Physics – Uspekhi **54** (5) 519 – 528 (2011)

METHODOLOGICAL NOTES PACS numbers: 03.65.Ta, **03.75.–b**, 07.79.Fc

Measurability of quantum fields and the energy–time uncertainty relation

M B Mensky

DOI: 10.3367/UFNe.0181.201105f.0543

Contents

Abstract. Quantum restrictions on the measurability of an electromagnetic field strength and their relevance to the energy – time uncertainty relation are considered. The minimum errors in measuring electromagnetic field strengths, as they were estimated by the author (1988) in the framework of the phenomenological method of restricted path integral (RPI), are compared with the analogous estimates found by Landau and Peierls (1931) and by Bohr and Rosenfeld (1933) with the help of certain measurement setups. RPI-based restrictions, including those of Landau and Peierls as a special case, hold for any measuring schemes meeting the strict definition of measurement. Their fundamental nature is confirmed by the fact that their associated field detectability condition has the form of the energy – time uncertainty relation. The weaker restrictions suggested by Bohr and Rosenfeld rely on an extended definition of measurement. The energy – time uncertainty relation, which is the condition for the electromagnetic field to be detectable, is applied to the analysis of how the near-field scanning microscope works.

1. Introduction

The most familiar difference between quantum mechanics and classical mechanics resides in the Heisenberg relation $\Delta q \Delta p \geqslant \hbar/2$, which imposes limitations on the measurability of the position and momentum of an elementary particle.[1] When applied to photons, this relation imposes quantum limitations on the measurability of some electromagnetic field characteristics and on the measurability limits of those facilities (for instance, optical microscopes) in which photons are used as microsensors of the measuring equipment.

In some cases, however, attempts to apply the Heisenberg relation to the quantum analysis of measurements involving electromagnetic or other fields do not meet with success. In particular, this is the case when measurements are made of the field strength and the dimension of the measurement region and the measurement time are extremely small. Other approaches have been proposed for the quantum analysis of such measurements.

In 1931, Landau and Peierls [3] proposed formulas for the minimal error in measuring electromagnetic field strength, which, in the authors' view, were absolute in the sense that they might not be overcome for the same geometrical characteristics of the spatio-temporal measurement domain. In 1933, employing a different measuring scheme, Bohr and Rosenfeld [4] derived other formulas which yielded milder limitations on the measurability of the field strength. According to these formulas, the measurement error may be made arbitrarily small by increasing the amount of charges and currents of the probe bodies. The absolute character of the limitations obtained by Landau and Peierls (LP) was thereby questioned by Bohr and Rosenfeld (BR).

In 1988, the author of the present paper considered quantum limitations on the measurability of electromagnetic field strengths using the restricted path integral (RPI) method and derived formulas which were consistent with the LP

M B Mensky P N Lebedev Physical Institute,
Russian Academy of Sciences,
Leninskii prosp. 53, 119991 Moscow, Russian Federation
Tel. (7-499) 132 67 43
Fax (7-495) 938 22 51
E-mail: mensky@lpi.ru

Received 24 August 2010, revised 6 December 2010
Uspekhi Fizicheskikh Nauk **181** (5) 543 – 552 (2011)
DOI: 10.3367/UFNr.0181.201105f.0543
Translated by E N Ragozin; edited by A Radzig

[1] This relation always holds true for the uncertainties of the coordinate and the momentum as characteristics of the particle state; however, when it is interpreted as the quantum limitation on a measurement (when the coordinate is measured accurate to Δq, the momentum acquires an uncertainty $\hbar/2\Delta p$, and vice versa), it may be violated under a specially selected measurement mode [1, 2].

formulas and generalized them [5, 6]. The phenomenological RPI method does not rely on some specific measuring scheme, and the conclusions drawn from this method are therefore true for any field strength measurements that correspond to the definition of measurement adopted under this approach. It turned out that the minimal errors in field strength measurements derived by the RPI method depend only on the volume of the spatio-temporal domain of the spacetime in which the measurement is made. In this sense, the resultant limitations on measurability are absolute.

There emerges an apparent contradiction between the BR formulas, on the one hand, and, on the other hand, the LP formulas and those which were obtained by the author using the RPI method. The scientific community supposedly adopted the conclusion made by Bohr and Rosenfeld and thought that the conclusions made by Landau and Peierls were erroneous. In our view, there is no error in the paper by LP as regards the limits of electromagnetic field measurability.[2] Different estimates of the limitations on field measurability, which have been obtained in a variety of papers, emerge because the definitions of the measurement notion itself are different in these papers.

The rigorous definition of measurement, which underlies the RPI method, was also adopted in the measurement model applied in the LP paper. This definition implies that the quantity being measured does not go beyond the limits of the measurement error anywhere in the spatio-temporal measurement domain. In the paper by BR, use was made of the broad interpretation of the very notion of measurement, which implies the possibility of significant perturbation of the measurable quantity in the measurement domain, provided that these perturbations in a sense compensate for each other in such a way that they do not make a large contribution to the value of this quantity averaged over the measurement domain. By taking advantage of such definition of measurement, Bohr and Rosenfeld derived limitations on the field measurability, which turned out to be milder than those derived by Landau and Peierls and those obtained later by the RPI method. Furthermore, the limitations on the field measurability obtained by Bohr and Rosenfeld are not inherently absolute (may be made arbitrarily mild by selecting the probe bodies).

As will be shown in the present paper, there is one more indication that the limitations obtained by LP and borne out (and generalized) by the author are inherently fundamental. The point is that the formulas for the minimal field strength measurement error derived in these works are linked to the energy–time uncertainty relation in a special way. Specifically, the expression for the minimal detectable field strengths, which follows from these formulas, turns out to be equivalent to the energy–time uncertainty relation.

By way of illustration, the energy–time uncertainty relation is employed to analyze the measurement procedure realized in the so-called near-field scanning microscope (NFSM). Unlike an ordinary optical microscope, the resolution of an NFSM is better than that imposed by the diffraction limit, i.e., the uncertainty in determining the position in this microscope is smaller than the wavelength of light utilized therein. Of significance in the NFSM are the processes occurring in the near-field zone, i.e., in the region around the probe with dimensions much smaller than the wavelength. It makes no sense to represent the field in this zone as an assemblage of photons, and conventional reasoning involving the application of the Heisenberg uncertainty relation to photons cannot be employed here to analyze the processes occurring in this zone. Instead, we may represent the process in the NFSM as the detection of the near field and apply to this process the limitations on electromagnetic field detectability.

