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Integral characteristics: a key to understanding structure formation in stochastic dynamic systems

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Contents

Abstract. **Some general problems concerning the stochastic approach are discussed in relation to parametrically excited stochastic dynamic systems described by partial differential equations. Such problems arise in hydrodynamics, magnetohydrodynamics, and astro, plasma, and radio physics and share the feature that the statistical characteristics of their solutions (moments, correlation and spectral functions, and so on) increasing exponentially with time, whereas some solution implementations lead to the formation of random structures with probability one as a result of clustering. The goal of this paper is to use the ideas of stochastic topography to find conditions under which such structures arise.**

"Look in the root!"
Koz'ma Prutkov

1. Introduction

Parametrically excited dynamical systems are encountered in all branches of physics. Their excitation is commonly associated with their property of being unstable as a consequence of fluctuations in the initial conditions. A similar situation also arises for deterministic initial conditions, but fluctuating system parameters. Dynamical systems can be described in these cases by ordinary and partial differential equations. A simple dynamical system corresponding to a log-normal random process $y(t;\alpha)$ is described by the ordinary first-order stochastic differential equation

$$\frac{d}{dt}\,y(t;\alpha) = \bigl(-\alpha + z(t)\bigr)\,y(t;\alpha)\,, \quad y(0;\alpha) = 1\,, \tag{1}$$

where $z(t)$ is a Gaussian *white noise* process with the parameters

$$\langle z(t)\rangle = 0, \quad B_z(t-t') = \langle z(t)\,z(t')\rangle = 2D\delta(t-t')\,.$$

The solution of Eqn (1) is given by

$$y(t;\alpha) = \exp\left(-\alpha t + \int_0^t d\tau\, z(\tau)\right). \tag{2}$$

The basic statistical properties of this process are well known; they are described in Ref. [1] and monographs [2–6]. More subtle properties, related to the formation of structures in dynamical systems and not covered in the literature, are considered in Sections 1.2 and 1.3. Figure 1 displays realizations of log-normal random processes $y(t)$ in Eqn (2) and $1/y(t)$ for the parameter ratio $\alpha/D = 1$ (the dashed curves show the functions $\exp(-Dt)$ and $\exp(Dt)$). The figure shows the presence of rare but strong fluctuations relative to the dashed curves toward both large values and zero. Such a property of random processes is called *intermittency* (see, e.g., Refs [7, 8]). It is noteworthy that this property is common to all random processes. The curve with respect to which the fluctuations are observed is referred to as the *typical realization curve* (see Section 1.2).

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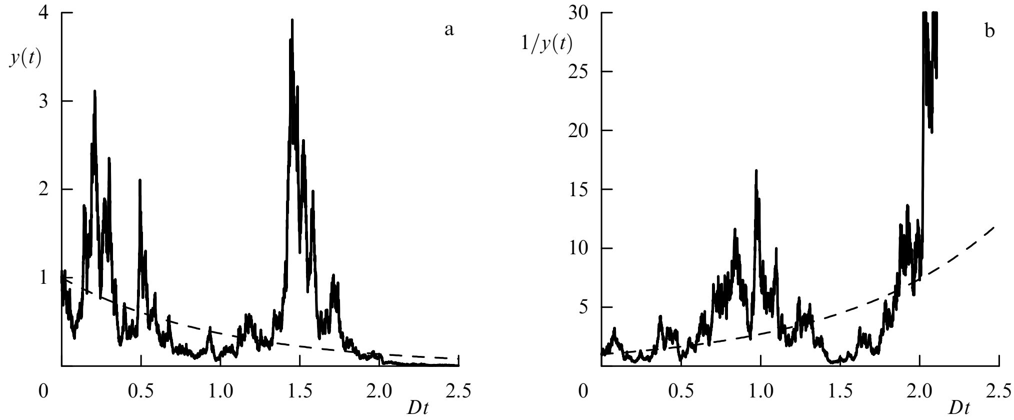

Figure 1. Realizations of log-normal processes $y(t)$ (a) and $1/y(t)$ (b) for the parameter ratio $\alpha/D = 1$.

Other simple examples are furnished by problems involving a stochastic oscillator with the fluctuating frequency $\omega(t)$ and decay coefficient $\lambda(t)$ [2–6, 9]:

$$m \frac{d^2}{dt^2} x(t) + \lambda(t) \frac{d}{dt} x(t) + \omega^2(t) x(t) = 0 \,,$$

$$x(0) = x_0 \,, \quad \frac{d}{dt} x(t)\bigg|_{t=0} = y_0 \,.$$

We note that the problem of an oscillator with a fluctuating mass was considered in Refs [10–12].

A stationary boundary value problem for a plane wave incident on a layer of a randomly layered medium $[L_0, L]$ is described by the equation

$$\frac{d^2}{dx^2} u(x) + k^2 \big[1 + z(x)\big] u(x) = 0 \,, \tag{3}$$

subject to the continuity conditions for the field and its derivative at the layer boundaries,

$$u(L) + \frac{i}{k} \frac{du(x)}{dx}\bigg|_{x=L} = 2 \,, \quad u(L_0) - \frac{i}{k} \frac{du(x)}{dx}\bigg|_{x=L_0} = 0 \,.$$

Figure 2 plots two realizations for the wave field intensity $I(x) = |u(x)|^2$ in a sufficiently thick layer of the medium, corresponding to two realizations of medium inhomogeneity generated numerically. The difference between them is that the functions $z(x)$ have different signs in the middle of the layer over the wavelength distance. This allows observing the influence of small medium detuning on the behavior of solution of the boundary value problem. We note that the tendency to steep exponential decay (with strong fluctuations toward increased as well as zero intensity values) brought about by multiple reflections of the wave in a chaotic inhomogeneous medium (*dynamic localization*) is explicit in Fig. 2. This phenomenon is commonly identified with the *Anderson localization* for wave eigenfunctions of the stationary one-dimensional Schrödinger equation with a random potential [13] (see also Ref. [14]). But this is not exactly correct because the solution of boundary value problem (3) is given by a superposition of all eigenfunctions.

For boundary value problem (3), the wave field intensity $I(x)$ for a half-space of a random medium is statistically equivalent to the random process $2y(t;\alpha)$ at $\alpha = D$ and its realization resembles a mirror reflection of Fig. 1a.

