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Acoustic gradient barriers (exactly solvable models)

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Abstract. This paper reviews the physical fundamentals and mathematical formalism for problems concerning acoustic waves passing through gradient wave barriers formed by a continuous one-dimensional spatial distribution of the density and/or elastic parameters of a medium in a finite-thickness layer. The physical mechanisms of such processes involve nonlocal (geometric) normal and anomalous dispersion determined by the profiles and geometric parameters of the gradient barrier. The relevant mathematics relies on exactly solvable gradient barrier models with up to three free parameters and on the auxiliary barrier method with which the exactly solvable models found can be used to build new, also exactly solvable, models for such barriers. The longitudinal and shear wave transmission spectra through the gradient barriers considered are presented, and the dependence of these spectra on the gradient and curvature of the density distribution and on the elastic parameters of the barrier is expressed using general formulas corresponding to the geometrical and abnormal geometric dispersion. Examples of reflectionless tunneling of sound through gradient barriers formed either by the elastic parameter distribution in an inhomogeneous layer or by curvilinear boundaries of a homogeneous layer are considered. It is also shown that by using subwavelength gradient barriers and periodic structures composed of them, phonon crystal elements can be fabricated.

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1. Introduction. Inhomogeneous acoustic media

This review is devoted to the physical foundations and mathematical apparatus of the theory of gradient acoustic barriers. Such barriers are formed by finite-thickness layers of an inhomogeneous elastic medium with continuous density and elastic modulus distributions of the medium inside the layer. The propagation of elastic waves in inhomogeneous natural media traditionally attracted attention in geophysical problems [1–3]. The advent of artificial materials (metamaterials [4–6]) in recent years and their extensive studies stimulated the development of the qualitatively new concept of gradient acoustic barriers based on the results of these investigations. This concept is being developed now in connection with the problems of sound reflection and transmission in layers of inhomogeneous alloys [7], composite materials [8], and spatially bounded porous structures [9], in particular, of subwave dimensions. The acoustic spectra of such layers can drastically differ from these spectra of natural and homogeneous media.

(1) Gradient acoustic barriers have characteristic frequencies determined by the shape of spatial profiles of the density and elastic properties of the barrier and its thickness. The influence of these frequencies on the propagation of waves leads to a strong geometrical or nonlocal frequency dispersion of the reflection and transmission spectra of the barrier produced artificially in the specified frequency range. Considerable amplitude–phase variations in the wave field structure can be produced at distances shorter than the wavelength (subwavelength barriers).

(2) The artificial dispersion controlled by the parameters of a gradient barrier can be both normal and anomalous. To provide the required dispersion in the specified frequency range, the material and thickness of the gradient barrier can be chosen to minimize the wave energy losses in this frequency range.

(3) The effects of artificial geometrical dispersion allow establishing analogies between the properties of gradient barriers in acoustics and optics [10]. In particular, acoustic barriers with artificial dispersion of the waveguide type open the possibility of reflectionless tunneling of longitudinal and transverse acoustic waves.

The physical features of the effects considered here are illustrated with the example of problems concerning the interaction of sound with gradient barriers in the simplest geometry. We assume that a plane acoustic wave is incident from the side $z < 0$ normally to the boundary of an isotropic layer coinciding with the plane $z = 0$; another boundary of the layer is formed by the plane $z = d$. It is known that in this configuration, two acoustic waves corresponding to the longitudinal and transverse modes can propagate in a homogeneous layer in the z direction. The velocities v_{l} and v_{t} of these modes and their wave numbers $k_{\mathrm{l,t}}$ for each frequency ω are given by [11]

$$v_{\mathrm{l}}^2 = \frac{E(1-\mu)}{\rho(1+\mu)(1-2\mu)}\,, \qquad v_{\mathrm{t}}^2 = \frac{E}{2\rho(1+\mu)}\,, \qquad k_{\mathrm{l,t}} = \frac{\omega}{v_{\mathrm{l,t}}}\,, \tag{1.1}$$

where E is the Young modulus, ρ is the density of the medium, and μ is Poisson's ratio. Sound dispersion in medium (1.1) is absent.

Unlike (1.1), the density ρ and quantities E and μ in the gradient layer under study depend on the coordinate z. These dependences can be conveniently represented by introducing dimensionless differentiable functions $F^2(z)$ and $W^2(z)$. For the density profile $\rho(z)$, we then assume

$$\rho(z) = \rho_0 F^2(z)\,, \qquad \rho\big|_{z=0} = \rho_0\,, \qquad F\big|_{z=0} = 1\,. \tag{1.2}$$

Shear waves can be conveniently described by relating the function $W^2(z)$ to the coordinate-dependent shear modulus $G(z)$:

$$G(z) = G_0 W^2(z)\,, \qquad G_0 = \frac{E}{1+\mu}\,, \qquad W\big|_{z=0} = 1\,. \tag{1.3}$$

The values of E, μ, and G_0 in (1.3) correspond to the barrier boundary $z = 0$. The function $W^2(z)$ for longitudinal waves is defined in Sections 3.1, 4.1, 4.2, and 6.1.

We note that the nonlocal acoustic dispersion in the gradient media considered here principally differs from the local spatial dispersion of sound in structured materials containing homogeneously distributed inclusions with elastic properties different from those of the main material [12, 13]. The elastic moduli of such homogeneous structured materials are characterized by two additional constants g and h having the dimension of length and related to the potential and kinetic energies of inclusions in the wave field. The phase velocities of longitudinal (V_{l}) and shear (V_{t}) waves defined in this approach depend on the corresponding wave numbers k_{l} and k_{t} [14]:

$$V_{\mathrm{l,t}} = v_{\mathrm{l,t}} \sqrt{\frac{1+g^2k_{\mathrm{l,t}}^2}{1+h^2k_{\mathrm{l,t}}^2}}\,.$$

Here, v_{l} and v_{t} are the phase velocities of longitudinal and shear waves in the absence of inclusions, Eqn (1.1). The local dispersion of the phase velocity determined by the characteristic size of inclusions is of interest for studying this size. For example, applied to the acoustics of solid porous biomaterials, this approach gives the estimate $g \approx h \approx 10^{-5}$ m [15].

The physical foundation of the gradient acoustic barriers under study is the geometric dispersion of sound caused by inhomogeneous profiles of the density and elastic properties of the material and the barrier thickness. The mathematical apparatus that allows determining and optimizing the contribution from these dispersion effects to the spectral characteristics of the barrier is based on exact solutions of the equations of gradient acoustics. Because the thickness of acoustic barriers can be comparable to or smaller than the wavelength, the required solutions are not related to the assumption of smallness or slow variation of the parameters of the medium or wave field. Correspondingly, the WKB approximation, the perturbation theory, or asymptotic methods [16] are not used in these problems. A number of exact solutions in the acoustics of inhomogeneous media are studied in monograph [17] generalizing the results known at the time of publishing (1989). By contrast, in this review, we mainly consider new analytic methods revealing the specific features of dispersion and tunneling in subwavelength gradient acoustic barriers produced by spatial distributions of the density and elasticity moduli of materials. Special attention is paid to the auxiliary barrier method representing a standardized algorithm for obtaining reflection spectra from new barriers based on the transformation of previously known analytic solutions.

Our aim is to find the laws of nonlocal dispersion and reflection and transmission spectra for waves in gradient acoustic barriers characterized by different distributions of parameters $F^2(z)$ and $W^2(z)$. Section 2 is devoted to the shifts of oscillation eigenfrequencies in solid and liquid layers caused by the density and elastic modulus gradients. Sections 3 and 4 show the influence of the curvature of the profiles of $F^2(z)$ and $W^2(z)$ on the reflection spectra of longitudinal and shear waves in gradient barriers. In Section 3, the effects of these profiles are considered separately, and Section 4 is devoted to reflection from complex barriers formed simultaneously by both profiles ('double' barriers). In these sections, the auxiliary barrier method is illustrated using a simple mathematical apparatus. The reflectionless sound tunneling through gradient barriers for different tunneling mechanisms is considered in Section 5. Examples of subwave gradient structures, in particular, periodic structures of interest for the development of phonon crystals, are discussed in Section 6. In conclusion, in Section 7, a number of urgent problems to be solved in this theory that has been developing in recent years are pointed out.

2. Eigenfrequencies of shear waves in gradient layers

Theoretical problems of sound propagation in elastic media are considered based on the equations of motion relating the displacement **u** of particles of a medium to the components σ_{ik} of the stress tensor [11]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ik}}{\partial x_k} . \tag{2.1}$$

Here, ρ is the density of the medium and x_k are coordinates; the density ρ and the tensor components σ_{ik} in gradient media continuously depend on coordinates x_k. We analyze an acoustic wave with frequency ω propagating along the z direction and characterized by the displacement $u_x \exp(-\mathrm{i}\omega t)$ of the medium in the direction $x \perp z$ (the shear wave) taking into account that Eqn (2.1) contains only one stress tensor component, $\sigma_{xz}(z)$ [11]. Representing this component with the help of the function $W^2(z)$ in the form

$$\sigma_{xz}(z) = \frac{E}{2(1+\mu)} \frac{\mathrm{d}u_x}{\mathrm{d}z} W^2(z) , \tag{2.2}$$

we can write equation of motion (2.1) as

$$W^2(z) \frac{\partial^2 u}{\partial z^2} + \frac{\omega^2}{v_0^2} F^2(z) u + 2 W W_z \frac{\partial u}{\partial z} = 0 , \tag{2.3}$$

where $u = u_x$. Here, $W_z = \partial W/\partial z$ and $v_0 = v_\mathrm{t}$, where v_t defined in (1.1) is the shear wave velocity on the layer boundary. The choice of model functions F and W in (2.3) is limited so far only by the conditions $F^2(0) = W^2(0) = 1$. Equation (2.3) is used in Section 3 for solving problems in gradient layer acoustics. In Sections 2.1 and 2.2, we discuss the frequency spectra of shear oscillations of these layers, which are determined by the density and shear modulus distributions.

2.1 Variable-density strings: a new look at the old problem

"Stretched strings should be always of special interest for a mathematician because it is they that were at the center of arguments between D'Alembert, Euler, Bernoulli, and Lagrange concerning the solution of partial differential equations" — thus in *The Theory of Sounds* [18] Rayleigh defined the role of the problem of oscillations of an elastic string in the development of mathematical physics. These words proved to be prophetic: in the years to follow, the equation of elastic oscillations of a thin homogeneous string became the standard equation for many problems in optics, radiophysics, and quantum mechanics. This equation, which follows from (2.3) with $F = W = \mathrm{const} = 1$, coincides with the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = 0 , \tag{2.4}$$

describing a bending wave propagating at the speed v along a thin string with a constant cross section S stretched by the force T, where [11]

$$v^2 = \frac{T}{S\rho} . \tag{2.5}$$

The spectrum of eigenfrequencies Ω_n of a homogeneous string stretched between points $z = 0$ and $z = d$, such that the string displacement at these points is zero, is described by the classical formula

$$\omega_n = \frac{v_0 \pi n}{d} , \qquad n = 1, 2, 3, \ldots . \tag{2.6}$$

Rayleigh extended the applicability limits of these expressions for perturbations and studied the oscillation spectrum of a "string with a linear density not quite constant" [18]. Assuming that density variations are very weak and using the perturbation theory, Rayleigh found small corrections to spectrum (2.6).

To illustrate the methods of gradient acoustics, it is useful to consider this classical problem again and to find the oscillation spectrum of a variable-density string without the assumption of small density variations. Assuming that $W = \mathrm{const} = 1$ in Eqn (2.3), we can rewrite this equation in the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + \frac{\omega^2}{v_0^2} F^2(z) u = 0 . \tag{2.7}$$

The dimensionless function $F^2(z)$ simulates the density distribution along the string. Equation (2.7) can be formally considered the wave equation for a medium in which the wave velocity depends on the coordinate as $v(z) = v_0/F(z)$. Solutions of such equations bear an essential dependence on the form of the function $F^2(z)$. For example, we consider a simple model having an exact solution in the form of elementary functions [15]

$$F(z) = \left(1 + \frac{s_1 z}{L_1} + \frac{s_2 z^2}{L_2^2}\right)^{-1} , \quad s_1 = 0, \pm 1 , \quad s_2 = 0, \pm 1 . \tag{2.8}$$

Profile (2.8) contains four free parameters: the characteristic lengths L_1, L_2 and s_1, s_2. The values $s_1 = -1$, $s_2 = 1$ correspond to a convex profile of $F(z)$, and $s_1 = 1$, $s_2 = -1$ correspond to a concave profile. We note that for $s_2 = 0$, profile (2.8) transforms into the Rayleigh model [18]

$$F(z) = \left(1 \pm \frac{z}{L}\right)^{-1} , \tag{2.9}$$

describing monotonic density variations, often used in the theory of waves in gradient media. This model is a particular case of more flexible model (2.8), which can be used to describe both monotonic and nonmonotonic (convex and concave) distributions.

We consider the lowest oscillations of a string of length d with a convex symmetric density profile $\rho(z)$ (Fig. 1). In this case, the function $F(z)$ in (2.8) satisfies the condition $F(0) = F(d) = 1$ and the profile maximum ρ_max is located at the point $z = 0.5d$. The unknown lengths L_1 and L_2 in model (2.8) are expressed in terms of the layer thickness d with the help of a dimensionless parameter y:

$$y = \frac{L_2}{2L_1} , \qquad L_2 = \frac{d}{2y} , \qquad L_1 = \frac{d}{4y^2} , \tag{2.10}$$

and the parameter y is related to the profile maximum ρ_max as

$$\rho_\mathrm{max} = \frac{\rho_0}{(1-y^2)^2} , \qquad y = \sqrt{1 - \sqrt{\frac{\rho_0}{\rho_\mathrm{max}}}} . \tag{2.11}$$

It follows from (2.11) that $0 \leqslant y^2 < 1$.

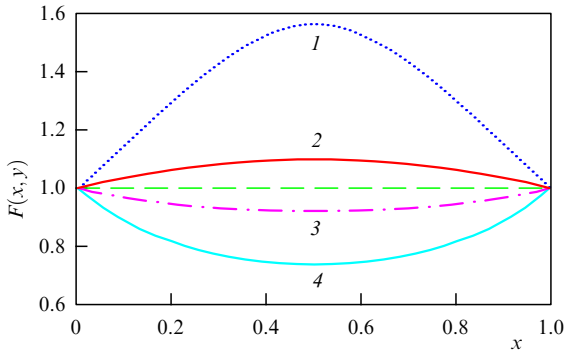

Figure 1. The profile of $F(x)$ in (2.8) in gradient barriers; $x = z/d$ is the normalized barrier thickness. Profiles *1* and *2* correspond to $s_1 = -1$ and $s_2 = 1$ (convex profile), and *3* and *4* correspond to $s_1 = 1$ and $s_2 = -1$ (concave profile). For curves *2* and *3*, the parameter $y = 0.3$, and for curves *1* and *4*, $y = 0.6$.