[2] It is another matter that a statement contained in the paper by LP turned out to be incorrect, namely the statement that the relativistic generalization of the quantum theory is impossible.

2. Restricted path integral (RPI) method

The phenomenological restricted path integral (RPI) method introduced by the author of Ref. [7], as applied to the problem of quantum field measurability, is based on the following simple considerations.

The dynamic properties of a free quantum field in a given spatio-temporal domain are described by the integral of the Feynman amplitude $\exp[(\mathrm{i}/\hbar)\,S(\sigma)]$ over all possible field configurations σ in this domain. The Feynman amplitude, in turn, depends on the field configuration σ and is defined as the imaginary exponential of the classical action $S(\sigma)$ of this field for its given configuration. This integral over the field configurations is analogous to the Feynman path integral in quantum mechanics and is quite often also referred to as the path integral.

The integral over all field configurations σ in a given spatio-temporal domain Ω defines the field dynamics only when the information about the true field configuration in this domain is fundamentally inaccessible. But when *the field* in the Ω domain *is measured*, which yields certain information about the state of this field in the given domain, *the integral should be limited* to the set Σ of such configurations σ which are compatible with the measurement result.

For instance, when the field strength is measured in the Ω domain, integration should be performed only over such configurations which are characterized by the strength values obtained in the measurement. In view of the finite measurement accuracy, the measurement result is characterized not by some specific strength value but by some interval of strength values.

In the general case, such a field strength interval should be specified at each point of the Ω domain. Only those configurations are then included in the restricted path integral, which are characterized at each point by the strength falling into the specified interval. For simplicity, it may be assumed that the interval of strength values is the same at each point of the Ω domain. For a relatively small domain Ω, this simplification is quite sufficient for obtaining estimates that are correct to an order of magnitude.

Therefore, the result of field measurement in the Ω domain is described by some family Σ of field configurations σ in this domain. By definition, Σ is the set of field configurations which are 'compatible with the measurement result'. Under the conditions when the field is measured, its dynamics are described by an integral $I(\Sigma)$ which is similar to the Feynman path integral, with integration taken not over all field configurations but only over the limited set Σ of configurations σ. This integral is termed the restricted path integral (RPI).[3]

[3] It is technically more convenient and more realistic (but equivalent for obtaining order-of-magnitude estimates) to define the restricted path integral by introducing under the Feynman path integral sign a cut-off functional which rapidly decreases beyond the Σ set. For the description of this technique and other details of RPI application, see the monograph [7].

The restricted path integral makes it possible not only to describe the field dynamics under measurement conditions, but also to find the probability distributions for different field measurement results by estimating the square of the modulus, $P(\Sigma) = |I(\Sigma)|^2$, of each of the integrals $I(\Sigma)$. Most probable are those Σ measurement results for which the values of $|I(\Sigma)|^2$ are highest. When $|I(\Sigma)|^2$ is much smaller than the peak values, the corresponding Σ measurement result appears with a very low probability.

This brings up the question: How broad or how narrow is the set of those Σ measurement results for which the probability of their occurrence is high enough? The answer to this question gives an estimate for the measurement error: the narrower the indicated set, the more accurate the measurement defines field configuration, i.e., the smaller the uncertainty of measurements.

The restricted path integral method [5, 6, 8, 9] shows that the minimal possible uncertainty of strength measurement for a quantum electromagnetic field depends on the 4-dimensional volume of the spatio-temporal measurement domain Ω. By the order of magnitude, this minimal uncertainty is estimated as

$$\Delta E_{\min} \sim \sqrt{\frac{\hbar}{\tau v}}, \tag{1}$$

where τ is the time scale of the spatio-temporal domain Ω in which the measurement is made, and v is the volume of its spatial section. A similar formula applies to magnetic fields, with the substitution of H for E:

$$\Delta H_{\min} \sim \sqrt{\frac{\hbar}{\tau v}}. \tag{2}$$

Formulas (1), (2) are consistent with the limitation on electromagnetic field measurability found by Landau and Peierls [3]. However, Landau and Peierls implied a specific measuring scheme, while formulas (1), (2) were derived by the author employing the RPI method without recourse to some specific measuring scheme, and therefore their applicability has been substantiated in a broader range of conditions (for more details about this, see Section 4).

3. Energy–time uncertainty relation

In Section 2, we considered quantum limitations on the measurability of electromagnetic field strength and derived expressions (1), (2) for the minimal uncertainty of field strength measurements. From these formulas we now move on to the formula for the minimal detectable magnitude of strength.

An electric field is detectable, or observable, when the magnitude of its strength E is higher than the minimal uncertainty $\Delta E_{\min}$ of field measurements dictated by the quantum nature of the field itself, the same applying to a magnetic field. Hence, the condition for electromagnetic field observability may be written in the form of inequalities

$$E \gtrsim \Delta E_{\min}\,, \quad H \gtrsim \Delta H_{\min}\,. \tag{3}$$

Upon substitution of expressions (1), (2) for the minimal uncertainty of field strength measurement into formula (3), we obtain the minimal magnitude of the field strength for which this field may be observed in the spatio-temporal domain of volume τv. By denoting the lowest observable strengths as E_{observ}, H_{observ}, for a field measurability condition we obtain

$$E \gtrsim E_{\text{observ}} \sim \sqrt{\frac{\hbar}{\tau v}}, \quad H \gtrsim H_{\text{observ}} \sim \sqrt{\frac{\hbar}{\tau v}}. \tag{4}$$

Taking the squares of the expressions for E_{observ}, H_{observ}, we arrive (once again neglecting a factor on the order of unity) at

$$(E^2 + H^2) \gtrsim (E^2_{\text{observ}} + H^2_{\text{observ}}) \sim \frac{\hbar}{\tau v}\,. \tag{5}$$

But quantity $E^2 + H^2$ is, by an order of magnitude, equal to the energy density of the electromagnetic field. By multiplying this expression by the volume v of the measurement domain, we therefore obtain the field energy $\mathcal{E}$ in this domain. As a result, relation (5) takes on the form

$$\mathcal{E} \gtrsim \mathcal{E}_{\text{observ}} \sim \frac{\hbar}{\tau}\,, \tag{6}$$

where $\mathcal{E}_{\text{observ}}$ is the lowest field energy at which this field is observable, provided the observation time (the time dimension of the spatio-temporal measurement domain) equals τ.

