METHODOLOGICAL NOTES

PACS numbers: 01.40. - d, 03.30. + p

DOI: 10.3367/UFNe.0181.201105g.0553

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A rigorous minimum-assumption derivation of the Lorentz transformation

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Abstract. The available derivations of the Lorentz transformation (LT) are of questionable validity because they introduce some assumptions in addition to Einstein's two fundamental postulates or, even if they do not do so, are highly abstract and abstruse (as is the case with two or three 'exact' derivations). The rigorous LT derivation proposed in this paper has only the constant speed of light and two thought experiments as its underlying assumptions. With the constant speed of light used to prove all the necessary assumptions, no additional assumptions are needed. Our systematic approach explains in a convincing way why stress is irrelevant to length contraction.

1. Introduction

Einstein established his remarkable special theory of relativity in 1905, deriving the Lorentz transformation (LT), which is the central part of his theory, from two simple postulates. However, Einstein made additional assumptions in deriving the LT [1, 2]. Therefore, his derivation is very difficult to accept absolutely, and various derivations of the LT have been published [3–7]. Nonetheless, in the existing derivations, the additional assumptions are not entirely abandoned. We list Einstein's two postulates and common additional assumptions.

• Postulate of relativity: The laws of physics are the same in all inertial frames of reference.

• Postulate of the constancy of the speed of light: The speed of light in the vacuum is the same in all inertial frames of reference and is independent of the motion of the source.

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Received 13 September 2010 Uspekhi Fizicheskikh Nauk 181 (5) 553-556 (2011) DOI: 10.3367/UFNr.0181.201105g.0553 Translated by M I Zel'nikov; edited by A M Semikhatov

• Assumption of the invariance of the relative velocity between two inertial coordinates. If Mary sees Henry is moving away from her with a constant velocity u, then Henry sees Mary moving away from him with the same velocity u.

• Assumption that lengths measured perpendicular to the relative motion of two frames do not contract; i.e., y and zcoordinates transform in a Galilean manner.

• Assumption that the coordinate transformation relations are linear.

• Assumption that space-time is homogeneous and isotropic.

Physicists studying the derivation of the LT are still attempting to eliminate the additional assumptions [8]. In particular, the first two assumptions are often used, that the relative velocity between two inertial systems is the same and the v and z coordinates transform in a Galilean manner. Some researchers and teachers believe that the arguments for the two assumptions are questionable, even if they may seem to be common sense. The invariance of the relative velocity between two inertial coordinates is an obvious case. Because neither the unit length nor the unit time is directly comparable in two systems, the relative velocities of the two systems might not be the same [9].

Here, we derive the LT with the fewest assumptions or postulates possible. Indeed, we use fundamental postulates to exactly prove the necessary assumptions. First, the invariance of the relative velocity between two inertial coordinates is exactly proved using the constancy of the speed of light and a thought experiment. Second, on the basis of the above straightforward method and results, length contraction and time dilation are directly derived. Finally, the LT is exactly formulated without uncalled-for assumptions. Our derivation of the LT is perhaps one of the most accurate derivations.

2. Proof of the invariance of the relative velocity between two inertial frames

Let K and K' be two inertial reference frames. The coordinate axes of the two fames are parallel. K' moves with a constant speed u relative to K in the positive direction along the common x-x' axis, as viewed from K. For simplicity, let the origins O and O' coincide at the initial instant t = t' = 0.

To prove the invariance of the relative velocity between two inertial frames, we consider a thought experiment. If a light source at rest at the origin O in K is flashed on and off rapidly at t = t' = 0, the constancy of the speed of light implies that observers in both K and K' see the wavefront moving outward from the respective origins with the speed c. When the wavefront arrives at the two points A and B on the x axis at a time t in K, points A', M', O', and B' on the x' axis coincide with respective points A, O, C, and B on the x axis, as seen from K. This is shown in Fig. 1, in which K' is at rest relative to K.



Figure 1. In *K*, the wavefront arrives at two points A and B on the *x* axis at the instant *t*.