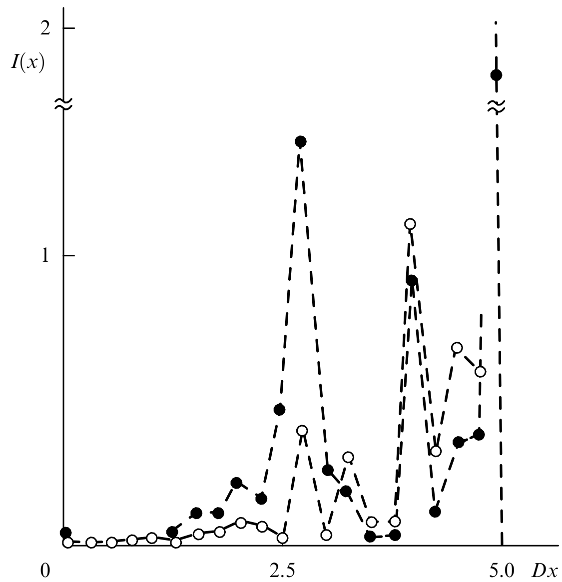

Figure 2. Numerical simulation of the dynamic localization of the plane wave intensity for two realizations of medium inhomogeneities.

As regards random fields and partial differential equations, the simplest problem (in formulation) is that of the diffusion of a particle in a random velocity field $\mathbf{u}(\mathbf{r}, t)$ with given statistical properties [2–6, 15]:

$$\frac{d}{dt} \mathbf{r}(t) = \mathbf{u}\big(\mathbf{r}(t), t\big) \,, \quad \mathbf{r}(0) = \mathbf{r}_0 \,. \tag{4}$$

Further, we can introduce the generalization of log-normal process (2) for a log-normal random field by the formula

$$f(\mathbf{r}, t; \alpha) = \exp\left(-\alpha t + \int_0^t d\tau \, z(\mathbf{r}, \tau)\right) , \tag{5}$$

where $z(\mathbf{r}, t)$ is a Gaussian random field, delta-correlated in time, with zero mean and the correlation function

$$\big\langle z(\mathbf{r}, t) \, z(\mathbf{r}', t') \big\rangle = 2D(\mathbf{r} - \mathbf{r}') \, \delta(t - t') \,. \tag{6}$$

This field satisfies the first-order differential equation

$$\frac{d}{dt} f(\mathbf{r}, t; \alpha) = \big\{-\alpha + z(\mathbf{r}, t)\big\} f(\mathbf{r}, t; \alpha), \quad f(\mathbf{r}, 0; \alpha) = 1 \,, \tag{7}$$

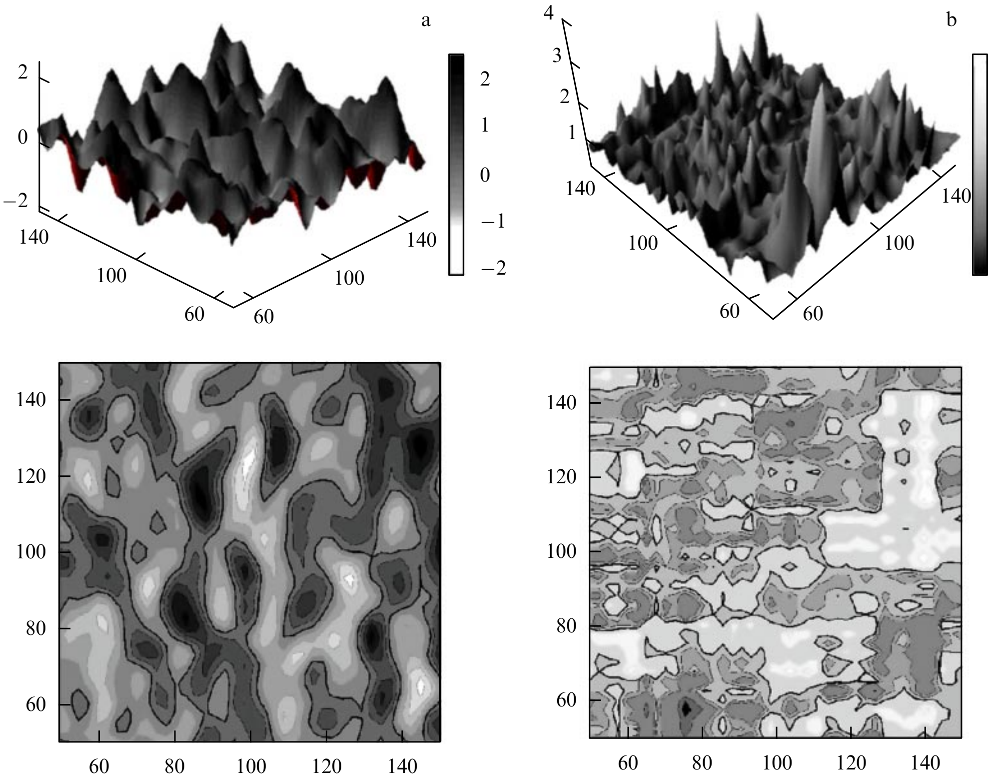

Figure 3. Realizations of (a) the Gaussian field $\ln f(\mathbf{r}, t)$ and (b) the log-normal field $f(\mathbf{r}, t)$ (10) and their topographic level lines. The thick curves in the bottom panels mark isolines that correspond to the field values 0 (a) and 1 (b).

which depends parametrically on the spatial location $\mathbf{r}$. All one-point statistical characteristics of this field are independent of $\mathbf{r}$; hence, if we are only interested in one-point characteristics, the random field $f(\mathbf{r}, t; \alpha)$ can be considered statistically equivalent to the random process $y(t; \alpha)$ in (2).

We note in that the aforementioned Refs [7, 8], the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}, t) = z(\mathbf{r}, t) f(\mathbf{r}, t) + \mu_f \Delta f(\mathbf{r}, t) , \tag{8}$$

where μ_f is the dynamical diffusivity coefficient for the field $f(\mathbf{r}, t)$, is regarded as a model problem. It is noteworthy that Eqn (8), containing the terms responsible for random reproduction and diffusion, is also relevant for problems occurring in biology and the kinetics of chemical and nuclear reactions (see, e.g., Ref. [16]).