Therefore, from expressions (1) and (2) for the lowest measurable field strengths it is easy to derive inequality (6), which coincides in form with one of the forms of the energy–time uncertainty relation (see Ref. [10]). This is testimony to the fundamental nature of formulas (1), (2) found with the aid of RPIs. We shall additionally consider the status of these formulas in Section 4, when comparing them, on the one hand, with similar (though less general) formulas derived in the paper by LP and, on the other hand, with quite a different estimate of the minimal uncertainties for field strength measurements, derived in the paper by BR.

4. Landau–Peierls and Bohr–Rosenfeld formulas: what is called a measurement?

Let us compare the limitations (1), (2) on the measurability of electromagnetic field strengths, which were derived in the framework of the RPI method, with the results of the LP [3] and BR [4] papers. This comparison elucidates the status of the formulas derived in these studies (see also Ref. [8]).

4.1 Limitations on field measurability and measurement models

In work that dates back to 1931 [3], Landau and Peierls derived formulas for the minimal uncertainties of measurements of electric and magnetic field strengths in the framework of simple models of their measurement. We give these formulas here and demonstrate that they are a special case of formulas (1), (2), which were later derived [5, 6] by the phenomenological RPI method (see Section 2).

For a model of electric field strength measurement, Landau and Peierls adopted the measurement of acceleration which a probe charge acquires under the action of the field. The uncertainty of measurement comprised the intensity of the field radiated by this charge due to its acceleration. The minimal uncertainty of electric field strength measurement turned out, according to LP, to be equal to

$$\Delta E_{\min\ \text{LP}} = \sqrt{\frac{\hbar}{c^3\tau^4}}. \tag{7}$$

Expression (7) results from the general formula (1) by substituting $v = \ell^3$, $\ell = c\tau$ (c is the speed of light). This

coincidence is not accidental; it is precisely these measurement parameters that should be substituted into the general formula to move to the measuring scheme adopted by LP.

Indeed, the radiation of accelerated probe charge induces an intensity perturbation, and the intensity of the radiation field, according to the ideology adopted by LP, should be included in the uncertainty of measurement. During the measurement time τ, the radiation propagates through a distance on the order of $c\tau$. Hence, a domain $\ell \sim c\tau$ in size should be considered to be the measurement domain, and this domain is on the order of $v \sim \ell^3$ in volume.

Apart from the indicated procedure of electric field measurement, Landau and Peierls considered the measurement of a magnetic field with the aid of a magnetic needle and arrived at the following formula for the minimal uncertainty of magnetic field measurement:[4]

$$\Delta H_{\min\ \mathrm{LP}} = \sqrt{\frac{\hbar}{c^3\tau^4}}\,, \tag{8}$$

This formula, like the previous one, is a special case of formula (2) for $v = \ell^3$, $\ell = c\tau$.

Unlike the paper by LP, no specific intensity measuring scheme is implied in the RPI-based approach, and the parameters τ and v, which enter into formulas (1) and (2), are independent of each other. In particular, these formulas do not imply that the dimension of the measurement domain, which is on the order of $\ell \sim v^{1/3}$, equals $c\tau$.

In response to the work by Landau and Peierls, in 1933 Bohr and Rosenfeld [4] considered another class of measuring models and obtained alternative formulas for the minimal uncertainty of field strength measurements. Let us discuss the difference between BR's approach and that of LP.

It was assumed in BR's paper that electric and magnetic field measurements make use of a system of probe bodies with large masses (so that their displacements are small) and high charges and currents (for measuring electric and magnetic fields, respectively). The field strength averaged over the domain occupied by this system of bodies is estimated from the motion of these probe bodies, resulting from the action of the field. To eliminate the contribution from the intrinsic field of the probe bodies to the average field under measurement, advantage is taken of another system of closely located bodies with charges and currents having opposite signs. However, the positions of the bodies which make up the second, auxiliary, system are fixed, i.e., remain invariable during measurements.

An analysis of the measurement made with the aid of this measuring scheme led to other formulas for the minimal uncertainties of field strength measurements than in the paper by LP. For an electric field (we do not consider a magnetic field explicitly, but the conclusions are the same in this case), this is the following:

$$\Delta E_{\min\ \mathrm{BR}} \sim \frac{\hbar}{\ell\tau Q}\,, \tag{9}$$

where Q is the total electric charge of the system of probe bodies, and ℓ is the dimension of the domain in which they are located.

[4] When the electric and magnetic fields are simultaneously measured, according to LP there emerges an additional inequality $\Delta E\,\Delta H > \hbar c/(c\tau)^2(\Delta l)^2$, where Δl is the distance between the probe charge and the magnetic needle. This inequality does not introduce additional limitations when $\Delta l > c\tau$.

Although Landau and Peierls considered a specific measuring scheme, they believed that their formula (7) gives a limitation of measurability, which may not be overcome by way of selection of probe bodies. The fundamental difference in formula (9) obtained by Bohr and Rosenfeld consists in the fact that it permits making the measurement error arbitrarily small by increasing the charge Q of the system of probe bodies. On the face of it, the result of Landau and Peierls concerning the existence of absolute (independent of the measurement model) limitations on the measurability of field strength may seem to be thereby disproved.

The RPI method is phenomenological and yields a limitation (1) on the field measurability, which is independent of the measurement model selected (see Section 2). The limitations obtained by this method should therefore be absolute (true for any measuring scheme for a given volume of the measurement domain). As shown above, formula (7) from the LP paper constitutes a special case of the general formula (1) obtained by the RPI method. It remains yet unclear why do formulas (1) and (7) disagree with formula (9) derived in the work by BR?

BR's inferences about the minimal uncertainty of measurement were different from the results of LP's work (and from the results obtained with the RPI method) because Bohr and Rosenfeld adopted a different definition for the very notion of measurement.

The same rigorous definition of measurement was implicitly adopted in the LP work, which explicitly appeared in the RPI method. Limitations (1) and (2) hold true for any measuring schemes corresponding to this definition. Adopted in the BR paper, which polemicized LP's conclusions, was a less restrictive definition of measurement, which naturally led to looser limitations on the field measurability.

What is the difference between these definitions of measurement?