Our first question is: when we know that K' moves with a constant speed u relative to K in the positive x-direction, as observed in K, how do we calculate the velocity u' of K relative to K', as seen from K'? To solve the problem, we let \overline{AB} , \overline{AO} , \overline{OC} , \overline{OB} , and \overline{CB} be the distances between the respective pairs of points in K. According to the geometry relations in Fig. 1, we obtain

$$\overline{\mathbf{AB}} = 2\overline{\mathbf{AO}} = 2\overline{\mathbf{OB}} = 2ct, \qquad (1)$$

$$\overline{\text{OC}} = ut = \frac{u\overline{\text{AB}}}{2c} \,, \tag{2}$$

where *c* is the speed of light in the vacuum.

We let $\overline{A'B'}$, $\overline{A'M'}$, $\overline{M'B'}$, and $\overline{M'O'}$ be the distances between the respective pairs of points in K'. In Fig. 1, they must be the *proper lengths* in K' because observers are at rest in K. Therefore, their moving lengths, as measured in K at the instant t, are respectively \overline{AB} , \overline{AO} , \overline{OB} , and \overline{OC} . Next, we let $\overline{OB} = \alpha \overline{M'B'}$, where α is an arbitrary factor. Because the points A' and M' are moving with the constant speed urelative to K, M' must move from O to B when A' moves from A to O. Therefore, the moving length of $\overline{A'M'}$ is also \overline{OB} . Because the moving length measured is independent of time, we still have $\overline{OB} = \alpha \overline{A'M'}$. Hence, $\overline{AO} = \alpha \overline{A'M'}$ and $AB = \alpha A'B'$. Therefore, the arbitrary factor α is independent of the proper length; that is, proportionality relations between any proper length in K' and the moving length in Kmust have a common linear factor α . This is basically an essential conclusion of the uniform linear motion. Thus, we have

$$\overline{AO} = \alpha \overline{A'M'}, \quad \overline{OB} = \alpha \overline{M'B'}, \quad \overline{AC} = \alpha \overline{A'O'},$$

$$\overline{CB} = \alpha \overline{O'B'}, \quad \overline{OC} = \alpha \overline{M'O'}, \quad \overline{AB} = \alpha \overline{A'B'}.$$
(3)

We next change the perspective. Observers at rest in K' see that the wavefront is emitted from the origin O' at t' = 0 and

moves to B' at time t'(B'). Similarly, the wavefront arrives at A' at the instant t'(A'). According to the constancy of the speed of light and Eqns (1)–(3), we can calculate t'(B') and t'(A'):

$$t'(\mathbf{B}') = \frac{\overline{\mathbf{O}'\mathbf{B}'}}{c} = \frac{(c-u)t}{\alpha c}, \qquad (4)$$

$$t'(\mathbf{A}') = \frac{\overline{\mathbf{A}'\mathbf{O}'}}{c} = \frac{(c+u)t}{\alpha c} \,. \tag{5}$$

Because $\overline{A'O'}$ is greater than $\overline{O'B'}$, t'(A') is greater than t'(B'). This is also the famous relativity of simultaneity. Combining the above equations, we have

$$t'(\mathbf{A}') - t'(\mathbf{B}') = \frac{2ut}{\alpha c} = \frac{u}{c^2} \overline{\mathbf{A}'\mathbf{B}'}.$$
 (6)

Expression (6) allows us to draw an important conclusion: the physical meaning of the expression is that we can calculate the time interval between any two events in K' when the two events occur at the two points in K simultaneously. In Fig. 1, because M' coincides with O and B' coincides with B at the instant t simultaneously in K, the time interval between the two points M' and B' is t'(M') - t'(B'), as measured in K'. In the same way, we have t'(O') - t'(B'). From expression (6), we obtain

$$t'(\mathbf{M}') - t'(\mathbf{B}') = \frac{u}{c^2} \overline{\mathbf{M}'\mathbf{B}'} = \frac{u}{c^2} \overline{\frac{\mathbf{OB}}{\alpha}} = \frac{ut}{\alpha c}, \qquad (7)$$

$$t'(\mathbf{O}') - t'(\mathbf{B}') = \frac{u}{c^2} \,\overline{\mathbf{O}'\mathbf{B}'} = \frac{(c-u)ut}{\alpha c^2} \,. \tag{8}$$

Solving the above equations for t'(M') and t'(O'), we have

$$t'(\mathbf{M}') = \frac{t}{\alpha}, \quad t'(\mathbf{O}') = \frac{1 - u^2/c^2}{\alpha} t.$$
 (9)