Replacing t with the imaginary time $\mathrm{i}\tau$, we arrive at the *Schrödinger equation* with a random potential $z(\mathbf{r}, \tau)$,

$$\frac{\partial}{\partial \tau} f(\mathbf{r}, \tau) = \mathrm{i} z(\mathbf{r}, \tau) f(\mathbf{r}, \tau) + \mathrm{i}\mu_f \Delta f(\mathbf{r}, \tau) , \quad f(\mathbf{r}, 0) = f_0(\mathbf{r}) . \tag{9}$$

At the initial stage of diffusion, the solution of problem (8) is given by function (5) with $\alpha = 0$, i e.,

$$f(\mathbf{r}, t) = \exp\left(\int_0^t \mathrm{d}\tau \, z(\mathbf{r}, \tau)\right) . \tag{10}$$

Introducing randomness into the medium parameters leads to stochastic behavior of the physical fields themselves. For example, individual realizations of the log-normal scalar two-dimensional field $f(\mathbf{R}, t)$ in (10), where $\mathbf{R} = (x, y)$, resemble a mountain landscape with randomly distributed peaks, trenches, ridges, and passes. Figure 3 presents examples of realizations of two random fields with different statistical structures simulated numerically.

An important issue in the theory of magnetohydrodynamic turbulence is the treatment of diffusion in the kinematic approximation of passive fields such as a tracer density field (particle concentration) and the magnetic field. Basic stochastic equations are then the continuity equation for the tracer density $\rho(\mathbf{r}, t)$,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \, \mathbf{u}(\mathbf{r}, t)\right) \rho(\mathbf{r}, t) = \mu_\rho \, \Delta\rho(\mathbf{r}, t) , \quad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r}) , \tag{11}$$

and the induction equation for a solenoidal magnetic field $\mathbf{H}(\mathbf{r}, t)$ [17],

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \, \mathbf{u}(\mathbf{r}, t)\right) \mathbf{H}(\mathbf{r}, t) = \left(\mathbf{H}(\mathbf{r}, t) \, \frac{\partial}{\partial \mathbf{r}}\right) \mathbf{u}(\mathbf{r}, t) + \mu_H \, \Delta\mathbf{H}(\mathbf{r}, t) ,$$
$$\mathbf{H}(\mathbf{r}, 0) = \mathbf{H}_0(\mathbf{r}) , \tag{12}$$

where μ_ρ is the dynamical diffusivity coefficient for the density, $\mu_H = c^2/(4\pi\sigma)$ is the dynamical diffusivity coefficient for the magnetic field, which is related to the medium conductivity σ, and $\mathbf{u}(\mathbf{r}, t)$ is the field of turbulent velocities, which are considered homogeneous in space (nonisotropic and possessing helicity in the general case) and stationary with the given statistical properties.

We note that Eqns (11) and (12) pertain to the Eulerian description of field dynamics, while Eqn (4) is characteristic for equations that correspond to Eqns (11) and (12) in the Lagrangian description.

We also specially mention that Eqn (12) describes the diffusion of a magnetic field in a three-dimensional ($d = 3$) space $\mathbf{r} = \{\mathbf{R}, z\}$ ($\mathbf{R} = \{x, y\}$) by the three-dimensional velocity field $\mathbf{u}(\mathbf{r}, t) = \{u_x, u_y, u_z\}$.

For a plane parallel flow, the velocity field $\mathbf{u}(\mathbf{R}, t) = \{u_x, u_y, 0\}$ is two-dimensional ($d = 2$). In this case, for the three-dimensional magnetic field $\mathbf{H}(\mathbf{R}, t) = \{\mathbf{H}_\perp, H_z\}$, where $\mathbf{H}_\perp(\mathbf{R}, t) = \{H_x, H_y\}$, Eqns (12) can be split. Namely, the two-dimensional component of the magnetic field in the plane $\mathbf{R}-\mathbf{H}_\perp(\mathbf{R}, t)$ is described by the equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{R}}\,\mathbf{u}(\mathbf{R}, t)\right)\mathbf{H}_\perp(\mathbf{R}, t) = \left(\mathbf{H}_\perp(\mathbf{R}, t)\,\frac{\partial}{\partial \mathbf{R}}\right)\mathbf{u}(\mathbf{R}, t) + \mu_H\,\Delta\mathbf{H}_\perp(\mathbf{R}, t)\,,$$
$$\mathbf{H}_\perp(\mathbf{R}, 0) = \mathbf{H}_{\perp 0}(\mathbf{R})\,,$$

and the magnetic field component $H_z(\mathbf{R}, t)$ obeys the continuity equation for a passive scalar

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{R}}\,\mathbf{u}(\mathbf{R}, t)\right)H_z(\mathbf{R}, t) = \mu_H\,\Delta H_z(\mathbf{R}, t)\,,$$
$$H_z(\mathbf{R}, 0) = H_{z0}(\mathbf{R})\,,$$

resembling Eqn (11) for the density field.

Another example of problems pertaining to the parametric excitation of dynamical systems is the propagation of monochromatic radiation in random multidimensional media. It is described by the complex-valued parabolic equation (see, e.g., [2–6, 18])

$$\frac{\partial}{\partial x}\,u(x, \mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_\mathbf{R}\,u(x, \mathbf{R}) + \frac{\mathrm{i}k}{2}\,\varepsilon(x, \mathbf{R})\,u(x, \mathbf{R})\,, \quad u(x, \mathbf{R}) = u_0(\mathbf{R})\,, \tag{13}$$

where x is the coordinate in the direction of wave propagation, $\mathbf{R}$ are the coordinates in the transverse plane, and $\varepsilon(x, \mathbf{R})$ is the deviation of the dielectric permittivity from unity. We mention that this equation also has the form of the nonstationary *Schrödinger equation*, similar to Eqn (9), with a random potential $\varepsilon(x, \mathbf{R})$, after replacing x with time.