To solve wave equation (2.7), it is convenient to introduce the function Ψ and variable η [10]:

$$u = \frac{\Psi}{\sqrt{F}}\,, \qquad \eta(z) = \int_0^z F(z_1)\,\mathrm{d}z_1\,. \tag{2.12}$$

After such a change of variables, Eqn (2.7) becomes an equation with constant coefficients in the η space [10]:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\eta^2} + q^2\Psi = 0\,, \quad q = \frac{\omega}{v_0}\,N_+\,, \quad N_+ = \sqrt{1 + S_1^2}\,, \quad S_1 = \frac{\Omega_1}{\omega}\,, \tag{2.13}$$

where Ω_1 is the characteristic frequency determined by the propagation time of the wave at the speed v_0 through a gradient layer with the thickness d and the geometric parameters [form factor $\theta_1(y)$]

$$\Omega_1 = \frac{v_0}{d}\,\theta_1(y)\,, \qquad \theta_1(y) = 2y\sqrt{1 - y^2}\,, \tag{2.14}$$

$$\eta = \int_0^z F(z_1)\,\mathrm{d}z_1 = \frac{L_2}{\sqrt{1 - y^2}}\arctan\frac{x\sqrt{1 - y^2}}{1 - xy}\,, \qquad x = \frac{z}{L_2}\,. \tag{2.15}$$

Writing the linearly independent solutions of Eqn (2.13) in the form $\sin(q\eta)$ and $\cos(q\eta)$, we can represent the solutions of (2.7) describing standing waves established in the string in the form

$$u = \frac{\sin(q\eta)}{\sqrt{F(z)}}\,. \tag{2.16}$$

The frequencies of standing waves are described by solutions (2.16), vanishing at the string endpoints $\eta = 0$ and $\eta(d)$; the value of $\eta(d)$ is then calculated from (2.16) as

$$\eta(d) = dA\,, \quad A = \left(2y\sqrt{1 - y^2}\right)^{-1}\arctan\frac{2y\sqrt{1 - y^2}}{1 - 2y^2}\,. \tag{2.17}$$

Substituting the values of q from (2.13) and $\eta(d)$ from (2.17) in the condition $\sin(q\eta) = 0$ and taking the characteristic frequency Ω_1 in (2.14) into account, we find the discrete mode spectrum of a gradient string with a 'convex' density distribution along the string specified by function (2.8) with $s_1 = -1$ and $s_2 = 1$:

$$\Omega_{+n} = \omega_n D_n\,. \tag{2.18}$$

Here, ω_n is eigenfrequency (2.6) of the homogeneous string and D_n is the dimensionless correction coefficient,

$$D_n = \sqrt{A^{-2} - \frac{4y^2(1 - y^2)}{\pi^2 n^2}}\,. \tag{2.19}$$

The mode spectrum of the string with 'concave' density profile (2.8) ($s_1 = 1$ and $s_2 = -1$) is determined similarly. From (2.13)–(2.15), we obtain

$$q = \frac{\omega}{v_0}\,N_-\,, \qquad N_- = \sqrt{1 - S_2^2}\,, \qquad S_2 = \frac{\Omega_2}{\omega}\,, \tag{2.20}$$

$$\Omega_2 = \frac{v_0}{d}\,\theta_2(y)\,, \qquad \theta_2(y) = 2y\sqrt{1 + y^2}\,. \tag{2.21}$$

Similarly to (2.11), the parameter y is related to the minimal density $\rho_{\max}$ in the layer:

$$y = \sqrt{\sqrt{\frac{\rho_0}{\rho_{\min}}} - 1}\,, \tag{2.22}$$

$$\eta(d) = dB\,, \qquad B = \left[2y\sqrt{1 + y^2}\,\right]^{-1}\ln\frac{y_+}{y_-}\,, \qquad y_\pm = \sqrt{1 + y^2} \pm y\,. \tag{2.23}$$

The mode spectrum Ω_{-n} of the string with 'concave' density profile (2.8) can be rewritten in form (2.18) by introducing a correction coefficient H_n:

$$\Omega_{-n} = \omega_n H_n\,, \qquad H_n = \sqrt{B^{-2} + \frac{4y^2(1 + y^2)}{\pi^2 n^2}}\,. \tag{2.24}$$

We note that the mode spectrum of variable-density string (2.8) was calculated without assuming that density variations are small. Plots of the correction coefficients D_n and H_n for different eigenfrequencies are shown in Fig. 2. In the limit of

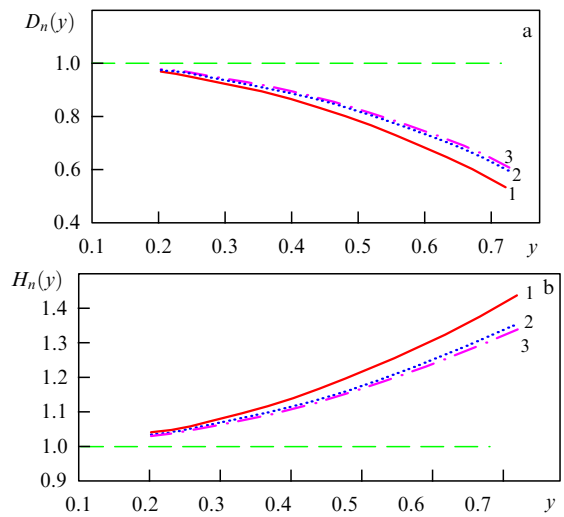

Figure 2. Correction factors (a) D_n in Eqn (2.19) and (b) H_n in Eqn (2.24) for the mode spectra of strings with inhomogeneous density distribution (2.8). Numbers 1, 2, 3 at the curves are mode numbers.

the vanishing inhomogeneity ($y \to 0$), it follows from (2.19) and (2.24) that

$$\lim D_n\big|_{y\to 0} = 1\,, \qquad \lim H_n\big|_{y\to 0} = 1\,, \qquad \lim \Omega_{+n}\big|_{y\to 0} = \lim \Omega_{-n}\big|_{y\to 0} = \omega_n\,. \tag{2.25}$$

As expected, spectra (2.18) and (2.24) of gradient strings are reduced in this limit to classical formula (2.6) for a homogeneous string.

2.2 Eigenfrequencies of a variable-shear modulus layer

In contrast to the oscillations considered in Section 2.1, we here consider eigenoscillations of a layer with constant density and with the shear modulus depending on the coordinate. Such a model is used, in particular, in geoacoustics to describe the propagation of seismic shear waves [18] in sedimentary rock layers on the seabed. In that case, depth variations in the rock density are neglected ($\rho = \rho_0$, $F = \text{const} = 1$) and the dependence of the shear modulus $G(z)$ on the depth z in a layer ($0 \leqslant z \leqslant d$) is given by the empirical formula

$$W = \left(1 + \frac{z}{L}\right)^q, \qquad 0 < q < 1\,. \tag{2.26}$$

Here, L is a characteristic length, q is a dimensionless parameter, and the depth z is measured downward from the surface of a sedimentary rock layer lying on a solid seabed.

To solve Eqn (2.3) in model (2.26), it is convenient to introduce a new variable x and function f [19]:

$$x = 1 + \frac{z}{L}\,, \qquad u = x^{1/2-q} f. \tag{2.27}$$

Such a change of variables reduces Eqn (2.3) to the form

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}f}{\mathrm{d}x} + f\left[\frac{\omega^2 L^2}{v_0^2 x^{2q}} - \frac{(1/2-q)^2}{x^2}\right] = 0\,, \tag{2.28}$$

where v_0 is the velocity of shear waves on the layer surface $z = 0$. To solve Eqn (2.28), it is necessary to introduce the new variable

$$x = y^m\,, \qquad m = \frac{2}{1-q}\,. \tag{2.29}$$

Substitution (2.29) transforms (2.28) into the equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}y^2} + \frac{1}{y}\frac{\mathrm{d}f}{\mathrm{d}y} + 4f\left(p^2 y^2 - \frac{s^2}{y^2}\right) = 0\,, \qquad p = \frac{\omega L}{v_0(1-q)}\,, \qquad s = \frac{q-1/2}{1-q}\,, \tag{2.30}$$

which becomes the Bessel equation after the change of variables $y^2 = u$. The medium displacement at the depth z in the shear wave field can be written in the form of a standing wave:

$$u = \left(1 + \frac{z}{L}\right)^{1/2-q} J_s\left[p\left(1 + \frac{z}{L}\right)^{1-q}\right]. \tag{2.31}$$

Assuming that the displacement of the medium at the water layer bottom ($z = d$) is zero, we can determine the eigenfrequencies of the layer in terms of the roots of the Bessel function $J_s(\chi_{sn}) = 0$ [17],

$$\omega_n = \frac{v_0(1-q)\chi_{sn}}{L(1+d/L)^{1-q}}\,, \qquad n = 1, 2, 3, \ldots\,. \tag{2.32}$$

For typical values $v_0 = 100$ m s^{-1}, $q = 0.6$, $d = 200$ m, and $L = 1$ m [19], expression (2.32) gives the resonance frequencies of the first modes lying in the frequency range of several hertz. In the particular case $q = 2/3$, it follows from (2.30) that $s = 0.5$. It is known that for half-integer values of the parameter s, the Bessel function is expressed in terms of elementary functions. For example, $J_{0.5}(x) = x^{-0.5}\sin x$; in this case, the roots are $\chi_{sn} = \pi n$ and oscillation spectrum (2.32) of a gradient layer consists of equally spaced frequencies.

Another interesting example of acoustic eigenfrequencies appearing in gaseous and liquid media with an inhomogeneous density $\rho(z)$ in the gravitational field with the gravity acceleration g are the Väisälä–Brunt frequencies Ω_{VB} (see, e.g., [20]):

$$\Omega_{\text{VB}}^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}z}\,. \tag{2.33}$$

Typical Ω_{VB} values are of the order of 10^{-2} Hz in the atmosphere and $10^{-3}-10^{-4}$ Hz in the ocean [20]. Internal gravitational waves at such frequencies play an important role in the dynamics of the atmosphere and ocean [20–22].

3. Nonlocal dispersion of gradient acoustic barriers: normal and anomalous dispersion

Unlike Section 2, where standing waves in elastic inhomogeneous media were discussed, this section is devoted to the interaction of traveling acoustic waves with gradient solid layers. The acoustics of structured solids containing inclusions of other materials was considered in [13]. The specific features of ultrasonic fields in such structures were studied in [14]. For arbitrary frequencies, the reflection and transmission spectra of waves in acoustic barriers formed by gradient isotropic subwavelength layers can have a strong frequency dispersion produced in the required wavelength range with the help of specially selected spatial distributions of the density $F^2(z)$ or elastic properties $W^2(z)$ across barriers. The distributions $F^2(z)$ and $W^2(z)$ are considered for which the wave field inside a barrier is described by exact analytic solutions of Eqn (2.3) (exactly solvable models). The required spectra are calculated from the continuity conditions for displacements and stresses at the barrier boundary. For the normal incidence of waves on the boundary $z = 0$, these conditions can be written as [8]

(a) the equality of displacements u_i,

$$u_i\big|_{z=-0} = u_i\big|_{z=+0}; \tag{3.1}$$

(b) the equality of normal stresses σ_{iz},

$$\sigma_{iz}\big|_{z=-0} = \sigma_{iz}\big|_{z=+0}\,. \tag{3.2}$$

For simplicity, we assume below that elastic media on the left and right of the barrier are homogeneous and identical. An exactly solvable model of such a barrier with an inhomogeneous density and homogeneous elastic properties is considered in Section 3.1 using the approach developed in Section 2.1. The opposite situation with variable elastic

properties and constant density of the medium is analyzed in Section 3.2 with the help of the special 'auxiliary barrier' method allowing the use of solutions obtained in Sections 2.1 and 3.1. Although such a separation of medium properties is conventional, it allows choosing the approach to the development of gradient acoustic materials with specified reflection and transmission spectra.

3.1 The passage of a longitudinal sound wave through a variable-density layer

Propagation of a longitudinal sound wave incident along the direction z normally to a variable-density layer can be studied using equations of motion (2.1) and (2.3), assuming that $u = u_z$, $\rho(z) = \rho_0 F^2(z)$, and $W = 1$. In this case, the only component σ_{zz} of the stress tensor in the right-hand side of (2.1) and the velocity v_0 equal to the longitudinal wave velocity v_1 in a homogeneous medium are determined by the known expressions [8]

$$\sigma_{zz} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)}\frac{\partial u}{\partial z}, \qquad v_0^2 = v_1^2 = \frac{E(1-\mu)}{\rho_0(1+\mu)(1-2\mu)}, \tag{3.3}$$

and Eqn (2.3) takes form (2.7).

From (2.7), we can find the reflection spectrum for a longitudinal wave incident from a homogeneous medium ($z \leqslant 0$) along the z axis on a gradient wave barrier (a variable-density layer). The properties of such spectra can be conveniently studied using the exactly solvable model in (2.8), which allows solutions in the form of elementary functions.

We first consider the normal incidence of a wave on a layer with thickness d with a convex symmetric density profile $\rho(z)$ (see Fig. 1). In this case, the condition $F(0) = F(d) = 1$ is satisfied for the function $F(z)$ and the profile maximum $\rho_{\max}$ is located at the point $z = 0.5d$. Because the differential equation describing the propagation of a longitudinal sound wave through a layer formally coincides in this case with (2.7), we can use the results of the analysis of that equation in Section 2.1 and also reduce (2.7) to simple wave equation (2.13) with constant coefficients in the η space. The parameters of this equation are defined in (2.14) and (2.15). But the solution of Eqn (2.13) should be sought not in the form of standing wave (2.16) but as a superposition of direct and backward monochromatic waves $\exp(\pm \mathrm{i}q\eta)$ traveling along the η axis in opposite directions:

$$u = \frac{A_\mathrm{r}\left[\exp(\mathrm{i}q\eta) + Q\exp(-\mathrm{i}q\eta)\right]}{\sqrt{F(z)}}. \tag{3.4}$$

Here, A_r is the wave field amplitude formed by the interference of the direct and backward waves. Because the wave is incident on the layer boundary $z = 0$, the quantity Q characterizes the contribution of the wave reflected from the rear boundary $z = d$ of the layer to the wave field and the variable η, determined by substituting (2.8) in (2.12), is expressed by (2.15).

