Our Smallest Companies Put the Most People to Work* (New York: Free Press, 1987)
2. Birch D, Medoff J, in *Labor Markets, Employment Policy, and Job Creation* (Eds L Solomon, A Levenson) (Boulder, Co.: Westview, 1994) p. 159
3. Davis S J et al. *Small Business Economics* **8** (4) 297 (1996)
4. Kirchhoff B A *Entrepreneurship and Dynamic Capitalism* (Westport, Conn: Praeger, 1994)
5. Audretsch D B *Small Business Economics* **18** (1 – 3) 13 (2002)
6. Delmar F, Davidsson P, Gartner W B *J. Business Venturing* **18** (2) 189 (2003)
7. Europe INNOVA (2008): Europe INNOVA Gazelles Innovation Panel. Summary and Conclusions from Panel Discussions, http://www.europe-innova.org
8. Henrekson M, Johansson D *Small Business Economics* **35** 227 (2010)
9. Chepurenko A Yu *Mir Rossii* (2) 22 (2008)
10. Yudanov A Yu *Opyt Konkurentsii v Rossii. Prichiny Uspekhov i Neudach* (The Experience of Competition in Russia: Causes of Success and Failure) (Moscow: Izd. INTRAS–KNORUS, 2007)
11. Yudanov A Yu *Voprosy Ekonomiki* (2) 85 (2007)
12. Dumnaya N N *Novaya Rynochnaya Ekonomika* (New Market Economy) (Moscow: MAKS-Press, 2009)
13. Kolodnyaya G V *Skvoz' Prizmu Praktiki*: *Neoinstitutsional'naya Teoriya Firmy* (Through the Prism of Practice: Neoinstitional Theory of the Firm) (Moscow: Finansy i Kredit, 2007)
14. Vin'kov A A, Gurova T I, Polunin Yu A, Yudanov A Yu *Ekspert* (10 (599)) 36 (2008)
15. Yudanov A Yu *Zh. Novoi Ekonom. Assotsiatsii* (5) 91 (2010)
16. Penrose E *Am. Econ. Rev.* **45** 531 (1955)
17. Penrose E *The Theory of the Growth of the Firm* (Oxford: Oxford Univ. Press, 1995)
18. Maevsky V I *Voprosy Ekonomiki* (11) 58 (2001)
19. Medovnikov D, Imamutdinov I "Proiti pubertatnyi period" ("To leave puberty behind") *Ekspert* (2 (641)) 24 (2009)
20. Hirsch-Kreinsen H, Jacobson D (Eds) *Innovation in Low-Tech Firms and Industries* (Cheltenham: Edward Elgar, 2008)
21. Polterovich V M *Voprosy Ekonomiki* (6) 4 (2009)

PACS numbers: 89.65.Gh, **89.75.–k**
DOI: 10.3367/UFNe.0181.201107h.0762

Equilibrium models of economics in the period of a global financial crisis

I G Pospelov

1. Economy as an example of a complex creative system

The economy is a subsystem of society that controls the production, distribution, and consumption of resources, goods, and services. The task of modern economics is extremely difficult. We speak here about producing several billion types of goods and distributing them among several billion individuals and entities. This is the reason why economics as a management system is always fairly decentralized. We recall that the USSR Planning Committee operated with approximately 2000 types of goods, while the actual list of types of goods exceeded a hundred million. We believe that the glaring mismatch of management and the growing complexity of economic links constituted the main reason for the demise of the idea of centralized planning.

To build a model of the economy, we need to face the *complexity of the system*. Complex systems are special, not just in consisting of a large number of elements but, above all, in their *uniqueness* and, perhaps most importantly, their *ability to undergo qualitative changes*. Consequently, when we study complex systems, we are actually following a single trajectory, which does not display statistical reproducibility and does not reveal the full potential of the system. The study of complex systems takes us beyond the empirical method, which underlies the triumph of science over the past 300 years. Models of physical systems are expected to explain the results of experiments and to predict the results of the planned ones; in contrast, models of complex systems are essentially built to replace the experiment. As a result, for a complex system, we obtain many models that cannot be derived as special cases of a universal 'supermodel.' Partial models describe different perspectives of the system. They operate with different sets of concepts and neglect by no means small deviations from the dependences that they take into account [1].

However, experience shows that models can go a long way in describing a complex system. A good model not only describes the behavior of the system under a current structure of relations but also contains a description of its own applicability limits and the limits of stability of the described structure. However, we have to be reconciled with the following facts:

- different models of a complex system cannot add up to a full adequate model, as happens with computer simulators of engineering systems;
- it is unlikely that we will someday be able to predict which structure arises in a complex system after the loss of stability by the previous structure. The model can reveal the real threat to the existing order of things, but it cannot predict how the crisis will be resolved.

I G Pospelov Dorodnitsyn Computation Center, RAS, Moscow, Russian Federation. E-mail: pospeli@ccas.ru

Uspekhi Fizicheskikh Nauk **181** (7) 762–767 (2011)
DOI: 10.3367/UFNr.0181.201107h.0762
Translated by V I Kisin; edited by A M Semikhatov

The oil crisis of 1975 is an example of an abrupt break in trends. Both history and physics claimed that production cannot grow faster than the consumption of energy. In fact, however, the living standards in the West grew by a factor of 1.5–2, while energy consumption per capita did not increase at all!

Another example is the current global economic crisis: everyone expected the crisis of possibilities (depletion of resources), but the crisis that actually occurred was that of needs (depletion of growth stimuli). Physically, there is no constraint on the economy of the 'golden billion,' but it nonetheless refuses to grow! In addition, the virtual economy proved more stable than the real one — foodstuffs, fuel, metals, and gold proved to be in excess and lost in price much more than did services and information. Even in the financial field, the item that for centuries was considered the most reliable investment — the mortgage business, i.e., credits to a real person pledging real property — has collapsed; at the same time, such credits as needed for launching web sites with advertising continue to be financed. The farther production is located in the technological chain from base industries, the less if suffers from a crisis.

This happens for the same reason that the doubling and tripling of energy and metal prices hardly affect car prices. The cost of design, quality control, and promotion make up a considerably greater part of the net cost of a modern automobile than the cost of materials. The new product for which people are prepared to pay (i.e., the value added) is currently created not so much on the field or in a factory as in the design office, in the department of technical control, and in the shops [2]. It is difficult to say now where all this may end up, especially if we consider S P Kapitsa's important observation that the current crisis coincides with a phenomenon unprecedented in human history: population growth is slowing down without famines and epidemics! We do not rule out the possibility that humanity is on the path toward zero growth and a purely 'spiritual' life. Not to the life, of course, that ecologists and moralists painted for us. What we are witnessing is the stabilization of aggregate material consumption plus the loss of interest that humankind has in the world outside, focusing its attention on inexhaustible variations of problems of interpersonal relations.[1]

This, however, is a remote future, at least because a well-settled system of economic mechanisms is operational only if the prospects of economic growth materialize. Therefore, either growth will be restored in a foreseeable future or we are in for a series of unsuccessful attempts to restore it. As long as new mechanisms adapted to zero growth have not been shaped and started to work, only processes of decline and restoration of growth can be modeled in seriousness.

2. System analysis of a developing economy

Numerous models of an economy and a plethora of methodological approaches to its modeling are available. The most popular models since the 1990s are the *Computable General Equilibrium models*, CGE (see, e.g., [4]), because it has been understood that taking only technological limitations (*equilibrium models* [5]), extrapolation of past trends (*econo-*

[1] It was already mentioned in [3] that the transition of civilization to the introvert phase could be an explanation for the phenomenon of 'silent cosmos,' which appears gradually stranger in view of the recent discoveries that seem to demonstrate a wide distribution of conditions suitable for life in outer space.

metric models [6]), and direct superposition of external constraints (*models of global dynamics* [7]) into account is insufficient for adequately describing the present-day economy. We need to remark here that in modern science, the term 'equilibrium' is given three originally different interpretations:

• *dynamic equilibrium — the balance of forces* acting on a system;

• *statistical equilibrium — the balance of probabilities of transitions* between states of a system;

• *economic equilibrium* — the balance of interests of the parties to a conflict.

There is something that these concepts share, but ignoring the differences among them would be an inadmissible vulgarization. We see below that economic equilibrium implies neither static behavior nor simple dynamics.

A new avenue of research emerged in 1975 at the Computing Center of the USSR Academy of Sciences (later, Russian Academy of Sciences): the *system analysis of an evolving economy* (SAEE), in which the methodology of mathematical modeling of complex systems, developed in the natural sciences, was synthesized with the achievements of modern economic theory [8]. The ideas of SAEE models are close to those of CGE models, but they pay more attention to the specifics of current economic relations; furthermore, we began our research some 15 years before the appearance of the first CGE models.

The study began with a model of a market economy, and in 1988, we constructed a model that reproduced the main qualitative features of the evolution of a planned economy. Therefore, an approach to analyzing the changes occurring in the economy had already been developed by the time the economy started to change in the USSR, and later in Russia. The short-term consequences were correctly predicted two years before the reform of 1992. Each of the subsequent models — the model of the high-inflation period 1992–1995, the model of the 'financial stabilization' period 1995–1998, which predicted the 1998 crisis, and the model of assessment of the prospects for economic development after the 1998 crisis — was based on a set of hypotheses concerning the nature of economic relations that dominated the corresponding period in Russia. It can be said that we have produced a real 'chronicle' of economic reforms in Russia that was written in the language of mathematical models. These models are described in detail in [9], and a detailed overview is given in [10].

These models allow understanding the internal logic of the evolution of economic processes. The experience accumulated in applying the models showed that they provide a reliable tool for analyzing macroeconomic regularities, as well as for predicting the effects of macroeconomic decisions *under the assumption of preservation of entrenched economic links*.

3. Model of intertemporal equilibrium of Russia's economy

In 2004, we pushed the CGE methodology aside and turned to the theoretically more consistent but conceptually and technically much more complicated construction of the *general intertemporal equilibrium with control of capital* (GEC). The GEC model is constructed not by the sequential addition of descriptions of individual economic processes but by specialization of the general scheme of operation of an ideal market economy. This scheme seems to be the most important discovery of mathematical economics over the entire 150 years of its existence. If we formally consider the task of economic planning in the interests of consumers subject to technological constraints of the individual production units and constraints on material balances over the entire economy, then the problem of *central planning* using the saddle-point theorem can be written as the *equilibrium problem in a game of independent macroagents each of which pursues its own interests* (equilibrium of perfect competition). The top five agents from the list below emerge in the process, as do *money and prices* [10]. Each of the GEC macroagent models solves the problem of optimally controlling the material and financial flows that the microagent manages.[2] We assumed that the macroagent knows the *correct prognosis of economic trends*, and these prognoses are determined by the conditions of approval of plans of agents within the complete system of material and financial balances. This constitutes the *Principle of Rational Expectations* — a strange principle that nevertheless works well in practical situations (Rational Expectations [11]).[3] Possible reasons for the applicability of this principle are discussed in Section 4.

An applied model of the GEC is obtained from this ideal construction by aggregating wealth, differentiating the money, introducing agents 'external' to the economy (see the last four items in the list given below), and, most of all, subjecting the actions of the agents to additional restrictions that reflect the 'observed rules of the game' in the economy.

The current version of the model of the Russian economy describes the real sector, which produces domestic and export products and consumes imported and domestic products, and the financial sector. The financial flows that accompany production, distribution, and consumption of products are described as the turnover of six financial instruments: *cash, payment accounts, correspondent accounts at the Central Bank (CB), bank loans, bank deposits, and bank deposits/loans in securities and foreign currency*. Products, labor, and financial instruments form a set of additive variables for which a complete system of balances is written in the model with the flows of financial instruments divided into legal and shadow ones. The movements of additive quantities are described as produced by nine macroagents:

(1) the *producer*, which represents nonfinancial commercial organizations;

(2) the *bank*, which represents financial commercial organizations;

(3) the *households*, which represent individuals — consumers and employed personnel;

(4) the *owner*, representing the physical and legal persons who control the movement of capital among domestic sectors of the economy and its flows outside the country;

(5) the *seller*, acting as a pure intermediary between consumers, producer, exporter, and importer;

[2] Both the experience of economic theory and a large amount of empirical data suggest that the rationality of behavior is typical of large groups of subjects that play a similar role in the economy, while the behavior of individuals, including the state, is chaotic and difficult to predict.

[3] Any alternative to this principle means that simulation must be executed separately for the economy evolving the way it does in reality, and separately for the picture of this evolution in the minds of subjects. This attitude appears to be not only hardly realizable, but equally too self-assured. The Principle of Rational Expectations assigns equal rights to the researcher and to the subjects under study: the model agents use the same model for their forecasts that we are constructing!

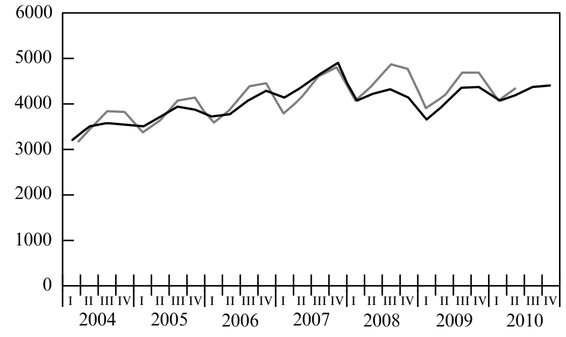

Figure 1. Real GDP, billion rubles in 2004 prices, per quarter.

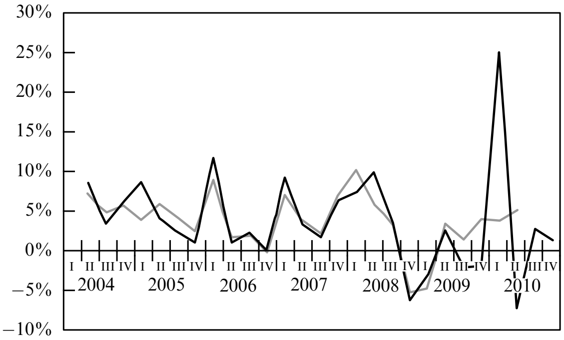

Figure 2. Quarterly rate of inflation in GDP, per quarter.

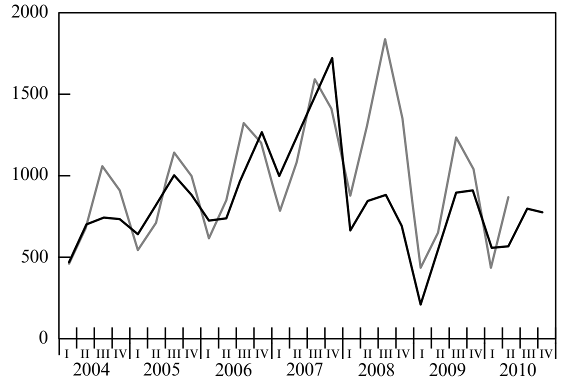

Figure 3. Real investment, billion rubles in 2004 prices.

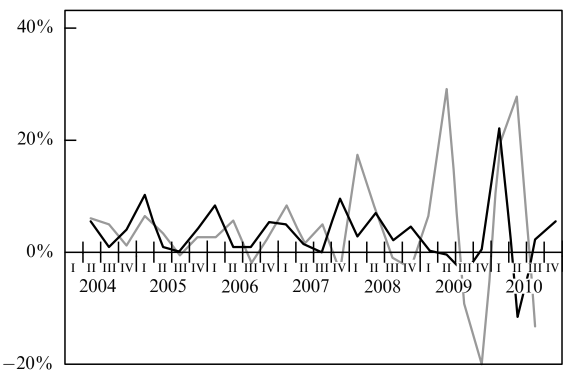

Figure 4. Quarterly inflation rate in capital investments.

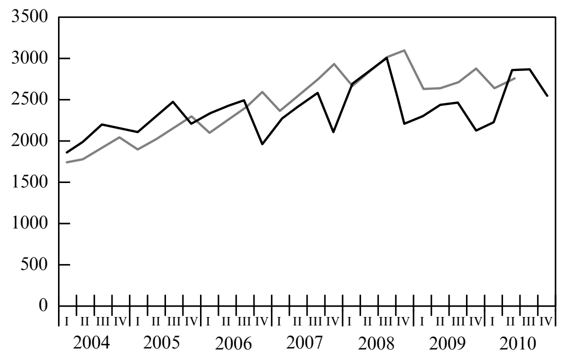

Figure 5. Quarterly real consumption, billion rubles in 2004 prices.

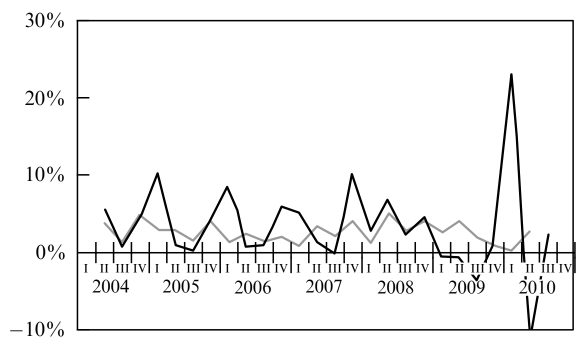

Figure 6. Quarterly rate of inflation in consumption.

(6) the *government*, whose work is represented in the model explicitly by an aggregated description of the Ministry of Finance and implicitly by establishing various parameters of economic policy (tax rates, government spending, regulations for reserves, etc.);

(7) the *Central Bank*, represented in the model by its functions of the issuer of national currency, the holder of currency reserves, the settlement center, and the lender to commercial banks;

(8) *exporter*;

(9) *importer*.

The initial structured representation of the model consists of 162 dynamic and finite nonlinear equations for which a *boundary value problem* is posed. *Indices of export and import prices, numbers of employees*, as well as the national economic policy (described using *government consumption, subsidies to the households, exchange rate, and the Central Bank discount rate* as variables) are treated as external variables. The set of equations of the model contains 50 constant parameters, of which 30 are identified regardless of the model (tax rates, the parameters of production functions, etc.), and the other 20 are used as fitting parameters. The model is identified using the official *unsmoothed quarterly statistics*.