If we introduce the wave field amplitude and phase using the formula

$$u(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\left[\mathrm{i}S(x, \mathbf{R})\right],$$

then the equation for the wave field intensity $I(x, \mathbf{R}) = |u(x, \mathbf{R})|^2$ assumes the form

$$\frac{\partial}{\partial x}\,I(x, \mathbf{R}) + \frac{1}{k}\,\nabla_\mathbf{R}\left\{\nabla_\mathbf{R} S(x, \mathbf{R})\,I(x, \mathbf{R})\right\} = 0\,, \quad I(0, \mathbf{R}) = I_0(\mathbf{R})\,. \tag{14}$$

Equation (14) coincides in form with the continuity equation (11) for the tracer field in a random potential flow.

The commonly used methods of statistical averaging (i.e., the computation of mean values, such as $\langle \rho(\mathbf{r}, t)\rangle$ and $\langle \mathbf{H}(\mathbf{r}, t)\rangle$, spatio-temporal correlation functions, such as $\langle \rho(\mathbf{r}, t)\,\rho(\mathbf{r}', t')\rangle$ and $\langle H_i(\mathbf{r}, t) H_j(\mathbf{r}', t')\rangle$, where $\langle \ldots \rangle$ denotes averaging over an ensemble of realizations of random parameters) smooth the qualitative features of individual realizations, and often the statistical characteristics obtained in this way have nothing in common with the behavior of these realizations.

Therefore, the statistical characteristics of the above type generally characterize 'global' spatio-temporal scales of the domain hosting stochastic processes, but tell us nothing about the details of process development inside it.

Dynamical systems (11)–(13) are conservative and preserve integral characteristics such as the total mass of a tracer $M = \int \mathrm{d}\mathbf{r}\,\rho(\mathbf{r}, t)$, the magnetic field flux $\int \mathrm{d}\mathbf{r}\,\mathbf{H}(\mathbf{r}, t)$, and the wave field power $I = \int \mathrm{d}\mathbf{R}\, I(x, \mathbf{R})$. Under the assumptions of homogeneous initial conditions $\rho_0(\mathbf{r}) = \rho_0$, $\mathbf{H}_0(\mathbf{r}) = \mathbf{H}_0$, and $u_0(\mathbf{R}) = u_0$ and random parameters statistically homogeneous in space, the following equalities emerge as a consequence of the conservative character of dynamical systems (11)–(13):

$$\langle \rho(\mathbf{r}, t)\rangle = \rho_0\,, \quad \langle \mathbf{H}(\mathbf{r}, t)\rangle = \mathbf{H}_0\,, \quad \langle I(x, \mathbf{R})\rangle = I_0 = |u_0|^2\,.$$

In the general case, under homogeneous initial conditions, the statistical mean, for example, $\int \mathrm{d}\mathbf{r}\,\langle f(\mathbf{H}(\mathbf{r}, t))\rangle$ for the magnetic field, translates into the expression $\langle f(\mathbf{H}(\mathbf{r}, t))\rangle$, i.e., a one-to-one correspondence exists:

$$\int \mathrm{d}\mathbf{r}\,\big\langle f\big(\mathbf{H}(\mathbf{r}, t)\big)\big\rangle \Leftrightarrow \big\langle f\big(\mathbf{H}(\mathbf{r}, t)\big)\big\rangle\,.$$

Consequently, the quantity $\langle f(\mathbf{H}(\mathbf{r}, t))\rangle$ is specific, related to unit volume and is therefore an integral quantity. All integral quantities are independent of advection of the respective fields described by a flux (divergence) term in respective equations.

We note that the integral quantities are the main ones characterizing dynamical systems as a whole. Indeed, all conservation laws in the mechanics and electrodynamics of condensed media are written for integral quantities.

A specific feature of Eqns (11) and (12) is the parametric excitation with time of both the density field $\rho(\mathbf{r}, t)$ (for a compressible fluid flow) and the magnetic field energy $E(\mathbf{r}, t) = \mathbf{H}^2(\mathbf{r}, t)$ (for a turbulent fluid flow), which is called the *stochastic dynamo* (see, e.g., review paper [19] and monographs [20–28]).

Such a parametric excitation is accompanied at the initial stages of dynamical system evolution by the temporal growth of all the statistical characteristics of the problem solution, for example, all moments of density $\langle \rho^n(\mathbf{r}, t)\rangle$ and the magnetic field energy $\langle E^n(\mathbf{r}, t)\rangle$, and also the growth with distance of the moments of radiation power $\langle I^n(x, \mathbf{R})\rangle$ in random media.

At the initial phases of development, the effects of dynamical diffusion are not essential for the density and magnetic field, and neglecting them, we arrive at the first-order partial differential equations

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r}, t)\right)\rho(\mathbf{r}, t) = 0\,, \quad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,, \tag{15}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r}, t)\right)\mathbf{H}(\mathbf{r}, t) = \left(\mathbf{H}(\mathbf{r}, t)\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}(\mathbf{r}, t)\,, \quad \mathbf{H}(\mathbf{r}, 0) = \mathbf{H}_0(\mathbf{r})\,. \tag{16}$$

Stochastic equation (16) is actively studied in connection with the problem of the small-scale magnetic field dynamo — the amplification of magnetic field fluctuations by random motions of a medium with given correlation properties. In this case, stochastic nonstationary phenomena such as mixing and clustering unfold in the phase and physical spaces. The clustering of any field (density, magnetic field energy, and so on) is identified with the emergence of compact areas with large values of this field, on the background of areas where these values are fairly low. The lifetime of the clusters is limited. Their spatial pattern is permanently changing. We note that the similarity between problems of the magnetic dynamo and waves in random media was mentioned in [23, p. 54]. The manifestations of the effects of magnetic field clustering are well known. Indeed, the bulk of the plasma current in a stellarator is filled with stochastic plasma structures characterized by large field values having non-Gaussian distributions. A similar picture is observed in a tokamak plasma [29]. Granulation in the solar photosphere with locally large values of magnetic fields, which are attributed to the action of turbulent motions, represents, perhaps, one more example of the clustering phenomenon [30, 31].

Discovering and describing these phenomena is only possible via the analysis of one-point (in time and space) probability densities for solutions of the equations given above, based on the ideas of statistical topography.