We next calculate the velocity u' at which K moves relative to K' in the negative direction along the common x-x' axis, as measured in K'. Because every point in K moves with a constant speed u' relative to K' in the negative x' direction, we only calculate the speed at which the origin O moves. The observers at rest in K' see that O coincides with O' when the flash is emitted from O' at t' = 0 and O coincides with M' at t'(M') in K'. Hence, the moving time interval of the origin O from O' to M' is t'(M'), and the moving distance of the origin O is $\overline{O'M'}$, as measured in K'. Therefore, the moving velocity of the origin O relative to K' must be $u' = \overline{O'M'}/t'(M')$. Combining Eqns (2), (3), and (9), we find

$$u' = \frac{\overline{\mathcal{O}'\mathcal{M}'}}{t'(\mathcal{M}')} = \frac{ut}{\alpha} \frac{\alpha}{t} = u.$$
(10)

We have thus proved the conclusion that the relative velocity between two inertial frames is invariant.

3. Derivation of length contraction and time dilation

To find the common proportionality factor α , we consider that observers at rest in K' see the wavefront arriving at B' at an instant t'(B'), as shown in Fig. 2. Our question is how to determine the position of A in K' at an instant t'(B'). Because A and A' coincide at t' = t'(A') in K', point A and a point E' between the two points A' and B' coincide at the instant



Figure 2. In K', the wavefront is observed at B' at the instant t'(B').

 $t' = t'(\mathbf{B}')$. In addition, the distance between A' and E' is $\overline{\mathbf{A}'\mathbf{E}'} = u[t'(\mathbf{A}') - t'(\mathbf{B}')]$.

From expression (6), we obtain

$$\overline{\mathbf{A}'\mathbf{E}'} = \frac{u^2}{c^2} \,\overline{\mathbf{A}'\mathbf{B}'}\,.\tag{11}$$

In Fig. 2, because the observers at rest in K' measure the length between the moving points A and B at the same instant, the distance between A and B is the proper length in K. Hence, $\overline{E'B'}$ is the moving length of \overline{AB} , as seen in K'. Because u' = u, the proportionality factor between the moving and proper lengths, as measured in K', is the same as the above α factor, as measured in K. Therefore, we have $\overline{E'B'} = \alpha \overline{AB}$. According to the geometric relations in Fig. 2, we have $\overline{A'E'} + \overline{E'B'} = \overline{A'B'}$. Combining the above equations, we obtain

$$\frac{u^2}{c^2} \overline{\mathbf{A}'\mathbf{B}'} + \alpha \overline{\mathbf{A}\mathbf{B}} = \overline{\mathbf{A}'\mathbf{B}'}.$$
 (12)

Because the scale factor α cannot be negative, solving equation (12) for α , we find

$$\alpha = \sqrt{1 - \frac{u^2}{c^2}} \,. \tag{13}$$

Ultimately, we have proved the length contraction effect; i.e., $\overline{AB} = \alpha \overline{A'B'}$. Next, we express the time dilation effect on the basis of Eqns (9) and (13); that is, $t'(O') = \alpha t$. Therefore, the length contraction effect and the time dilation effect have been derived without any superfluous assumptions.

4. Derivation of the Lorentz transformation

To derive the Lorentz coordinate transformation, we refer to Fig. 3, in which the initial conditions are the same as in Fig. 1. For simplicity, we first consider the one-dimensional case. We assume the time and space coordinates of an arbitrary point on the common x - x' axis to be P(x, t) and P'(x', t') in K and K', respectively. Thus, P(x, t) and P'(x', t') coincide at an instant t, and the distance from O to N at the instant t is ut, as seen from K. The coordinate x' from O' to P' is a proper length in K'; hence, in K, the distance from N to P is $\alpha x'$. Therefore, the distance from O to P is x:

$$x = ut + \alpha x'. \tag{14}$$

Solving the equation for x', we obtain the first equation of the Lorentz coordinate transformation:

$$x' = \frac{x - ut}{\alpha} \,. \tag{15}$$



As seen from K at the instant a

Figure 3. In K, an event occurs at a point on the common x - x' axis at the instant t.