The reflection spectrum of the wave reflected from the gradient layer is found from continuity conditions (3.1) and (3.2) at the layer boundaries $z = 0$ and $z = d$. Representing a longitudinal wave incident from a homogeneous medium ($z \leqslant 0$, density ρ_1, wave velocity v_1) on the layer boundary $z = 0$ in the form $u = A_\mathrm{i}\exp\left[\mathrm{i}\omega(z/v_1 - t)\right]$ and introducing the complex reflection coefficient R, we can write these boundary conditions as

$$A_\mathrm{i}(1+R) = A_\mathrm{r}(1+Q), \tag{3.5}$$

$$\mathrm{i}\omega\rho_1 v_1(1-R)A_\mathrm{i} = A_\mathrm{r}\rho_0 v_0^2\left[-\frac{1+Q}{2L_1} + \mathrm{i}q(1-Q)\right]. \tag{3.6}$$

From (3.5) and (3.6), we find

$$R = \frac{\mathrm{i}\alpha + \gamma/2 - \mathrm{i}N_+(1-Q)(1+Q)^{-1}}{\mathrm{i}\alpha - \gamma/2 + \mathrm{i}N_+(1-Q)(1+Q)^{-1}}, \tag{3.7}$$

where α is the ratio of acoustic impedances $I_{1,2}$ of the adjacent media ($I = \rho v$) and γ is a dimensionless parameter:

$$\alpha = \frac{\rho_1 v_1}{\rho_0 v_0}, \qquad \gamma = \frac{v_0}{\omega L_1} = \frac{2Sy}{\sqrt{1-y^2}}. \tag{3.8}$$

The quantity Q in (3.7) characterizing the backward wave amplitude is found from boundary conditions similar to (3.1) and (3.2) on the $z = d$ surface. Assuming for simplicity that the medium in the region $z \geqslant d$ is the same as in the region $z \leqslant 0$, we can write

$$Q = -\frac{\exp(2\mathrm{i}q\eta_0)(\gamma/2 - \mathrm{i}\alpha + \mathrm{i}N_+)}{\gamma/2 - \mathrm{i}\alpha - \mathrm{i}N_+}, \qquad \eta_0 = \eta(d). \tag{3.9}$$

Finally, substituting (3.9) in (3.7), we obtain an explicit expression for the complex reflection coefficient $R = |R|\exp(\mathrm{i}\phi_\mathrm{r})$:

$$R = \frac{\tan(q\eta_0)(\alpha^2 + \gamma^2/4 - N_+^2) + \gamma N_+}{\tan(q\eta_0)(\alpha^2 - \gamma^2/4 + N_+^2) - \gamma N_+ + 2\mathrm{i}\alpha\left[N_+ + (\gamma/2)\tan(q\eta_0)\right]}, \tag{3.10}$$

$$\tan\phi_\mathrm{r} = \frac{2\alpha\left[N_+ + (\gamma/2)\tan(q\eta_0)\right]}{\gamma N_+ - (\alpha^2 - \gamma^2/4 + N_+^2)\tan(q\eta_0)}, \tag{3.11}$$

$$\eta_0 = \eta(d) = \frac{L_2}{\sqrt{1-y^2}}\arctan\frac{2y\sqrt{1-y^2}}{1-2y^2}, \qquad q\eta_0 = \frac{N_+}{S}\arctan\frac{2y\sqrt{1-y^2}}{1-2y^2}. \tag{3.12}$$

Expressions (3.10)–(3.12) solve the problem of the reflection of a monochromatic longitudinal wave from gradient layer (2.8) with a convex density profile (curve *1* in Fig. 1); in this case, the dependence $q = q(\omega)$ specified in (2.13) corresponds to normal dispersion.

The reflection spectra $|R(S)|^2$ of longitudinal waves for the anomalous and normal nonlocal dispersions of gradient barriers are presented in Figs 3a and 3b. Spectra for the impedance $\alpha = 1.25$ are shown in Fig. 3c. Figure 3d shows the phase $\phi_\mathrm{r}(S)$ of the reflection coefficient $R(S) = |R|\exp[\mathrm{i}\phi_\mathrm{r}(S)]$ for $\alpha = 0.3$ and $y = 0.45$ for $m = 1$ (curve *1*) and $m = 3$ (curve *2*), where m is the number of barriers. Figure 3e shows the transmission spectrum $|T(S)|^2$ for $m = 1$ (curve *1*), $m = 3$ (curve *2*), and $m = 6$ (curve *3*), with $\alpha = 0.3$ and $y = 0.45$ (anomalous dispersion). It can be seen from Fig. 3e that as the number of gradient barriers increases, the number of wave frequencies at which almost 100% transmission of the wave through a multilayer periodic structure is realized also increases considerably.

Before analyzing this result, it is useful to consider a similar problem of reflection from a layer with the concave

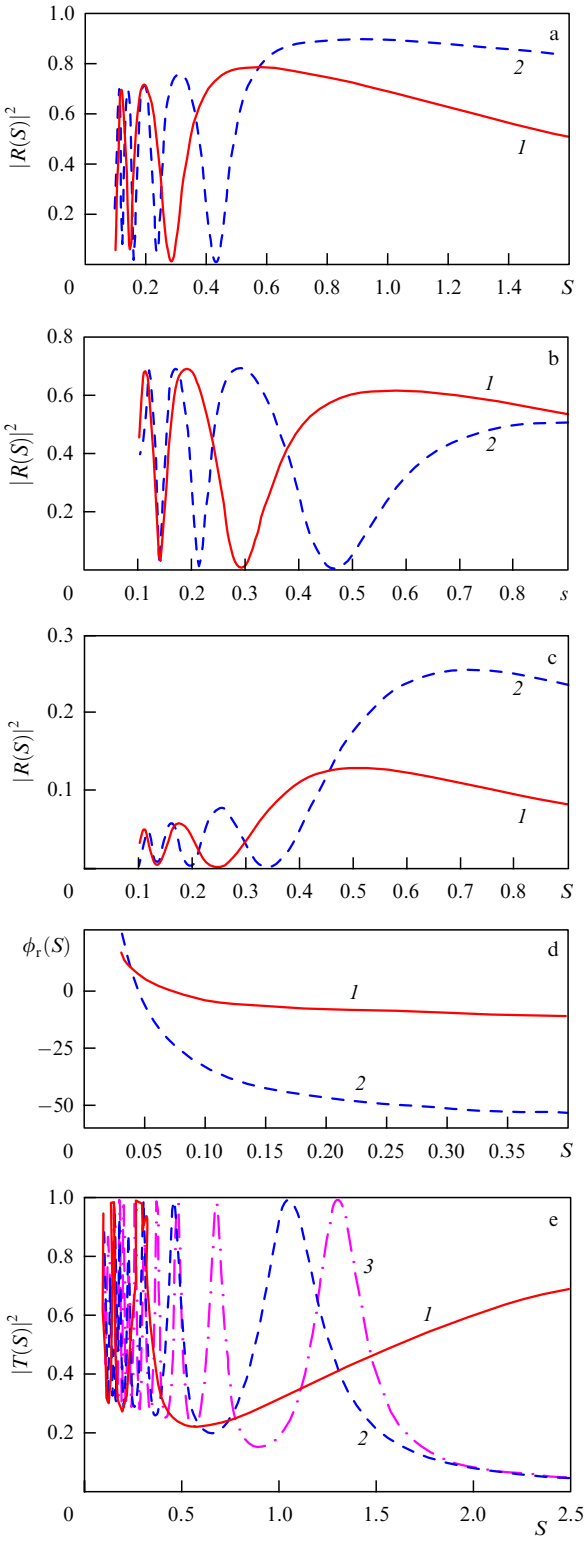

Figure 3. Reflection spectra $|R(S)|^2$ for longitudinal waves for different dispersions of gradient barriers formed by the density distribution: (a) anomalous dispersion with the parameters $\alpha = 0.3$, $y = 0.45$ (curve *1*), and $y = 0.7$ (curve *2*); (b) normal nonlocal dispersion for $\alpha = 0.3$, $y = 0.45$ (curve *1*), and $y = 0.7$ (curve *2*); (c) normal dispersion for $\alpha = 1.25$, $y = 0.45$ (curve *1*), and $y = 0.7$ (curve *2*), with a concave profile. (d) The phase $\phi_r(S)$ of the reflection coefficient $R(S) = |R| \exp[i\phi_r(S)]$ for $\alpha = 0.03$, $y = 0.45$; $m = 1$ (curve *1*) and $m = 3$ (curve *2*). (e) Transmission spectra $|T(S)|^2$ for $\alpha = 0.3$ and $y = 0.45$ (anomalous dispersion) for the number of barriers $m = 1$ (curve *1*), $m = 3$ (curve *2*), and $m = 6$ (curve *3*).

density profile described by (2.8) with $s_1 = 1$ and $s_2 = -1$ (curve *2* in Fig. 1). Reflection coefficients are calculated the same in both cases. The solution of wave equation (2.13) for the concave profile is again represented in form (3.4), where the 'wave number' q is defined, unlike (2.13), by expressions (2.20) corresponding to anomalous dispersion. The characteristic frequencies Ω_2 and Ω_1 correspond to different form factors defined (together with the frequencies) in (2.21) and (2.14). The parameter y is related to the minimal density $\rho_{\min}$ in the layer, Eqn (2.22). The coordinate η in (2.15) for this profile geometry is calculated from the expression

$$\eta = \frac{L_2}{2\sqrt{1+y^2}} \ln \frac{1+xy_+}{1-xy_-}, \qquad x = \frac{z}{L_2}, \qquad y_\pm = \sqrt{1+y^2} \pm y, \qquad y_+ y_- = 1. \tag{3.13}$$

Finally, the reflection coefficient for the concave density profile is found similarly to (3.10):

$$R = \frac{\tan(q\eta_0)(\alpha^2 + \gamma^2/4 - N_-^2) - \gamma N_-}{\tan(q\eta_0)(\alpha^2 - \gamma^2/4 + N_-^2) + \gamma N_- + 2i\alpha\left[N_- - (\gamma/2)\tan(q\eta_0)\right]}. \tag{3.14}$$

The parameters γ and $q\eta_0$ in (3.14) are given by

$$\gamma = \frac{2Sy}{\sqrt{1+y^2}}, \qquad q\eta_0 = \frac{N_-}{S} \ln \frac{y_+}{y_-}. \tag{3.15}$$

If the reflection coefficient R is known, then the wave energy transmission coefficient through the gradient barrier is determined from the expression

$$|T|^2 = 1 - |R|^2. \tag{3.16}$$

We note that if the longitudinal wave velocity $v_0 = v_l$ in (3.3) is replaced with the shear wave velocity $v_0 = v_t$ in (1.1), expressions (3.10) and (3.14) can be used for calculating reflection coefficients for shear waves. The analysis of longitudinal sound waves in a gradient layer described above can be used in a number of problems in the acoustics of gradient media considered in Sections 3.2, 4–6.

3.2 Reflection spectra of shear waves in a variable-shear modulus layer. The auxiliary barrier method

The reflection of shear waves from a medium with a spatially distributed shear modulus can be studied using Eqn (2.3). To separate the effects caused by this distribution, we assume that the medium density is independent of the coordinates ($\rho = \rho_0$) and consider the normal incidence of radiation on the layer boundary $z = 0$. In this case, the only component of the stress tensor entering (2.3) is written as

$$\sigma_{xz} = \left[\frac{E}{2(1+\mu)}\right]_0 W^2(z) \frac{\partial u_x}{\partial z}, \tag{3.17}$$

and Eqn (2.3) takes the form

$$\frac{d^2u}{dz^2} + \frac{\omega^2}{v_0^2} \frac{u}{W^2(z)} = -\frac{2W_z}{W} \frac{du}{dz}. \tag{3.18}$$

According to (1.1), the shear wave velocity v_0 in (3.18) is equal to the transverse velocity, $v_0 = v_t$. Equation (3.18) differs by its right-hand side from Eqn (2.13) in the problem for a

variable-density medium, and is therefore solved using a special algorithm based on the auxiliary barrier method. This method involves the following stages.

(1) Differentiation with respect to z in (3.18) is replaced by differentiation with respect to a new variable η, which is now, unlike (2.12), defined by the relation

$$\mathrm{d}\eta = \frac{\mathrm{d}z}{W^2(z)}\,. \tag{3.19}$$

Passing to the variable η eliminates the right-hand side of Eqn (3.18):

$$\frac{\mathrm{d}^2u}{\mathrm{d}\eta^2} + \frac{\omega^2}{v_0^2}\,W^2(z)u = 0\,. \tag{3.20}$$

The displacement u in (3.20) depends on two variables, z and η. To solve this equation, it is necessary to specify the function $W^2(z)$ and express it in terms of η. In particular, Eqn (3.20) is reduced to Eqn (2.13) solved previously by introducing an auxiliary barrier $F^2(\eta)$ in the η space:

$$W^2(z) = F^2(\eta)\,. \tag{3.21}$$

The function $F^2(\eta)$ in (3.21) can be chosen arbitrarily. But if it is taken in form (2.8) with z replaced by η, we can use ready-made solution (2.13). We can write the function $F^2(\eta)$ corresponding, for example, to the convex profile ($s_1 = -1$, $s_2 = 1$) in the form

$$F^2(x) = (1 - 2yx + x^2)^{-2}\,, \qquad x = \frac{\eta}{L_2}\,. \tag{3.22}$$

The characteristic lengths L_1 and L_2 and the parameter y in (3.22) are unknown.

(2) Substituting expressions (3.22) and (3.21) in (3.19) and using the condition $\eta\,|_{z=0} = 0$ that follows from (3.22), we can find the dependence of z on x by integrating (3.19),

$$\frac{z(x)}{L_2} = \frac{1}{2(1-y^2)^{3/2}}\left\{\arctan\frac{x-y}{\sqrt{1-y^2}} + \arctan\frac{y}{\sqrt{1-y^2}} + \sqrt{1-y^2}\left[y + \frac{x-y}{1-y^2+(x-y)^2}\right]\right\}. \tag{3.23}$$

To find y in (3.23), we note that according to (3.21), the convex profile $F^2(x)$ corresponds to the convex profile $W^2(x)$ and the maximum of the convex profile $F^2_{\max} > 1$ corresponding to the maximum of the profile $W^2_{\max} = F^2_{\max}$. Substituting the value $F^2_{\max} = (1-y^2)^{-2}$ from (2.15), we find

$$y = \sqrt{1 - \frac{1}{W_{\max}}}\,. \tag{3.24}$$

The parameter x in (3.23) can be easily found by substituting (3.22) in (3.21), solving the resulting equation for x, and replacing y by (3.24):

$$x(W) = \sqrt{1 - \frac{1}{W_{\max}}} \pm \sqrt{\frac{1}{W} - \frac{1}{W_{\max}}}\,. \tag{3.25}$$

Expressions (3.23)–(3.25) implicitly define the coordinate dependence of the shear modulus inside the barrier $W^2(z)$; as follows from (3.22), the variable x ranges the interval $0 \leqslant x \leqslant 2y$. In this case, $z(0) = 0$ and the barrier width d, determined by the distance between the points where $W(0) = 1$ and $W(2y) = 1$, is related to the characteristic size L_2:

$$d = L_2 z(2y) = 2L_2B_1\,, \tag{3.26}$$

$$B_1 = \frac{1}{(1-y^2)^{3/2}}\left(y\sqrt{1-y^2} + \arctan\frac{y}{\sqrt{1-y^2}}\right). \tag{3.27}$$

Thus, knowing the width d and height $W_{\max}$ of the barrier $W^2(z)$ specified implicitly, we can find the parameters of the auxiliary barrier $F^2(\eta)$ in (3.22) specified explicitly. The height and width of the auxiliary barrier are $W_{\max}$ and d_1, and the characteristic lengths L_1 and L_2 are expressed in terms of the width d and the parameter y:

$$d_1 = 2yL_2 = \frac{yd}{B_1}\,, \qquad L_1 = \frac{d}{4yB_1}\,, \qquad L_2 = \frac{d}{2y}\,. \tag{3.28}$$

The convex barrier $W^2(z)$ and corresponding auxiliary barrier (3.23), which are characterized by anomalous dispersion, are shown in Fig. 4a.