Some results of calculations are shown in Figs 1–12. The lighter curves plot series of statistical data and the darker ones are calculated curves from the first quarter of 2004 to the fourth quarter of 2010. We note that this entire set of trajectories represents *a single economic equilibrium*.

It is obvious that the model reproduces the statistics satisfactorily, including the phases of fluctuations, the decline as a result of the crisis, and the difference in inflation rates for different products. The model *describes the phenomenon that distinguishes the Russian crisis from the crisis in all other countries*: everywhere, a decline in production is accompanied by deflation, whereas in our case, by inflation!

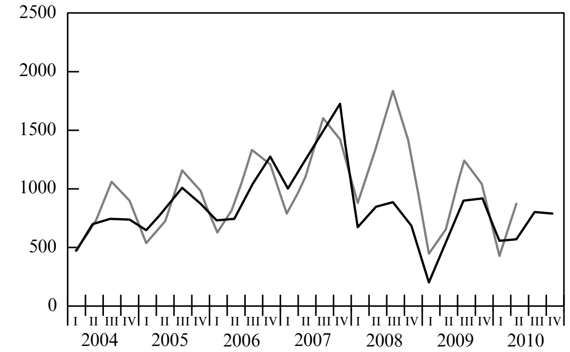

Figure 7. Quarterly real exports, billion rubles in 2004 prices.

Figure 8. Quarterly real imports, billion rubles in 2004 prices.

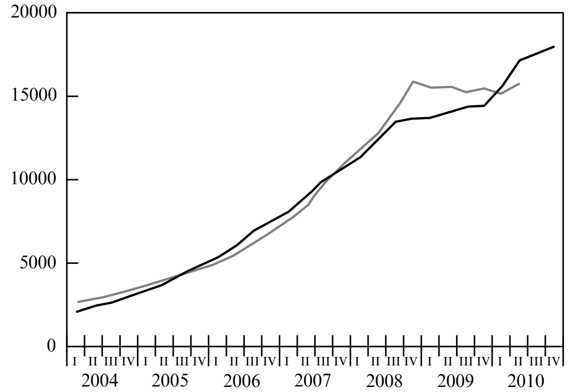

Figure 9. Credits to legal entities, billion rubles.

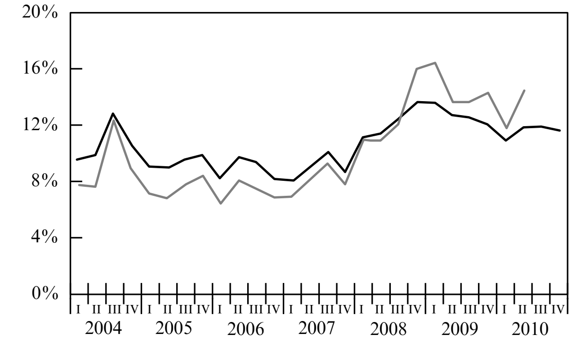

Figure 10. Annual interest on loans.

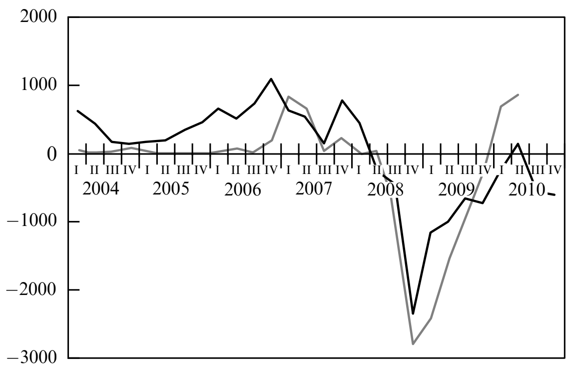

Figure 11. Net deposits of banks at the Central Bank, billion rubles.

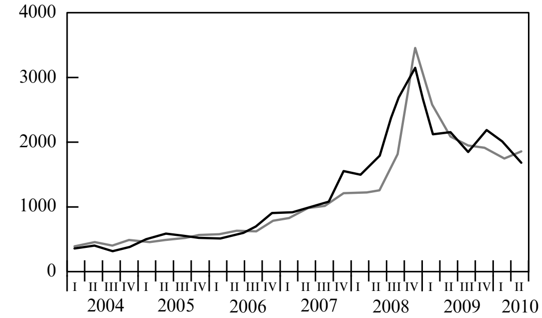

Figure 12. Liquid assets of banks, billion rubles.

4. Economics and physics: why similar approaches yield different results

In modeling the economy, we can successfully use the approaches established in theoretical physics long ago: variational principles, principles of symmetry (both exact and broken), separation of variables into intensive and extensive, etc. Nevertheless, our experience shows that the similarity in approaches does not imply a qualitative similarity in the behavior of models of physical and economic systems.

In *physics*, the extensive (additive) quantities are the *masses* of different forms of matter, different types of *charges*, all forms of *energy, entropy, momentum, angular momentum*, and so on. Their changes are described by transfer equations or reaction–diffusion equations.

Extensive quantities in *economics* are the *accumulated* wealth and financial instruments. Their movements are described by equilibrium equations, not in space but in a set of economic agents.

The most important for *physics* are conserved extensive quantities. In modern formal financial systems, a conserved quantity is the algebraic sum of reserves of any financial instrument. The aggregate turnover of a financial instrument grows only at the expense of the so-called credit emission — simultaneous growth in assets (positive stocks) and liabilities (negative stocks). Consequently, formal laws of conservation of financial instruments are less useful than conservation laws in physics. For example, it would be naive to expect that with the collapse of one market the money would flow out and transfer to another. In a crash, the liabilities and assets cancel each other, and turnovers slide downwards on all markets.

The symmetries typical of natural systems are translational and rotational, while the economy reveals scale symmetries. The best proof of this is the fact that we usually characterize changes in the physical world by their speed and changes in economic indicators by their rates (logarithmic time derivatives). This means that in the former case, the absolute scales of quantities are essential, while in the latter case they are not. As a result, in physics, the 'favorite' (i.e., the most illustrative) solutions are uniform motions at constant speed, while in economics, these are self-similar solutions in which extensive quantities increase exponentially, i.e., at a constant rate. The vast majority of the conclusions of economic theory have been obtained by comparative analysis of self-similar solutions of simple models.[4]

The variational principle in physics 'controls' *the system as a whole*, while in economic models, *each agent has its own variational principle*. Even more important is the significant difference among the topological structures of these principles. Application of the variational principle always leads to a Hamiltonian system of equations of motion, and this motion is on the surface of a constant Hamiltonian function.

In physics, the Hamiltonian function is, roughly speaking, downward convex. Therefore, its stable *critical points* are essentially *centers* around energy minima, and typical motions reduce to rotations, vibrations, and windings on tori. In general, these movements demonstrate neutral stability, i.e., they shift on the whole by a distance of the order of the increment in the initial conditions.

With the characteristic orientation of the *economy to maximization of capitalization*, utility, profit, etc., the Hamiltonian function is convex downward in 'momenta' and convex upward in 'coordinates,' and all its *critical points* are *saddle-shaped*. As a result, any economically meaningful movements of the system are close to stable separatrices of saddles. These solutions are unstable with respect to the initial values of the momenta (which in addition are unobservable) but depend weakly on the initial conditions for coordinates and on perturbations in the distant future. For a Hamiltonian system, we have to solve not a Cauchy problem but a *boundary value problem*. The ensuing results are known as *turnpike theorems*. They give us hope that models like intertemporal equilibrium models will be true in the mid-term, regardless of the accuracy of predictions for the distant future.

Our main result obtained in recent years was the discovery of a strong turnpike property: *even though we allow agents in the model to know the future, this knowledge proves useless for them as regards choosing the optimal behavior*. Because this property holds, it removes all objections to the application of the principle of rational expectations. In other words, the model reduces to a conventional dynamic system. However, the property itself requires an explanation.

The key here is that a strong turnpike effect is observed not in the model in general, at the level of formulas, but only if the parameters are correctly identified [12]. We need to recall here that economics as a management system is meant not only to coordinate the actions of billions of people but also to do it in a way that allows people to make a reasonable choice in most cases, without complex calculations. Therefore, even widely familiar economic mechanisms may not work, owing to complexity and risk. We can assume that at any given time, the economic system selects and engages a set of mechanisms that do not require detailed calculations for reasonable solutions. Consequently, by describing the mechanisms in the model 'true to life,' not true to textbooks, we arrive at a model with a strong turnpike property.

All this somewhat resembles the anthropic principle well familiar to physicists: the Universe appears to the observer as harmonious and 'adapted' to him or her because no observer could appear in a differently arranged universe.

This work was supported by the RFBR (No. 09-01-13534-ofi_tc), by Programs of Basic Research OMN RAN No. 3 and RAN No. 14, and the Finance Market Fund, Norway (the project Stochastic Dynamics of Financial Markets).

[4] Incidentally, in a historical perspective, the 'economic exponent' of the industrial society continues to break through seemingly fairly objective external constraints, while forecasts found by using models that attempt to include specific limits to growth — from the predictions of T Malthus to those of D Meadows — proved unsuccessful.

References

1. Polterovich V M "Krizis ekonomicheskoi teorii" ("Crisis of economic theory") *Ekonomicheskaya Nauka Sovrememmoi Rossii* (1) 46 (1998)
2. Pospelov I G *Ekologiya i Zhizn'* (3) 18 (2003)
3. Shklovskii I S *Vselennaya. Zhizn'. Razum* (The Universe. Life. Mind) (Moscow: Nauka, 1980)
4. Makarov V L, Preprint No. WP/99/069/ (Moscow: CEMI RAS, 1999)
5. Leont'ev V V *Ekonomicheskoe Esse* (Essays in Economics) (Moscow: Polit. Literatura, 1990)
6. Greene W H *Econometric Analysis* (Upper Saddle River, NJ: Prentice Hall, 2003)
7. Meadows D H *The Limits to Growth* (New York: Universe Books, 1972)
8. Krasnoshchekov P S, Petrov A A *Printsipy Postroeniya Modelei* (Principles of Building Models) (Moscow: Fazis, 2000)
9. Petrov A A, Pospelov I G, Shananin A A *Opyt Matematicheskogo Modelirovaniya Ekonomiki* (Experience in Mathematical Modeling of Economy) (Moscow: Energoatomizdat, 1996)
10. Andreev M Yu et al. *Tekhnologiya Modelirovaniya Ekonomiki i Model' Sovremennoi Ekonomiki Rossii* (Modeling Technology of the Economy and Model of a Modern Russian Economy) (Moscow: MIFI, 2007)
11. Hildenbrand W, Sonnenschein H (Eds) *Handbook of Mathematical Economics* Vol. 4 (Amsterdam: North-Holland, 1991)
12. Andreev M Yu, Pil'nik N P, Pospelov I G *Zh. Novoi Ekonom. Assotsiatsii* (3 – 4) 72 (2009)

PACS numbers: 89.65.Gh, **89.75.–k**
DOI: 10.3367/UFNe.0181.201107i.0767

On econophysics and its place in modern theoretical economics

D S Chernavskii, N I Starkov, S Yu Malkov, Yu V Kosse, A V Shcherbakov

1. Introduction

Theoretical economics has the same goals as other theoretical fields:

(1) Description of an object (system) in the language of mathematical methods.

D S Chernavskii, N I Starkov Lebedev Physical Institute, Russian Academy of Sciences, Moscow, Russian Federation
E-mail: chernav@lpi.ru
S Yu Malkov, Yu V Kosse Academy of Military Science, Moscow, Russian Federation
A V Shcherbakov 'Kurs' Corporation, Moscow, Russian Federation

Uspekhi Fizicheskikh Nauk **181** (7) 767 – 773 (2011)
DOI: 10.3367/UFNr.0181.201107i.0767
Translated by V I Kisin; edited by A M Semikhatov

(2) Description of the system responses to external stimuli.

(3) Prognosis of the system behavior under constant external conditions.

Even though economics is traditionally regarded as belonging to the humanities, theoretical economics operates with mathematical and physical concepts and methods. In this sense, theoretical economics is an interdisciplinary field. As a rule, cross-disciplinary areas (biophysics, physical chemistry, etc.) succeed in not breaking the ties to their progenitors or other sciences. There are exceptions to this rule, however.

We are currently facing a crisis in theoretical economics [1]. There are several (at least two) different avenues of research that start with different basic assumptions (axioms), use different mathematical methods, and in the end arrive at different solutions of the problems formulated above.

Nevertheless, the fates of people, countries, and perhaps the world may depend on the solutions chosen in the current period of crisis. This, in the nutshell, is what makes the situation critical.

We now discuss these fields of research.

I. The approach that is known best (especially outside Russia) is the so-called neoclassical approach, considered to be 'mainstream' [2]. The principal features of the mainstream are:

(1) an individual is an elementary unit of society. An individual fabricates products with a view to gain maximum profit. Individuals consume products with a view to extract maximum benefit for themselves. The main function considered by classical economics is the utility function. It is assumed that the individual is 'rational' and is capable of identifying the most useful set of benefits (material ones first and foremost). Differences in customs ('mentalities') in different countries and societies are not taken into consideration;

(2) self-organization of individuals in a society leads to the occurrence of a stationary (equilibrium) state in which demand (for goods, labor, money, etc.) is counterbalanced by supply. It is assumed that the society (state) is formed as a result of the assembly of individuals and that this does not generate new qualities that are not deducible from the qualities of individuals.

It seems that the purpose of the mainstream is to build (even if only in one's mind) a stable ideal society in which the interests of everyone would be satisfied to the maximum, and that this society would exist forever (like a utopian 'City of the Sun').

Mainstream systems often use dogmas such as:

(1) the state should not intervene in the economy. This provision has not been proved (hence, 'dogma') and cannot be proved because neither total noninterference nor total intervention are possible. The important factor is the degree of intervention, which depends on the situation;

(2) the stationary and equilibrium state of the market economy is unique. This position is also an unproved dogma and has been recently thrown into doubt. Nevertheless, practically no study is available in the mainstream on the dynamics of transitions from state to state.

The strength of the mainstream is that it involves large teams of high-level professionals. The main achievement of the mainstream is the analysis of the optimal steady state of the market economy.

The weakness of the mainstream is that it is isolated from other natural sciences and has in fact degenerated into Herman Hesse's 'glass beads game.' Internal consistency and conformity with the axioms are regarded as the quality criteria for assessing a solution of a problem. The condition of the results matching reality are typically not discussed. Solutions of the above-mentioned urgent problems are not considered within the mainstream, and these problems are not posed.

II. A new approach that appeared in theoretical economics relatively recently is econophysics (or physical economics). This field attacks a group of various problems.

(1) Evolutionary economics emerged in the early twentieth century in the work of Schumpeter [3]. It was further developed in many papers (see [4, 5] and the references therein). In fact, it was Schumpeter who drew attention to the fact that economies are not static but evolving systems. For a while, this approach evolved without using mathematical tools. Currently, it does have a mathematical apparatus to work with — the same as in other currently evolving systems (and in synergetics).

(2) Mathematical modeling (based on the theory of dynamic systems) of both macroeconomic and microeconomic processes [this field is often called economy synergetics (Zhang [6])].

(3) Analysis of stock series and of their properties.

The last three approaches to economics are grouped together by similarity of the systems of the main concepts, the nature of the tasks, and the methods of solving them. The economy is treated in all three as an evolving system, and all descriptions work with the same tool, the theory of dynamical dissipative systems. In other words, theoretical economics is not treated as an isolated science; rather quite the opposite is true: it joins the family of natural sciences. At the same time, econophysics is a fully defined field of science in this family.

The problem that plays a special role here is the dualism of evolution: on the one hand, society must preserve the stored information, while on the other hand, it must create new information. These two problems are complementary (in the sense of complementarity defined by Niels Bohr). In physics, complementary problems have been solved in many cases, while in economics this is a task for the future.

We thus see that econophysics does not offer anything very new (in comparison with other natural sciences). The important side of physics in this context is that it deals both with nature and with modern mathematics, and as a result it exercises disciplined thinking and a critical attitude toward dogmas. For example, physics both accepts the emergence (and disappearance) of several stationary states and widely uses this scenario [7, 8].

From this perspective, economics is a very interesting and important application area for physics.

In reality, the boundary between neoclassical economics and econophysics is somewhat blurred. Many economists use both of these approaches in different papers.

The purpose of this talk is to select examples of mathematical models in economics and discuss their results.

2. Basic concepts and models. The demand function

Demand plays an important role in the economy. Demand depends on the needs of individuals; but the behavior of others (including the media) plays at least as important a role. In other words, demand is a collective behavioral response of

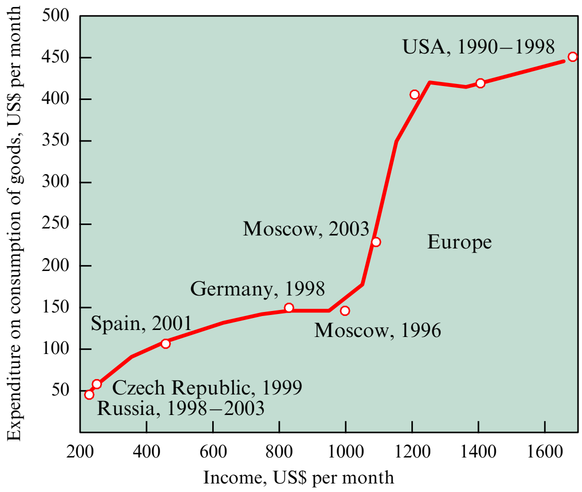

Figure 1. Empirical plot of demand function (empirical data) [9].

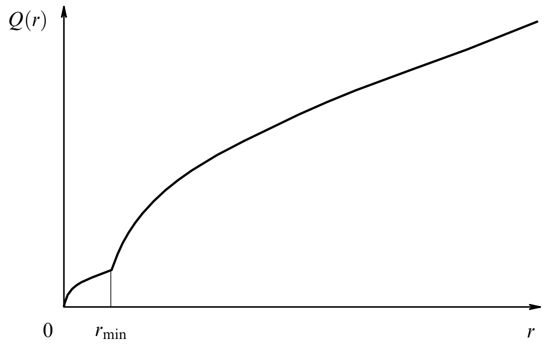

Figure 2. Analytic representation of the demand function.

human society. Demand is described by the demand function, of which there are several versions.