Because O' and P' respectively coincide with N and P at the instant t, as seen from K, the difference between t'(O') and t'(P') = t' is defined by Eqn (6):

$$t'(O') - t' = \frac{u}{c^2} O'P' = \frac{u}{c^2} x'.$$
 (16)

Considering $t'(O') = \alpha t$ and Eqn (15) and solving Eqn (16) for t', we obtain the second equation of the Lorentz coordinate transformation:

$$t' = \frac{t - ux/c^2}{\alpha} \,. \tag{17}$$

Finally, we consider how the transformation relations between y and y' and between z and z' can be derived from the constancy of the speed of light. We consider another thought experiment. As we discussed previously in relation to Fig. 1, we suppose that a wavefront is emitted from O' at t' = 0 and moves in the positive direction along the y' axis, as viewed from K'. After the wavefront travels a distance y', it is reflected by a mirror and returns to O' at an instant t'. Hence, the wavefront moves the total distance 2y' and the time interval is t'. Therefore, according to the constancy of the speed of light, we find

$$y' = \frac{ct'}{2} \,. \tag{18}$$

The same propagation of the wavefront may also be considered relative to K, in which case the constancy of the speed of light must also be satisfied. The round-trip time measured in K is a different interval t. In K, the wavefront leaves and returns at different points on the x axis. During the time t, the origin O' moves a distance ut relative to K. The distance between the mirror and origin O is y when the two origins O' and O coincide. Therefore, according to the constancy of the speed of light, we have

$$\sqrt{y^2 + \left(\frac{ut}{2}\right)^2} = \frac{ct}{2} \,. \tag{19}$$

Substituting Eqn (18) and $t' = \alpha t$ in Eqn (19), we find

$$y' = y. (20)$$

In the same way, we obtain the transformation relation between z and z':

$$z' = z \,. \tag{21}$$

We have thus exactly derived the LT. Equations (15), (17), (20), and (21) are the direct Lorentz coordinate transforma-

tion. The so-called inverse Lorentz coordinate transformation can be found from the direct LT by straightforward calculation.

5. Conclusion

The theory of relativity is the greatest theory in physics, and the LT is at its heart. But the theory of relativity is sometimes criticized. One doubt is that too many assumptions are made, especially in the derivation of the LT. To address these concerns, researchers have studied the derivation of the LT without using uncalled-for assumptions. However, no one has successfully solved the problem. We derived the LT exactly.

We note that we have not made any assumptions except for the constancy of the speed of light and uniform linear motion in the above derivation of the LT. Otherwise, our derivation is simple and clear.

Our approach to the derivation of the length contraction has revealed its intrinsic properties; that is, when we see a few points on a moving rod simultaneously, we see each of the points has a different time coordinate in terms of the intrinsic time of the rod. Therefore, although we see that distance contractions occur between these points, there is no stress effect because they are considered to have various intrinsic times. In other words, no stress is involved in length contraction. Figuratively, we can visualize yesterday's right hand touching today's left hand.

Our systematic approach will be important in the study, application, and teaching of the special theory of relativity.

The author expresses his gratitude to Prof. Wang Qiang for the useful discussions.

References

- 1. Einstein A *Relativity: The Special and General Theory* (New York: Methuen, 1916) p. 71
- 2. Einstein A *The Meaning of Relativity* (Princeton: Princeton Univ. Press, 1922) p. 24
- Jackson J D Classical Electrodynamics 3rd ed. (New York: John Wiley, 1999) Sect. 11.3
- 4. Bergmann P G "The special theory of relativity", in *Encyclopedia of Physics* Vol. IV (Ed. S Flügge) (Berlin: Springer-Verlag, 1962) p. 109
- 5. Bergmann P G *Introduction to the Theory of Relativity* (London: Butterworths, 1958) p. 33
- Denisov A A, Teplitsky E Sh Usp. Fiz. Nauk 176 857 (2006) [Phys. Usp. 49 831 (2006)]
- 7. Hassani S Eur. J. Phys. 29 (1) 103 (2008)
- 8. Escalona H, Franco R J A J. Vectorial Relat. 2 (4) 15 (2007)
- 9. Fock V A *The Theory of Space, Time and Gravitation* 2nd ed. (Oxford: Pergamon Press, 1964) p. 30