(3) To calculate the refection coefficient of the barrier $W^2(z)$, it is necessary to find the field inside the barrier described by Eqn (3.20). Under condition (3.21), Eqn (3.20) coincides formally with (2.13). Introducing the variable τ similarly to (2.12),

$$\tau(\eta) = \int_0^{\eta} F(\eta_1)\,\mathrm{d}\eta_1\,, \tag{3.29}$$

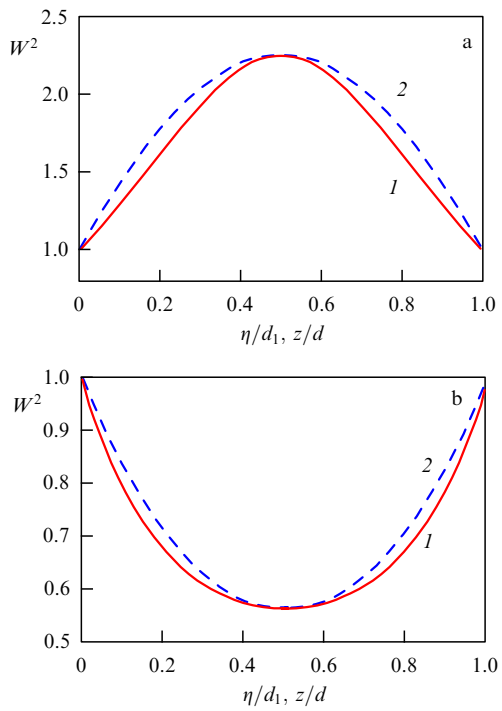

Figure 4. Gradient barriers formed by shear modulus distributions in parametric form (curves *2*) and the corresponding auxiliary barriers (curves *1*): (a) $s_1 = 1$ and $s_2 = -1$, (b) $s_1 = -1$ and $s_2 = 1$. Barriers (3.23) in Fig. 4a are characterized by anomalous dispersion, and barrier (3.33) in Fig. 4b is characterized by normal dispersion. The distances η and z along the horizontal axes are respectively normalized to d_1 and d.

we can write the solution of Eqn (3.20) in the form similar to (3.4),

$$u = \frac{A_{\mathrm{r}}\left[\exp\left(\mathrm{i}q\tau\right) + Q\exp\left(-\mathrm{i}q\tau\right)\right]}{\sqrt{F(\eta)}}\,. \tag{3.30}$$

Continuing this analogy, we can find the reflection coefficient R for the shear modulus inhomogeneity inside barrier (3.23) obtained by transforming the profile $F^2(x)$ in (2.8). This coefficient is expressed by the same formula (3.10) as the reflection coefficient $F^2(z)$ in (2.8) of the inhomogeneity of density. In this case, the characteristic frequency Ω_1 entering the parameter S differs from (2.14) by its form factor $\theta_1(y)$ caused by the ratio of barrier widths,

$$\theta_1(y) = \frac{2}{1-y^2}\left(y\sqrt{1-y^2} + \arctan\frac{y}{\sqrt{1-y^2}}\right). \tag{3.31}$$

(4) Reflection from the concave profile $W^2(z)$ characterized by the minimum $W_{\min}$ can be studied by choosing the concave profile of the auxiliary barrier $F^2(x)$ with $s_1 = 1$ and $s_2 = -1$. Repeating the analysis in (3.21)–(3.25), we find the parameter

$$y = \sqrt{W_{\min}^{-1} - 1} \tag{3.32}$$

and the implicit expression for the shear modulus profile inside the barrier

$$\frac{z(x)}{L_2} = \frac{1}{(1+y^2)^{3/2}}\left\{\operatorname{artanh}\frac{x-y}{\sqrt{1+y^2}} + \operatorname{artanh}\frac{y}{\sqrt{1+y^2}} + \sqrt{1+y^2}\left[y + \frac{x-y}{1+y^2-(x-y)^2}\right]\right\}, \tag{3.33}$$

$$x(W) = \sqrt{\frac{1}{W_{\min}} - 1} \pm \sqrt{\frac{1}{W_{\min}} - \frac{1}{W}}. \tag{3.34}$$

The auxiliary barrier width d_2 and characteristic lengths L_1 and L_2 can be expressed, similarly to (3.28), in terms of the width d of the barrier $W^2(z)$ and the parameter y in (3.32):

$$d_2 = \frac{yd}{B_2}\,, \qquad L_1 = \frac{d}{4yB_2}\,, \qquad L_2 = \frac{d}{2y}\,, \tag{3.35}$$

$$B_2 = \frac{1}{(1+y^2)^{3/2}}\left(y\sqrt{1+y^2} + \operatorname{artanh}\frac{y}{\sqrt{1+y^2}}\right). \tag{3.36}$$

The concave barrier $W^2(z)$ and the corresponding auxiliary barrier (3.33) characterized by normal dispersion are shown in Fig. 4b.

The reflection coefficient for the concave barrier is calculated from expression (3.14), where the parameter y is defined in (3.32), and the frequency Ω_2 is given by (2.21), but with a different form factor $\theta_2(y)$:

$$\theta_2(y) = \frac{2}{1+y^2}\left(y\sqrt{1+y^2} + \operatorname{artanh}\frac{y}{\sqrt{1+y^2}}\right). \tag{3.37}$$

We note that reflection coefficients for shear waves obtained in Section 3.2 can also be used for normally incident longitudinal waves, assuming that $v_0 = v_{\mathrm{l}}$ in all expressions, where v_{l} is longitudinal wave velocity (1.1).

(I) The main results in this section are the expressions for the reflection coefficients for longitudinal and shear waves reflected from inhomogeneous wave barriers formed by spatial distributions of the density and elastic properties. These expressions are obtained based on exact analytic solutions of the wave equation for a gradient medium without using any assumptions about the smallness or slow variations in the field and medium. The expressions include the contributions to sound reflection caused not only by the difference of acoustic impedances [the parameter α in (3.10) and (3.14)] but also by the gradient and curvature of the normalized density profile $F^2(z)$ depending on the characteristic lengths L_1 and L_2. As the inhomogeneity weakens ($L_1 \to \infty$, $L_2 \to \infty$), the parameters y and γ and the characteristic frequencies Ω_1 and Ω_2 tend to zero, while expressions (3.10) and (3.14) transform into the known expression for the reflection of normally incident sound from a homogeneous layer:

$$R = \frac{\tan\delta(\alpha^2-1)}{\tan\delta(\alpha^2+1) + 2\mathrm{i}\alpha}\,, \qquad \delta = \frac{\omega d}{v_0}\,. \tag{3.38}$$

(II) It is important that the analysis of sound reflection from gradient barriers involves the characteristic frequencies Ω_1 and Ω_2 determined by the travel times of waves with velocities v_0 through a gradient barrier of width d and by the geometric parameters $\theta_1(y)$ and $\theta_2(y)$ of the layer. In the expressions for N_+ in (2.13) and N_- in (2.20), whose structure resembles that of refractive indices in the electrodynamics of dielectrics with anomalous and normal (waveguide) dispersion, these frequencies characterize the nonlocal dispersion of the acoustic medium.

The nonlocal artificial dispersion formed by the geometric parameters of the barrier allows selecting a spectral range for a specified frequency band far from the absorption band of the acoustic medium.

(III) Within a unified approach, the auxiliary barrier method reveals the similarity and differences of reflection spectra caused by physically different gradient structures (for example, inhomogeneities of the density and elastic parameters of the medium). In this approach, the reflection spectra of acoustic waves reflected from barriers with the normal and anomalous nonlocal dispersion are described by general expressions (3.10) and (3.14), which are valid after the substitution of the corresponding values of the parameter y and form factors $\theta_{1,2}(y)$. This generality can be extended, as is shown in Section 4, to other classes of acoustic barriers.

4. Propagation of sound through gradient solid structures: combined variable-density and elasticity dispersion effects

Unlike the sound dispersion caused by either the density distribution $F^2(z)$ or the elastic parameter distribution $W^2(z)$ considered in Section 3, the dispersion of gradient barriers discussed here depends on spatial distributions of the density and the elastic parameters of the medium simultaneously. Such combined dependences attract attention in the acoustics of organic materials [23], biological microstructures [24], and composite and granulated metamaterials [25, 26]. The combined action of these mechanisms leads to competing dispersion effects in sound reflection and transmission spectra of gradient barriers. Because both these effects are simultaneously manifested in one barrier, we can speak about 'double' barriers and their complicated spectra.

Some specific features of the formation of such spectra can be distinguished by considering two qualitatively different problems:

(i) finding the reflection spectrum of a barrier in which the formal relation between the distributions of the density $F^2(z)$ and elastic moduli $W^2(z)$ in (2.3) is absent; moreover, changes in $F^2(z)$ and $W^2(z)$ inside the barrier can be different and even opposite;

(ii) finding the spectral characteristics of the gradient barrier in which the distributions $F^2(z)$ and $W^2(z)$ are functionally related; by properly selecting these distributions, it is possible to optimize barrier parameters to obtain the required reflection spectrum.

These problems are considered in Sections 4.1 and 4.2.

4.1 'Double' gradient barrier: optimal barrier parameters for the specified dispersion

We consider a shear wave inside the gradient layer described by Eqn (2.3) and introduce a new variable η by formula (3.19). Then Eqn (2.3) takes the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} + \frac{\omega^2}{v_0^2} F^2(z) W^2(z)\, u = 0\,. \tag{4.1}$$

Describing the distribution $F^2(z)$ and $W^2(z)$ inside a barrier of width d with the help of characteristic lengths l_1 and l_2,

$$W(z) = 1 + \frac{z}{l_1}\,, \qquad F(z) = \frac{1}{1 + z/l_2}\,, \tag{4.2}$$

we can study the effects caused by the increase or decrease in the density and elastic parameters inside the 'double' barrier in the general form, considering both positive and negative values of the lengths l_1 and l_2 in (4.2) independently. To distinguish these lengths, related to models of different physical quantities, from the lengths L_1 and L_2 characterizing the distribution of one quantity, for example, the density in model (2.8), the former are written in lowercase letters.

Expressing the variable η explicitly in terms of z with the help of (3.19),

$$\eta = \frac{z}{1 + z/l_1}\,, \tag{4.3}$$

and representing the functions $W(z)$ and $F(z)$ in (4.2) with the help of (4.3) as functions of η, we find

$$F(z)\, W(z) = U(\eta) = \frac{1}{1 + \eta/l}\,, \tag{4.4}$$

$$l = \frac{l_1 l_2}{l_1 - l_2}\,. \tag{4.5}$$

Substituting (4.4) in (4.1), we can rewrite this equation in the η space in a form similar to (2.7),

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} + \frac{\omega^2}{v_0^2} U^2(\eta)\, u = 0\,. \tag{4.6}$$

This equation is simple to solve using the algorithm that was already applied in Sections 2 and 3. Introducing the new variable

$$\tau = \int_0^{\eta} U(\eta_1)\, \mathrm{d}\eta_1 = l \ln \frac{l_1(z + l_2)}{l_2(z + l_1)}\,, \tag{4.7}$$

we can represent the solution of Eqn (4.6) in the form of direct and backward waves traveling along the τ axis:

$$u = \frac{A_{\mathrm{r}}\left[\exp(\mathrm{i}q\tau) + Q\exp(-\mathrm{i}q\tau)\right]}{\sqrt{U(\eta)}}\,. \tag{4.8}$$

The 'wave number' q in (4.8) corresponds to the waveguide-type dispersion in a gradient layer:

$$q = \frac{\omega}{v_0}\sqrt{1 - \frac{\Omega^2}{\omega^2}}\,, \qquad \Omega = \frac{v_0}{2l}\,. \tag{4.9}$$

The characteristic frequency Ω in (4.9) depends via the parameter l on the spatial scales l_1 and l_2 of variations of the density and elastic properties. Taking boundary conditions (3.1) and (3.2) at the barrier boundary $\eta = 0$ $(z = 0)$ into account, we find the expression for the reflection coefficient R. Unlike reflection coefficient (3.7) calculated for normal dispersion, R in the problem considered here corresponds to waveguide dispersion (4.9):

$$R = \frac{\mathrm{i}\alpha - \gamma/2 - \mathrm{i}N(1 - Q)(1 + Q)^{-1}}{\mathrm{i}\alpha + \gamma/2 + \mathrm{i}N(1 - Q)(1 + Q)^{-1}}\,, \tag{4.10}$$

$$N = \sqrt{1 - S^2}\,, \qquad S = \frac{\Omega}{\omega}\,. \tag{4.11}$$

But the parameter Q, which in (4.8) describes the contribution of the backward wave to the field inside the barrier $U(\eta)$, should be calculated again because this barrier, unlike that in (2.8), is asymmetric, $U(\eta = 0) \neq U(\eta_0)$, where the coordinate η_0 corresponds to the rear boundary of the barrier $z = d$:

$$\eta_0 = \eta(d) = \frac{d}{1 + d/l_1}\,, \qquad U_0 = U(\eta_0) = \frac{l_1(d + l_2)}{l_2(d + l_1)}\,. \tag{4.12}$$

Writing the coordinate τ in (4.7) corresponding to the rear boundary of the barrier as

$$\tau_0 = \tau(d) = l \ln U_0\,, \tag{4.13}$$

and using the relations

$$\frac{\mathrm{d}\eta}{\mathrm{d}z} = \frac{1}{W^2(z)}\,, \qquad \frac{\mathrm{d}\tau}{\mathrm{d}z} = \frac{F(z)}{W(z)}\,, \tag{4.14}$$

which follow from distributions (4.2), we write the conditions for the continuity of displacements and stresses at this boundary:

$$\frac{A_{\mathrm{r}}\left[\exp(\mathrm{i}q\tau_0) + Q\exp(-\mathrm{i}q\tau_0)\right]}{\sqrt{U_0}} = A_2\,, \tag{4.15}$$

$$\frac{A_{\mathrm{r}}}{\sqrt{U_0}}\left\{-\frac{\mathrm{i}\chi}{2}\left[\exp(\mathrm{i}q\tau_0) + Q\exp(-\mathrm{i}q\tau_0)\right] + N\left[\exp(\mathrm{i}q\tau_0) - Q\exp(-\mathrm{i}q\tau_0)\right]\right\} = \alpha\beta A_2\,, \tag{4.16}$$

$$\beta = \frac{W(d)}{F^3(d)} = \left(1 + \frac{d}{l_1}\right)\left(1 + \frac{d}{l_2}\right)^3\,. \tag{4.17}$$

Here, A_2 is the transmitted wave amplitude, Ω is the characteristic frequency defined in (4.9), and α is the ratio of impedances in (3.8). In boundary conditions (4.15) and (4.16), it is assumed for simplicity that the density and elastic

parameters of the media on the left and right of the barrier are the same.