(1) The demand function for a given commodity (or a group of similar commodities) is the amount of product Q consumed per unit time as a function of the amount of money U available to the consumer or their income D and the commodity price p. Because both of these are arbitrary, the demand function depends on the ratio U/p (or D/p). The quantity Q can be expressed in natural units (items, kilograms), but more often the quantity Qp is used, which is expressed in monetary units. As an example, Fig. 1 plots an empirical demand function [9] based on the data published by statistics offices of various countries. Nonmonotonicity (the presence of a so-called 'beak') is an important property of this function. We see below that the beak plays an important role.

(2) Macroeconomics operates with a joint aggregation, which incorporates first-priority goods (food, clothing, housing), durable goods (cars, television sets, etc.), and elitist goods. In this case, it is more convenient to use the demand function $Q(U/p)$.

The demand function $Q(U/p)$ (Fig. 2) can be represented in an analytic form as [10]

$$Q(r) = Q_1 \frac{r}{r + r_1} + \Theta(r - r_{\min}) \left[Q_2 \frac{r - r_{\min}}{r - r_{\min} + r_2} + \varepsilon (r - r_{\min}) \right], \qquad (1)$$

where

$$\Theta(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0. \end{cases}$$

The parameters Q_1, Q_2, r_1, r_2, $r_{\min}$, and ε have the following meaning. The parameter Q_1 corresponds to the total supply of goods of vital importance; r_1 is the value of purchasing power at which these needs are half-satisfied; $r_{\min}$ is the critical level of purchasing power: at a purchasing power less than $r_{\min}$, the consumer does not buy durable goods. The magnitude of $r_{\min}$ depends on the consumer psychology. Large $r_{\min}$ signify that people are more inclined to satisfy their essential needs, i.e., to live a simpler life. Small $r_{\min}$ signal that consumers prefer to live in a 'modern way,' even by cutting their food consumption. The parameter Q_2 corresponds to complete satisfaction in durable goods, i.e., to having bought the entire 'gentleman's set of goods': the full set of household conveniences, a car, a country house, etc. The parameter r_2 characterizes humanity's desire to look worthy of the title of gentleman. If r_2 is small, the individual who succeeded in accumulating the amount $U \approx r_{\min}/p$ seeks to immediately spend the money on acquiring the gentleman's set. If r_2 is large, the individual behaves in the opposite way: modestly and frugally even if the accumulations are $U > r_{\min}/p$. The parameter ε reflects the syndrome of 'ever-growing needs of an individual,' i.e., the inability to stop the spending spree for luxury items when the money is there to spend.

To summarize, the parameters of the demand function reflect the human factor, i.e., the consumer psychology.

The factor important for the model is the collective behavior of a large group of consumers, and, in this sense, the parameters of the demand function are of a social and psychological nature, i.e., they reflect the customs and rules of conduct that have evolved in the community.

In general, these parameters are different in different countries; demand functions may also differ significantly. The sigmoid form of the demand function plays a very important role for the model. Its effect depends on the parameters $r_{\min}$ and r_2: if $r_{\min}$ is small while r_2 is large, the demand function becomes smooth and everywhere convex. The sigmoid form (the beak in Fig. 2) disappears. The parameters of the demand function may change over time, albeit slowly, for example, when one generation is replaced by another (we discuss the role of these changes later).

3. Production function

The production function $F(r)$ is the amount of commodities produced (per unit time) depending on invested funds (current assets). These last variables are more conveniently expressed not in monetary units but as the ratio of the amount of money to the weighted average price of the product p: $r = U/p$. The role of inflation is excluded from consideration. Current assets include variable costs (proportional to the volume of production) and fixed costs (the costs of sustaining and modernizing production, i.e., of research and development). Three segments can be identified in the production function:

(1) segment of constant returns — investment is proportional to revenue;

(2) segment of decreasing returns — society displays no demand for the excess of produced commodities;

(3) segment of increasing returns — the demand for a commodity (typically an innovative one) keeps increasing and the enterprise receives superprofit.

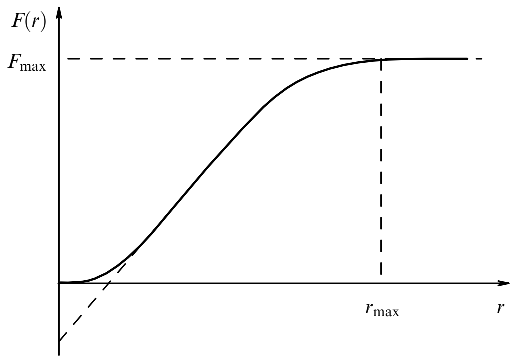

Figure 3. The production function.

The production function and the above segments are plotted in Fig. 3.

4. Basic models

In every science, there are two stages of model building [11]: first, the so-called basic (elementary) models are formulated. They operate with a small number of variables and equations and describe the essential features of the phenomenon (oscillations, crises, etc.). Next, basic models become more complex to reflect specific conditions, are modified or combined, and transform into imitation models. Physical economics has developed a set of basic models. We discuss them briefly.

(1) The logistic model, which contains a single equation of the form

$$\frac{dx}{dt} = a(x - x^2)\,. \qquad (2)$$

This model is used to describe the development of a company, the evolution of a species (in biology), demographic processes, etc. It includes two stages: the first is exponential growth (or an even steeper growth with a sharper peak [12]) and the second is reaching the steady-state mode reflecting external constraints.

(2) The basic model of competition among arbitrary portions of information, which contains two (or more) equations of the form [13, 14]

$$\frac{du_i}{dt} = \frac{1}{\tau_i}\,u_i - \sum_{j\neq i}^{n} b_{i,j}\,u_j u_i - a_i u_i^2 + D_i\,\Delta u_i\,, \quad i, j = 1, 2, \ldots, n\,, \qquad (3)$$

where u_i is the number of carriers of the ith portion of information; the first term in the right-hand side is the reproduction of the ith portion of information, the second is the interaction among carriers of different portions of information, the third represents external constraints, and the last term represents the migration of information carriers in space.

It is assumed that the interaction is antagonistic, i.e., all carriers attempt to keep their information, force it on the other carriers, and create new (proprietary) information.

Model (3) was used to describe the emergence of a universal genetic code in biology [13, 14]. In economics, it was used to describe the competition among firms (in particular, between innovators and conservatives) [15], the role of advertising [16], and the interaction among the major currencies on the external trade market [17]. In addition, the same model was used to describe processes in history (the formation of large states) [18]: it was demonstrated that the importance of ideological (informational) factors is on a par with economic factors.

(3) The covert bankruptcy model, which has the form [19, 20]

$$\frac{dM}{dt} = -\frac{M}{\tau} + p_m Q_0\,\frac{P}{P_0 + P} - \frac{p}{\tau_p}\,P - \kappa\,,$$

$$\frac{dP}{dt} = -pQ_0\,\frac{P}{P_0 + P} + \frac{M}{p\tau}\,,$$

where M are current assets and P is the quantity of goods in stock. The parameters are as follows: Q_0 is the maximum demand for the product, p_m is the market price, p is the production cost, τ is the length of the production cycle, τ_p is the duration of storage of goods, and κ is the fixed cost.

The model describes both the stable state of a company and the bifurcation, i.e., the transition to the state of bankruptcy. Simulated bankruptcy develops slowly at first, but then quickly reaches the critical phase. This model was used to discuss ecological and economic problems on a global scale. Especially important is the point at which resources are close to running out while the cost of their regeneration by waste reprocessing increases [20].

(4) The base model of the transition from the high production (HP) to the low production (LP) state; in fact, a model of the crisis, which is akin to the model of a phase transition. In econophysics, this model was made possible by dropping the dogma of the uniqueness of the state of the market. A macroeconomic imitation model of modern Russia has been created using this base model as a basis; we discuss it in more detail in Section 5.

5. Macroeconomic model of modern Russia

The purpose of the model is to describe what happened as a result of price liberalization and what the possible options are for further development. The model was built in the 1990s and was published in *Physics–Uspekhi* [7]. The crisis of the 1990s was then interpreted as a phase transition from the HP state to the LP state as a result of price liberalization.

The model has been improved and extended in recent years [10]. In this period, it has become clear that the results depend strongly on the external situation and the decisions of the government of Russia. At the present moment, the model is not intended to provide a long-term prognosis. We go even further: such a prognosis is simply impossible because the economy of Russia (and of the rest of the world, too) has moved close to an unstable state. As a consequence, the decisions taken by governments (of every country) produce important effects; however, neither those who take those decisions nor those who are trying to implement them understand what specific consequences to expect.

The proposed model was created to serve as a tool to support decision making. A model can do that if it is capable of answering questions like: what would happen if...? In addition, the model should be capable of short-term forecasting (like answering 'what would happen if we do nothing at all').

We do not reproduce the model here in full (it has been published); we only note some recent results.

(1) Visual representation of the state of the economy is shown in Fig. 4. Plotted along the ordinate axis are the demand function $Q(gr)$ and $F(r)\mu$, where g is the fraction of

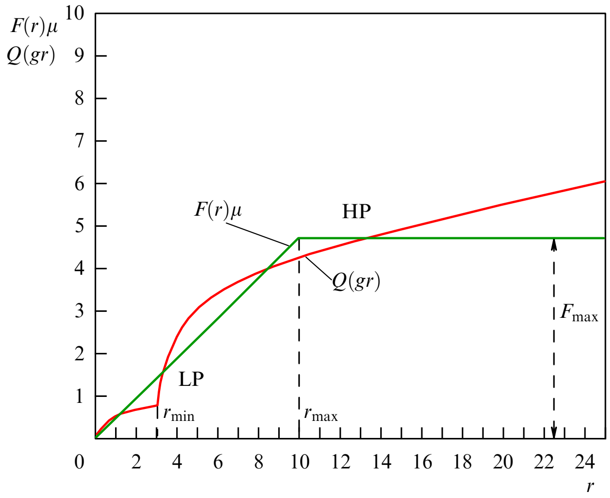

Figure 4. Balance diagram.

funds assigned by property owners to their personal needs, $F(r)$ is the production function, and μ is the level of costs [7]. Their intersections are stationary states. There are three of them: HP, LP, and the collapse of production. As these functions (their parameters) change, transitions (of the phase-transition type) become possible.

The following remarks are in order:

— the situation in Russia is truly variegated. Production functions in different regions and different companies are different, and therefore a range of functions can be shown in the diagram, covering companies both in their HP state (the so-called gazelles [21]) and approaching bankruptcy;

— generally, the demand function changes with the change of generation.

Figure 5a plots the demand function corresponding to the 'frugal' generation, which spends money to buy long-life commodities under the condition that they are available. The curve clearly shows that only the HP state is implemented in this situation. The demand function in Fig. 5b represents generations born into a prosperous life and wanting to have everything — all at once. Clearly, the economy then plunges into the LP state.

Long-term periodic changes (of the order of 10–20 years) are known in economics as the Kondratieff cycles. It is possible that one of the causes of these cycles is the process outlined above. In fact, Kondratieff himself connected cycles with the innovative activity of people (leaving aside the psychology and the demand function, although these factors also have a bearing on the issue.)

6. Crisis of 2008

The crisis engulfed the whole world, but its causes were different in the United States and in Russia.

In Russia, prices for raw materials and products of natural monopolies (NMs) started to grow in 2004. Prices of goods manufactured domestically grew more slowly (they followed inflation). Profitability fell and approached the critical value (the beak). The price equation showed decreasing stability and pointed to an Andronov–Hopf bifurcation (i.e., to an oscillation mode). If this tendency had continued, an industrial crisis in Russia would have been inevitable (this has been mentioned at conferences on numerous occasions [21] and also appeared in publications).

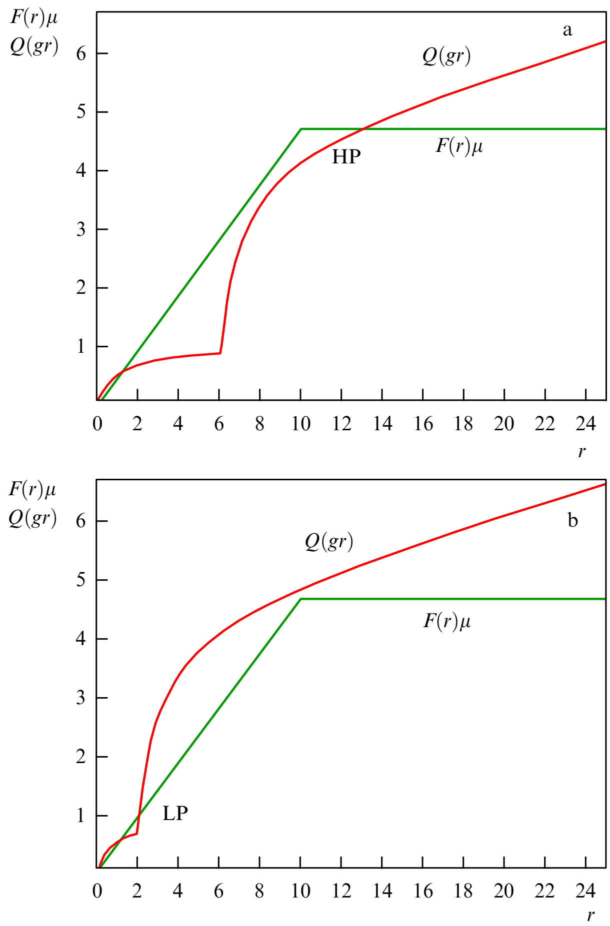

Figure 5. The 'phase transition' in response to a change in the demand function in the case of (a) a frugal generation and (b) a generation that grew up in prosperity.

In the U.S., the crisis was financial in nature (falling stocks and a crisis of commercial banks). The first and foremost to suffer from it were Russian financiers: U.S. banks demanded that they return loans and refused to offer new loans to repay debts. Russia's administration supported Russian financiers at the expense of funds withdrawn from the real sector. As a result, Russian industrialists were deprived of working capital for six months (the second half of 2008). Profitability being low, this led to the industrial crisis [22]. Figure 6 plots official data from 2003 to 2009 (quarterly, in rubles).

We see that the general trend of the gross domestic product (GDP) was positive prior to August 2008 (against the background of seasonal fluctuations). This does not contradict the assertion that the profitability of the manufacturing sector declined over these years; the aggregate GDP also includes services and the production of raw materials (where prices had risen), and this component is responsible for the overall positive trend (as we see from Fig. 6). The dashed curve represents the calculated data without the withdrawal of working capital in 2009. It is clear that in this case, there would have been no crisis in Russia (i.e., it would have happened a little later). As a result, the crisis did hit Russia 'unexpectedly' at the end of 2008, and the total GDP declined by nearly half. The GDP has increased subsequently, but has not reached the previous level.

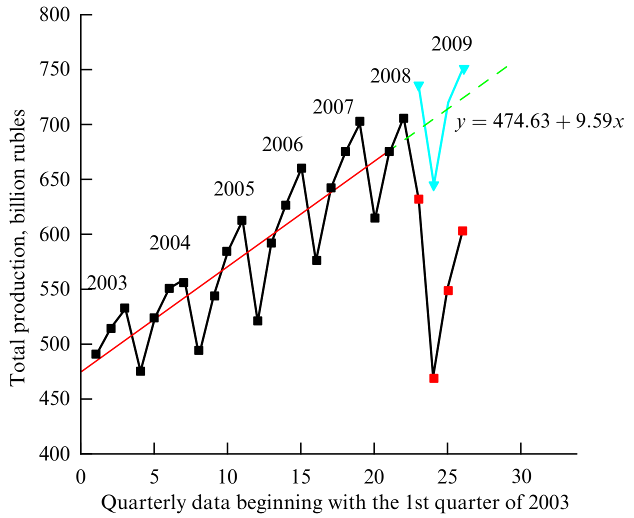

Figure 6. GDP dynamics in the interval 2003–2009: ■, ● — statistical data, ▼ — trend for 2008 (IV quarter) and 2009 (I–III quarters), straight line is the trend, with seasonal variations not taken into account.

7. Post-crisis measures

Anti-crisis measures have been proposed since the crisis (both by the government and by other organizations). We now discuss anti-crisis measures in the framework of our model.

(1) Partial state regulation of prices of raw materials, goods, and services of the NM. This factor enters our model as the parameter of profitability, and depends on inflation, global prices, and state-imposed stabilization of prices for a certain part of production. Figure 7 shows the results of GDP calculations for the manufacturing industries, for different levels of state regulation of prices. The parameter a is the fraction of the output of the NM whose prices are frozen at this (current) level and do not grow. The lower curve corresponds to no regulation, and the upper curve corresponds to the total freezing of prices. The starting point was August 2008.

The real GDP declined by about 30% and has stayed at this level ever since. Corresponding to this in the model is the freezing of prices at 75%. In reality, the falling of global prices also played its role and price freezing need not have been so harsh.

Figure 8 plots the results of model calculations and post-crisis statistical data. It is evident that the country has not yet emerged from the crisis (is still in depression) and, as we noted above, the process is oscillatory.

(2) The challenge of modifying the taxation system (lowering the value-added tax (VAT) and income tax). In our model, this is the parameter κ, determining profitability. Figure 9 plots model calculations of GDP growth resulting from tax cuts.

The figure shows that reducing the tax by 3–4% is already sufficient for leaving the crisis behind. A bifurcation is the moment when the taxation system changes.

(3) Implications of the increasing costs of the military-industrial sector (MIS). Most of the output of the MIS never reaches the internal market; hence, it generates no profit. Bearing these costs is nevertheless necessary due to non-economic reasons. The issue is: what damage will they do to the economy and what inflation will they cause? Figure 10 plots the calculated inflation as a function of the investment in

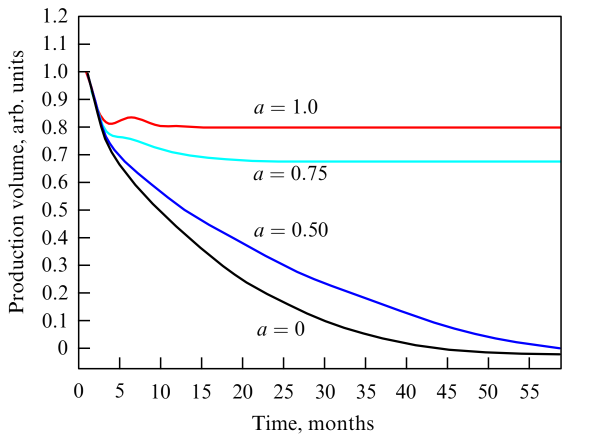

Figure 7. GDP dynamics for varying degrees of regulation of basic prices a.