1.1 Elements of the statistical topography of random fields

In the statistical topography of random fields, as in the topography of mountain terrains, the main object of study is the system of contours — isolines (in the two-dimensional case) or isosurfaces (in the three-dimensional case) defined by the equality $f(\mathbf{r}, t) = f = \text{const}$.

To analyze the system of contours (we limit ourselves in this section to two dimensions for simplicity, $\mathbf{r} = \mathbf{R}$), it is convenient to introduce the Dirac delta function concentrated on these contours,

$$\varphi(\mathbf{R}, t; f) = \delta\big(f(\mathbf{R}, t) - f\big)\,, \tag{17}$$

termed the *indicator function*.

Function (17), for example, allows expressing quantities such as the total area of regions bounded by isolines within which the random field $f(\mathbf{R}, t)$ exceeds a given level f, i.e., $f(\mathbf{R}, t) > f$,

$$S(t; f) = \int \theta\big(f(\mathbf{R}, t) - f\big)\,\mathrm{d}\mathbf{R} = \int_f^\infty \mathrm{d}f' \int \mathrm{d}\mathbf{R}\,\varphi(\mathbf{R}, t; f')\,,$$

and the total 'mass' of the field contained in these regions,

$$M(t; f) = \int f(\mathbf{R}, t)\,\theta\big(f(\mathbf{R}, t) - f\big)\,\mathrm{d}\mathbf{R} = \int_f^\infty f'\,\mathrm{d}f' \int \mathrm{d}\mathbf{R}\,\varphi(\mathbf{R}, t; f')\,,$$

where $\theta(f(\mathbf{R}, t) - f)$ is the Heaviside function.

The mean of indicator function (17) over an ensemble of realizations of the random field $f(\mathbf{R}, t)$ defines the one-point (in time and space) probability density [2–6, 32]

$$P(\mathbf{R}, t; f) = \big\langle \delta\big(f(\mathbf{R}, t) - f\big)\big\rangle\,;$$

hence, the ensemble means of $S(t; f)$ and $M(t; f)$ are directly defined by this probability density:

$$\big\langle S(t; f)\big\rangle = \int_f^\infty \mathrm{d}f' \int \mathrm{d}\mathbf{R}\,P(\mathbf{R}, t; f')\,,$$

$$\big\langle M(t; f)\big\rangle = \int_f^\infty f'\,\mathrm{d}f' \int \mathrm{d}\mathbf{R}\,P(\mathbf{R}, t; f')\,.$$

Additional information on the detailed structure of the field $f(\mathbf{R}, t)$ can be retrieved by including its spatial gradient $\mathbf{p}(\mathbf{R}, t) = \boldsymbol{\nabla} f(\mathbf{R}, t)$ into consideration. For example, the quantity

$$l(t; f) = \int \mathrm{d}\mathbf{R}\,\big|\mathbf{p}(\mathbf{R}, t)\big|\,\delta\big(f(\mathbf{R}, t) - f\big) = \oint \mathrm{d}l \tag{18}$$

gives the total length of contours. The integrand in Eqn (18) is described by the augmented indicator function

$$\varphi(\mathbf{R}, t; f, \mathbf{p}) = \delta\big(f(\mathbf{R}, t) - f\big)\,\delta\big(\mathbf{p}(\mathbf{R}, t) - \mathbf{p}\big) \tag{19}$$

and the mean of $l(t; f)$ [see Eqn (18)] is related to the joint one-point (in time and space) probability density of the field $f(\mathbf{R}, t)$ and its gradient $\mathbf{p}(\mathbf{R}, t)$ given by ensemble averaging of indicator function (19), i.e., by the function

$$P\big(\mathbf{R}, t; f, \mathbf{p}\big) = \big\langle \delta\big(f(\mathbf{R}, t) - f\big)\,\delta\big(\mathbf{p}(\mathbf{R}, t) - \mathbf{p}\big)\big\rangle\,.$$

Invoking second derivatives allows estimating the total number of contours $f(\mathbf{R}, t) = f = \text{const}$ with the help of the approximate formula (up to isolines that are not closed)

$$N(t; f) = N_{\text{in}}(t; f) - N_{\text{out}}(t; f) = \frac{1}{2\pi}\int \mathrm{d}\mathbf{R}\,\kappa(t, \mathbf{R}; f)\,\big|\mathbf{p}(\mathbf{R}, t)\big|\,\delta\big(f(\mathbf{R}, t) - f\big)\,, \tag{20}$$

where $N_{\text{in}}(t; f)$ and $N_{\text{out}}(t; f)$ are the numbers of contours for which the vector $\mathbf{p}$ is directed along the inner and outer normals and $\kappa(t, \mathbf{R}; f)$ is the curvature of an isoline.

We note that for a spatially homogeneous field $f(\mathbf{R}, t)$, when the one-point probability densities $P(\mathbf{R}, t; f)$ and $P(\mathbf{R}, t; f, \mathbf{p})$ are independent of $\mathbf{R}$, statistical means of all expressions (without integration over $\mathbf{R}$) describe specific (per unit area) values of these quantities. For example, the specific mean area $\langle S(t; f)\rangle$ over which the random field $f(\mathbf{R}, t)$ exceeds a given level f coincides with the probability of the event $f(\mathbf{R}, t) > f$ at any spatial point, i.e.,

$$\big\langle S(t; f)\big\rangle = \big\langle \theta\big(f(\mathbf{R}, t) - f\big)\big\rangle = \operatorname{Prob}\big\{f(\mathbf{R}, t) > f\big\}\,,$$

and therefore the mean specific area offers a geometric interpretation of the probability of the event $f(\mathbf{R}, t) > f$, which is apparently independent of the point $\mathbf{R}$.

In this case, by virtue of homogeneity and, in some other cases, isotropy, it may turn out that spatial derivatives of the field $f(\mathbf{R}, t)$ do not correlate with the field $f(\mathbf{R}, t)$ itself. Then the problem of computing the means of the functionals described above is essentially simplified, being reduced to computing the corresponding moment functions of the derivatives of $f(\mathbf{R}, t)$. All statistical characteristics considered above are integral and characterize the dynamics of stochastic systems as a whole in the entire space.

Furthermore, in the analysis of one-point statistical characteristics, it is convenient to use the statistical equivalence of the random field $f(\mathbf{R}, t)$ to a certain random process $y(t)$ with the same statistical characteristics. For such random processes, an important role is played by the concept of a *typical realization curve* and some properties of log-normal random process.