Determining the parameter Q from the boundary conditions

$$Q = -\frac{\exp(2\mathrm{i}q\tau_0)(\chi/2 - \mathrm{i}\beta + \mathrm{i}N)}{\chi/2 - \mathrm{i}\beta - \mathrm{i}N}, \qquad \chi = \frac{v_0}{l\omega}, \tag{4.18}$$

and substituting this in (4.10), we find the reflection coefficient

$$R = \frac{t(\alpha\beta - \chi^2/4 - N^2) + \mathrm{i}\left[(\chi t/2)(\alpha+\beta) + N(\alpha-\beta)\right]}{t(\alpha\beta + \chi^2/4 + N^2) + \mathrm{i}\left[(\chi t/2)(\alpha-\beta) + N(\alpha+\beta)\right]}, \qquad t = \tan(q\tau_0), \qquad q\tau_0 = \frac{N}{2S}\ln U_0. \tag{4.19}$$

Expression (4.19) for R is written for positive parameters l in (4.5). It can be seen from (4.5) that a value $l > 0$ is possible for three density and shear modulus profiles (4.2):

$$\begin{aligned} &(1)\ l_1 > l_2,\ l_1 > 0,\ l_2 > 0;\\ &(2)\ l_1 > l_2,\ l_1 < 0,\ l_2 < 0;\\ &(3)\ l_1 < l_2,\ l_1 < 0,\ l_2 > 0. \end{aligned} \tag{4.20}$$

Each of the combinations 1–3 in (4.20) determining the value of β in (4.17) corresponds to its own reflection coefficient. The reflection coefficient can also be calculated by (4.19) for $l < 0$; in this case, combinations of parameters similar to (4.20) are also possible, the variable τ remains positive, and the parameter χ, according to (4.11), should be taken with the opposite sign, $\chi \to -\chi$. Finally, in the particular case where $l_1 = l_2$, by passing to the limit $l \to \infty$ in (4.5) and (4.20), we find

$$U \to 1, \quad \Omega \to 0, \quad \tau = \eta, \quad \chi = 0, \quad N = 1, \qquad q\tau_0 = \frac{\omega d}{v_0}\left(1 + \frac{d}{l_1}\right)^{-1}. \tag{4.21}$$

Substituting the quantities from (4.21) in (4.19), we obtain the reflection coefficient in this limit case. We note that the nonlocal barrier dispersion vanishes in this limit ($\Omega = 0$, $N = 1$).

For a specified thickness d of the 'double' barrier and the cut-off frequency Ω in (4.9), i.e., under the condition that one of the lengths l_1 or l_2 is chosen arbitrarily (with $l = \text{const}$), this possibility is of interest for optimization of the barrier parameters. We note that expression (4.9) encompasses different types of the combined influence of variations in the density and elastic properties, described by models (4.2), on the reflection of normally incident shear waves from a 'double' gradient layer. These models also allow finding a similar expression for the reflection coefficient of longitudinal waves for arbitrary positive and negative parameters l_1 and l_2.

4.2 Reflection spectra of shear waves in a gradient layer for consistent density and elasticity distributions

In this section, we consider an acoustic barrier in which variations of the density and elastic parameters, unlike in Section 4.1, are characterized by the same normalized distribution $F^2(z) = W^2(z)$. In this case, Eqn (4.1) reduces

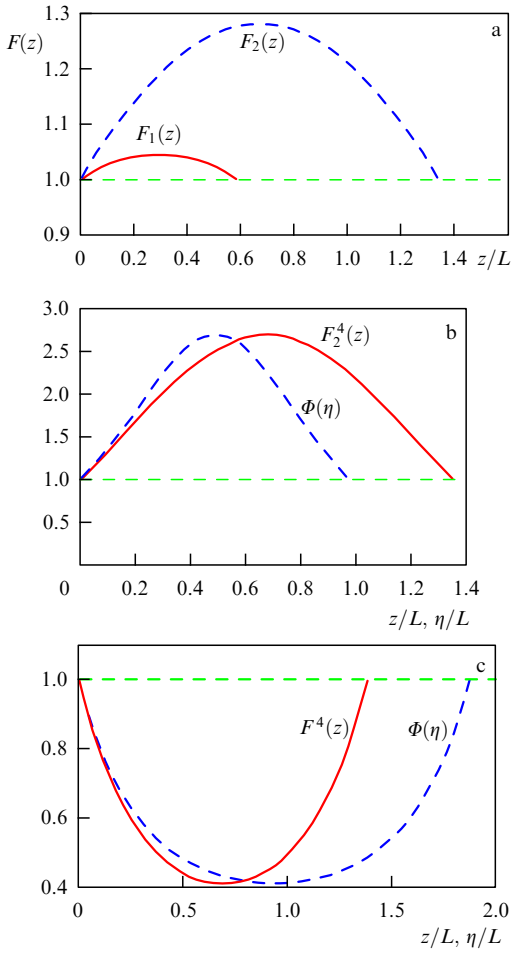

Figure 5. Gradient barriers formed by the 'consistent' density and shear modulus distributions $F^2(z)$ and $W^2(z)$ when $F^2(z) = W^2(z)$, with anomalous (4.23) and normal (4.31) nonlocal dispersions. (a) Plots of $F_1(z)$ and $F_2(z)$ corresponding to the parameters $M_1 = 0.3$ and $M_2 = 0.8$. (b) The auxiliary $\Phi(\eta) = F^2(z(\eta))$ and the main convex barriers for $M = 0.8$. (c) The auxiliary $\Phi(\eta)$ and the main $F^4(z(\eta))$ concave barriers for $M = 0.6$.

to the equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} + \frac{\omega^2}{v_0^2} F^4(z)\, u = 0. \tag{4.22}$$

Equation (4.22) can be conveniently solved by the auxiliary barrier method described in Section 3.2.

We first consider a convex profile $F(z)$ containing two free parameters: the characteristic length L and a dimensionless parameter M (Fig. 5a, different values of M),

$$F(z) = \cos\frac{z}{L} + M\sin\frac{z}{L}, \qquad 0 \leqslant \frac{z}{L} \leqslant \pi. \tag{4.23}$$

The value of η can be found by substituting the function $W^2 = F^2$ in (3.19):

$$\eta = \frac{Lt}{1 + Mt}, \qquad t = \tan\frac{z}{L}. \tag{4.24}$$

Using (4.23) and (4.24), we can express $F^2(z)$ in (4.23) in terms of η:

$$F^2(z) = \left[1 - \frac{2M}{L}\eta + \frac{1 + M^2}{L^2}\eta^2\right]^{-1}. \tag{4.25}$$

It is important that the function $F^2(z)$ written in form (4.25) coincides with the frequently used model $F(\eta)$ in (2.8) if we set $s_1 = -1$ and $s_2 = 1$ in (2.8) and find the characteristic lengths L_1 and L_2 and the parameter y of model (2.14) by comparing it with (4.25):

$$\frac{2M}{L} = \frac{1}{L_1}\,, \quad \frac{1+M^2}{L^2} = \frac{1}{L_2^2}\,, \quad y = \frac{L_2}{2L_1} = \frac{M}{\sqrt{1+M^2}} < 1\,. \tag{4.26}$$

The function $F^2(\eta)$ thus defined forms an auxiliary convex barrier (Fig. 5b) allowing the representation of Eqn (42) in the form coincident with (2.7):

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\eta^2} + \frac{\omega^2}{v_0^2}\, F^2(\eta)\, u = 0\,. \tag{4.27}$$

The maxima of the $F^4(z)$ and $F^2(\eta)$ barriers respectively located in the z and η spaces are equal to $F^2_{\max} = (1+M^2)^2$. The width d of symmetric barrier (4.25) in the z space, determined from the condition $F(0) = F(d) = 1$, is given by

$$d = L \arctan \frac{2M}{1-M^2}\,. \tag{4.28}$$

The width d_1 of the auxiliary barrier in the η space is determined from condition (2.10),

$$d_1 = 2yL_2 = \frac{2ML}{1+M^2}\,. \tag{4.29}$$

Comparing (4.28) and (4.29) shows that $d_1 < d$ for convex barriers (Fig. 5b).

By reducing Eqn (4.22) for the barrier to form (4.27) coincident with (2.7), we can use the solution of (2.7) to obtain the reflection coefficient of the gradient barrier given by the distribution $F^2(z) = W^2(z)$ in form (3.10), where the characteristic frequency Ω_1 and the phase shift $q\eta_0$ are given by

$$\begin{gathered} \Omega_1 = \frac{v_0}{d}\,\theta_1(M)\,, \qquad \theta_1(M) = \arctan\frac{2M}{1-M^2}\,, \\ q\eta_0 = \frac{\omega N_+ d}{v_0}\,, \qquad N_+ = \sqrt{1+S^2}\,, \qquad S = \frac{\Omega_1}{\omega}\,. \end{gathered} \tag{4.30}$$

The auxiliary barrier method also allows finding the reflection spectrum under the condition $F^2(z) = W^2(z)$ for a concave profile containing two free parameters L and M, as in (4.23):

$$F(z) = \cosh\frac{z}{L} - M\sinh\frac{z}{L} = W(z)\,. \tag{4.31}$$

Using the algorithm developed in (4.24)–(4.30) for a convex profile and substituting (4.31) in (3.19), we introduce the new variable

$$\eta = \frac{Lt}{1-Mt}\,, \qquad t = \tanh\frac{z}{L}\,. \tag{4.32}$$

Expressing the function $F^2(z)$ in terms of η, we obtain the concave profile of the auxiliary barrier

$$F^2(z) = \left(1 + \frac{2M}{L}\,\eta - \frac{1-M^2}{L^2}\,\eta^2\right)^{-1}. \tag{4.33}$$

Profile (4.33) coincides with model (2.8) if the characteristic lengths L_1 and L_2 and the parameter y of model (2.8) are defined by the expressions

$$\frac{2M}{L} = \frac{1}{L_1}\,, \quad \frac{1-M^2}{L^2} = \frac{1}{L_2^2}\,, \quad y = \frac{M}{\sqrt{1-M^2}}\,. \tag{4.34}$$

The widths d of barrier (4.31) and d_1 of auxiliary barrier (4.33) are

$$d = L\,\mathrm{artanh}\,\frac{2M}{1+M^2}\,, \qquad d_1 = \frac{2ML}{1-M^2}\,. \tag{4.35}$$

The minima of the barrier $F^2(z)$ in (4.31) and of auxiliary barrier (4.33) coincide $[F^2_{\min} = (1-M^2)^2]$, while the widths d and d_1, unlike those for a convex barrier, are related by the opposite inequality $d_1 > d$.

The reflection coefficient for concave profile (4.31) can be calculated from expression (3.14) found previously for the barrier $F^2(\eta)$, by using the expressions

$$\begin{gathered} \Omega_2 = \frac{v_0}{d}\,\theta_2(M)\,, \qquad \theta_2(M) = \mathrm{artanh}\,\frac{2M}{1+M^2}\,, \\ q\eta_0 = \frac{\omega N_- d}{v_0}\,, \qquad N_- = \sqrt{1-S^2}\,, \qquad S = \frac{\Omega_2}{\omega}\,. \end{gathered} \tag{4.36}$$

Gradient barriers formed by shear modulus distributions defined parametrically and the corresponding auxiliary barriers are presented in Fig. 4. Barriers (3.23) are characterized in Fig. 4a by anomalous dispersion, while barrier (3.33) in Fig. 4b has normal dispersion. The distances η and z along horizontal axes are respectively normalized to d_1 and d. Gradient barriers formed by the 'consistent' density $F^2(z)$ and shear modulus $W^2(z)$ distributions with $F^2(z) = W^2(z)$ and with anomalous (4.32) and normal (4.31) nonlocal dispersion are presented in Fig. 5. We note that the reflection spectra for these models are given by general expressions (3.10) and (3.14), while the phase shifts $q\eta_0 \propto N_\pm$ entering these expressions are determined by the characteristic frequencies $\Omega_{1,2}$ of nonlocal dispersion, which are described for barriers with convex and concave inhomogeneity profiles by similar expressions that differ only by geometric factors, for example, $\Omega_{1,2} = (v_0/d)\theta_{1,2}$. Notably, upon reflection from concave profiles (3.14) in the low-frequency region $\omega < \Omega_2$, $S > 1$, $N_-^2 < 0$, the phase shift $q\eta_0$ becomes imaginary. Peculiar effects appearing in gradient acoustics in this spectral range are considered in Section 5.

5. Sound tunneling in a nonlocal dispersion medium

Tunneling is a fundamental phenomenon in the dynamics of waves of different physical natures. The first steps in the investigation of this phenomenon were taken in optics in 1908, when Eikhenval'd showed theoretically [27] by solving the Maxwell equations that in the case of total internal reflection of light incident on the interface of two transparent media, the light field partially penetrates into the boundary region of the reflecting medium, decreasing exponentially at a distance of the order of the wavelength. This effect was confirmed experimentally by Mandelstam and Zeleni in 1910 [28]. The interest in tunneling effects increased after Gamow published his famous 1928 paper [29], where he explained the nuclear α decay by the tunneling of de Broglie waves describing the propagation of α particles through a potential barrier surrounding the atomic nucleus.

The probabilities of tunneling transitions of particles through potential barriers calculated according to the Gamow concept are used for solving many quantum mechanical problems [30]. Later, similar models were used in a number of radiophysical and electrodynamic problems of inhomogeneous plasmas [31].

But the models of homogeneous potential barriers describing an exponentially small transmission (a rectangular barrier and δ potential) used in these solutions could not be used for studying the efficient and, in particular, reflectionless energy transfer during the tunneling of waves through inhomogeneous potential barriers (see Fig. 2). Such processes, which have attracted recent attention to gradient optics [32], the electrodynamics of metamaterials [33], and the radiophysics of guiding systems [34], illustrate the generality of tunneling regimes for fields of different physical natures described by the wave equation.

This generality allows formulating the problem of sound tunneling, also described by the wave equation, through gradient acoustic barriers. It is known that the sound transmission spectra of homogeneous acoustic barriers with a waveguide dispersion (acoustic waveguides) are characterized by a cut-off frequency Ω_2 dividing the wave spectrum into two parts, traditionally called the transparency region ($\omega > \Omega_2$, $S < 1$, $N_-^2 > 0$) and the nontransparency region ($\omega < \Omega_2$, $S > 1$, $N_-^2 < 0$). In contrast, in gradient barriers, sound can propagate efficiently, sometimes totally, through a barrier in the nontransparency region. Such effects, which are caused by the interference of the forward and backward waves inside the barrier, can appear both for a certain profile of the sound speed inside an inhomogeneous barrier and inside a homogeneous barrier bounded by the walls of a certain profile. These physically different tunneling effects are considered in Sections 5.1 and 5.2.