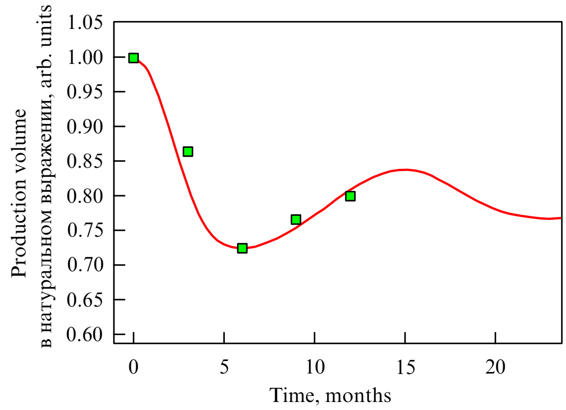

Figure 8. GDP dynamics (ignoring the trend): ■ — empirical data, the curve — model calculation taking price regulation into account for $a = 0.45$.

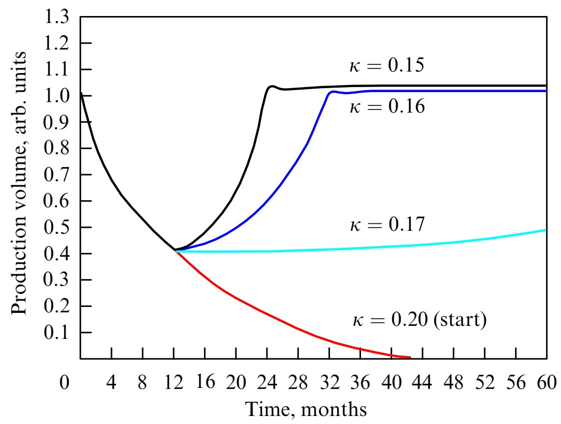

Figure 9. Response of GDP dynamics during crisis to change in taxation.

the MIS, the increase in pensions, and the increase in government-paid salaries.

Figure 10 shows that inflation caused by MIS costs is considerably smaller than could be expected. The reason is that not all investments in the MIS take part in the monetary market (in the form of salaries). They are partly channeled into the industrial sector as payment for parts for weaponry. Additionally, the amount of MIS costs is a small fraction of the budget in comparison with the amounts

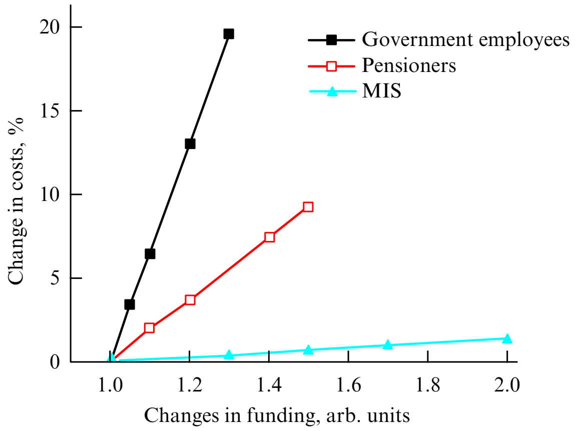

Figure 10. Changes in price inflation affected by additional funding of: ■ — government employees, □ — pensioners, ▲ — MIS industrial plants.

required to pay pensions and the salaries of government employees.

8. Conclusion

The study outlined in this paper belongs to a broad field of research known under several titles: 'self-organization' (I R Prigozhin), 'synergetics' (H Haken), and 'complexity.' The basic models mentioned above have in fact been investigated and developed in more detail than can be concluded from the present paper. Not all of them have been elaborated to the status of imitation models capable of addressing real current problems for specific conditions.

The need to reach this stage is obvious now, and research in this direction is being actively pursued. For example, papers [23, 24] present models of global dynamics. The model of 'strife of currencies' has been developed and refined for three or more participants. This is especially important for planning the introduction of a single trading currency for mutual settlements among the BRIC countries (Brazil, Russia, India, and China).

The basic model implies that the situation is far from simple and requires model calculations.

The basic model of covert failure is directly related to the global situation. The fact is that we are now supporting our lives not by producing commodities but by utilizing 'stored' materials. In other words, we are already in a state of covert bankruptcy, but have not noticed it yet. The basic model implies that the transition to an open bankruptcy may be 'unexpected' and very abrupt. It must be anticipated in time, which is why a mathematical model is needed.

The examples above do not exhaust all the problems of the economics of our time. But the basic models outlined above (and their combinations) provide a basis for modeling practically any of the pressing problems.

We have presented the model of modern macroeconomics of Russia, not as a basis model but already as an imitation one. Consequently, it claims to yield a description and a forecast of events to come. The reader will judge how successful the prognosis proves to be.

It is important that the model describes not so much a prediction as the system response to one specific external factor or another (events that have not yet occurred but may occur). In other words, the model can serve as a tool to support decision making by leaders guiding the economy of the country.

Moreover, the model is sufficiently complete, i.e., it provides information on the income and savings distribution among the population (both of these are very polarized in today's Russia), the demand for goods of different categories, etc. In other words, it allows piecing together an economic portrait of Russia as it is. It goes without saying that the model is upgraded each year (by taking the varying parameters into account). Special attention is paid to the question of how close (or how far) the country is from a bifurcation point.

This paper provides answers to only some of the possible "what if" questions. Nevertheless, econophysics has a mathematical basis for answering any such question.

We avoided touching on the issues concerned with the behavior of share prices in stock exchanges. These aspects are the subject of two other papers [25, 26] presented in this issue of *Physics–Uspekhi*.

This work was supported by the RFBR grant 08-06-00319.

References

1. Polterovich V M *Ekonomicheskaya Nauka Sovremennoi Rossii* (1) 46 (1998)
2. Hodgson G M *Evol. Inst. Econ. Rev.* **4** (1) 7 (2007) [Translated into Russian: *Ekonom. Vestn. Rostov. Gos. Univ.* **6** (2) 8 (2008)]
3. Schumpeter J *Theorie der wirtschaftlichen Entwicklung* (The Theory of Economic Development) (Leipzig: Duncker & Humblot, 1912) [Translated into English (New Brunswick, NJ: Transaction Books, 1983); Translated into Russian (Moscow: Progress, 1982)]
4. Maevsky V I, in *5-i Mezhdunar. Simp. po Evolyutsionnoi Ekonomike* (5th Intern. Symp. on Evolutionary Economics) (Moscow: Institut Ekonomiki RAN, 2004) p. 91
5. Maevsky V I *Vvedenie v Evolyutsionnuyu Makroekonomiku* (Introduction to Evolutionary Macroeconomics) (Moscow: Yaponiya Segodnya, 1997)
6. Zhang W-B *Synergetic Economics: Time and Change in Nonlinear Economics* (Berlin: Springer-Verlag, 1991) [Translated into Russian (Moscow: Mir, 1999)]
7. Chernavskii D S, Starkov N I, Shcherbakov A V *Usp. Fiz. Nauk* **172** 1045 (2002) [*Phys. Usp.* **45** 977 (2002)]
8. Robinson J *The Economics of Imperfect Competition* (London: Macmillan and Co., 1948) [Translated into Russian (Moscow: Progress, 1986) p. 101]
9. Kachalov I, http://www.foodmarket.spb.ru/current.php?article=781
10. Chernavskii D S et al. *Strategicheskaya Stabil'nost'* (1) 37 (2004)
11. Chernavskii D S, Chernavskaya N M *Biofizika* **54** 139 (2009) [*Biophysics* **54** 103 (2009)]
12. Kapitza S P, Kurdyumov S P, Malinetskii G G *Sinergetika i Prognozy Budushchego* (Synergetics and Prognostication) (Moscow: Nauka, 1997)
13. Chernavskii D S, Chernavskaya N M *J. Theor. Biol.* **50** 13 (1975)
14. Chernavskii D S *Sinergetika i Informatsiya* (Synergetics and Information) (Moscow: URSS, 2004)
15. Chernavskii D S, Shcherbakov A V, Zul'pukarov M-G M "Model' konkurentsii" ("Model of competition"), Preprint No. 64 (Moscow: M V Keldysh Applied Mathematics Institute, 2006)
16. Zul'pukarov M-G M, in *Trudy 7-go Mezhdunar. Simp. po Evolyutsionnoi Ekonomike* (Proc. of the 7th Intern. Symp. on Evolutionary Economics) (Ed. V I Maevsky) (Moscow: Izd. Instituta Ekonomiki RAN, 2009)
17. Chernavskii D S, Starkov N I, in *Trudy 9-go Mezhdunar. Simp. po Evolyutsionnoi Ekonomike* (Proc. of the 9th Intern. Symp. on Evolutionary Economics) (Ed. V I Maevsky) (Moscow: Izd. Instituta Ekonomiki RAN, 2010)
18. Chernavskii D S et al, in *Istoriya i Sinergetika: Matematicheskie Modeli Sotsial'noi, Ekonomicheskoi i Kul'turnoi Dinamiki* (History and Synergetics: Mathematical Models of Social, Economic and Cultural Dynamics) (Eds S Yu Malkov, A V Korotaev) 2nd ed. (Moscow: URSS, 2010) p. 142

19. Chernavskaya N M, Shchepetov D S, Vasil'eva L Yu, in *Fundamental'nye Fiziko-Matematicheskie Problemy i Modelirovanie Tekhniko-Tekhnologicheskikh Problem* (Collected Papers: Fundamental Physics and Mathematics Problems and Modeling of Technical and Technological Problems) Issue 7 (Ed. L A Uvarova) (Moscow: Yanus-K, 2004) p. 263
20. Chernavskaya N M et al. *Ekologo-Ekonomicheskaya Deyatel'nost' Predpriyatii pri Tekhnogeneze i Chrezvychainykh Situatsiyakh* (Ecological and Economic Activities of Enterprises in Technogenesis and in Emergency Situations) (Moscow: Yanus-K, 2010)
21. Yudanov A Yu *Opyt Konkurentsii v Rossii. Prichiny Uspekhov i Neudach* (The Experience of Competition in Russia. Causes of Success or Failure) (Moscow: Izd. INTRAS–KNORUS, 2007)
22. Shcherbakov A V *Kabeli and Provoda* (5) 6 (2008)
23. Akaev A A *Dokl. Ross. Akad. Nauk* **434** 749 (2010) [*Dokl. Math.* **82** 824 (2010)]
24. Akaev A A, Sadovnichii V A *Dokl. Ross. Akad. Nauk* **435** 320 (2010) [*Dokl. Math.* **82** 978 (2010)]
25. Vidov P V, Romanovsky M Yu *Usp. Fiz. Nauk* **181** 774 (2011) [*Phys. Usp.* **54** 749 (2011)]
26. Dubovikov M M, Starchenko N V *Usp. Fiz. Nauk* **181** 779 (2011) [*Phys. Usp.* **54** 754 (2011)]

PACS numbers: **02.50.–r**, 89.65.Gh
DOI: 10.3367/UFNe.0181.201107j.0774

Nonclassical random walks and the phenomenology of fluctuations of securities returns in the stock market

P V Vidov, M Yu Romanovsky

1. Introduction. Experimental facts observed in the fluctuations of securities returns

The logarithmic return of shares and stock indices, $S(\Delta t)$, measured over a time interval Δt is defined as

$$S(\Delta t) = \ln \frac{Y(t+\Delta t)}{Y(t)}\,, \tag{1}$$

where $Y(t)$ is the price of a share or the value of an index at time t. It was the subject of systematic study already at the time of L Bachelier [1]. Several facts have been established by experimental studies of share return in international financial markets.

First, for shares of the largest U.S. companies on the time interval from 1994 to 1995, the cumulative distribution function of probability of a fluctuation greater than x, and also smaller than $-x$, is well described by a power-law function of the form [2]

$$\Phi(x) \approx \begin{cases} x^{-3}\,, & S(\Delta t) > x\,, \\ -x^{-3}\,, & S(\Delta t) < -x\,. \end{cases} \tag{2}$$

Similar results were obtained for the shares of German [3], Norwegian [4], French, Japanese, Swiss, and British [5] companies, as well as stock indices [6].

P V Vidov INTRAST Management Company, Moscow, Russian Federation
M Yu Romanovsky A M Prokhorov General Physics Institute, Russian Academy of Sciences, Moscow, Russian Federation
E-mail: myrom@gpad.ac.ru, slon@kapella.gpi.ru

Uspekhi Fizicheskikh Nauk **181** (7) 774–778 (2011)
DOI: 10.3367/UFNr.0181.201107j.0774
Translated by V I Kisin; edited by A Radzig

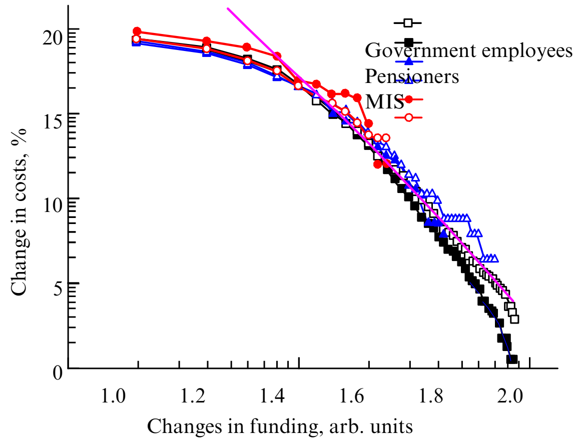

Figure 1. Cumulative distributions of normalized returns (see text) of Sberbank ordinary shares for various Δt. One-minute positive fluctuations — light squares, one-minute negative fluctuations — shaded squares; one-hour positive fluctuations — light triangles, one-hour negative fluctuations — shaded triangles; daily positive fluctuations — light circles, daily negative fluctuations — shaded circles. Bold straight line shows the dependence x^{-3}. One-minute, one-hour and daily data were obtained on trading days 10.01.2009–10.02.2009, 01.09.2008–30.09.2009, 23.01.2006–30.09.2009 (MMVB stock exchange).

Russian stocks ('blue chips') exhibit similar behavior (2). Figure 1 plots the cumulative distribution of returns for positive (black symbols) and negative (white symbols) fluctuations in Sberbank shares. The straight line in Fig. 1 plots the law x^{-3}. The ordinate is the cumulative distribution function, while the abscissa is the return normalized to the appropriate experimentally calculated root-mean-square return. We obtained similar curves for shares of other Russian companies, too. Figure 2 plots the distribution function of fluctuations of the Russian RTS stock index. It is clearly seen that all the plots of cumulative distributions resemble one another. At the same time, return curves for larger Δt lie somewhat higher than return curves for lower Δt (see also paper [5]).

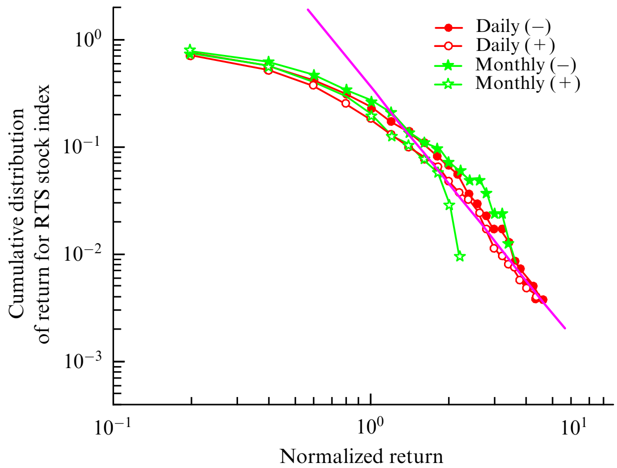

Figure 2. Cumulative distributions of normalized returns (see text) for the RTS stock index at various Δt: daily positive fluctuations — light circles, daily negative fluctuations — shaded circles; monthly positive fluctuations — light stars, monthly negative fluctuations — shaded stars. Bold straight line shows the dependence x^{-3}. Daily data were obtained on trading days 09.01.1995–27.06.2007 (RTS stock exchange), monthly data — on trading days 09.01.1995–20.10.2010.

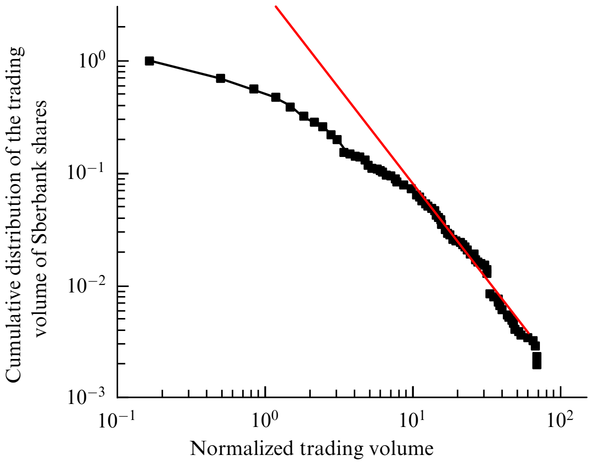

Figure 3. Cumulative distributions of the trading volume in one tick for Sberbank shares on 21.11.2007. The straight line traces the 'tail' curve $x^{-\zeta}$, where $\zeta = 1.7$.

Second, the distribution $Q(x)$ of the number of shares traded in one transaction (one tick) definitely falls within the Lévy range, i.e., the asymptotic ('tail') part of the distribution is well described by a $x^{-\zeta}$ curve, where $0 < \zeta < 2$ and we are dealing with the cumulative distribution function (see paper [7] and the discussion in Refs [8, 9]). The parameters $1.45 < \zeta < 1.63$ were obtained using a number of statistical methods applied to the same sample of shares of major U.S. market caps (see also Ref. [10]), $\zeta \approx 1.58$ for the shares of the 85 largest companies trading on the London Stock Exchange (LSE) in 2001–2002, and $\zeta \approx 1.53$ for the shares of 13 highest market caps included on the CAC 40 (Paris stock market) index.