4.2 What does it mean to measure the field strength?

Let us discuss the definition problem for the notion of measurement by the example of electric field strength measurement (see also Ref. [8]). We begin with the approach relying on restricted path integrals (RPIs) and then compare it with the LP and BR approaches.

When calculating the uncertainty of measurements by the RPI method (see Section 2), integration is performed only over those *field configurations which are compatible with the measurement result*.

Let us assume that a measurement is made of the field strength and the result is expressed as $E \pm \Delta E$. In this case, every configuration which participates in the integration describes a field with a strength no lower than $E - \Delta E$ and no higher than $E + \Delta E$ at each point of the spatio-temporal measurement domain Ω.

This signifies that the quantity under measurement (in this case, the electric field strength) corresponds to the result obtained in the measurement at each point of the measurement domain and throughout the period of measurement. Only physical processes of such a kind which satisfy this requirement are termed measurements under the RPI-based approach. This defines more precisely the very notion of 'measurement', as it is perceived under this approach.

The measurement thus defined does not move out the value under measurement beyond the limits of the measure-

ment error. This measurement may be referred to as *nonperturbative up to the measurement error*.[5]

In the calculation performed by Landau and Peierls, measurement is, in fact, understood in the same way. Indeed, the field measurement in the LP study is made by way of observation of how a charged probe body moves in this field. In this case, the field induced by the probe charge is taken into account when estimating the uncertainty of measurement.[6] If the field strength measurement yields the $E \pm \Delta E$ result, this signifies that the total field strength (with the inclusion of how this strength is distorted due to the presence of the probe charge) does not go beyond the limits of the interval $(E - \Delta E, E + \Delta E)$ in the measurement domain throughout the measurement period.

Therefore, the measurement result $E \pm \Delta E$ found with the aid of the procedure described in the LP paper sets the limits in which the true electric field strength (with the inclusion of all factors involved in the measurement procedure) remains throughout the spatio-temporal domain Ω wherein the measurement takes place. This corresponds to the definition of measurement adopted in the framework of the RPI method.

Thus, both LP's calculation and the phenomenological RPI-based approach rely on the same definition of measurement, which we referred to as a measurement which is nonperturbative up to the measurement error. It therefore comes as no surprise that the results of calculations of the minimal possible uncertainty of measurement under these two approaches, formulas (1), (2) and (7), (8), respectively, are also consistent with each other. They express (to an order of magnitude) limitations on the field measurability, which are true for any measurement understood in the sense indicated above. These limitations are absolute in the sense that the uncertainty of measurement is defined by the parameters τ, v of the measurement domain and may not be done zero for given values of these parameters.

In the work by Bohr and Rosenfeld [4], use is made of a less restrictive definition of measurement. According to this definition, measurement is made of the field averaged over some domain, while inside this domain the field may strongly depart from the average value (these departures cancel out on averaging). The existence of such departures is evident even from the description of the system of probe bodies. In the measuring scheme employed in BR's paper, an opposite charge with a fixed position is located alongside any probe charge. Because the charges are large, a strong field emerges between these two charges (the probe charge and the compensating one). By the definition of the quantity under measurement, this field is disregarded in the estimation of the uncertainty of measurement and may go far beyond the limits of this uncertainty.

Therefore, the result $E \pm \Delta E$ of the electric field strength measurement in the scheme of Bohr and Rosenfeld should indicate the average value of the field strength, but the true field (prior to averaging) is nonuniform, with the difference between the maximal and minimal values of the field strength being far greater than the ΔE value, which is adopted as the uncertainty of measurement.[7] No wonder that the minimal uncertainty of measuring field strength, according to BR's formula (9), lowers with increasing charge of the system of probe bodies: the field induced by this charge inside the measurement domain, which rises with increasing charge, is not included in the uncertainty of measurement.

Bohr and Rosenfeld [4] therefore admit of such ways of field strength measurement that perturb this field more strongly during the measurement than is supposed by the uncertainty of measurement. This definition admits of a substantially broader class of measuring schemes than does the rigorous definition adopted under the RPI method and in LP's work. That is why the limitations obtained by Bohr and Rosenfeld turned out to be looser. Furthermore, under this definition it turned out that there is no absolute limitation at all: by way of raising the charge of the system of probe bodies, the measurement error (9) can be made arbitrarily small, even when the spatio-temporal domain in which the measurement is made is fixed.

The conclusion that there is no absolute limit on field measurability, which is a matter of principle for the work by Bohr and Rosenfeld, is true for their broad interpretation of the notion of measurement, but is not true for the more rigorous definition which necessarily emerges under the RPI approach and is adopted in LP's work. Under this definition of the notion of measurement, the absolute limitations of the field measurability are given by formulas (1), (2).

5. Field measurability and near-field scanning microscope

In this section we shall consider a somewhat unusual example of the application of the foregoing formulas for the quantum limitations on the measurability of the electromagnetic field. It will be shown that these formulas may be employed in the analysis of one of the types of optical microscopes which overcome the diffraction limit of resolution. The case in point is the so-called near-field microscope.[8] This is a scanning optical microscope which determines the positions of density inhomogeneities on a plane with an error smaller than the wavelength of light.

The positions of the substrate molecules in an NFSM are localized due to the fact that the electromagnetic field produced by the probe interacts with substrate molecules in a domain of size ℓ smaller than the wavelength characteristic of this field. As a result, the position of the molecules relative to the probe is determined with a resolution $\ell \ll \lambda$, which goes beyond the diffraction limit. We shall analyze one of the types of such microscopes using the energy–time uncertainty relation for an electromagnetic field. To do this, the process occurring in the NFSM, whose purpose is to measure the positions of molecules, will be interpreted as the measurement (detection) of the electromagnetic field produced by the probe.

[5] Nonperturbative measurement, or more precisely quantum nondemolition measurement, is that which does not perturb altogether the quantity under measurement but perturbs only the observable one which is canonically conjugate to the quantity being measured [11].

[6] By lowering the charge value (for finite dimensions of the probe body) it is possible to decrease the contribution of the Coulomb field, but the radiation field has to be taken into account, because it is generated by the charge acceleration employed to estimate the field intensity.

[7] Incidentally, substantiation of the measurement procedure in BR's paper (as regards the issue of how the systems of probe bodies and compensating bodies interact with the field) relies heavily on the linearity of the electromagnetic field (the field induced by two sources is equal to the sum of the fields induced by each of these sources). For a nonlinear field (for instance, a gravitational field), the question is much more complicated.