5.1 Reflectionless acoustic tunneling through gradient wave barriers

In Sections 2–4, we considered applications of model equation (2.7) to a number of problems related to propagation of sound through gradient acoustic barriers formed by inhomogeneous distributions of the density and elastic parameters of a medium. In the case of the normal frequency dispersion of the barrier ('concave profile'), reflection spectrum (3.14) that we obtained is valid for high frequencies $\omega \geqslant \Omega_2$, where Ω_2 is the cut-off frequency determined by general expression (2.21) with the form factor θ_2 described by (2.21) or (3.37), depending on the problem geometry. In the low-frequency region $\omega < \Omega_2$, model equation (2.13) can also be used; but the spatial field structure inside the barrier is then described by nonperiodic solutions of (2.13); in this case, we can speak about sound tunneling. The simplest example of such tunneling is the propagation of a longitudinal wave incident on the $z = 0$ plane normally to gradient layer (2.8) with $s_1 = 1$ and $s_2 = -1$ (concave density profile). In this case, the solution of wave equation (2.13) is written in a form different from (3.4):

$$u = \frac{A_r\left[\exp(-p\eta) + Q\exp(p\eta)\right]}{\sqrt{F(z)}}\,, \qquad p = \frac{\omega}{v_0}N\,, \quad N = \sqrt{S^2 - 1}\,. \tag{5.1}$$

The subsequent analysis is performed according to the scheme in Section 3.2. The reflection coefficient R is determined from the boundary conditions on the $z = 0$ plane:

$$R = \frac{\mathrm{i}\alpha - \gamma/2 + N(1-Q)(1+Q)^{-1}}{\mathrm{i}\alpha + \gamma/2 - N(1-Q)(1+Q)^{-1}}\,. \tag{5.2}$$

Calculating Q from conditions on the rear boundary $z = d$ of the barrier and substituting in (5.2), we obtain

$$R = \frac{t(\alpha^2 + \gamma^2/4 + N^2) - \gamma N}{t(\alpha^2 - \gamma^2/4 - N^2) + \gamma N + 2\mathrm{i}\alpha(N - \gamma t/2)}\,, \tag{5.3}$$

$$\gamma = \frac{2Sy}{\sqrt{1+y^2}}\,, \quad t = \tanh(p\eta_0)\,, \quad p\eta_0 = \sqrt{1 - S^{-2}}\,Y\,, \tag{5.4}$$

$$Y = \ln\frac{y_+}{y_-}\,, \qquad y_\pm = \sqrt{1+y^2} \pm y\,. \tag{5.5}$$

If the wave frequency is equal to the cut-off frequency ($S = 1$), the reflection coefficient is obtained from (5.3) by passing to the limit $S \to 1$:

$$R\big|_{S\to 1} = \frac{Y\left[\alpha^2 + y^2/(1+y^2)\right] - 2y/\sqrt{1+y^2}}{Y\left[\alpha^2 - y^2/(1+y^2)\right] + 2y/\sqrt{1+y^2} + 2\mathrm{i}\alpha(1 - yY/\sqrt{1+y^2})}\,. \tag{5.6}$$

Finding $|R|^2$ from (5.3), we can obtain the energy transmission coefficient $|T|^2 = 1 - |R|^2$ of the barrier:

$$|T|^2 = \frac{4\alpha^2 N^2(1-t^2)}{\left[t(\alpha^2 - \gamma^2/4 - N^2) + \gamma N\right]^2 + 4\alpha^2(N - \gamma t/2)^2}\,. \tag{5.7}$$

The plots of $|T(S)|^2$ are shown in Fig. 6. The transmission coefficient T can also be written in complex form $T = |T|\exp(\mathrm{i}\phi_t)$, where the phase ϕ_t of the wave that tunnels through the barrier is given by

$$\tan\phi_t = \frac{t(\alpha^2 - \gamma^2/4 - N^2) + \gamma N}{2\alpha(N - \gamma t/2)}\,. \tag{5.8}$$

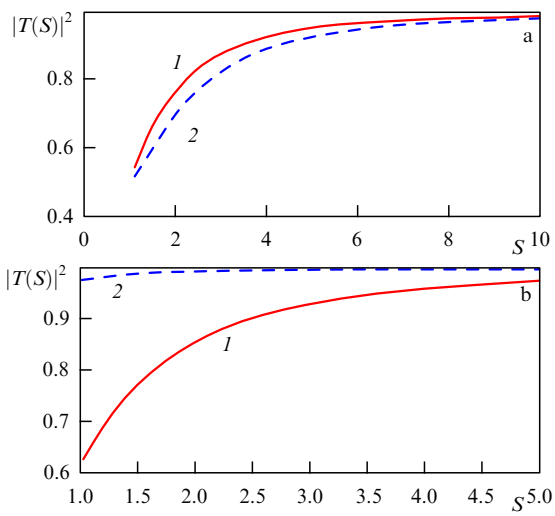

Figure 6. Transmission spectra $|T(S)|^2$ in (5.7) for shear waves tunneling through a single-layer gradient barrier formed by a density distribution: (a) $\alpha = 0.3$, curves *1* and *2* correspond to $y = 0.45$ and 0.7; (b) $y = 0.3$, curves *1* and *2* are respectively constructed for $\alpha = 0.3$ and 1.25.

The transmission coefficient for the cut-off frequency can be obtained from (5.7) by passing to the limit $S \to 1$:

$$|T|^2\Big|_{S\to 1} = 4\alpha^2\left\{\left[Y\left(\alpha^2 - \frac{y^2}{1+y^2}\right) + \frac{2y}{\sqrt{1+y^2}}\right]^2 + 4\alpha^2\left(1 - \frac{yY}{\sqrt{1+y^2}}\right)^2\right\}^{-1}. \quad (5.9)$$

The reflection coefficient for a system of m adjacent identical barriers is obtained by successively using boundary conditions for each interface between two adjacent barriers (3.1) and (3.2). The calculations again lead to expressions (5.4) and (5.7), generalized with the help of the substitution

$$t \to t_m = \tanh(mp\eta_0)\,, \qquad mp\eta_0 = mY\sqrt{1 - S^{-2}}\,. \quad (5.10)$$

In expressions (5.6)–(5.8) corresponding to the limit $S \to 1$, the combined effect of m identical barriers is taken into account by the replacement $Y \to mY$.

A fundamentally important effect inherent in gradient acoustic barriers with waveguide dispersion is the possibility of reflectionless tunneling at some frequency $\omega_0 \leqslant \Omega_2$ of the wave $(R(\omega_0) = 0,\ |T(\omega_0)|^2 = 1)$. The condition for the occurrence of this regime can be found from (5.3) and (5.9):

$$\tanh\left(mY\sqrt{1 - S^{-2}}\right) = \frac{\gamma N}{\alpha^2 + \gamma^2/4 + N^2}\,. \quad (5.11)$$

The spectra of the waves tunneling through the system of gradient acoustic barriers (5.7)–(5.9) depend on the ratio α of the impedances of the barrier and surrounding medium, on the geometric parameter γ, and on the number m of barriers. These dependences are shown in Fig. 7 for $m = 2$. It follows from Figs 6 and 7 that the transmission of single barriers monotonically increases with decreasing the frequency. But the influence of the second barrier (see Fig. 7) complicates the interference structure of the field and leads to the formation of the transmission maximum $|T|^2 = 1$ and a decrease in transmission to zero with increasing the frequency. We note the appearance of transparency windows with a finite spectral width in the tunneling regime, which correspond to high transmission coefficients. For example, according to Fig. 7a, $|T|^2$ reaches values $|T|^2 > 0.9$ in the spectral range $1.22 < S < 1.48$.

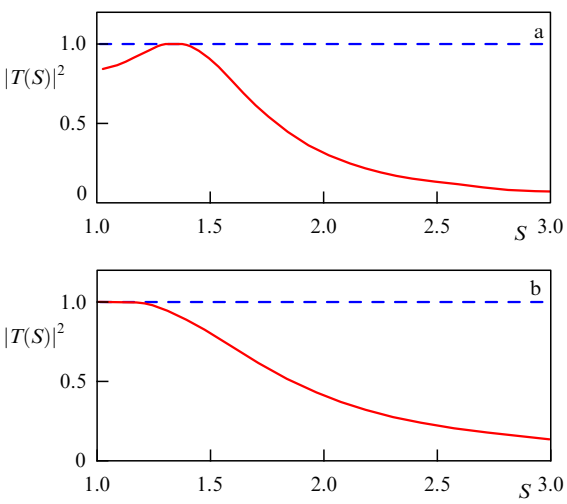

Figure 7. Transmission spectra $|T(S)|^2$ for shear waves tunneling through a composite gradient barrier formed by two identical density distributions (5.11), $m = 2$, $y = 0.577$; (a) $\alpha = 0.2925$; (b) $\alpha = 0.4515$. At the points $S = 1.32$ in Figs 7a and $S = 1$ in Fig. 7b, reflectionless tunneling is attained.

5.2 Longitudinal sound tunneling through a homogeneous medium in a channel with a variable cross section

The description of sound propagation in tubes, funnels, and concentrators often involves problems related to acoustic processes in waveguide systems with a variable cross section $F(z)$, where z is the coordinate measured along the system axis. The peculiarities of such processes can be found by analyzing a simple problem of longitudinal sound propagation in the irregular waveguide shown in Fig. 8: a tube with a cylindrical cross section has a narrowing in the region $0 \leqslant z \leqslant d$ (region II); the cross-sectional area in this region changes continuously according to a law $F(z)$; for $z \leqslant 0$ and $z \geqslant d$ (regions I and III), the tube cross section F_0 is constant, $F(0) = F(d) = F_0$. Region II is separated from regions I and III by thin soundproof membranes located in the $z = 0$ and $z = d$ planes; all the regions are filled with a continuous medium characterized by a density ρ_1 and longitudinal speed of sound v_1 (constant-cross-section regions I and III) and ρ_0 and v_0 (variable-cross-section region II). For a convenient comparison of the results obtained in this section with those in Section 4.1, the parameters of region II are indicated by subscripts 0.

If the characteristic radius of the tube is large compared to the wavelength $(r_0(z) \gg \lambda)$ and the tube cross section changes sufficiently slowly $(\mathrm{d}r_0/\mathrm{d}z \ll 1)$, then the propagation of a longitudinal sound wave along the z axis of this system is described by the Webster equation [35]

$$\frac{\partial^2 P}{\partial z^2} + \frac{\partial}{\partial z}\left(\ln F\,\frac{\partial P}{\partial z}\right) - \frac{1}{v_0^2}\,\frac{\partial^2 P}{\partial t^2} = 0\,, \quad (5.12)$$

where $P(z, t)$ is the acoustic pressure. Effects described by this equation, which was proposed as early as 1919, are still being analyzed [36–38]. The effects of nonlocal dispersion and tunneling of sound through a variable-cross-section region can also be studied using the Webster equation. To find an exact analytic solution of this equation for a monochromatic wave with frequency ω, we introduce a new function $f(z)$ instead of pressure:

$$P(z, t) = \frac{f(z)}{\sqrt{F(z)}}\exp(-\mathrm{i}\omega t)\,. \quad (5.13)$$

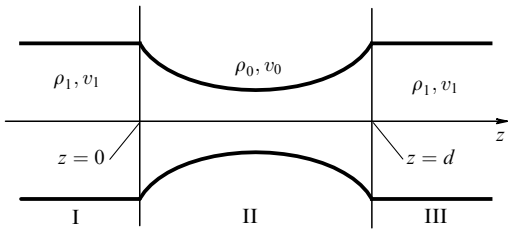

Figure 8. Narrowing in a channel filled with different liquids allows the reflectionless tunneling of longitudinal sound through region II that has a constant liquid density and is bounded by curvilinear walls.

Substituting (5.13) in (5.12), we obtain the equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + f\left[\frac{\omega^2}{v_0^2} - \frac{1}{2}\frac{\mathrm{d}^2 F}{\mathrm{d}z^2} + \frac{1}{4F^2}\left(\frac{\mathrm{d}F}{\mathrm{d}z}\right)^2\right] = 0\,, \tag{5.14}$$

which does not contain the first derivative of the unknown function. The function $F(z)$ describing a change in the tube cross section in (5.14) is not yet defined here.

We consider a symmetric concave profile $F(z)$ reducing Eqn (5.14) to the wave equation with constant coefficients:

$$\begin{gathered} F(z) = F_{\mathrm{m}} \cosh^2\left[\left(\frac{2z}{d} - 1\right)A\right], \qquad F(0) = F(d) = 1\,, \\ F\left(\frac{d}{2}\right) = F_{\mathrm{m}} < 1\,, \qquad A = \operatorname{arcosh}\frac{1}{\sqrt{F_{\mathrm{m}}}}\,. \end{gathered} \tag{5.15}$$

Here, d is the length of the narrowed part and F_{m} is the minimal cross-sectional area. Substituting this $F(z)$ in (5.14), we obtain the simple wave equation for the function $f(z)$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \frac{\omega^2}{v_0^2} N_-^2 f = 0\,, \quad N_-^2 = 1 - S^2\,, \quad S = \frac{\Omega}{\omega}\,, \tag{5.16}$$

$$\Omega = \frac{v_0}{d}\,\theta\,, \quad \theta = 2\operatorname{arcosh}\frac{1}{\sqrt{F_{\mathrm{m}}}} = 2\ln\frac{1+\sqrt{1-F_{\mathrm{m}}}}{\sqrt{F_{\mathrm{m}}}}\,. \tag{5.17}$$

Here, Ω is the characteristic frequency for the narrowed region. Equation (5.16) resembles the wave equation for a waveguide or plasma, with Ω playing the role of the cut-off frequency dividing the spectral regions of propagating ($\omega > \Omega$) and tunneling ($\omega < \Omega$) waves. It is important in this consideration that the medium in region II representing an acoustic barrier is assumed homogeneous and nondispersive, while the cut-off frequency is caused by nonlocal dispersion depending on the geometric parameters, the barrier thickness d and the minimal narrowed area F_{m}.