For Russian blue chips we have obtained indices in the range $1.6 < \zeta < 1.7$, depending on the particular security. Figure 3 plots the cumulative distributions of the trading volume in one tick for Sberbank shares on 21.11.2007.

Obviously, these correlations are valid for shares only. The situation with index returns is somewhat more complicated. The returns of the indices can naturally depend on the volume of trading in shares included in the index. However, it would be difficult to verify this assumption experimentally.

Finally, it is well known that the process $S(t)$ is delta-correlated in time:

$$B(\Delta t) = \left\langle S(t)\, S(t + \Delta t) \right\rangle \sim \delta(\Delta t)\,, \tag{3}$$

for all shares [11]. This statement was tested for Russian blue chips for various values of Δt, including the smallest interval available to us, namely 1 min. The following result was obtained in all cases: the value of the correlation function (3) tends to zero at the first nonzero measurement point Δt. A similar correlation function for the indices takes the form $\sim \exp(-t/\tau_{\text{corr}})$ [6], where the correlation time for the S&P 500 index (one of the most popular indices of the U.S. stock market) is about 4 min [6], and for the Russian RTS index it is 0.85 min [12]. Therefore, the behavior of share and index returns resembles a random process with independent increments.

2. Brownian motion and Gaussian random walk

Random walk is an attractive visually convincing model of a random process with independent increments. Formally, the problem of a random walk is posed as follows. Find the probability density that a particle, after N jumps from the starting point (for this point we can choose, without loss of generality, the point of origin of the coordinates) in space of some dimensionality G, will find itself at a distance in the range from $\mathbf{R}$ to $\mathbf{R} + \Delta\mathbf{R}$. Each ith jump can be made in an interval of length (in the model G-dimensional space) from $\mathbf{r}_i$ to $\mathbf{r}_i + \Delta\mathbf{r}_i$ with probability $\tau(\mathbf{r}_i)$. All jumps are independent random variables.

The method of solving this problem is well known (Chandrasekhar's scheme [13]). Let us assume that

$$\mathbf{R} = \sum_{i=1}^{N} \mathbf{r}_i\,. \tag{4}$$

Given that the probability density function $\tau(\mathbf{r}_i)$ possesses moments of all orders, we have

$$W_1(R) \to \frac{1}{\sqrt{2\pi N\langle r^2\rangle}} \exp\left(-\frac{R^2}{2N\langle r^2\rangle}\right). \tag{5}$$

Putting now $N\langle r^2\rangle = Dt$ (with D being the diffusion coefficient), we obtain the standard solution for the classical one-dimensional diffusion of a Brownian particle, in which its mean square displacement (variance) from the starting point is proportional to $t^{1/2}$.

The most important requirement in the Chandrasekhar scheme [13] is the existence of all moments of the jump law, even though only the second moment appears in expression (5). It seems likely that the jump law which is the 'slowest' in falling off to infinity and has all-orders finite moments is the Subbotin distribution [14]: $p(x) \sim \exp(-x^{\alpha})$, $\alpha > 0$ (in fact, only slightly greater than 0).

3. The Lévy walk

We shall analyze the one-dimensional random walk with the law of the elementary jump $\tau(\mathbf{r}_i)$, which allows normalization even though it does not have all finite moments. The simplest approximation is provided by the power law which assumes boundedness and smoothness for small jumps (at the zero point):

$$\tau(\mathbf{r}_i) = \frac{C_1}{(z^2 + r_i^2)^{\beta}}\,. \tag{6}$$

Here, C_1 is a constant determined by the normalization condition, $C_1 = 2\Gamma(\beta) z^{2\beta-1}/\pi^{1/2}\Gamma(\beta - 1/2)$, where $\Gamma(\beta)$ is Euler's gamma function, $\beta > 1/2$, and z is a constant interpretable as the characteristic length of the jump. Law (6) is therefore scaleless only for big jumps with $r \gg z$; in this case, it is reducible to a Pareto type law [15]: $\tau(r_i) \sim C_1/r_i^{2\beta}$. The distribution function for the law of the jump (6) reduces to the Lévy function

$$W_1(R) = \frac{1}{\pi}\int_0^{\infty} \cos(KR) \times \exp\left[-N(Kz)^{2\beta-1}\,\frac{\Gamma(3/2-\beta)}{2^{2\beta-1}\Gamma(\beta+1/2)}\right] \mathrm{d}K\,. \tag{7}$$

In principle, there is no need to demand that law (6) be identical for all jumps — the values of z can all be different (z_i); in this case, the quantity $Nz^{2\beta-1}$ in formula (7) should be replaced with the expression $\sum_i z_i^{2\beta-1}$.

The distribution law of Lévy random walk is characterized by a slowly decaying asymptotics, i.e., by a significant number of large fluctuations. Indeed, the asymptotics of function (7) is

$$W_1(R \to \infty) \approx \frac{\Gamma(2\beta) \sin\left[(\pi/2)(2\beta - 1)\right]}{\pi\rho^{2\beta}}\,, \qquad \rho = \frac{R}{R_0}\,,$$
$$R_0 = \frac{z}{2}\left[N\,\frac{\Gamma(3/2 - \beta)}{\Gamma(\beta + 1/2)}\right]^{1/(2\beta-1)}, \qquad (8)$$

i.e., the asymptotics of the Lévy distribution falls within the range from $1/\rho$ to $1/\rho^3$. The Lévy distribution has one very interesting property. By dividing asymptotics (8) by the asymptotics of the jump law (6), we obtain

$$\frac{W_1(R \to \infty)}{\tau(r \to \infty)} = \frac{N r^{2\beta}}{R^{2\beta}}\,. \qquad (9)$$

This expression means that large fluctuations may occur as a result of a single jump ($R = r$ at $N = 1$).

4. Truncated Lévy walk

The main difference between the truncated Lévy random walk [16, 17] and the Gaussian random walk lies in the thick tails, i.e., a great number of large fluctuations R. The law of the jump for truncated Lévy distribution is the same law (6), where now $\beta > 3/2$ (we continue to consider one-dimensional random walk). Under these conditions, the law has at least a second moment. For small fluctuations, up to $R \sim 10z$, these distributions are well approximated by a corresponding Gaussian function:

$$W_1^{\mathrm{G}}(R) = \sqrt{\frac{\beta - 3/2}{\pi N z^2}}\,\exp\left(-\frac{\beta - 3/2}{N z^2}\,R^2\right). \qquad (10)$$

This fact is an expression of the central limit theorem (CLT) for such random processes [18]: the Gaussian function describes fluctuations up to the magnitudes of $(N \ln N)^{1/2}$ greater than the characteristic average value z [19]. Sometimes, the result is referred to as the Chebyshev theorem; it holds true for any $\beta \geqslant 2$ [20].

To determine the behavior of truncated Lévy distributions in the range of large fluctuations $R \geqslant (N \ln N)^{1/2} z$, we need to find the exact form, so far unknown, of the asymptotics of the distribution function. It can be shown exactly that the asymptotic behavior of the density distribution of truncated Lévy walk can be described for any β by the law (Fig. 4)

$$W_1(R) \underset{R\to\infty}{\longrightarrow} \frac{2^{\beta} z^{2\beta-1} N}{\pi(2\beta - 3) R^{2\beta}}\,. \qquad (11)$$

Furthermore, distribution function (11) describes not only an infinitely divisible process [21], but also a stable one. Large R fluctuations after a single jump (9) are possible for truncated distributions only at $\beta = 2$, unlike fluctuations described by the Lévy function, which are possible for any $1/2 < \beta < 3/2$.

Let us trace now how the root-mean-square deviation changes with time. We obtain

$$\langle R^2 \rangle = \frac{N z^2}{2\beta - 3}\,. \qquad (12)$$

The law of truncated Lévy walk [asymptotics (10), (11)] can be normalized to the mean square of R (12). In this case, all the Gaussian asymptotes (for small R) for every β become identical. At the same time, the asymptotes (11) change to $\sim N^{-1/2}$. Figure 5 plots the cumulative distribution function of truncated Lévy walk at $\beta = 2$. The difference among the curves for different values of N is clearly seen. Cumulative distributions behave similarly for all values of β.

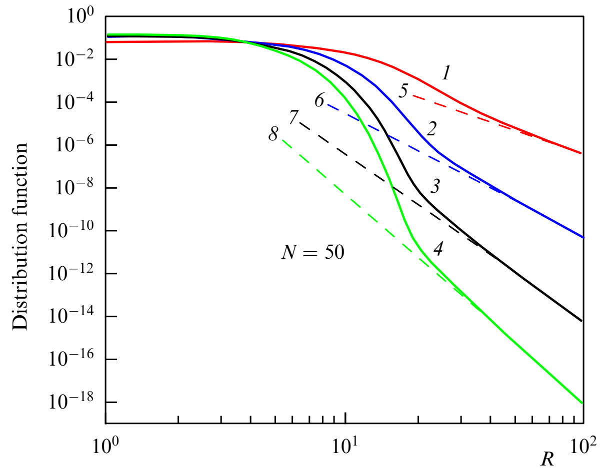

Figure 4. Exact normalized distribution functions for $\beta = 2$ (curve *1*), $\beta = 3$ (curve *2*), $\beta = 4$ (curve *3*), and $\beta = 5$ (curve *4*) as a function of jump length R normalized to z. Dashed lines *5–8* are the corresponding asymptotes for large R.

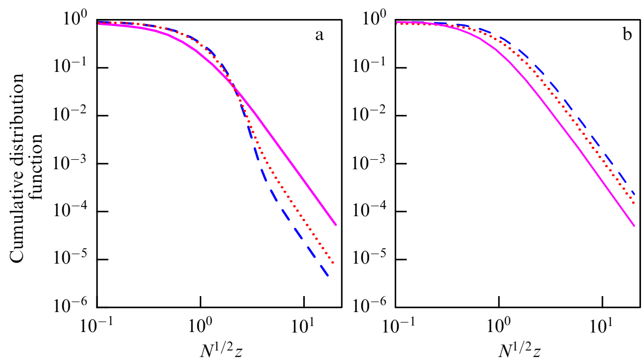

Figure 5. (a) Cumulative distribution function of truncated Lévy walk at $\beta = 2$, normalized to $R = N^{1/2}z$. Solid curve — $N = 1$, dashed curve — $N = 60$, and dotted curve — $N = 450$. The number of jumps corresponds to the ratio between 10-minute, one-hour, and one-day returns. (b) Cumulative distribution function at $\beta = 2$, normalized to R with $\delta = 2.7$. Solid curve — $N = 1$, dashed curve — $N = 60$, and dotted curve — $N = 450$. The number of jumps corresponds to the ratio between 10-minute, one-hour and one-day returns.

5. Comparison with experimental data

The form of the distribution of the truncated Lévy walk obtained as a result of implementing the scheme with the law of single jump (6) corresponds, therefore, at $\beta = 2$ to expression (1); however, in this case there are differences for various values of N — something we do not observe in real situations (see Introduction). To eliminate this discrepancy, the walk scheme needs to be corrected. First we ask a question: What in the experiment corresponds to a single transaction — to the so-called tick. Now we need to answer the following question: Is a tick a single jump in the random walk scheme?

At $\beta = 2$, the variance simply equals $N^{1/2}z$. Experimentally, $N \sim t$, where t is the frequency of fixing the values of the return. Hence, it should be possible to determine the

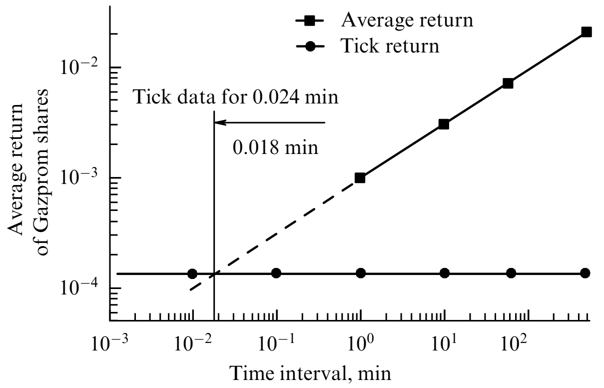

Figure 6. The average return of Gazprom shares for several t (squares connected by the solid line). Marked on the abscissa are time intervals at which the values of return are fixed. Dashed line is an extrapolation of data to small t. Horizontal solid line marks the level of root-mean-square tick return. Intersection of the two lines gives $t = 0.018$ min, the average time between two consecutive ticks is 0.024 min, and the difference comprises 33%. The tick data were obtained from the results of trading on 15.01.2008 on the MMVB stock exchange; 1-minute, 10-minute, one-hour, and daily data are the results of trading on 10.01–10.02.2009, 07.01–30.09.2009, 01.09.2008–30.09.2009, and 23.01.2006–30.09.2009, respectively.

minimum time t that corresponds to the least possible interval between instants of fixing the return, i.e., the interval between two consecutive ticks. On the one hand, this interval is a random quantity. Experimentally, it should not be difficult to find its mean value. On the other hand, in terms of the model this mean value should correspond to the average return of a tick, i.e., to the value of z (see Section 4). It is possible to plot mean returns for different time intervals t. Theoretically, by virtue of formula (12) this curve should exhibit the form $\sim t^{1/2}$. The plot of this function should definitely start from the level of tick return. We can experimentally compare the theoretical minimal time interval, dictated by the point of intersection of the curve of the root-mean-square returns with the level of the root-mean-square tick return (Fig. 6), with the average time interval between two consecutive ticks. The difference between the theoretically predicted minimum time interval and the experimentally obtained average time between two ticks for Gazprom shares is large compared with the difference for the stocks of other companies on the Russian market — 33%. The minimum difference between these values is observed for the shares of Sberbank — only 3%.

It is nearly certain that a tick is a single jump in the random walk scheme. The first obvious possibility of modifying the model boils down to an attempt of applying the random walk scheme with continuous time (Continuous Time Random Walk, CTRW) [22]. Indeed, the time intervals between two successive ticks can vary in a wide range. The distribution of these intervals for the U.S. stock market is known [23], with appropriate distribution function falling off with a decrease in Δt as $(\Delta t)^{4.4}$. Presumably, no new results can be obtained by taking into account the time interval between transactions due to the presence of mathematical expectation of the time interval between ticks.

Another possibility of modification of the truncated Lévy random walk scheme is the use of the power-law correlation of standard deviation of z and the average volume of one transaction. Our modification is limited to the assumption that each standard deviation of z in the walk scheme [see the law of jump (6)] is a random variable z_i proportional to the volume of the ith transaction (ith tick). We are returning to the second experimentally identified property described in the Introduction. We actually utilize the quite familiar, widely used stock exchange rule: 'the volume of trading drives the price' [24, 25].

This modification signifies that we are introducing the dependence of the probability distribution function of single fluctuations $\tau_i(r_i)$ on another random variable z_i. In this case, the model again resembles the CTRW model. The problem of direct application of CTRW scheme lies in the fact that the final distribution function for R will depend on the set of random variables $\{z_i\}$. For example, the distribution function of truncated Lévy random walks at $\beta = 2$ is found in the form

$$W_{\beta=2,\,z_i}(R) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}K \exp(\mathrm{i}KR)\prod_{i=1}^{N}\exp(-Kz_i)(1+Kz_i)\,. \tag{13}$$

Since all variables $\{z_i\}$ have the distribution function $\sim x^{-\delta}$ for large z_i, where $\delta \sim 2.5-2.7$ ($\delta = \zeta - 1$), formula (13) can be averaged over each z_i. This result will nevertheless be wrong since the final expression for function (13) averaged in this manner will disagree with the experimentally observed data, namely it will not be proportional to R^{-4} for large R.

It seems that using the simple CTRW scheme is possible, at least for asymptotic values of function (13), because for large R we have

$$W_{\beta=2,\,z_i}(R) = \frac{4\sum z_i^3}{\pi R^4}\,, \tag{14}$$

and the set of random variables $\{z_i\}$ only returns a single random variable: $\sum z_i^3$. Unfortunately, the distribution function of the probability density of this variable on the tails assumes the form $\sim x^{-2/3-\delta/3}$. There is no mathematical expectation for this function, which is necessary for the application of the CTRW scheme [22]. In fact, the variables $\{z_i\}$ are included explicitly in expression (13) in various combinations: $\sum z_i^2$, $\sum z_i^3$, etc. Each of these combinations is in itself a random variable. Because the asymptotic distribution function is a function of the sum of cubed z_i, we can conclude that the sum converges to the Lévy distribution (see Ref. [13]). Only this condition ensures the stability of $\sum z_i^3$ as new terms are added to the sum.

Consequently, the CTRW method must be generalized to the case of the absence of the conditional mean [of the random variable $\sum z_i^3$ in formula (14)] (see Ref. [26]). As in the case of the Lévy distribution (7), expression (13) can be examined for dependence on N, i.e., on renormalization. If the quantity R in formula (13), or the corresponding asymptotic cumulative distribution

$$\Phi_{\beta=2,\,z_i}(R) \simeq \frac{4\sum z_i^3}{3\pi R^3} \tag{15}$$

is renormalized to the standard deviation $\left(\sum z_i^2\right)^{1/2}$, as we do in all experiments, the result is the scaling dependence of expression (15) in the form $N^{-1/2}$ for $\sum z_i^3 \sim N$ [see formula (11) and Fig. 5a). At the same time, the dependence of $\sum z_i^3$ on N looks different because the distribution function of the random variable $\sum z_i^3$ converges to the Lévy distribution (see above). The end result for the Lévy distribution function is $\sum z_i^3 \sim N^{3/(\delta-1)}$, and the final observed dependence of distribution (15) on N after renormalization of the actual profitability R to the experimentally obtained standard deviation takes on the form

$$\Phi^{\text{renorm}}_{\beta=2,\,z_i}\left(\frac{R}{\sqrt{\sum z_i^2}}\right) \sim N^{3/(\delta-1)-3/2}\,. \tag{16}$$

If $\delta \sim 2.5-2.7$, we obtain dependences (16) in the range from $N^{0.5}$ to $N^{0.27}$ (Fig. 5b). It is seen that the standard experimental renormalization provides a weak dependence of all return distribution functions on the number of jumps (ticks) N. Notice that such dependences (16) as a function of N are similar to the experimental results [6] and to our results, too, obtained for the Russian stock market, where we observed weak dependences on N: the returns increase as N increases, in contrast to what we observe in Fig. 5a for the simple scheme of truncated Lévy random walks.