1.2 Typical realization curve of a random process

Statistical characteristics of a process $z(t)$ at a fixed time instant t are described by the probability density $P(z, t)$ and the probability distribution function $F(Z, t) = \int_{-\infty}^{Z} \mathrm{d}z'\,P(z', t)$.

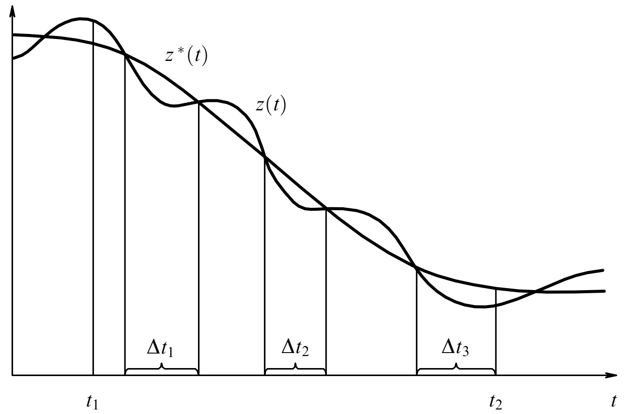

Figure 4. Toward defining a typical realization curve of a random process.

A deterministic curve $z^*(t)$ is called the typical realization of a random process $z(t)$ if it is the *median of the probability distribution function* and is determined as a solution of the algebraic equation

$$F\big(z^*(t), t\big) = \frac{1}{2} \,. \tag{21}$$

This means, on the one hand, that $\mathrm{Prob}\,\{z(t) > z^*(t)\} = \mathrm{Prob}\,\{z(t) < z^*(t)\} = 1/2$ for any t. On the other hand, the median has a specific property that for any interval (t_1, t_2), the random process $z(t)$ 'winds' around the curve $z^*(t)$ such that the mean time that the inequality $z(t) > z^*(t)$ holds coincides with the time during which the opposite inequality $z(t) < z^*(t)$ holds (Fig. 4), i.e.,

$$\left\langle T_{z(t)>z^*(t)} \right\rangle = \left\langle T_{z(t)<z^*(t)} \right\rangle = \frac{1}{2}(t_2 - t_1) \,.$$

Typical realization curve (21) for a Gaussian random process $z(t)$ coincides with the mean of the process $z(t)$, i.e., $z^*(t) = \langle z(t) \rangle$, while the typical realization curve for a log-normal random process $y(t) = \exp\,(z(t))$ is defined by the equality $y^*(t) = \exp\,\langle z(t) \rangle = \exp\,\langle \ln y(t) \rangle$.

The typical realization curve is therefore a deterministic curve with respect to which the intermittency is unfolding. But it carries no information about the amplitudes of excursions of the random process relative it.

1.3 Some properties of a log-normal random process

We defined a log-normal random process by formula (2). The one-time probability density for this process $P(t; y) = \langle \delta(y(t) - y) \rangle$ obeys the Fokker–Planck equation

$$\frac{\partial}{\partial t} P(y, t; \alpha) = \alpha \frac{\partial}{\partial y} y P(y, t; \alpha) + D \frac{\partial}{\partial y} y \frac{\partial}{\partial y} y P(y, t; \alpha) \,,$$
$$P(y, 0; \alpha) = \delta(y - 1) \,, \tag{22}$$

whose solution, naturally, depends on the parameter α:

$$P(y, t; \alpha) = \frac{1}{2y\sqrt{\pi D t}} \exp\left\{ -\frac{\ln^2 \left[y \exp\,(\alpha t) \right]}{4Dt} \right\} . \tag{23}$$

We mention that the one-time probability density of the random process $\tilde{y}(t; \alpha) = 1/y(t; \alpha)$ is also log-normal and is described by the formula

$$P(\tilde{y}, t; \alpha) = \frac{1}{2\tilde{y}\sqrt{\pi D t}} \exp\left\{ -\frac{\ln^2 \left[\tilde{y} \exp\,(-\alpha t) \right]}{4Dt} \right\} , \tag{24}$$

which is obtained from (23) by reversing the sign of α. For definiteness, we assume that $\alpha > 0$. Plots of logarithmically normal probability densities (23) and (24) for $\alpha/D = 1$ at dimensionless times $\tau = Dt = 0.1$ and 1 are shown in Fig. 5. We point out that the examples of realization of the random processes $y(t)$ and $\tilde{y}(t)$ in Fig. 1 are given just for this case.

These probability distributions have totally different structures. Their common feature is only the appearance of long shallow *tails* for $\tau = 1$, signaling an increased role of strong excursions of the processes $y(t; \alpha)$ and $\tilde{y}(t; \alpha)$ (see Fig. 1) in forming the one-time statistics. Figure 6 shows the behavior of the probability density for these processes for $\tau = 1$ on the logarithmic scale. Their probability distribution function, according to Eqns (23) and (24), is given by

$$F(y, t; \alpha) = \int_{-\infty}^{y} \mathrm{d}y'\, P(t; y') = \mathrm{Prob}\,\big(y(t; \alpha) < y\big)$$
$$= \mathrm{Pr}\left\{ \frac{1}{\sqrt{2Dt}} \ln \left[y \exp\,(\pm \alpha t) \right] \right\} , \tag{25}$$

where

$$\mathrm{Pr}\,(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{d}x \exp\left(-\frac{x^2}{2} \right) \tag{26}$$

is the *probability integral*. Obviously, $\mathrm{Pr}\,(\infty) = 1$ and $\mathrm{Pr}\,(0) = 1/2$.