We consider the tunneling of longitudinal sound through region II ($0 \leqslant z \leqslant d$). The pressure P in this region can be represented with the help of the forward and backward waves that are solutions of Eqn (5.16):

$$\begin{gathered} P_2 = \frac{A_{\mathrm{r}}\left[\exp(-pz) + Q\exp(pz)\right]}{\sqrt{F(z)}}\,, \\ p = \frac{\omega}{v_0}N\,, \quad N = \sqrt{S^2 - 1}\,. \end{gathered} \tag{5.18}$$

To simplify the notation, the factor $\exp(-\mathrm{i}\omega t)$ is omitted hereafter. The pressure P and velocity V of the medium in a sound wave field are related by the equations

$$\rho\frac{\partial V}{\partial t} = -\frac{\partial P}{\partial z}\,, \qquad V = -\frac{\mathrm{i}}{\omega\rho}\frac{\mathrm{d}P}{\mathrm{d}z}\,. \tag{5.19}$$

Substituting the pressure distribution P_2 from (5.18) in (5.19), we find the velocity distribution of the medium in region II:

$$\begin{aligned} V_2 = \frac{\mathrm{i}A_2}{v_0\rho_0}\frac{1}{\sqrt{F}}\Big\{&\frac{F_z}{2F}\frac{v_0}{\omega}\left[\exp(-pz) + Q\exp(pz)\right] \\ &+ N\left[\exp(-pz) - Q\exp(pz)\right]\Big\}\,, \qquad F_z = \frac{\mathrm{d}F}{\mathrm{d}z}\,. \end{aligned} \tag{5.20}$$

The pressure and velocity distributions in sound fields in regions in front of ($z \leqslant 0$) and behind ($z \geqslant d$) the narrowing are described by

$$\begin{gathered} P_1 = A_1\left[\exp(\mathrm{i}k_0 z) + R\exp(-\mathrm{i}k_0 z)\right]\,, \\ V_1 = \frac{A_1}{v_1\rho_1}\left[\exp(\mathrm{i}k_0 z) - R\exp(-\mathrm{i}k_0 z)\right]\,, \end{gathered} \tag{5.21}$$

$$P_3 = A_3\exp\left[\mathrm{i}k_0(z-d)\right]\,, \qquad V_3 = \frac{A_3}{v_1\rho_1}\exp\left[\mathrm{i}k_0(z-d)\right]\,. \tag{5.22}$$

Here, A_1, A_2, and A_3 are the respective amplitudes of the incident, tunneling, and transmitted waves, and the parameter Q in (5.20) characterizes the contribution of the backward wave in the sound field inside the narrowing.

The reflection coefficient for sound waves in region II can be calculated by using the continuity conditions for pressure and velocity at the region boundaries $z = 0$ and $z = d$. The ratio $F_z/(2F)$ in (5.20) is determined at the boundaries of region II as

$$\left.\frac{F_z}{2F}\right|_{z=0} = -\frac{2A}{d}\sqrt{1-F_{\mathrm{m}}} = -\left.\frac{F_z}{2F}\right|_{z=d}\,. \tag{5.23}$$

Using the values of pressure and velocity on both sides of the boundary $z = 0$ given in (5.18), (5.20), and (5.21), we can write the continuity conditions at this boundary relating the complex reflection coefficient R and the parameter Q:

$$A_1(1+R) = A_2(1+Q)\,, \tag{5.24}$$

$$\frac{\mathrm{i}A_1(1-R)}{v_1\rho_1} = \frac{A_2}{v_0\rho_0}\left[S\sqrt{1-F_{\mathrm{m}}}\,(1+Q) - N(1-Q)\right]\,. \tag{5.25}$$

The reflection coefficient R found from system (5.24), (5.25) is

$$R = \frac{\mathrm{i}\alpha_1 - S\sqrt{1-F_{\mathrm{m}}} + N(1-Q)(1+Q)^{-1}}{\mathrm{i}\alpha_1 + S\sqrt{1-F_{\mathrm{m}}} - N(1-Q)(1+Q)^{-1}}\,, \tag{5.26}$$

where α_1 is the impedance ratio described, unlike α in (3.8), by the expression

$$\alpha_1 = \frac{1}{\alpha} = \frac{\rho_0 v_0}{\rho_1 v_1}\,. \tag{5.27}$$

The parameter Q can be found from boundary conditions at the boundary of region II ($z = d$):

$$Q = -\exp(-2pd)\,\frac{\alpha_1 - \mathrm{i}S\sqrt{1-F_{\mathrm{m}}} - \mathrm{i}N}{\alpha_1 - \mathrm{i}S\sqrt{1-F_{\mathrm{m}}} + \mathrm{i}N}\,. \tag{5.28}$$

Substituting Q (5.28) in (5.26), we obtain the reflection coefficient

$$\begin{aligned} R = &\left\{t\left[\alpha_1^2 + S^2(1-F_{\mathrm{m}}) + N^2\right] - 2SN\sqrt{1-F_{\mathrm{m}}}\right\} \\ &\times\Big\{t\left[\alpha_1^2 - S^2(1-F_{\mathrm{m}}) - N^2\right] \\ &+ 2SN\sqrt{1-F_{\mathrm{m}}} + 2\mathrm{i}\alpha_1\left(N - S\sqrt{1-F_{\mathrm{m}}}\,t\right)\Big\}^{-1}\,, \end{aligned} \tag{5.29}$$

$$t = \tanh(pd)\,, \qquad pd = \frac{2NA}{S}\,. \tag{5.30}$$

Expressions (5.29) and (5.30) demonstrate a peculiar effect of the similarity of gradient wave barriers of different physical natures. These expressions describe the reflection of sound during tunneling through a flat homogeneous layer

bounded by a curvilinear boundary $F(z)$ in (5.15). The transmission of waves through region II determined by the expression $|T|^2 = 1 - |R|^2$ depends on the geometric parameters $F_{\rm m}$ and A in (5.15). On the other hand, the substitutions

$$1 - F_{\rm m} = \frac{y^2}{1+y^2}\,, \qquad 2A = \ln\frac{y_+}{y_-}\,, \qquad \alpha_1 = \alpha^{-1} \tag{5.31}$$

made in (5.29) and (5.30) transform these expressions into relations (5.3)–(5.5) describing another physical situation of sound tunneling through a gradient barrier separating homogeneous media. If a waveguide has several identical adjacent narrowings ($m > 1$), the reflection coefficient should be calculated by making replacement (5.10) in (5.30).

With relations (5.31), we can use the reflection spectra calculated for sound tunneling through gradient barriers to analyze the tunneling of longitudinal acoustic waves through narrowings in waveguides. For example, the spectrum in Fig. 6 ($y = 0.3$, $\alpha = 1.25$, $m = 1$) also describes the tunneling effect for an irregular waveguide (see Fig. 8) with the narrowing area $F_{\rm m} = 0.917$, $\alpha_1 = 0.8$, and $A = 0.59$. In this case, identical values of the normalized frequency S correspond to the same transmission $|T|^2$. This equality remains valid for all sound frequencies ω and the parameters of the narrowed region, the velocity v_0 and thickness d, connected by the phase relation

$$\frac{\omega d}{v_0} = \frac{2A}{S}\,. \tag{5.32}$$

A similar comparison for $y = 0.577$, $\alpha_1 = 0.2925$, and $m = 2$ (see Fig. 7) shows, for example, that transparency windows can appear during reflectionless tunneling ($|T|^2 = 1$, $S = 1.28$) through an irregular acoustic waveguide (see Fig. 8) with two identical narrowings and the parameters $F_{\rm m} = 0.75$, $\alpha_1 = 3.42$, and $A = 1.099$. This effect appears at all frequencies satisfying condition (5.32). This similarity again illustrates the analogy between tunneling processes for waves of different physical natures.

6. Gradient elements of phonon crystals

The increasing recent interest in the physics and technology of phonon crystals is related to the development of new-generation control systems for acoustic wave fluxes [39, 40]. As in the manufacturing of photonic crystals for optics and radiophysics, these acoustic developments involve artificial inhomogeneous and composite materials that are not encountered under natural conditions. By continuing this analogy, we can point out that along with the case of waves normally incident on a gradient layer discussed in Sections 3–5, the propagation of waves along the surface of a gradient medium is also of interest for a number of structures. In this case, we are dealing with surface waves in gradient structures. Below, we briefly consider some properties of gradient acoustic barriers (Section 6.1) and periodic structures consisting of such barriers (Section 6.2), which are of interest for the manufacturing of phonon crystals.

6.1 Subwavelength acoustic barriers

The reflection spectra of different gradient acoustic barriers considered in Sections 3–5 demonstrate some general properties of these barriers produced by density or elastic parameter inhomogeneities or by the combined effect of these inhomogeneities.

(1) The reflection and transmission coefficients for sound propagating through a homogeneous nonabsorbing layer depend in the case of normal incidence only on the layer thickness d and the acoustic impedance contrast $I = \rho v$ inside and outside the layer. By contrast, in the case of incidence on a gradient layer, the reflected wave is formed due to the interference of waves reflected with their own values of the amplitude and phase at each point inside the layer. A thin layer with a thickness smaller than the wavelength can then make a certain contribution to the reflected wave structure. The thickness of such layers can be estimated from reflection spectra $R(S)$ (see, e.g., Figs 3 and 10). Using the values of the dimensionless frequency $S = \Omega/\omega$ on the abscissas of these plots and expressions for the characteristic frequencies Ω_1 in (2.14) and Ω_2 in (2.21), we find the ratio

$$\frac{\lambda}{d} = \frac{2\pi S}{\theta}\,, \tag{6.1}$$

where, according to Section 3.1, the form factors are $\theta = \theta_1(y) = 2y\sqrt{1-y^2}$ or $\theta = \theta_2(y) = 2y\sqrt{1+y^2}$ for the density inhomogeneity shown in Fig. 1. The form factors $\theta_1(y)$ in (3.31) and $\theta_2(y)$ in (3.37) in the case of shear modulus inhomogeneities (see Fig. 5) are given in Fig. 9. Substitution of these values in (6.1) shows that the thickness d has subwave values for different wavelengths: $d \approx (0.2-0.3)\lambda$. Such dimensions are promising for manufacturing miniature phonon crystals.

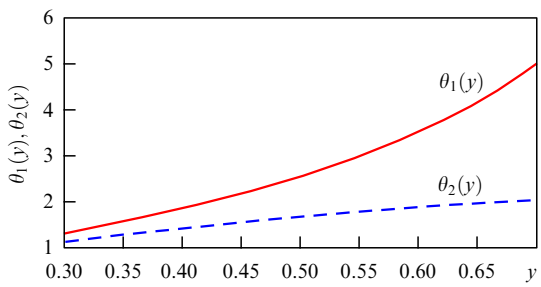

Figure 9. Form factors $\theta_1(y)$ in (3.31) and $\theta_2(y)$ in (3.37) for gradient barriers formed by the shear modulus inhomogeneity.

(2) The transmission spectra for waves in the tunneling regime for some sets of parameters are of interest for the manufacturing of efficient gradient acoustic reflectors in the specified frequency interval. Using Fig. 7, we can plot the reflection coefficient $|R(\delta)|^2 = 1 - |T(\delta)|^2$ (Fig. 10). It follows from Fig. 10 that the reflection coefficient of a barrier consisting of two identical adjacent layers ($m = 2$) with a thickness d each, which are characterized by concave density profile (2.8) and the value $y^2 = 1/3$ and are surrounded by an elastic medium (the impedance ratio is α), vanishes for $\delta = 1.32$; in this case, the form factor is $\theta = \theta_2 = 2y\sqrt{1+y^2}$, Eqn (2.21). We now consider a configuration in which this gradient barrier is replaced by a homogeneous layer that similarly to the gradient barrier has the thickness $2d$ at the same values of the frequency ω, velocity v_0, and impedance ratio α. Rewriting expression (6.1) for the gradient barrier in the form

$$\frac{\omega d}{v_0} = \frac{\theta_2}{S}\,, \tag{6.2}$$

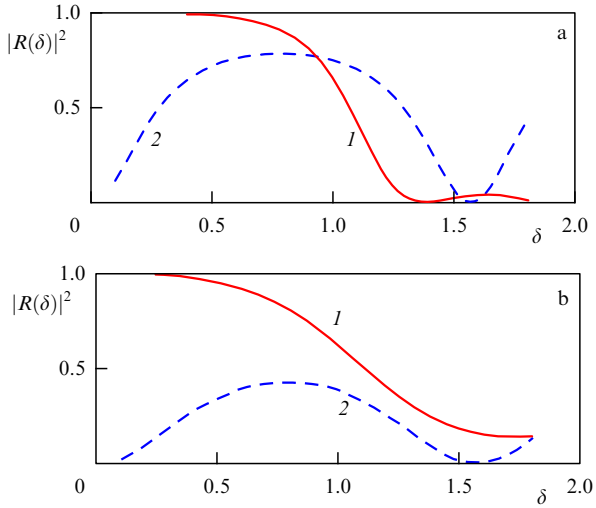

Figure 10. Reflection coefficient of a barrier with a thickness $2d$ as a function of the dimensionless frequency $\delta = \omega d/v_0$: (a) $\alpha = 0.25$; (b) $\alpha = 0.4595$. Curve *1*: the reflection coefficient of a composite gradient barrier formed by two identical layers with a thickness d, Eqns (5.7) and (5.9), $m = 2$, $y = 0.75$; curve *2*: reflection coefficient (3.38) of a homogeneous barrier with the thickness $2d$. Reflection from the gradient barrier in some regions of δ is stronger than that from the homogeneous barrier with the same thickness and the same velocity v_0 and parameter α.

we find the reflection coefficient $|R|^2$ of the given homogeneous layer for comparison. Substituting the parameter

$$\delta = \frac{2\omega d}{v_0} = \frac{4y\sqrt{1+y^2}}{S} \tag{6.3}$$

in expression (3.38) describing reflection from such a layer with thickness $2d$, we calculate the reflection coefficients $|R|^2$ for the values corresponding to Fig. 8 ($y^2 = 1/3, \alpha = 0.2925$).

It can be seen from Fig. 10 that the reflection coefficient of the gradient barrier in the low-frequency region considerably exceeds that of a homogeneous layer with the same parameters. Such a dispersion of the gradient barrier is of interest for the manufacturing of acoustic filters and selectively reflecting surfaces.

(3) In the case of certain relations between parameters of the gradient barrier, its nonlocal dispersion can vanish. For example, for a simple barrier formed by the convex density profile $F^2(z)$, the characteristic frequency Ω_1 in (2.14) vanishes for $y = 1$, i.e., $L_2 = 2L_1$; in this case, the wave number q is given by the standard expression $q = \omega/v_0$.

A more complicated case is described by Eqn (4.2). In this case, the propagation of a wave through a 'double' barrier depends on the density distribution $F^2(z)$ and the shear modulus distribution $W^2(z)$. In the particular case $F^2(z)W^2(z) = 1$, the solution of (4.2) has the form

$$u = \exp\frac{\mathrm{i}\omega\eta}{v_0} + Q\exp\left(-\frac{\mathrm{i}\omega\eta}{v_0}\right). \tag{6.4}$$

For the density profile $F(z) = \cos(z/L) + M\sin(z/L)$ in (4.23) instead of (4.24), we obtain the expression

$$\eta = \frac{L\tan(z/L)}{1 + M\tan(z/L)} \tag{6.5}$$

for the variable η in (6.4). The wave described by (6.4) and (6.5) propagates in the 'double' barrier $F^2(z)W^2(z) = 1$ with a variable phase velocity but without dispersion distortions.

6.2 Periodic gradient structures

Acoustic metamaterials with a periodic structure are often simulated as composite media consisting of isotropic matrices with periodic inclusions of concentrated masses, elastic elements (springs), and cavities with different shapes and volumes [41]. By contrast, in this section, we point out the possibility of producing periodic structures considered in Sections 3–5 from gradient acoustic barriers providing controllable reflection and transmission of wave fluxes in the specified spectral range. Traditional multilayer structures manufactured for this purpose contain alternating layers of materials with different sound speeds $v_{\mathrm{l,t}}$ and different thicknesses d. Gradient layers forming a periodic structure can differ not only in $v_{\mathrm{l,t}}$ and d but also in the density and elastic parameter distributions $F^2(z)$ and $W^2(z)$ inside each layer. The reflection and transmission spectra of such a periodic structure related to the discontinuities of the gradient and curvature of $F^2(z)$ and $W^2(z)$ at the layer boundaries can considerably differ from the corresponding spectra of a single barrier.