Notice that the established dependence on N occurs only for the profitability of the stock. The possible dependence of the index returns on the volume of trading in shares listed in the index may have a different form (from the law $x^{-\zeta}$). If this law does not fall within the Lévy range $0 < \zeta < 2$ and ζ is greater than 2 (by only a little), then the cumulative distribution of the index returns on long time intervals (16 days as in Ref. [6], positive monthly returns of the RTS index in Fig. 2: the last two points) can converge to Gaussian one (see formula (16) for $\delta \sim 3-4$). These distributions will look similar to those shown in Fig. 5a for large N, not like the same curves in Fig. 5b.

6. Conclusion

The introduction of the law of jump of (6) type allows one to consider in a unified analytical manner both the ordinary and the truncated Lévy walks. The truncated Lévy walk asymptotically manifested the same properties of stability and scalability as the ordinary random walk. Analytical asymptotes were obtained for the truncated random walk and scaling laws were established. It turned out that the asymptotic truncated Lévy walk possesses characteristic scaleless distribution $\sim R^{-2\beta}$, which is also typical of the asymptotes of the 'pure' random Lévy walk but, in contrast to the latter, decays faster with increasing R. Therefore, the truncated Lévy walk, together with the pure random walk, covers the entire class of Pareto distributions [15].

We can assume that the law $\sim 1/x^3$ for the cumulative distribution function of share and index fluctuations is universal. Such a distribution can be obtained by using the scheme of random walks (jumps) with the law of single jump (6) only at $\beta = 2$. This means that the law of jumps at such a value of β, namely

$$\tau_i(r) = \frac{4z_i^3}{\pi(z_i^2 + r^2)^2}\,, \tag{17}$$

is also universal. Here, the value of z_i represents some characteristic return used for normalization. This result can be considered as proof of the existence of a microscopic law of return fluctuations on the stock market. Therefore, the prices of all shares (indices act essentially as baskets of shares and their behavior is similar) perform 'jumps' for different 'distances' at constant probabilities. The microscopic law (17) explains the phenomenology of the law $\sim 1/x^3$ [2].

Apparently, the existence of strict laws of a single jump (16) is possible for two reasons. First, the probability distributions of fluctuations of returns should have a second moment, i.e., have variance. In the final analysis, this requirement is a reflection of the limited amount of money available. Second, the distribution function must have the same asymptotics as the law of the jump, i.e., a nonzero probability of large fluctuations resulting from a single jump must exist. All Lévy functions meet the second requirement but not the first. Only distribution function (12) at $\beta = 2$ satisfies both conditions.

A simple definition of the z_i variable as a characteristic length of a jump cannot provide an exact explanation for the dependence of the normalized distribution functions and cumulative distributions on N. A modification of the random walk scheme is provided through the introduction of the dependence of $\{z_i\}$ on the number of shares traded in one transaction (tick), because the correspondence of one tick to one jump is an experimentally verified fact (see also Ref. [12]). In this case, the distribution function of the quantity $\sum z_i^3$ converges to the Lévy function with the Lévy index $(\beta-1)/3$. The final dependence of the cumulative distribution functions on the number of ticks (jumps) falls into the range from $N^{0.5}$ to $N^{0.27}$. Russian stocks exhibit weaker dependence than shares from the USA and shares traded in the LSE and the Paris Stock Exchange. We conclude that the presented final random walk scheme looks like the CTRW scheme lacking the conditional expectation (for the quantity $\sum z_i^3$).

This work was partially supported (M.Yu.R.) by RFBR grant 10-07-00202.

References

1. Bachelier L, Thèses Fac. Sci. Paris Le Grade de Docteur Sci. Math. (Paris: Gauthier-Villars, 1900); *Ann. Sci. Ècole Normale Supérieure* **3** (17) 21 (1900); *Theory of Speculation* (Princeton: Princeton Univ. Press, 2006)
2. Gopikrishnan P et al. *Eur. Phys. J. B* **3** 139 (1998)
3. Lux T *Appl. Financ. Econ.* **6** 463 (1996)
4. Skjeltorp J A *Physica A* **283** 486 (2000)
5. Loretan M, Phillips P C B *J. Empirical Finance* **1** 211 (1994)
6. Gopikrishnan P et al. *Phys. Rev. E* **60** 5305 (1999)
7. Plerou V, Stanley H E *Phys. Rev. E* **76** 046109 (2007)
8. Rácz É, Eisler Z, Kertész J *Phys. Rev. E* **79** 068101 (2009)
9. Plerou V, Stanley H E *Phys. Rev. E* **79** 068102 (2009)
10. Gopikrishnan P et al. *Phys. Rev. E* **62** R4493 (2000)
11. Mantegna R N, Stanley H E *An Introduction to Econophysics: Correlations and Complexity in Finance* (New York: Cambridge Univ. Press, 2000)
12. Romanovsky M Yu, Vidov P V, Pyrkin V A *Komp'yut. Issled. Modelirovanie* **2** 219 (2010)
13. Chandrasekhar S *Rev. Mod. Phys.* **15** 1 (1943)
14. Subbotin M T *Mat. Sb.* **31** (2) 296 (1923)
15. Pareto V *Cours d'Economie Politique* (Lausanne: F. Rouge, 1986, 1897)
16. Koponen I *Phys. Rev. E* **52** 1197 (1995)
17. Mantegna R N, Stanley H E *Phys. Rev. Lett.* **73** 2946 (1994)
18. Feller W *An Introduction to Probability Theory and its Applications* (New York: Wiley, 1970, 1971) [Translated into Russian (Moscow: Mir, 1984)]
19. Bouchaud J-P, Potters M *Theory of Financial Risk and Derivative Pricing: from Statistical Physics to Risk Management* 2nd ed. (Cambridge: Cambridge Univ. Press, 2003)
20. Vidov P V, Romanovsky M Yu *Phys. Wave Phenomena* **17** 218 (2009)
21. Khintchine A *Mat. Sb.* **2(44)** (1) 79 (1937)
22. Scher H, Montroll E W *Phys. Rev. B* **12** 2455 (1975)
23. Gopikrishnan P et al. *Physica A* **287** 362 (2000)
24. Karpoff J M *J. Finance* **41** 1069 (1986)
25. Brailsford T J *Accounting Finance* **36** 89 (1996)
26. Sokolov I M *Phys. Rev. E* **63** 011104 (2000)

PACS numbers: **05.40.–a**, 05.45.Df, 89.65.Gh
DOI: 10.3367/UFNe.0181.201107k.0779

Econophysics and the fractal analysis of financial time series*

M M Dubovikov, N V Starchenko

1. Introduction

The term '*econophysics*' was coined in 1995 by H Eugene Stanley as a common name for research in which methods of statistical physics were applied to analyzing the behavior of financial markets. Such research efforts were stimulated most of all by the revolution in computer technologies, which by that time had led to the creation of huge readily accessible arrays of financial data which had been painstakingly accumulated from the middle of 1980s. Later on, the term began to be used in a wider context, indicating that a paper on economics or another social science had been written by a physicist. Since 2002, such publications have begun to appear regularly in all the major general physical journals, such as *Reviews of Modern Physics*, *Physical Review E*, *Physical Review Letters*, and some others. At this time, a course on econophysics is being taught in the West in the most prestigious universities, and a section dedicated to it has grown to become an integral part of major annual international and national conferences on social sciences [e.g., ESHIA (Economic Science with Heterogeneous Interacting Agents), AKSOE (Arbeitskreis Physik Sozio-Ökonomischer Systeme), and others]. The first All-Russia Congress of Econophysics was convened in June 2009 in Moscow.

The first econophysics work whose popularity far transcended the bounds of any one field of science was the publication by Mantegna and Stanley in *Nature* [1]. In essence, this work developed the rather old idea of Benoit Mandelbrot concerning the *Lévy flight* [2] so as to make it to agree with new empirical data.

This article is devoted to the development in the same field of another of Mandelbrot's seminal ideas, which was also first advanced in paper [2] in the study of financial time series. Subsequently, this idea has been successfully applied in a number of very different fields of physics [3].

Ever since the 1950s, experts have been quite familiar with the proposition that movements of prices of most financial tools over various time and price scales look very similar. An observer cannot identify from the shape of the charts if the data describe weekly, daily, or hourly fluctuations [3]. In today's language, the indicated self-similarity signifies that financial time series are *fractals* [4]. The main characteristic of such structures is, as we know, the fractal dimension D. In the case of chaotic time series, this indicator defines the Hurst index H ($D = 2 - H$), which is a measure of persistence in a time series (the ability to sustain a certain trend). However, an impossibly large representative scale is required for a reliable calculation of D (as well as H), which excludes any chance of using D as an indicator defining the local dynamics of a time series.

In this paper we introduce new fractal parameters: the *dimension of minimum covers* and the related *index of fractality*. It has been rigorously proved in the principal order in δ (here, δ is the minimum scale of partitioning the time series) that for $\delta \to 0$ the dimension of the minimum cover is identical to D. By the example of financial time series, it has been proved that the amount of data contained in the minimum scale required for determining the introduced indicators with an acceptable accuracy is less by two orders of magnitude than the corresponding scale for the determination of the Hurst index H. This makes it possible to consider the index of fractality as a local indicator of stability of the time series. An empirical justification of the concept of *stability* on the financial market has been proposed, based on the index of fractality. An effect of enhancement of large-scale fluctuations and suppression of small-scale oscillations has been revealed; it was used to build an indicator of strong fluctuations in the global financial market.

2. Fractal structures

1. Objects for which the methods of *classical* analysis proved totally unsuitable (such as the Cantor set, the Weierstrass function, and the Peano curve) were found in mathematics for the first time at the end of the 19th century. All of them were built using very simple rules of iterative procedure, and all possessed scalable self-similarity (consisted of parts similar to the whole). By the beginning of the 20th century, the number of such objects became sufficiently large, and to analyze them, Felix Hausdorff offered in 1919 his definition of the dimension of a compact set in an arbitrary metric space [5]. Hausdorff noticed that if these sets are covered by spheres with a radius δ, the minimum number $N(\delta)$ of such spheres will grow with diminishing δ by the power-law dependence

$$N(\delta) \sim \left(\frac{1}{\delta}\right)^D . \qquad (1)$$

Notice that the power exponent D is typically calculable exactly. This was the exponent that Hausdorff called 'dimension'.[1] If we now take the logarithms of both sides of this expression and rewrite them in the form of an equality for D, we obtain the exact definition of the Hausdorff dimension:

$$D = \lim_{\delta \to 0} \left[\frac{\ln N(\delta)}{\ln(1/\delta)}\right] . \qquad (2)$$

For sets that are familiar in classical calculus (e.g., smooth curves or surfaces), the exponent D coincides with the

* Dedicated to the memory of Benoit Mandelbrot (20.11.1924–14.10.2010).

M M Dubovikov, N V Starchenko INTRAST Management Company, Moscow Engineering Physics Institute, Moscow, Russian Federation
E-mail: Dubovikov@intrast.ru

Uspekhi Fizicheskikh Nauk **181** (7) 779 – 786 (2011)
DOI: 10.3367/UFNr.0181.201107k.0779
Translated by V I Kisin; edited by A Radzig

[1] This quantity is sometimes called the ultimate capacity (see, e.g., monograph [6]) perceiving here by the Hausdorff dimension d_H the critical value of the argument of the function

$$m(p) = \sup_{\varepsilon>0} \inf_{\{A_i^\varepsilon\}} \sum (\mathrm{diam} A_i^\varepsilon)^p ,$$

which possesses the following property: $m(p) = \infty$ for $p < d_H$, and $m(p) = 0$ for $p > d_H$. Here A_i^ε is the coverage of the original set by the family of sets A_i^ε with a diameter less than ε. Such a dimension is of more recent origin and extends to an unbounded set. As a rule, $d_H = D$, but counterexamples have been found. One can only state in the general case that $d_H \leqslant D$.

topological dimension D_T equal to the minimum number of coordinates necessary to describe such sets (e.g., one coordinate is sufficient to describe the line, two coordinates to describe the surface, and three coordinates to describe the body [2]). It was found that for the nonclassical sets mentioned above, the Hausdorff dimension (typically a fractional quantity) is always greater than the topological dimension D_T. The last property was later used by Mandelbrot for one of the possible definitions of the fractal, which states that a fractal is a set where $D > D_T$ [3].

It should be noted that if the original set is immersed in an Euclidean space, then other approximations of the set cover by simple shapes (e.g., cells) of size δ can be used instead of covering this set by spheres. In addition, new fractal dimensions (cellular, internal, etc.) appear, along with the original spherical dimension D; they usually coincide as limiting values for $\delta \to 0$. However, the rates of convergence to this limit may vary considerably for these dimensions.

Consider, for instance, the *Sierpiński carpet*, which is constructed as follows. Take a unit square, and in the first step divide it into nine equal squares, of which the middle one is thrown out (Fig. 1a). In the next step, this procedure is repeated with all remaining squares, and so forth. In the limit, the set obtained by an iterative procedure is known as the Sierpiński carpet (it can be shown that $D_T = 1$ for this object). Notice that in constructing model fractals, a set consisting of $N(\delta)$ elementary simplexes of linear size δ usually emerges in the nth iteration step. Mandelbrot called this set the pre-fractal of the nth generation. This set for the Sierpiński carpet consists of $N(\delta) = 8^n$ cells with a side length $\delta = (1/3)^n$. If now we use pre-fractals in definition (2) instead of covering set by spheres, the dimension D can be calculated in a straightforward manner. Indeed, passing to the limit $n \to \infty$ in our case, we obtain from formula (2):

$$D = \lim_{\delta \to 0} \left(\frac{\ln 8^n}{\ln 3^n} \right) = \lim_{n \to \infty} \left(\frac{n \ln 8}{n \ln 3} \right) = \frac{\ln 8}{\ln 3} \, (\approx 1.89) \, .$$

The result will not change if spheres are chosen instead of cells. However, the characteristics of the algorithms of direct calculation of these two dimensions are very different. In order to show this, we construct for each dimension the plot of the function $N(\delta)$ at $\delta = (1/3)^n$ on a double logarithmic scale (Fig. 1b). On this scale, all power-law functions are linear, and the exponent D is defined as the slope of the regression line corresponding to the plot. For cellular covers (pre-fractals), all points of the plot of the function $N(\delta)$ lie on one straight line. This means that the function $N(\delta)$ *rapidly reaches the asymptotic power mode* (1), which allows us to get the value of D already in the first iteration step. If we use spheres instead of cells for calculating D, the corresponding plot becomes closer to power law (1) only asymptotically as $\delta \to 0$. A more profound analysis reveals that the above property of $N(\delta)$ for pre-fractals of the Sierpiński carpet emerges due to the fact that cellular cover is in a sense a minimal cover in each iteration step. Therefore, it is precisely the minimality of the covers which is the reason why the appropriate function determined by the covers and being used to calculate the dimension D grows rapidly and reaches the power-law asymptotic mode. As will be shown in Section 3, this principle allows straightforward generalization to the case of chaotic time series.

2. Objects with a nontrivial Hausdorff dimension were for a long time regarded only as a figment of the sophisticated mathematical intellect. These days, largely through the efforts of Benoit Mandelbrot, we know that fractals are all around us. Some fractals are continually changing, like moving clouds or flickering flame, while others preserve the structure created in the process of evolution, as happened with coastlines, trees, or our vascular systems. The real range of scales in which fractal structures are observed stretches from intermolecular distances in polymers to distances between clusters of galaxies in the Universe.

We need to point to the main features of natural fractals that distinguish them from model ones. *First*, natural fractals are never strictly symmetrical. Self-similarity holds for them only on average. *Second*, calculations of the dimensions of natural fractals inevitably exclude scales that are smaller than a certain minimum scale δ_0 of the structure. This means that power law (1) manifests itself as an 'intermediate asymptotics' (as $\delta \to 0$, the scale considered is much smaller than a certain characteristic scale but greater than the minimum scale δ_0). *Third*, for natural fractals there is no system of pre-fractals. Therefore, the system of approximations by simplexes, required for the construction of the function $N(\delta)$ when $\delta \to 0$, is in the general case fairly arbitrary. Consequently, the computation of the dimension D as the slope of the regression line $N(\delta)$ on the double logarithmic scale needs a large amount of data, since the function $N(\delta)$ usually converges to the power law (1) very slowly.

We will show, nevertheless, in Section 3 that for computing the dimensions of fractal time series it is possible to construct a sequence of minimal covers similar to the sequence of pre-fractals of a Sierpiński carpet.

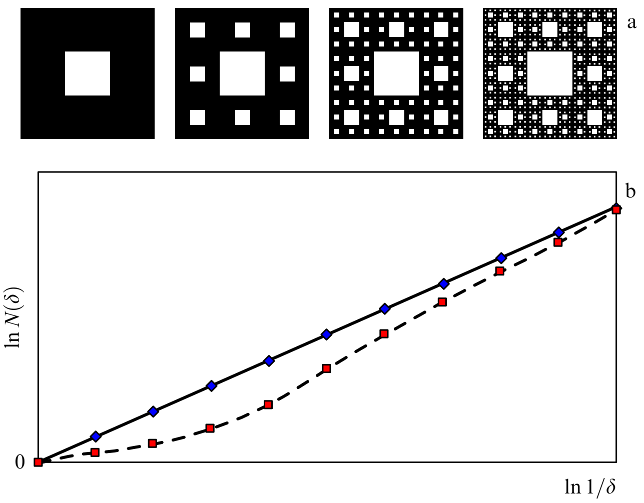

Figure 1. Pre-fractals of four generations for the Sierpiński carpet (a), and the function $N(\delta)$ in double logarithmic scale (b) for cellular (solid line) and Hausdorff (dashed curve) dimensions.