[8] Two terms are used in English: near-field scanning optical microscope (NSOM) and near-field scanning microscope (NFSM).

5.1 Microscope with an error smaller than the wavelength of light

As is commonly known, the resolution ℓ of an ordinary optical microscope is limited by values on the order of the wavelength λ of light used in it. This is attributed to photon diffraction, which is in essence a consequence of the Heisenberg uncertainty relation. On the face of it, for this reason the wavelength λ sets the absolute limit for the resolution of a microscope which operates by light. However, this is not so, and at present there exist scanning optical microscopes which offer an uncertainty of measurement much smaller than the wavelength, $\ell \ll \lambda$. In such a microscope, light is delivered to the subject under investigation (a substrate) via a thin probe, which may be positioned near any point of the substrate with the aid of a special mechanical device. All surface points of the material under study are investigated by scanning, i.e., by displacing the probe sequentially along all these points.

One way to achieve a smaller measurement error than the wavelength is evident and involves making the probe–substrate spacing and the aperture of the light-feeding channel much narrower than λ. Such a microscope is schematized in Fig. 1. The electromagnetic radiation emanates from a probe of smaller-than-wavelength diameter, penetrates through a relatively thin substrate, and is recorded on its opposite side. The recorded radiation intensity is proportional to the substrate density at the point to which the probe was brought (alternatively, it is possible to record the radiation that reflects from the substrate and re-enters the probe). The resolution of a microscope so designed is approximately equal to the aperture of the probe due to the geometry of the whole structure.

There is another type of high-resolution optical microscope with a probe in the form of a needle with a very sharp tip. In lieu of the aperture, the crucial role in the operation of such a microscope is played by *the near field* of the radiation emanating from the probe. It is limited to a small domain about the probe tip, which is smaller in dimension than the radiation wavelength. In this domain, the field is still void of its typical wave character.

In microscopes of this type, the radiation emanating from the very narrow region at the probe tip excites substrate molecules located near the probe tip, while the resultant radiation of these molecules (fluorescence) is detected by a photomultiplier (Fig. 2). The resolution of this microscope is determined by the dimension ℓ of the 'activation domain' about the probe, which is well known to be shorter than the wavelength λ of its emanating radiation.

The processes occurring in microscopes of this type were analyzed in a recent paper by I S Osad'ko [12]. He showed that the molecules located in the near-field zone (at a shorter-than-wavelength distance from the probe tip) are excited much more strongly than the molecules in the far-field zone. That is why the photomultiplier will detect the fluorescence of the molecules located in the near-field zone and will 'overlook' the very weak fluorescence of the molecules residing in the far-field zone, although the radiation emanating from the probe tip spans all of them. This results in a resolution which may be tens of times higher than in conventional optical microscopes.

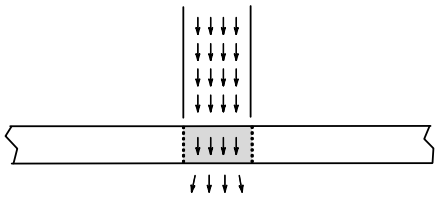

Figure 1. Geometrical way of providing high microscope resolution: the field transmitted through a thin substrate is detected; the resolution is determined by the aperture.

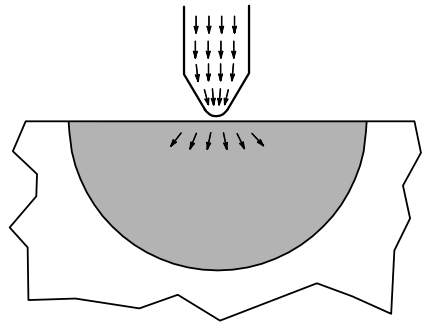

Figure 2. The principle of NFSM operation: a sharpened probe and a very small 'activation domain' in which the field intensity produced by the probe suffices to excite molecules (this domain is hatched); the radiation of substrate molecules in this domain is detected, and the NSOM resolution is determined by its size ℓ, which is much shorter than λ.

Thus, a qualitative answer to the question about the cause of high NFSM resolution (which overcomes the diffraction limit) is quite simple: an NFSM harnesses for a 'probe' the near field, which is not wave type in character, unlike the far field. Molecules are excited in the near-field zone, which is shorter than λ in size and where the ordinary notion that the substrate molecules interact with separate electromagnetic field quanta (photons) is inapplicable. This is the reason why neither the Heisenberg relation nor the laws of wave optics may be applied to analyze NFSMs.

Therefore, the reason for the high resolution of NFSMs is clear. This brings up the question: Is it possible to analyze it proceeding from general quantum laws like the uncertainty relation?

We shall show that an analysis of this kind is possible if the NFSM operation is considered as a field intensity measurement effected by exciting a molecule residing in this field rather than a molecule position measurement with the aid of the field (as a part of the measuring device, its 'microsensor').

5.2 NFSM operation as field measurement

Since our intention is to analyze the quantum limitations on the measurements occurring in an NFSM, we should turn our attention to the quantum theory of measurements. To do this requires specifying what precisely is subjected to measurement in this case, and what is the character of this measurement (what information results from this measurement). Initially, we make use of the fact that in the NFSM, like in any system wherein quantum measurements are made, there is a certain symmetry between the system that measures and the system subject to the measurement. For this reason, the process occurring in the NFSM may be interpreted in two ways: as the measurement (of the position) of substrate molecules with the help of the field, and as the measurement of field (intensity) with the help of the molecules. Using the latter interpretation, we shall discuss the specific features of the field measurement resulting in this way.

5.2.1 System under measurement and measuring system. In an NFSM there are two physical systems: the field and the substrate molecule, the former being usually considered as a part of the measuring instrument (sensor), and the latter as the system under measurement. When reasoning in the framework of the quantum theory of measurements, we should consider both of these subsystems as quantum systems.

The interaction between these subsystems, as it always is in a quantum measurement, leads to the quantum correlation (entanglement) of their states [13]. In this case, the situation is symmetric relative to these two quantum systems, and with equal right we may treat the molecule as the sensor of the measuring system, and the field as the system under measurement. By considering next the correlation of the states of these two quantum systems with the state of the macroscopic part of the instrument, we can say that the state of the instrument is indicative of the state of each of the two quantum systems.

We shall explain this with the aid of quite general formulas employed in the quantum theory of measurements, without referring in detail to the specific case of an NFSM as a measuring instrument.