For example, we consider a periodic structure composed of m identical adjacent barriers with variable density (2.8) and anomalous geometric dispersion (Fig. 11a). The densities of any adjacent barriers coincide on their common boundary $z = d$ ($F^2(d) = 1$), while the density gradients experience a discontinuity at this boundary determined from (2.8). This discontinuity on crossing the boundary along the wave propagation direction is

$$\operatorname{grad} F^2\big|_{z=d-0} - \operatorname{grad} F^2\big|_{z=d+0} = -\frac{8y^2}{d}. \tag{6.6}$$

Using the continuity condition at the interface between two adjacent layers and assigning the number $m = 1$ to the extreme layer on the distant side of the structure (along the wave propagation direction), we can obtain a recursive relation for the parameter Ω_m describing the contribution of the backward wave in the mth layer,

$$Q_m = \exp\left[2\mathrm{i}(m-1)q\eta_0\right]Q_0. \tag{6.7}$$

Here, Q_0 is equal to Q defined in (3.9). Substituting Q_m given by (6.7) instead of Q in (3.9) shows that expression (3.10) for

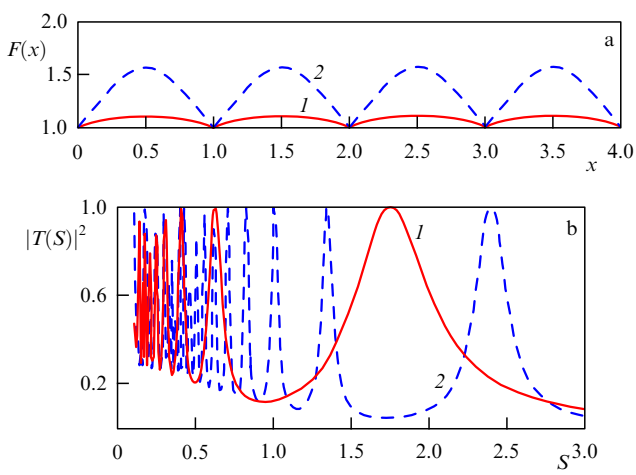

Figure 11. (a) Periodic structure made of four convex gradient barriers (curve *1* corresponds to $y = 0.3$, curve *2* to $y = 0.6$). (b) Transmission spectra of a periodic structure consisting of three (curve *1*) and five (curve *2*) gradient barriers for $\alpha = 0.2925$ and $y^2 = 1/3$.

the reflection coefficient of a single barrier with anomalous dispersion can be generalized to the case of reflection from a periodic system of m such barriers (see Fig. 11) by replacing

$$\tan(q\eta_0) \to \tan(mq\eta_0) \tag{6.8}$$

in (3.10). It follows from Fig. 11b that as the number m of gradient barriers increases, the number of frequencies ω at which the wave completely passes through a periodic structure with subwave inhomogeneities increases.

A periodic structure composed of m gradient barriers (2.8) with normal dispersion [a concave profile $F^2(z)$] is characterized by a different discontinuity of grad F^2, which is equal to $8y^2/d$. In this case, replacement (6.8) in expression (3.14) obtained for a single barrier with normal dispersion leads to an expression for the reflection coefficient of the periodic structure consisting of m such barriers. The reflection spectra of such gradient structures in Fig. 3 show that unlike the transmission bands of structures composed of homogeneous layers, the transmission bands of gradient periodic structures with boundaries defined by the condition $R = 0$ have different widths in different parts of the spectrum.

The complex reflection coefficient for a wave tunneling through a structure containing m barriers is obtained by the similar substitution

$$\tanh(p\eta_0) \to \tanh(mp_0) \tag{6.9}$$

in expressions (5.4) and (5.8). The transmission spectra of waves tunneling through such structures are presented in Fig. 7.

Figures 3 and 7 demonstrate a strong dependence of the amplitudes of waves propagating through gradient periodic structures on the number m of barriers even for small m. Such a dependence, especially in the wave tunneling region, in conjunction with the subwave thickness of gradient acoustic barriers is of interest for the creation of subwave phonon crystal elements.

7. Conclusions

The dynamics of wave processes in gradient media involve a number of effects common for waves of different natures. A typical example of such a generality is the above-mentioned tunneling of waves in electrodynamics, acoustics, and quantum mechanics. In this connection, we also note the analogy between the 'double' gradient acoustic barrier (see Section 4.1) and a superhigh-frequency transmission line with distributed parameters. It is known [42] that the distribution of the current I and voltage V in a transmission line without losses with distributed parameters is described by the system of equations

$$\frac{\partial V}{\partial z} + L(z)\,\frac{\partial I}{\partial t} = 0\,, \qquad \frac{\partial I}{\partial z} + C(z)\,\frac{\partial V}{\partial t} = 0\,, \tag{7.1}$$

where $L(z) = L_0 F^2(z)$ and $C(z) = C_0 W^{-2}(z)$ are the self-induction and capacity distributions per transmission line unit length, which depend on the coordinate z along the line. From system (7.1), introducing the generating function Ψ such that

$$V = \frac{W^2(z)}{C_0}\,\frac{\partial \Psi}{\partial z}\,, \qquad I = -\frac{\partial \Psi}{\partial t}\,, \tag{7.2}$$

we obtain Eqn (2.3) derived for the description of a displacement in a gradient layer with the density and shear modulus distributions $F^2(z)$ and $W^2(z)$ across the layer. This analogy is of interest for the simulation of complex acoustic fields in inhomogeneous media with the help of a radio engineering transmission line.

To explain the physics of the interaction of acoustic fluxes with gradient barriers, we here used the simplest one-dimensional model of this interaction, assuming the normal incidence of the flux, simple laws of nonlocal dispersion, and exactly solvable models for plane waves. We note that some fundamentally important problems remain beyond the framework of this model.

(I) One such problem is the problem of point sources and acoustic beams of nonplane waves and momenta in inhomogeneous media. One of the few exact results not restricted by the assumptions of one-dimensional plane waves and homogeneous media describes the refraction of an acoustic flux at the interface of two liquids with densities ρ_1 and ρ_2 [43],

$$\rho_1 \tan\phi_1 = \rho_2 \tan\phi_2\,, \tag{7.3}$$

where ϕ_1 and ϕ_2 are the angles between the flux vectors and the normal to the interface between the liquids. Some exactly solvable models of two-dimensional and three-dimensional wave beams are discussed in monograph [44]. But the physical concepts of nonlocal dispersion and tunneling for such wave-field configurations have not been developed yet.

(II) In this review, we have considered the problems of wave propagation in gradient media in the direction of density change (grad F^2) or of the elastic parameter change (grad W^2). We note that the different geometry of gradient acoustic problems corresponding to the propagation of waves perpendicular to grad F^2 and grad W^2 is also a 'hot' problem having numerous applications in radiophysics and electronics. This field configuration is involved in the studies of surface waves in gradient media [45]. Recently, surface wave fields in gradient media characterized by their own inhomogeneity scale have attracted special interest. Information on the spatial structure and spectral properties of such fields is finding numerous applications, both for probing natural geophysical media [46] and for manufacturing artificial materials for the absorption and conversion of acoustic waves [47].

(III) Another problem is the 'acoustic mask.' Advances in the theory of gradient dielectric layers covering an opaque target, capturing the incident electromagnetic wave, and minimizing scattering by this target ('invisible target') [48] stimulated corresponding developments in acoustics [49]. For example, physical foundations for such an 'acoustic mask' were developed in [50] in the particular case of a cylindrical body on a water surface. The model of an acoustic metamaterial with anisotropic density and elasticity consisting of springs with different masses and rigidities was proposed in [51]. However, this promising theory, which does not repeat the 'transformation optics' developed for the corresponding electrodynamic problems [52], is now only taking its first steps. We also note that the further development of the resonance tunneling of waves through gradient barriers in acoustics will involve the study of the role of nonlinear effects, some of which have been investigated in monograph [53]. In addition, it is very important to consider the influence of large-scale flows on the resonance tunneling of waves in the atmosphere and ocean. For example, in the

presence of such flows in the atmosphere, the propagation of internal gravitational waves (IGWs) from the troposphere zones of crisis processes such as hurricanes and earthquakes, to ionosphere altitudes followed by the generation of indicators and precursors of these crisis phenomena in the ionosphere is possible only for horizontal lengths of the IGW modes no less than ≈ 30 km [54].

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References

1. Slawinski M A *Seismic Waves and Rays in Elastic Media* (Amsterdam: Pergamon, 2003)
2. Zaman F D, Al-Muhiameed Z I A *Appl. Acoust.* **61** 427 (2000)
3. Taroyan V, Erdélyi R *Proc. Int. Astron. Union* **3** 86 (2007)
4. Sheng P et al. *Physica B* **338** 201 (2003)
5. Milton G W, Briane M, Willis J R *New J. Phys.* **8** 248 (2006)
6. Chen H, Chan C T *Appl. Phys. Lett.* **91** 183518 (2007)
7. Lee S-J et al. *Appl. Phys. Lett.* **82** 2133 (2003)
8. Ben-Amoz M *Z. Angew. Math. Phys.* **27** 83 (1976)
9. Chakraborty A *J. Acoust. Soc. Am.* **123** 56 (2008)
10. Shvartsburg A B, Kuzmiak V, Petite G *Phys. Rep.* **452** 33 (2007)
11. Landau L D, Lifshitz E M *Teoriya Uprugosti* (Theory of Elasticity) (Moscow: Nauka, 1987) [Translated into English (Oxford: Pergamon Press, 1986)]
12. Mindlin R D *Int. J. Solids Struct.* **1** 417 (1965)
13. Erofeyev V I *Wave Processes in Solids with Microstructure* (River Edge, NJ: World Scientific, 2003)
14. Bennett T, Gitman I M, Askes H *Int. J. Fract.* **148** 185 (2007)
15. Vavva M G et al. *J. Acoust. Soc. Am.* **125** 3414 (2009)
16. Kravtsov Yu A *Akust. Zh.* **14** 1 (1968) [*Sov. Phys. Acoust.* **14** 1 (1968)]
17. Brekhovskikh L M, Godin O A *Akustika Sloistykh Sred* (Acoustics of Layered Media, I, II) (Moscow: Nauka, 1989) [Translated into English (Berlin: Springer-Verlag, 1990, 1992)]
18. Strutt J W (Baron Rayleigh) *The Theory of Sounds* (London: Macmillan and Co., 1937) [Translated into Russian (Moscow: GITTL, 1940)]
19. Godin O A, Chapman D M F *J. Acoust. Soc. Am.* **106** 2367 (1999)
20. Miropol'skii Yu Z *Dinamika Vnutrennikh Gravitatsionnykh Voln v Okeane* (Dynamics of Internal Gravitational Waves in Ocean) (Liningrad: Gidrometeoizdat, 1981)
21. Talipova T G, Pelinovsky E N, Petrukhin N S *Okeanologiya* **49** 673 (2009) [*Oceanology* **49** 622 (2009)]
22. Gossard E E, Hooke W H *Waves in the Atmosphere* (Amsterdam: Elsevier, 1975) [Translated into Russiqn (Moscow: Mir, 1978)]
23. Schoenberg M, Sen P N *J. Acoust. Soc. Am.* **73** 61 (1985)
24. Artan R, Altan B S *Int. J. Solids Struct.* **39** 5927 (2002)
25. Guck J et al. *Phys. Rev. Lett.* **84** 5451 (2000)
26. Aleshin V, Gusev V, Tournat V *J. Acoust. Soc. Am.* **121** 2600 (2007)
27. Eikhenwald A A *Zh. Russk. Fiz.-Khim. Obshch. Ch. Fiz.* **41** 131 (1909)
28. Mandelstam L I, Zeleni P *Ann. Phys.* **31** 58 (1910)
29. Gamow G *Z. Phys.* **51** 204 (1928)
30. Feynman R P, Hibbs A R *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill, 1965)
31. Mikhailovskii A M *Elektromagnitnye Neustoichivosti Neodnorodnoi Plazmy* (Electromagnetic Instabilities in an Inhomogeneous Plasma) (Moscow: Eneroatomizdat, 1991) [Translated into English (Bristol: IOP Publ., 1992)]
32. Shvartsburg A B, Petite G *Opt. Lett.* **31** 1127 (2006)
33. Shalaev V M *Nature Photon.* **1** 41 (2007)
34. Shvartsburg A B et al. *Phys. Rev. E* **78** 016601 (2008)
35. Eisner E *J. Acoust. Soc. Am.* **41** 1126 (1967)
36. Webster A G *J. Audio Eng. Soc.* **25** 24 (1977)
37. Forbes B J *Phys. Rev. E* **72** 016627 (2005)
38. Rudenko O V, Shvartsburg A B *Akust. Zh.* (2010) (in press)
39. Chen H, Chan C T *Appl. Phys. Lett.* **91** 183518 (2007)
40. Bliokh K Yu, Bliokh Yu P *Usp. Fiz. Nauk* **174** 439 (2004) [*Phys. Usp.* **47** 393 (2004)]
41. Cummer S A et al. *Phys. Rev. Lett.* **100** 024301 (2008)
42. Jackson J D *Classical Electrodynamics* 3rd ed. (New York: Wiley, 1999) [Translated into Russian (Moscow: Mir, 1965)]
43. Godin O A *J. Acoust. Soc. Am.* **125** EL117 (2009)
44. Brekhovskikh L M, Godin O A *Acoustics of Layered Media II: Point Sources and Bounded Beams* 2nd ed. (Berlin: Springer, 1999)
45. Biryukov S V et al. *Poverkhostnye Akusticheskie Volny v Neodnorodnykh Sredakh* (Surface Acoustic Waves in Inhomogeneous Media) (Moscow: Nauka, 1991)
46. Chen C-H et al. *Phys. Rev. Lett.* **103** 238101 (2009)
47. Sluys L J, de Brost R, Mühlhaus H B *Int. J. Solids Struct.* **30** 1153 (1993)
48. Dubinov A E, Mytareva L A *Usp. Fiz. Nauk* **180** 475 (2010) [*Phys. Usp.* **53** 455 (2010)]
49. Norris A N *Proc. R. Soc. London A* **464** 2411 (2008)
50. Farhat M et al. *Phys. Rev. Lett.* **101** 134501 (2008)
51. Milton G W *New J. Phys.* **9** 359 (2007)
52. Leonhardt U *Science* **312** 1777 (2006)
53. Gurbatov S N, Rudenko O V, Saichev A I *Volny i Struktury v Nelineinykh Sredakh bez Dispersii* (Waves and Structures in Nonlinear Media without Dispersion) (Moscow: Fizmatlit, 2008)
54. Erokhin N S, Mikhailovskaya L A, Shalimov S L *Geofiz. Issled.* (7) 53 (2007)