We note that for $z < 0$, function (26) can be written as

$$\mathrm{Pr}\,\big(-|z|\big) = \frac{1}{\sqrt{2\pi}} \int_{|z|}^{\infty} \mathrm{d}x \exp\left(-\frac{x^2}{2} \right) = 1 - \mathrm{Pr}\,\big(|z|\big) \,. \tag{27}$$

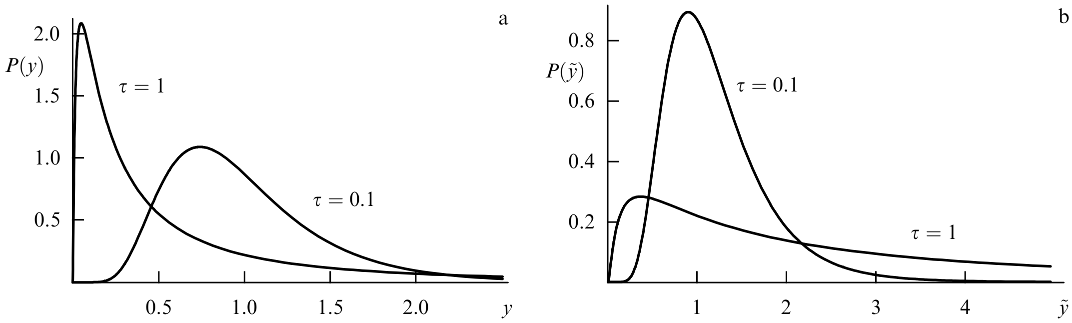

Figure 5. Log-normal probability distributions (23) (a) and (24) (b) for $\alpha/D = 1$ and the dimensionless time $\tau = 0.1$ and 1.

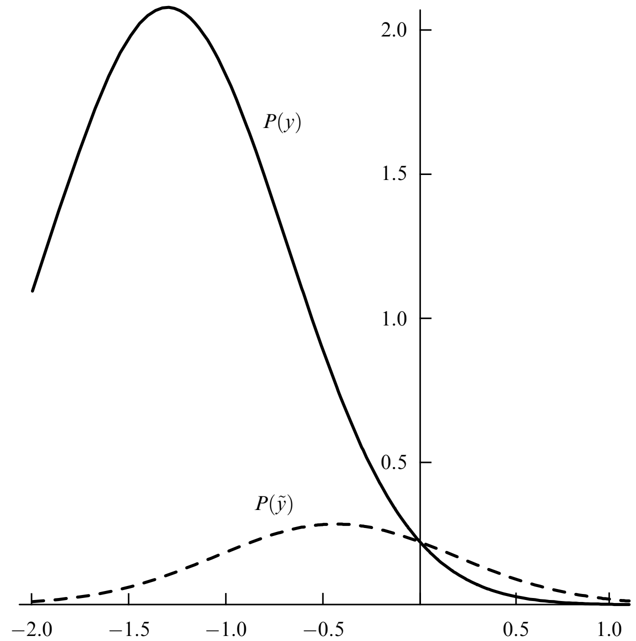

Figure 6. Probability densities of the processes $y(t;\alpha)$ (solid curve) and $\tilde{y}(t;\alpha)$ (dashed curve) for $\tau = 1$ on the logarithmic (decimal) scale.

Starting with expressions (26) and (27), we readily find the asymptotic form of the probability integral as $z \to \pm\infty$:

$$\Pr(z)_{z\to\infty} \approx 1 - \frac{1}{z\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right),$$
$$\Pr(z)_{z\to-\infty} \approx \frac{1}{|z|\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right). \tag{28}$$

It is also straightforward, starting with Eqn (22), to write equations for moment functions of the processes $y(t;\alpha)$ and $\tilde{y}(t;\alpha)$. The solutions of these equations (for $\tau = Dt$)

$$\left\langle y^n(\tau; a)\right\rangle = \exp\left[n\left(n - \frac{\alpha}{D}\right)\tau\right],$$
$$\left\langle \frac{1}{y^n(\tau; a)}\right\rangle = \exp\left[n\left(n + \frac{\alpha}{D}\right)\tau\right], \quad n = 1, 2, \ldots, \tag{29}$$

grow exponentially with time following practically the same laws. Hence, despite the apparent difference between probability densities for the quantities $y(t;\alpha)$ and $1/y(t;\alpha)$, their moment functions have a very similar form.

Using Eqn (22), it is a simple exercise to derive the equalities

$$\left\langle \ln y(t;\alpha)\right\rangle = -\alpha t, \qquad \left\langle \ln\frac{1}{y(t;\alpha)}\right\rangle = \alpha t. \tag{30}$$

Correspondingly, the parameter α in Eqn (30) is the Lyapunov characteristic exponent for the log-normal random process $y(t)$ (see, e.g., Refs [14, 33, 34]).

In addition to the Lyapunov exponent, we can also compute the *typical realization curves* characterizing the behavior of a random process in individual realizations. For log-normal processes $y(t;\alpha)$ and $\tilde{y}(t;\alpha)$ (assuming $\alpha > 0$), these curves coincide with the Lyapunov exponent and are described by the formulas

$$y^*(t) = \exp\left\langle \ln y(t)\right\rangle = \exp(-\alpha t),$$
$$\tilde{y}^*(t) = \exp\left\langle \ln \tilde{y}(t)\right\rangle = \exp(\alpha t), \tag{31}$$

which represent an exponentially decaying curve for the process $y(t;\alpha)$ and an exponentially growing one for $\tilde{y}(t;\alpha)$, despite the similarity in temporal growth laws for all moment functions of these quantities. Just these curves are plotted in Fig. 1 (dashed lines).

For the parameter $\alpha/D = 1$, the mean of the process $y(t;D)$ is independent of time and is equal to unity. But the probability that, for example, the inequality $y(t;D) < 1$ holds for $Dt \gg 1$ rapidly tends to unity, according to Eqn (25), following the law

$$\text{Prob}\left\{y(t;D) < 1\right\} = \Pr\left(\sqrt{\frac{Dt}{2}}\right)$$
$$= 1 - \frac{1}{\sqrt{\pi D t}} \exp\left(-\frac{Dt}{4}\right),$$

i.e., over most of the time axis, plots of the process realizations lie below its mean $\langle y(t;D)\rangle = 1$, while the respective probability of the event $\text{Prob}\{y(t;D) > 1\}$ tends to zero, although statistical moments of the process $y(t;D)$ are defined by its large outbursts.

We note that stationary probability distributions exist for areas $S_n(x) = \int_0^\infty d\tau\, y^n(\tau;\alpha)$, in particular, for $\alpha = D$: $P(S_1) = 1/(DS_1^2)\exp[-1/(DS_1^2)]$. This implies that large fluctuations of the log-normal process $y(t;\alpha)$ for $\alpha > 0$ are sufficiently narrow.