The states of the molecule and the field will be denoted by the letters ψ and φ, respectively, and the state of the macroscopic part of the instrument by the letter Φ (with the corresponding subscript on each of these letters). Then, the final state (after measurement) of the compound system consisting of all subsystems is described by the formula[9]

$$\Psi = \sum_i \psi_i \varphi_i \Phi_i \,, \tag{10}$$

in which the two subsystems, ψ and φ, enter symmetrically. When interpreting this formula it is valid to say (i) that the result of the measurement of system ψ due to its interaction with sensor φ is expressed in state Φ_i, or (ii) that the result of the measurement of system φ due to its interaction with sensor ψ is expressed in state Φ_i.

By applying this reasoning to the case at hand we can say that the NFSM photomultiplier measures the parameters of the substrate molecule due to its interaction with the field; however, with equal right we may believe that it measures the parameters of the field due to its interaction with the substrate molecule. In truth, the parameters of the states of both subsystems are measured, which are correlated with the instrument as well as with each other.

This offers a fresh possibility for analyzing an NFSM as a measuring instrument. Specifically, for this purpose we may employ quantum limitations on the measurability of the electromagnetic field and, in particular, the energy–time uncertainty relation for the field.

5.2.2 What is measured in an NFSM, and how? To correctly apply the general formulas for quantum limitations on measurements, we must specify what is measured in an NFSM, and how.

The main purpose of the optical microscope is to determine the positions of substrate density nonuniformities. This is done by scanning the probe position and observing the fluorescence of substrate molecules for each position of the probe. The fluorescence intensity increases with the number of molecules that experience excitation, i.e., with the number of molecules that find themselves in the activation zone (the domain in which the intensity of the electromagnetic field suffices to excite molecules). The microscope resolution is equal to the dimension ℓ of this zone which does not go beyond the near-field zone [12], so that certainly $\ell < \lambda$. The fluorescence intensity depends on the number of molecules that find themselves in the activation zone, i.e., on the density of substrate substance in this zone.

Let us consider the same process from a different standpoint. The observation of the fluorescence of substrate molecules will be considered as a signal about the detection of the electromagnetic field exciting these molecules. When the fluorescence is observed, this signifies that the field has been detected in the domain of size ℓ. The parameters of the field, in particular, its intensity amplitude E, are *a priori* known and remain invariable (because the probe structure is known, as are the conditions for radiation emission from the probe). That is why the presence of fluorescence is the signal that this field has been detected in the domain of size ℓ near the probe.

The question we shall discuss below is: Is this mode of electromagnetic field observation at variance with quantum limitations on the measurability of this field? The parameters required for this analysis are the volume v of the domain in which the field detection takes place, the magnitude of intensity E of the field which is to be detected in this domain, and the measurement time τ. These parameters will be borrowed from Ref. [12].

In agreement with what was said in the foregoing, we assume that the volume v to an order of magnitude equals

$$v \sim \ell^3 \,. \tag{11}$$

For the near-field intensity, we adopt the estimate given in Ref. [12]:

$$E_{\mathrm{N}} \sim \frac{d}{r^3} \,, \tag{12}$$

where d is the dipole moment of the emission domain at the probe tip. Accordingly, the field intensity in the domain v will be estimated by order of magnitude as[10]

$$E \sim \frac{d}{\ell^3} \,. \tag{13}$$

Lastly, the field measurement time[11] is taken to be equal to the molecular excitation time. According to Ref. [12], it is equal (by order of magnitude and under the assumption that the finiteness of the energy level width may be neglected) to

$$\tau \sim \frac{\hbar}{Ep} \,, \tag{14}$$

where p is the dipole moment of the molecule (corresponding to the transition which takes place in its excitation).

Notice that we estimate the volume in which the field is measured (detected) as the volume v of the whole molecular activation domain. This is due to the fact that the molecule

[9] We represent the set of measurement results as a discrete set, which enables a more lucid representation of the structure of the state, and in doing so does not change the heart of the matter.

[10] To perform a more accurate calculation requires taking into account the field nonuniformity in the activation zone, i.e., formula (12). This is basically possible, but here we shall restrict ourselves to a crude estimate (13), which would suffice for calculations accurate to the order of magnitude.

[11] This term should not be understood literally in our case. This will be discussed in greater detail in Section 5.4.

position in the activation domain is not measured, so that the detection of fluorescence signifies that the field is present in the activation domain, i.e., somewhere in volume v.

Therefore, we are facing the problem of ascertaining that the detection of field (13) in the domain of volume (11) during the time period (14) does not contradict the general limitations existing in quantum mechanics. We shall see that there is no contradiction indeed, and in certain respects we shall define more accurately the characteristics of the measurement procedure occurring in NFSMs.

5.3 Analysis of near field detectability in an NFSM

To analyze the operation of an NFSM, we shall take advantage of the energy–time uncertainty relation in the form of relation (6) or (5).

The parameters characterizing the NFSM operation were discussed in Section 5.2.2 with reference to review [12]. For the subsequent discussion it is significant that the fluorescence of substrate molecules permits detecting the probe-produced field in the domain of volume $v \sim \ell^3$, where ℓ is the dimension of the near-field zone. In this case, the time expended for this detection is $\tau \sim \hbar/Ep$ (for more on this issue, see also Section 5.4). So, the procedure of field detection is confined to the 4-dimensional domain of volume $\tau v \sim \hbar\ell^3/Ep$.

We shall apply the energy–time uncertainty relation written in the form of relation (6) or (5) to the procedure realized in an NFSM. The procedure may be referred to as the procedure for detecting an electric field, because the magnetic field in the near-field zone of the NFSM is much weaker than the electric field [12] and the substrate molecules are excited under the action of only the electric field. On the strength of formula (5), for those parameters τ, v that are characteristic of NFSMs, the electric field is detectable if its strength satisfies condition (4):

$$E \gtrsim E_{\text{observ}} \sim \sqrt{\frac{\hbar}{\tau v}} \sim \sqrt{\frac{Ep}{\ell^3}}.$$

Hence, we obtain the field observability condition in the form

$$E \gtrsim \frac{p}{\ell^3}\,. \tag{15}$$

It is evident that the near-field intensity (13) in the NFSM satisfies the field detectability condition for

$$d \gtrsim p\,. \tag{16}$$

This inequality signifies that quantum limitations on the detectability of an electromagnetic field (expressed in the form of the energy–time uncertainty relation) permit detecting the near field in an NFSM when the dipole moment of the emission domain at the probe tip exceeds the molecular dipole moment (which is, naturally, fulfilled under ordinary conditions).[12]