[2] This approach can be generalized (e.g., for an arbitrary compact set) in at least two ways [7]. The first approach is based on the fact that any two closed disjoint subsets of the original set of dimension $n+1$ can be split with a partition of dimension n. The dimension is introduced here by induction. The second method is based on the fact that the minimum multiplicity of the cover of a set of dimension n by closed sets with an arbitrarily small diameter equals $n+1$. Multiplicity is understood here to equal the maximum number of the cover elements having nonempty intersections.

3. The dimension of the minimum cover. The index of fractality

1. *Chaotic time series* form the most important class of natural fractals. Such series with an extremely irregular behavior are found in the observations of various natural, social, and technological processes. Some of these processes are traditional (geophysical, economic, medical), and some were discovered fairly recently (daily variations in crime level or in traffic accidents in an administrative region, fluctuations in the number of hits of certain sites on the Internet, etc.). Such series of data are usually generated by complex nonlinear systems of various natures. However, all of them behave in essentially the same characteristic manner within a certain range of scales. The easiest method of studying the fractal structure of these series is based on calculating the cellular dimension D_c. To find D_c, one divides the plane, on which the diagram of the time series is defined, into cells of size δ. Then, for different δ we plot the function $N(\delta)$ which is equal to the number of cells of size δ that contain at least one point on the diagram. The dimension D_c is found from the slope of the regression line $N(\delta)$ on the double logarithmic scale. It is readily shown that $D_c = D$. The fractal dimension for *chaotic* series happens to be especially important because this indicator is closely related to the Hurst exponent (index) H which is usually calculated using the normalized amplitude range and, as we know, is an indicator of *persistence* (ability to sustain trends) of a time series. Notice that if $H > 0.5$, the series is persistent (it is likely that the movement of the series in a certain direction on an interval will initiate movement in the same direction on the next interval). If $H < 0.5$, the series is *antipersistent* (it is likely that the movement of the series in a certain direction on an interval will initiate movement in the opposite direction on the next interval). Finally, if $H \approx 0.5$, the series has *zero persistence* (the motion of the series on any interval is independent of its motion on the previous interval).

More than ten different algorithms for the calculation of this indicator were created later owing to its importance [8–11]. It seems that the simplest method for calculating the exponent H is based on the formula

$$\left\langle |f(t+\delta) - f(t)| \right\rangle \sim \delta^{H} \quad \text{as} \quad \delta \to 0\,, \tag{3}$$

where angle brackets denote averaging over the time interval, and $f(t)$ is the value assumed by the time series at the instant of time t. The exponent H is found from the corresponding regression line. It is easy to show that for Gaussian random processes $H = 2 - D$. Virtually all experts agree that this relation has a wider range of applicability, since it has been confirmed for all the observed chaotic time series in all those cases in which both indicators are accurately determinable. Also, all difficulties associated with the computation of dimension D are transferred to the algorithms for the calculation of the exponent H. Thus, any reliable determination of H requires a representative scale of several thousand data sets. As a rule, a time series changes the parameters of its behavior on such a long scale many times, which greatly devalues the analysis of time series with the aid of the Hurst exponent H. As we saw for the dimension D, this difficulty stems from the fact that the convergence of the corresponding function to power law (3) for $\delta \to 0$ unfolds extremely slowly. To overcome this obstacle, it is possible to follow the analogy of how this is done in the case of the Sierpiński carpet and to determine the sequence of approximations of a series, which consists of minimal covers for any fixed δ. Indeed, if we multiply both sides of formula (1) by δ^2, the definition of the dimension can be rewritten as a power law for the approximation area $S(\delta)$:

$$S(\delta) \sim \delta^{2-D} \quad \text{for} \quad \delta \to 0\,. \tag{4}$$

Notice that, in contrast to formula (1), this form does not require that the simplexes of which each individual approximation consists be identical. It would be sufficient for them to have one and the same geometric factor δ. It is this circumstance that allows us to use approximations which are minimal covers.

2. Indeed, let a function $y = f(t)$ having not more than a finite number of points of discontinuity of the first kind be defined on a segment $[a, b]$: it is natural to consider precisely such functions as model ones, e.g., for financial time series. We introduce a uniform partition of the segment, $\omega_m = [a = t_0 < t_1 < \ldots < t_m = b]$, where $t_i - t_{i-1} = \delta = (b-a)/m$, $(i = 1, 2, \ldots, m)$. We cover the graph of this function with rectangles in such a way that this cover is the minimum area in the class of covers by rectangles with base δ (Fig. 2). Then the height of the rectangle on the segment $[t_{i-1}, t_i]$ equals the amplitude $A_i(\delta)$ which is the difference between the maximum and minimum values of function $f(t)$ on this segment. We now introduce a quantity

$$V_f(\delta) \equiv \sum_{i-1}^{m} A_i(\delta)\,. \tag{5}$$

The total area $S_\mu(\delta)$ of the minimum cover can then be written as $S_\mu(\delta) = V_f(\delta)\,\delta$. Consequently, formula (4) implies that

$$V_f(\delta) \sim \delta^{-\mu} \quad \text{for} \quad \delta \to 0\,, \tag{6}$$

where $\mu = D_\mu - 1$. We call the dimension D_μ *the dimension of the minimal covers*. To appreciate the differences between D_μ and other dimensions, especially the cellular dimension D_c, we construct the cellular partition of the plane of the graph of the function $f(t)$ as shown in Fig. 2. Let $N_i(\delta)$ be the number of cells that cover the plot of $f(t)$ on the segment $[t_{i-1}, t_i]$. It is seen from the figure that

$$0 < N_i(\delta)\,\delta^2 - A_i(\delta)\,\delta < 2\delta^2\,. \tag{7}$$

Divide this inequality by δ and sum up over i, taking into account Eqn (5). As a result, we have

$$0 < N(\delta)\,\delta - V_f(\delta) < 2(b-a)\,, \tag{8}$$

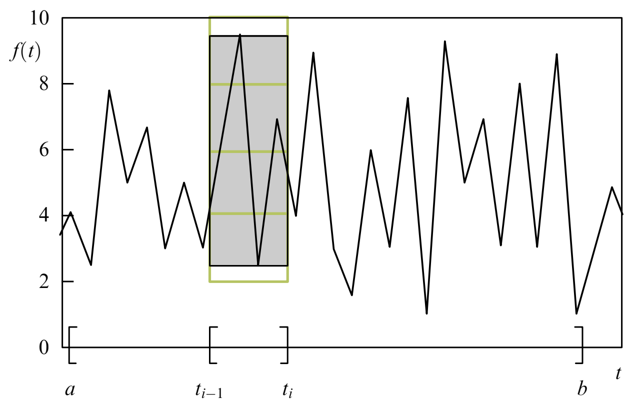

Figure 2. The minimal (shaded rectangle) and cellular (light rectangle) covers of the function $f(t)$ on the interval $[t_{i-1}, t_i]$ of length δ.

where $N(\delta) = \sum N_i(\delta)$ is the total number of cells of size δ that cover the plot of function $f(t)$ on the segment $[a, b]$. Passing to the limit $\delta \to 0$ and taking formula (6) into account, we obtain

$$N(\delta)\,\delta \sim V_f(\delta) \sim \delta^{-\mu} = \delta^{1-D_\mu}\,. \tag{9}$$

On the other hand, it follows according to formula (4) that

$$N(\delta)\,\delta = S_{\mathrm{c}}(\delta)\,\delta^{-1} \sim \delta^{1-D_{\mathrm{c}}}\,. \tag{10}$$

Hence, $D_{\mathrm{c}} = D_\mu$. Note, however, that despite this equality, the minimal and cellular covers for real fractal functions may provide different convergences of the quantity $S(\delta)$ to the asymptotic mode (4), and this difference may be quite large. Next, since $D_{\mathrm{c}} = D_\mu = D$, $\mu = D_\mu - 1$ and since $D_{\mathrm{T}} = 1$ for the one-dimensional function, we have $\mu = D - D_{\mathrm{T}}$. In this case, therefore, the index μ can naturally be called the *index of fractality*. In what follows, we will analyze financial time series and regard this index as the main fractal indicator.

4. Financial time series. Problems of identification and prediction

1. The most popular representatives of fractal time functions are financial time series (first and foremost, series of stock prices and currency rates). There is reliable numerical evidence of the fractal structure of such series [12, 13]. Theoretically, fractality is usually linked to the fact that investors with different investment horizons (from several hours to several years) must be active in the market for sustaining its stability. This is the factor that produces scaling invariance (absence of a singled out scale) of price series over the corresponding time interval [14, 15].

As an example, a database was investigated which included price series for shares of thirty companies included in the Dow Jones Industrial Index (DJII) from 1970 to 2002. Each series contained about eight thousand records. Each record corresponded to a running day of trading and included four values: the lowest and highest prices, and the opening and closing prices. In the literature, financial series are usually represented using the *Japanese candles chart*. A fragment of such a series for the Coca-Cola Co. is displayed in Fig. 3a. To simplify the analysis, only the last $2^{12} = 4096$ records for each company were considered. To compute the variation index μ, the sequence of m nested divisions ω_m for $m = 2^n$ ($n = 0, 1, 2, \ldots, 12$) were used. Each division consisted of 2^n intervals containing 2^{12-n} trading days. For each division ω_m, the value of $V_f(\delta)$ was calculated by formula (6). Here, $A_i(\delta)$ equals the difference between the highest and lowest prices over the interval $[t_{i-1}, t_i]$ (thus, if $\delta = \delta_0$, $A_i(\delta)$ equals the difference between the highest and lowest prices over one day). A typical example of the behavior of $V_f(\delta)$ on the double logarithmic scale for the Microsoft company is illustrated in Fig. 4. We see that the data lie with amazing precision on a straight line, except for the last two points, which deviate from the linear mode and exhibit a 'break'. To find the value of μ from these data should exclude the last two points and determine the regression line. At the confidence level

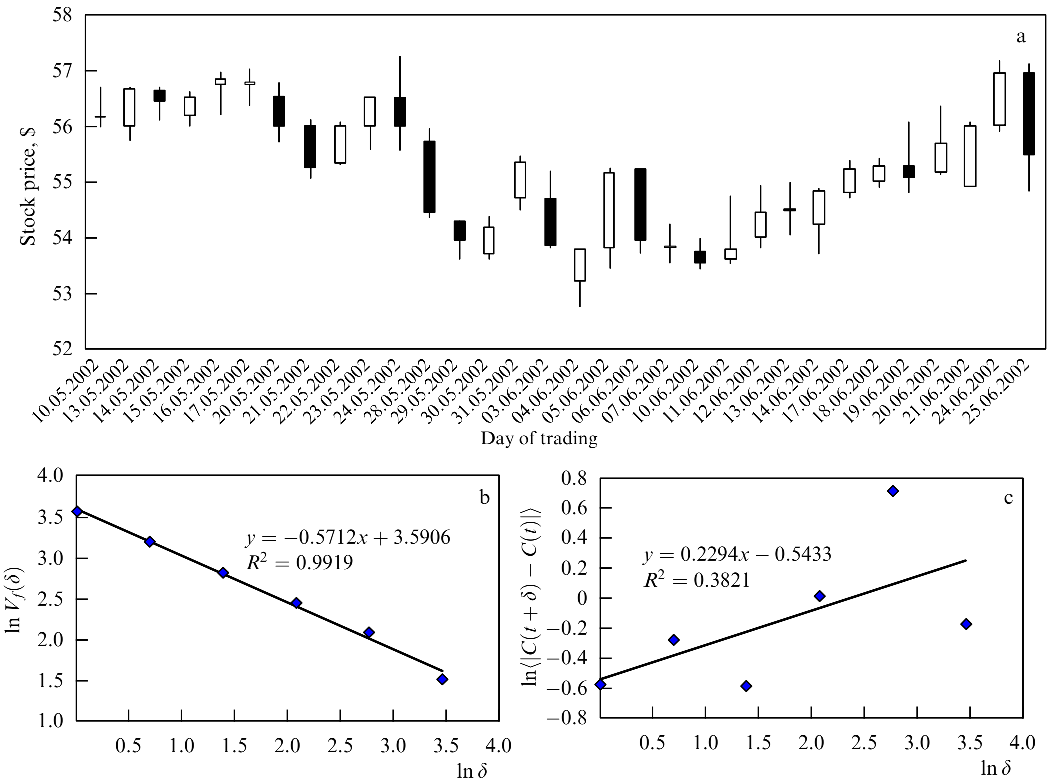

Figure 3. (a) Typical *Japanese-candles* financial series in the interval of 32 days (the graph of Coca-Cola share prices was used). Each of the rectangles (known as the candle body) with two vertical bars above and below (known as candle shadows) symbolizes price fluctuations during one day of trading. The top point of the upper shadow indicates the highest day price, and the bottom of the lower shadow is the lowest day price. The upper and lower boundaries of the candle body show the opening price and closing price on the trading day. The white (black) color of the candle body indicates that the closing price was above (below) the opening price. (b) The result of calculation of $V_f(\delta)$ on the double logarithmic scale for the presented time series. The relation $y = ax + b$ was calculated by the method of least squares; here $\mu = -a$. (c) The result of calculations of $\langle |C(t+\delta) - C(t)| \rangle$ for the same series, and the corresponding function $y = ax + b$, for $H = a$.

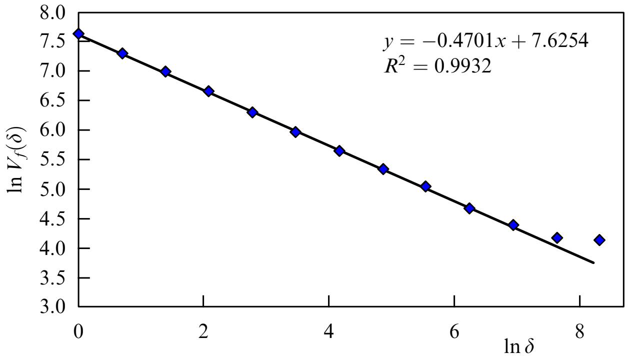

Figure 4. The result of calculation of $V_f(\delta)$ and the corresponding relation $y = ax + b$ for the time series of Microsoft stock prices in the interval of 4096 days.

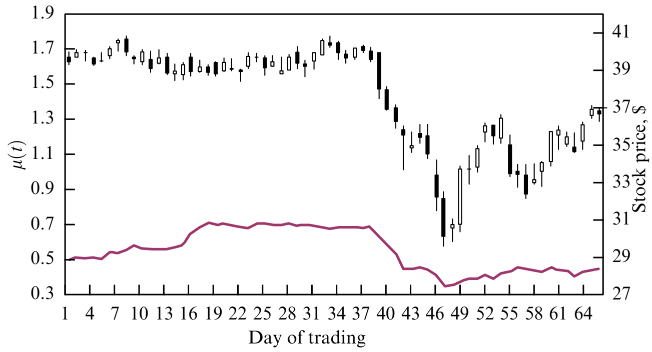

Figure 5. Daily prices of Exxon Mobil Corporation shares (Japanese candles, right-hand scale) and plot of the function $\mu(t)$ (solid curve, left-hand scale).

$\alpha = 0.95$, we have in the above example $\mu = 0.472 \pm 0.008$ and $R^2 = 0.999$. Here, R^2 is the determination index for the regression line. A comparison of this algorithm of computation of D ($D = \mu + 1$) and, correspondingly, H ($H = 2 - D = 1 - \mu$) with standard algorithms for computing these indices shows that the results are consistent with acceptable accuracy. However, the values of $V_f(\delta)$ on double logarithmic scale fall appreciably more accurately onto a straight line (except for the last two points) than those values corresponding to other algorithms, which also allows us to determine the characteristic scale on which the break of the linear mode occurs.

Now we need to point out that for each of the 30 companies the plot of $V_f(\delta)$ on the double logarithmic scale fits the straight line, almost as accurately, on all shorter representative intervals too, down to 32 days, and sometimes even down to 16 days. Note that on intervals shorter than 500 days the break on the linear part of the plot, as a rule, disappears.

A typical example of the behavior of the function $V_f(\delta)$ on a segment of financial time series 32 days long (Fig. 3a) is given in Fig. 3b. If $\alpha = 0.95$, we obtain $\mu = 0.571 \pm 0.071$, $R^2 = 0.992$. For comparison, Fig. 3c displays an example of the behavior of $\langle |C(t+\delta) - C(t)| \rangle$ on the same segment (we use here $32 + 1 = 33$ close prices $C(t)$ and averaging is carried out in nonintersecting intervals $\delta = 2^n$ in length, where $n = 0$, 1, 2, 3, 4, 5). In this case, $H = 0.229 \pm 0.405$, $R_{\rm H}^2 = 0.382$. It immediately becomes evident that the calculation of the index H over this interval is simply meaningless.

The conclusion is that the fact that the quantity $V_f(\delta)$ rapidly reaches the asymptotic power-law mode makes it possible to reliably calculate the index of fractality μ over short intervals, as well. Further analysis showed that the power law for the function $V_f(\delta)$ fits the results with remarkable precision in the range of scales from several minutes to several years. It was understood that this property helps achieve significant progress in solving the two main problems of time series analysis: identification and prediction.

2. The problem of identification usually consists in determining the state of the system (the macrostate of the time series) on the basis of the observed values of the series in some local range. Specialists identify three types of local states for financial time series: trend (movement directed upward or downward), flat (relatively stable state), and random walk (intermediate state between trend and flat). In order to correlate the value of μ with the states of the financial time series, we introduce a function $\mu(t)$ as such value of μ that can still be calculated at an acceptable accuracy on a minimal interval τ_μ foregoing time t. If the argument t is continuous, we could choose for such an interval an arbitrarily small one. However, since in practical cases the time series always has a minimum scale (in our case, it spans one day), τ_μ is of finite length (in our case, we take $\tau_\mu = 32$ days). Such a function $\mu(t)$ was constructed for each of the companies in the Dow Jones index.