Let us now consider the case where the substrate thickness a is smaller than the dimension ℓ of the zone of molecular activation (Fig. 3). In this case, for a volume of the measurement domain we must substitute a quantity on the order of $v \sim a\ell^2$. Then, $\tau v \sim \hbar a\ell^2/Ep$, and the near field detectability condition becomes

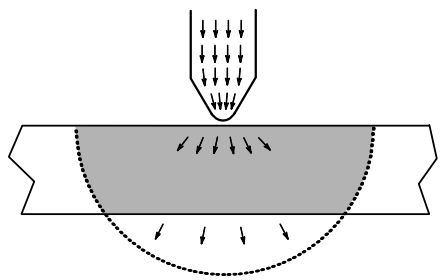

Figure 3. Substrate of thickness $a < \ell$.

$$\frac{d}{p} \gtrsim \frac{\ell}{a}\,. \tag{17}$$

Inequalities (16), (17) may be replaced with a more concrete statement that the signal-to-noise ratio[13] for thick and thin substrates assumes the forms

$$\left.\frac{E_{\text{N}}^2}{E_{\text{observ N}}^2}\right|_{v\sim\ell^3} \sim \frac{d}{p}\,, \quad \left.\frac{E_{\text{N}}^2}{E_{\text{observ N}}^2}\right|_{v\sim a\ell^2} \sim \frac{d}{p}\frac{a}{\ell}\,, \tag{18}$$

respectively.

Recall that the volume v of the measurement domain in the above calculation was taken to be equal to the volume ℓ^3 (or, to $a\ell^2$ in the latter case) of the molecular activation domain rather than to the volume of one molecule which actually reacts to the existence of the field. This is because the location of the molecule inside the activation domain is not measured. That is why under the specific conditions which correspond to the NFSM design the detection of molecular radiation merely provides the information that the field has been detected somewhere in the activation domain, i.e., in the domain where one of the molecules could have become excited. Outside this domain molecules cannot become excited at all or, more properly, their excitation probability is much lower, and we neglect it.

Consider a purely hypothetical situation (presently impossible) wherein more exact information about the position of precisely that molecule which reacted to the field is available. This would imply that the NFSM structure has been changed and one more microscope, whose resolution is better than the resolution of the NFSM in its present-day unchanged form, has been introduced into it. This, in turn, would signify the acquisition of more exact information about the state of the molecules and of the quantum field. As a consequence, the minimal unavoidable uncertainties of measurement of these quantum systems would also become larger. To state it in different terms, this would be an entirely different instrument, which is to be differently calculated assuming a different value for the size of the measurement domain.

5.4 Once more on the 'measurement time' in an NFSM

We shall explain below some features of the notion 'measurement time' for an NFSM, and the relation of this notion to

[12] By the detectability of the field is meant here its detection with a sufficiently high efficiency. So, the observation of the radiation emitted by molecules with a probability on the order of unity signifies that the field of the given configuration exists in the given volume. When inequality (16) is not fulfilled, this probability will become lower progressively with decreasing d.

[13] By definition, we assume that the field is detectable (to be more precise, the efficiency of its detection is high enough) when the signal-to-noise ratio is greater than or on the order of unity. In this case, the term noise is used in reference to the quantum uncertainty of measurements, which arises due to unavoidable quantum fluctuations (the mechanism of their action comprises the so-called dark current of a photomultiplier).

what spatial domain should be considered as the measurement domain. For definiteness, we shall consider only the case of a thick substrate as in the first part of the previous section (the case of a thin substrate is treated in a similar way with somewhat different formulas leading to the same conclusions).

Let ℓ be the dimension of the near-field zone, i.e., the domain around the probe in which the probe radiation field has the form $E_{\mathrm{N}} \sim d/r^3$. For estimates accurate to an order of magnitude, it may be assumed that its intensity in the near-field zone is on the order of $E_{\mathrm{N}} \sim d/\ell^3$. According to Ref. [12], the molecular excitation time in the near-field zone is $\tau_{\mathrm{N}} \sim \hbar/E_{\mathrm{N}}p$. We assume, as in the first part of the previous section, that $v \sim \ell^3$ to obtain the time of the molecular excitation in the near-field zone:

$$\tau_{\mathrm{N}} \sim \frac{\hbar\ell^3}{pd} \,. \tag{19}$$

In the far-field zone, according to Ref. [12], the field strength takes the form $E_{\mathrm{F}} \sim d/\lambda r^2$. We consider a small part of the far-field zone (nearest to the probe) with a size on the order of λ and reason in the same way to obtain the time of molecular excitation in this zone:

$$\tau_{\mathrm{F}} \sim \frac{\hbar\lambda^3}{pd} \,. \tag{20}$$

Since $\ell < \lambda$, the excitation time in the far-field zone is much longer than in the near-field zone:

$$\frac{\tau_{\mathrm{F}}}{\tau_{\mathrm{N}}} \sim \left(\frac{\lambda}{\ell}\right)^3 \gg 1 \,. \tag{21}$$

What does this relation signify from the physical standpoint? We consider this question in several stages, beginning with a situation not always real but quite simple for understanding.

Let us assume that at a certain instant of time the radiation from the probe is turned on and the photomultiplier begins to record the fluorescence of molecules. After excitation, a molecule will return to the ground state as time $\tau_{\mathrm{radiation}}$ passes and in doing so will emit fluorescent radiation, which will be recorded by the photomultiplier within a time $\tau_{\mathrm{detection}}$. Attaining the highest microscope resolution requires that all of these processes take place equally fast: $\tau_{\mathrm{radiation}} \sim \tau_{\mathrm{detection}} \sim \tau_{\mathrm{N}}$. Then, the duration $T_1 = \tau_{\mathrm{N}} + \tau_{\mathrm{radiation}} + \tau_{\mathrm{detection}}$ of the entire cycle (excitation, emission, detection) is on the order of τ_{N}. In the course of one cycle, the majority of molecules in the near-field zone make a contribution to the radiation emission and to the photomultiplier signal; however, on the strength of inequality (21) hardly any of the molecules in the far-field zone have time even to become excited. Evidently, the intensity of the signal recorded by the photomultiplier will be proportional to the number of molecules in the near-field zone. This permits estimating the substrate density in the domain of size ℓ about the probe. That is why, the near-field zone of size ℓ determines the microscope resolution.

Usually, the measurement procedure lasts much longer than the period T_1. It is valid to say that the measurement is performed continuously. Let us assume that the measurement time T is nevertheless finite but is much longer than $T_1 \sim \tau_{\mathrm{N}}$. Then, each molecule in the near-field zone will manage to become T/τ_{N} times excited during this period and subsequently emit a photon, which will be recorded. On the strength of relation (21), the number of excitation-emission-detection cycles for every molecule in the far-field zone will be many times smaller: $T/\tau_{\mathrm{F}} = (\ell/\lambda)^3 (T/\tau_{\mathrm{N}}) \ll (T/\tau_{\mathrm{N}})$.

This reasoning signifies that the probability of radiation emission per unit time from the far-field zone (a small part of the far-field zone, of size λ) is $(\ell/\lambda)^3$ times lower than the probability of emission from the near-field zone. The far-field zone layers which are more remote from the probe radiate with progressively lower probability as the distance from the probe increases. From the practical standpoint, it is valid to say that the fluorescent radiation is emitted only from the near-field zone. This is the reason why the microscope resolution is determined by the dimension ℓ of the near-field zone.

Therefore, in the unusual situation occurring in the application of the energy–time uncertainty relation to the analysis of an NFSM, the notion of 'measurement time' as it enters into the general quantum limitations should not be perceived literally. The measurement is performed continuously, but it may be represented as a multiple repetition of one measurement cycle. That which figures as the 'measurement time' in the general formulas for field measurability is merely some parameter characterizing continuous measurement, which may be qualitatively characterized as the 'duration of one measurement cycle'.

6. Concluding remarks

In the quantum theory, the limitations on measurability are most often characterized quantitatively by the Heisenberg uncertainty relation or its analog pertaining to a different pair of canonically conjugate observables. For certain types of measurements, however, one has to use either a more sophisticated form of the Heisenberg type uncertainty relation or other relations which are substantially different in form. Either of the examples is found in monograph [7]. In particular, the *action uncertainty principle* applies for a continuous measurement (i.e., a measurement which lasts in time and whose result is expressed by a time function) (see Refs [7, 9]).

In this paper, we considered formulas (1), (2) for the minimal error in measuring electromagnetic field intensity, which stem from the quantum nature of the field. These formulas, whose special case was derived by Landau and Peierls in 1931, were proved by the author in the general case employing the restricted path integral method.

We discussed at length whether the aforementioned limitations on field measurability are inherently absolute, as believed by Landau and Peierls, or not, as considered by Bohr and Rosenfeld (1933). We discussed this issue in detail and showed that the limitations found by LP (like the more general formulas (1), (2)) are indeed absolute in character (i.e., are independent of the measuring scheme) if the rigorous definition of measurement is adopted. This definition supposes that the true field corresponds to the measurement result (accurate to the uncertainty of measurement) throughout the spatio-temporal domain in which the measurement takes place. In contrast to LP, BR adopted a broad interpretation of the notion of measurement to arrive, in the framework of this definition, at looser limitations, which are not inherently absolute.

Furthermore, in this paper we showed that the field observability (detectability) condition following from formu-

las (1), (2) is equivalent to the energy–time uncertainty relation. Evidently, this is an additional indication of the fundamental character of these formulas.

To illustrate how quantum limitations on the field detectability operate in the situation where the Heisenberg uncertainty relation is inapplicable, we analyze in the framework of these limitations the operation of a near-field scanning microscope (NFSM). For this purpose, the process in the NFSM is represented as the detection of the field radiated by the probe. We apply the energy–time uncertainty relation in the form of expression (5) or (6) to analyze this process and emphasize that the NFSM resolution, which overcomes the diffraction limit, is consistent with the limitations on electromagnetic field measurability in the form of the energy–time uncertainty relation.

Acknowledgments. The author is grateful to Yu I Vorontsov for helpful discussions, which facilitated the elimination of ambiguities and inaccuracies in the text of the manuscript, and to I S Osad'ko for the elucidation of certain aspects of the operation of a near-field microscope.

References

1. Vorontsov Yu I *Usp. Fiz. Nauk* **175** 1053 (2005) [*Phys. Usp.* **48** 999 (2005)]
2. Mensky M B *J. Russ. Laser Res.* **27** 332 (2006)
3. Landau L, Peierls R *Z. Phys.* **69** 56 (1931) [Translated into English: in Landau L D *Collected Works* Vol. 1 (Moscow: Nauka, 1969) p. 56]
4. Bohr N, Rosenfeld L *Klg. Danske Videnskab Selskab., Math. Fys. Medd.* **12** (8) 3 (1933); *Phys. Rev.* **78** 794 (1950)
5. Mensky M B *Ann. Physik* **500** 215 (1988), the volume number is indicated according to the unitary volume numbering adopted for the Wiley site, originally Vol. 45
6. Menskii M B *Teor. Mat. Fiz.* **80** 29 (1989) [*Theor. Math. Phys.* **80** 689 (1989)]
7. Mensky M B *Quantum Measurements and Decoherence: Models and Phenomenology* (Dordrecht: Kluwer Acad. Publ., 2000) [Translated into Russian: *Kvantovye Izmereniya i Dekogerentsiya: Modeli i Fenomenologiya* (Moscow: Fizmatlit, 2001)]
8. von Borzeszkowski H-H, Mensky M B *Phys. Lett. A* **188** 249 (1994); quant-ph/0007092
9. Mensky M B *Continuous Quantum Measurements and Path Integrals* (Bristol: IOP Publ., 1993) [Japanese extended translation (Kyoto: Yoshioka, 1995)]
10. Vorontsov Yu I *Usp. Fiz. Nauk* **133** 351 (1981) [*Sov. Phys. Usp.* **24** 150 (1981)]
11. Braginskii V B, Vorontsov Yu I *Usp. Fiz. Nauk* **114** 41 (1974) [*Sov. Phys. Usp.* **17** 644 (1975)]
12. Osad'ko I S *Usp. Fiz. Nauk* **180** 83 (2010) [*Phys. Usp.* **53** 77 (2010)]
13. Menskii M B *Usp. Fiz. Nauk* **173** 1199 (2003) [*Phys. Usp.* **46** 1163 (2003)]