Figure 5 shows a typical fragment of the price series of one of these companies together with the function $\mu(t)$ calculated for this fragment. Suffice it to throw a quick glance at Fig. 5 to understand that the index μ has a direct bearing on the states of the time series. Indeed, $\mu(t) > 0.5$ in the interval between the 1st and the 39th days, where prices are relatively stable (flat). Further, simultaneously with the unfolding of the trend state in the price chart, $\mu(t)$ drops sharply to values below 0.5, and finally, after the 56th day when the prices are in an intermediate state between trend and flat, $\mu(t)$ returns to a value of $\mu \approx 0.5$. The original series thus becomes more stable as μ increases. Also, if $\mu > 0.5$, the flat state is observed, and if $\mu < 0.5$, the trend state is observed. Finally, if $\mu \approx 0.5$, then the series resides in the random walk state, which is intermediate between the trend and a flat states. Such a correlation between the value of μ and the behavior characteristics of the original time series was observed for all investigated series. A theoretical basis for this correlation can be found, for instance, in paper [16]. We will show below how the function $\mu(t)$ can be used to justify the classical theory of finance.

3. The basic model of financial time series is the random walk model.[3] Rethinking this model led to the concept of the *effective* market (Effective Market Hypothesis, EMH) on which the price fully reflects all available information. For such a market to exist, it is sufficient to assume that it has a large number of fully informed, rational agents with uniform preferences, which instantly adjust the prices and bring them into equilibrium. It is natural that the basic model of such a

[3] The first random walk model [17] was constructed by Luis Bachelier in 1900 (five years before Albert Einstein proposed his model of Brownian motion), who used it for describing the behavior of stock prices on the Paris Stock Exchange. Many of the results linked to this model, which were later obtained by other authors (Chapman–Kolmogorov equation, martingale theory, Black–Scholes equation), were already implied in Bachelier's paper.

market is the random walk model. It should be noted that all the main results of the classical theory of finance [portfolio theory, CAPM (Capital Asset Pricing Model), Black–Scholes model, etc.] have been obtained within the framework of precisely this approach. At present the "the concept of effective market continues to play a dominant role both in financial theory and in financial business" [20].

However, by the beginning of the 1960s some empirical studies showed that large fluctuations of yield series occur much more often than could be expected on the basis of the normal distribution (the problem of 'fat tails'), plus these large changes usually followed one another (effect of volatility clusterization). Mandelbrot [2] was one of the first to severely criticize the above concept. Indeed, if we calculate the value of the exponent H ($H = 1 - \mu$ in our case) for some share, then in all likelihood (see the beginning of this section), this value will differ from $H = 0.5$, which corresponds to the random walk model.

The reader will recall that two postulates lie at the base of this model. First, the price increments [4] in any time interval have a normal (Gaussian) distribution; this follows from the central limit theorem and is obtained as a result of summation of a sufficiently large number of independent random variables with finite variance. Second, these increments are statistically independent in disjoint intervals. It was the rejection of the first postulate, while maintaining the second one, that led Mandelbrot to consider a random process which he called the Lévy flight [2]. The rejection of the second postulate, while maintaining the first, led him to introducing the concept of *generalized Brownian motion* (Fractional Brownian Motion) [21].

The behavior of a time series for which $H \neq 0.5$ can be described using any of these processes. For the ideological base, people typically use *the concept of fractal market* (Fractal Market Hypothesis, FMH), which is usually considered as an alternative to EMH. This concept assumes that the market comprises a wide range of agents with different investment horizons and, therefore, with different preferences. These horizons vary from one minute for *intraday* traders to several years for banks and corporations. The stable equilibrium in this market is the regime for which the mean yield is independent of scale, except for a multiplication by the appropriate scale coefficient [2]. Since this coefficient has an undefined power exponent, we are actually dealing with a whole class of regimes, each of which is determined by its specific value of the index H. Consequently, the value of $H = 0.5$ is fully equivalent to any other value ($0 < H < 1$). Similar arguments caused serious doubts about the reality of equilibrium on the stock market (see, e.g., Refs [20–24]) and, hence, about the validity of the modern theory of finance.

Investigation of the function $\mu(t)$ using the initial basis (see the beginning of the section), as well as of Russian (included in the MMVB index) and American (included in the DJII index) companies, together with the corresponding indices for the last ten years, makes it possible to clearly show that the value $H = 0.5$ is a distinguished one.

Figure 6 displays typical probability distributions of the values of index μ for a time series of one of the shares included in the DJII index on the intervals of different lengths (from 8 to 256 days). All distributions are asymmetric. This means

[4] Various modifications of the Bachelier model [18, 19] typically operate with logarithms of price increments instead of increments themselves. This difference is not significant for us in this context.

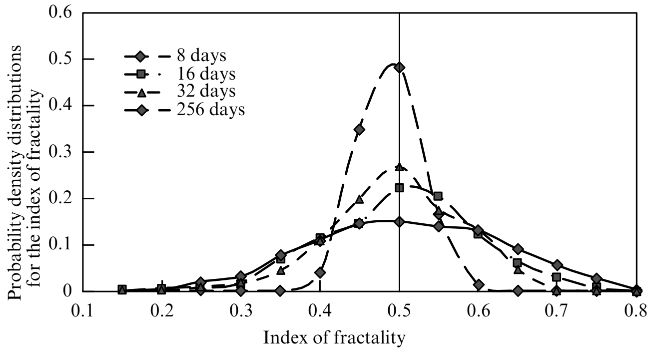

Figure 6. Probability density distributions for the values of the index μ for a series of daily changes in Ford share prices over a period from 03.01.2000 to 30.11.2010 (2745 records all-in-all, of trading days, each storing information on opening and closing prices, as well as daily records of the highest and lowest prices). The function $\mu(t)$ was calculated from the foregoing intervals containing from 8 to 256 values, and then empirical probability densities were constructed in each case from the values of μ.

that the average value of the index of fractality for this stock differs from $\mu = 0.5$ in appropriate intervals. However, all these distributions have the *principal mode* precisely at this value of μ.

To a first approximation, the following general pattern is observed in all series. The function $\mu(t)$ performs quasiperiodic oscillations around the position $\mu = 0.5$ between the values of $\mu < 0.5$ and $\mu > 0.5$. The mode of the time series is continuously changing, from the trend state via the random walk to flat and then back. For each series, the states with relatively stable values of μ (see Fig. 5) emerge time and again, then disappear. Among these states, the mode $\mu = 0.5$ occupies an obviously privileged position. For each time series, it is the longest in all intervals containing 8 or more points.

It should be emphasized that the agent-oriented interpretation of the price fluctuations may vary greatly on different scales. Thus, for example, agents' intraday behavior is apparently very close to rational behavior on a small scale when more than 50% of transactions are concluded (on U.S. markets) by trading robots. Unlike this, an essential role on scales from several days to several months is played by the social psychology, which always involves an irrational element. Incidentally, the unchanging nature of these fluctuations is reproduced on all scales, beginning with the shortest. This last remark points to a conjecture that some common mechanism of retardation, accompanying most decision-making processes, constitutes the nature of such fluctuations. But the principal state is, nevertheless, the random walk, which remains to act as the main attraction regime on all scales.

4. Generally, the prediction problem seeks to determine certain qualitative or quantitative parameters of the future behavior of a time series on the basis of the entire array of historical data. The most interesting in this situation is the problem of determining the earliest precursors of the critical behavior of a time series. We shall consider one approach to solving this problem. Starting with Eqn (5), we introduce the average amplitude $A(\delta)$ using the formula

$$A(\delta) \equiv \langle A_i(\delta) \rangle = m^{-1} V_f(\delta) \,. \qquad (11)$$

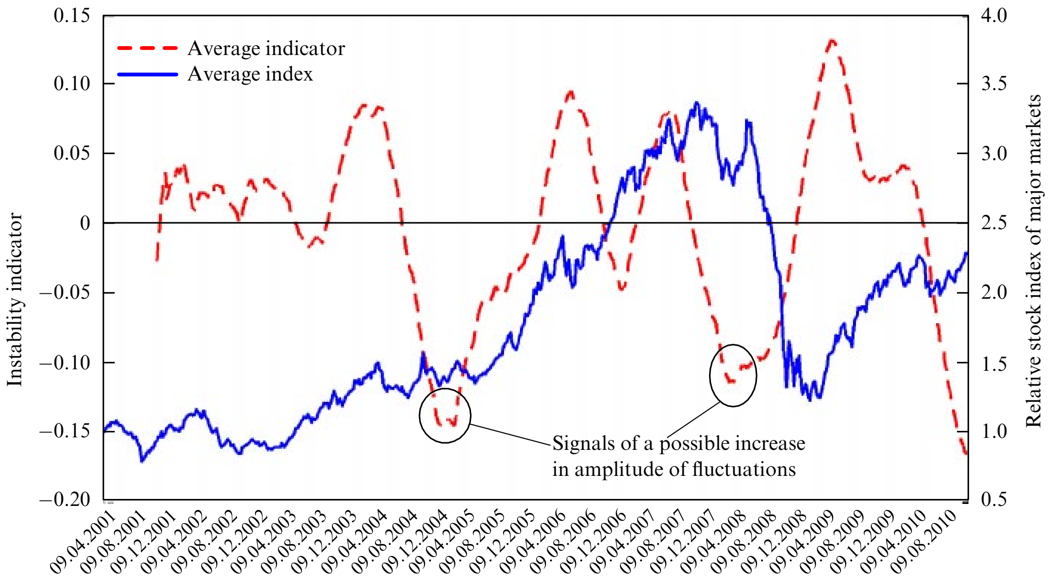

Figure 7. The aggregated index in relative units (solid curve, right-hand scale) and the appropriate indicator of instability centered about the mean value (dashed line, left-hand scale).

Multiplying Eqn (5) by $m^{-1} \sim \delta$ and substituting into Eqn (6), we obtain

$$A(\delta) \sim \delta^{H_\mu} \quad \text{for} \quad \delta \to 0\,, \tag{12}$$

where $H_\mu \equiv 1 - \mu$. Comparing H_μ with Fig. 5, we obtain a visual confirmation that this index is a measure of persistence for a series (see item 3 of this section) and a direct generalization of the Hurst exponent H to small intervals. In item 3 of this section we have seen that the index μ essentially imposes a certain method of functional integration of the time series, and that it is then clear that the series corresponding to the random walk acquires the maximum weight. It then becomes possible to build a number of distributions, including conditional ones, over μ. It is becoming clear that the effect of enhancement of large-scale fluctuations, and at the same time a certain weakening of small-scale ones, plays a special role here. This effect manifests itself because, first, the power law for the function $A(\delta)$ (as well as for the function $V_f(\delta)$) holds in an immense range of scales: from several minutes to several years. Second, the power-law function possesses an important property: the slower it decays (in comparison with a function with a different power exponent) for $\delta \to 0$, the faster it grows for $\delta \to \infty$. This fact implies that a change in the regime of the system caused by an abrupt drop in μ (increase in the index H_μ) leads to a further suppression of small-scale fluctuations and at the same time to intensification of large-scale fluctuations in the series. This means that an abrupt drop in small-scale fluctuations at present may, under certain conditions, be a precursor to strong large-scale fluctuations in the future. Testing over the entire database mentioned above has demonstrated that this effect manifests itself at a probability of about 70–80%. Note that this percentage grows even higher in those cases in which the impact of external factors can be reduced to a minimum.

Figure 7 plots the indicator (dashed curve) which was constructed on the basis of this effect in the INTRAST Management Company in 2007. The solid curve represents the initial series, namely, the aggregated index including the stock indices of both developed and developing countries (one from each country).[5] This approach excludes the factor of mutual influences exerted by stock markets of different countries against each other and produced by intercountry flows of capital on the global financial market. Figure 7 gives evidence that twice after 2001 the indicator revealed an abrupt drop in small-scale fluctuations. The first time it happened was in December 2004, and it was followed by steep growth of all indices half a year later; the growth lasted for about two years. The second time it happened was in April 2008 after which — also about half a year later — the crisis triggered a sharp drop in all indices. Moreover, we see from this figure that a new signal is being actively formed now (08.11.2010), which is a precursor of intense fluctuations of the stock market in the mid-term (from six months to one year).[6] Even though the indicator says nothing about the direction of this strong movement, the information received may prove to be sufficient, e.g., for building a successful asset management strategy on the stock market.

To complete this section, several words are in order about the effect of intensification of large-scale fluctuations as the small-scale ones decrease. In essence, this effect signifies that the trends in complex systems (natural, social, technological), which form very slowly and imperceptibly but show more-than-average implacability, often grow globally with time and dictate the main vector of evolution of such systems. Note that the well-known *calmness effect* (i.e., suppression of the high-frequency noise component) which usually precedes natural disasters (e.g., earthquakes) is a particular manifestation of this effect. Therefore, in their evolution, many global trends do resemble *the mustard seed* (of the Gospel parable): "...Although it is the smallest of all seeds, when it is fully grown it is larger than the garden plants and becomes a tree, and the birds in the sky come and nest in its branches" (Matthew, 13:32).

[5] USA, Germany, France, Japan, Russia, Brazil, China, Korea.

[6] This prediction is in clear contradiction to the general expectation of 'a slow emergence out of recession'.

5. Conclusion

To recapitulate, new fractal indices have been proposed for a one-dimensional fractal function $f(t)$: its dimension D_μ and the related index μ. The limiting value of dimension D_μ for $\delta \to 0$ coincides with the usual fractal dimension D. Numerical calculations carried out for stock price series have shown that the application of minimal covers leads to rapid convergence to a power-law asymptotic behavior of the function $V_f(\delta)$ with respect to δ. This is the reason why the representative scale required to determine these parameters with acceptable accuracy contains less data by two orders of magnitude than, for example, the scale determining the Hurst exponent H. This makes it possible to treat the index μ as a local characteristic and to introduce the function $\mu(t)$ which is an indicator of local stability of the time series: the greater μ, the more stable the series. It has been shown, using a very rich empirical data array, that the index of fractality, in essence, defines a natural way of integrating over all possible price trajectories (starting from the shortest). It turns out that trajectories corresponding to random walk have the greatest weight. This fact can well be regarded as a justification for the modern theory of finance. Finally, an early precursor of strong fluctuations in stock markets was built, based on the effect of enhancement of large-scale fluctuations accompanied with suppression of small-scale ones.

Benoit Mandelbrot, who should rightly be considered one of the main predecessors of econophysics, had the notoriety of 'iconoclast, a dynamiter of foundations' and earned complete rejection by some among the economics community. He was one of the most ardent critics of the modern theory of finance based on the concept of general equilibrium from its conception, and was seeking an acceptable alternative to it until the last days of his life. Econophysics tries to propose an alternative to the concept of general equilibrium in a similar manner, but now in the framework of the entire theory of economics. Nevertheless, it was indeed Mandelbrot who was able to work out a system of concepts which allows, as we have shown, not only generating an efficient prognosis but also proposing, after appropriate modifications, what at the moment appears to be the only empirical justification of the *classical* theory of finance.

In conclusion, the authors take this opportunity to express their profound gratitude to V M Polterovich for unflinching support, helpful advice, and valuable comments.

References

1. Mantegna R N, Stanley H E *Nature* **376** 46 (1995)
2. Mandelbrot B *J. Business* **36** 394 (1963)
3. Mandelbrot B B *The Fractal Geometry of Nature* (San Francisco: W.H. Freeman, 1982)
4. Mandelbrot B *Les Objets Fractals* (Paris: Flammarion, 1975) [Translated into English: *Fractals* (San Francisco: W.H. Freeman, 1977)]
5. Hausdorff F *Math. Ann.* **79** 157 (1919)
6. Malinetskii G G, Potapov A B, Podlazov A V *Nelineinaya Dinamika*: *Podkhody, Rezul'taty, Nadezhdy* (Nonlinear Dynamics. Approaches, Results, Expectations) (Moscow: URSS, 2006)
7. Aleksandrov P S, Pasynkov B A *Vvedenie v Teoriyu Razmernosti* (Introduction to Dimension Theory) (Moscow: Nauka, 1973)
8. Brown J et al., Working Papers in Economics 08/13 (Christchurch, New Zealand: Univ. of Canterbury, Department of Economics and Finance, 2008)
9. Clegg R *Int. J. Simulation*: *Syst., Sci. Technol.* **7** (2) 3 (2006)
10. Mishura Y *Stochastic Calculus for Fractional Brownian Motion and Related Processes* (New York: Springer, 2007)
11. Brodu N, nlin/0511041
12. Kantz H, Schreiber T *Nonlinear Time Series Analysis* (Cambridge: Cambridge Univ. Press,1997)
13. Abarbanel H D I *Analysis of Observed Chaotic Data* (New York: Springer, 1996)
14. Peters E E *Fractal Market Analysis*: *Applying Chaos Theory to Investment and Economics* (New York: J. Wiley, 1994) [Translated into Russian (Moscow: Internet-Trading, 2004)]
15. Peters E E *Chaos and Order in the Capital Markets* (New York: Wiley, 1996) [Translated into Russian (Moscow: Mir, 2000)]
16. Dubovikov M M, Starchenko N S, Dubovikov M S *Physica A* **339** 591 (2004)
17. Bachelier L, in *The Random Character of Stock Market Prices* (Ed. P H Cootner) (Cambridge, Mass.: M.I.T. Press, 1964) p. 17
18. Kendall M G *J. R. Statistical Soc.* **96** 11 (1953)
19. Samuelson P A *Industrial Management Rev.* **6** 13 (1965)
20. Shiryaev A N *Osnovy Stokhasticheskoi Finansovoi Matematiki* (Essentials of Stochastic Financial Mathematics) Vol. 1 (Moscow: Fazis, 1998)
21. Mandelbrot B B, Van Ness J W *SIAM Rev.* **10** 422 (1968)
22. Shiryaev A N *Teor. Veroyatn. Primenenie* **39** 5 (1994) [*Theor. Probab. Appl.* **39** (1) 1 (1994)]
23. Shiryaev A N *Obozrenie Priklad. Promyshl. Mat.* **1** 780 (1994)
24. Polterovich V M *Ekonom. Nauka Sovremennoi Rossii* (1) 46 (1998)