If $\alpha = 0$, then $y^*(t) = 1$, which corresponds to the random process $y(t) = \exp(w(t))$, where $w(t)$ is the Wiener process. In this case, the random processes $y(t)$ and $1/y(t)$ are statistically equivalent, while their probability density and probability distribution functions take the form

$$P(y,t) = \frac{1}{2y\sqrt{\pi D t}} \exp\left(-\frac{\ln^2 y}{4Dt}\right),$$
$$F(y,t) = \Pr\left(\frac{1}{\sqrt{2Dt}} \ln y\right). \tag{32}$$

Before passing to the analysis of problems described by partial differential equations, we consider a simple three-dimensional model of a random anisotropic velocity field that admits analytic treatment and consequently allows following the 'live' dynamics of these fields (the dynamics of clustering) on plots.

2. A simple anisotropic three-dimensional model of clustering of density and magnetic energy fields

To consider clustering of the density field in the framework of Eqn (15), a simple stochastic three-dimensional model of the velocity field was proposed in [35] in the form

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{v}(t)\, f(\mathbf{kr}), \tag{33}$$

where

$$f(\mathbf{kr}) = \sin 2(\mathbf{kr}), \tag{34}$$

and $\mathbf{v}(t)$ is a Gaussian random stationary vector process with the correlation tensor

$$\langle v_i(t)\, v_j(t') \rangle = 2\sigma^2 \delta_{ij} \tau_0\, \delta(t - t')\,, \quad (35)$$

where σ^2 is the variance for each velocity component and τ_0 is its temporal correlation radius. We note that this representation of the velocity field corresponds to the first term in the expansion of the magnetohydrodynamic velocity field in harmonics commonly used in numerical simulations of the problem.

Selecting the x axis in the direction of the vector $\mathbf{k}$, we see that in this model, the velocity field depends only on a single spatial variable, $f(\mathbf{kr}) = f(kx)$, and is therefore divergent and anisotropic.

2.1 Dynamical model description

2.1.1 Clustering of the density field. In this model, the dynamic equation for the density field with the homogeneous initial condition $\rho(\mathbf{r}, 0) = \rho_0$ leads to essential simplification and assumes the form

$$\frac{\partial}{\partial t}\, \rho(x, t) + v_x(t)\, \frac{\partial}{\partial x}\, f(kx)\, \rho(x, t) = 0\,. \quad (36)$$

The density field depends only on x and with function (34), the solution of Eqn (36) is described by the formula [2–5, 32, 35]

$$\frac{\rho(x, t)}{\rho_0} = \frac{1}{\exp\big(T(t)\big) \cos^2(kx) + \exp\big(-T(t)\big) \sin^2(kx)}\,, \quad (37)$$

where

$$T(t) = 2k \int_0^t \mathrm{d}\tau\, v_x(\tau) \quad (38)$$

is the Wiener random process with the parameters following from Eqn (35):

$$\langle T(t) \rangle = 0\,, \qquad \sigma_T^2(t) = 8k^2 \sigma^2 \tau_0 t\,. \quad (39)$$

Figure 7 presents a portion of the realization of the random process $T(t)$ obtained by numerical integration of Eqn (38) for a certain realization of the random process $v_x(t)$ used to compute the field $\rho(x, t)$ in accordance with Eqn (37). Figure 8 shows computation results for the spatio-temporal evolution of the realization of the Eulerian density field $1 + \rho(x, t)/\rho_0$ (the unit is added to eliminate problems with density values close to zero in logarithmic scaling) in dimensionless variables

$$t \to k^2 \sigma^2 \tau_0 t\,, \quad x \to kx\,, \quad \langle v_i(t)\, v_j(t') \rangle \to 2\delta_{ij}\, \delta(t - t')\,. \quad (40)$$

It clearly shows a gradual concentration of the density field in narrow vicinities of the points $x \approx 0$ and $x \approx \pi/2$, i.e., the formation of clusters in which the relative density value reaches the order of $10^3 - 10^4$, while it stays practically equal to zero over the remaining space. Notably, at instants t such that $T(t) = 0$, the realization of the density field passes through the initial homogeneous state.

We also note that the quantity given by Eqn (37), after averaging over the spatial variable,

$$\overline{\left(\frac{\rho(x, t)}{\rho_0}\right)} = 1\,,$$

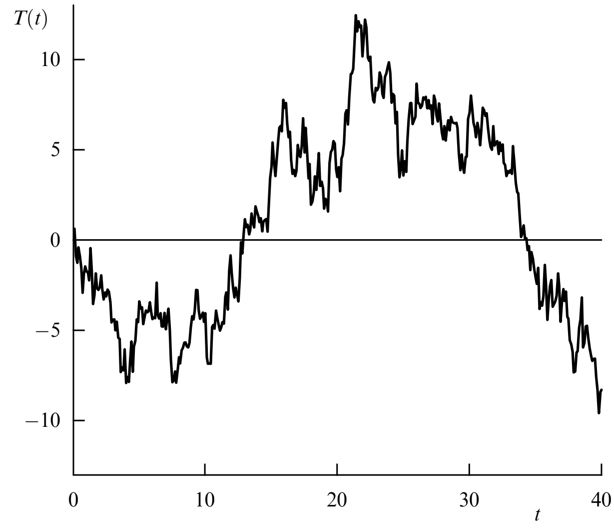

Figure 7. A portion of the realization of the random process $T(t)$.

is independent of the random factor $T(t)$, while for the squared density field, we obtain the expression

$$\overline{\left(\frac{\rho^2(x, t)}{\rho_0^2}\right)} = \frac{1}{2}\left[\exp\big(T(t)\big) + \exp\big(-T(t)\big)\right]$$

that almost always grows with time. For a Gaussian random process $v_x(t)$ with correlation function (35), upon averaging the last equality over an ensemble of realizations of the Wiener process $T(t)$, we use Eqn (39) to find the expression exponentially growing with time:

$$\left\langle \overline{\frac{\rho^2(x, t)}{\rho_0^2}} \right\rangle = \exp\left(\frac{1}{2}\, \sigma_T^2(t)\right) = \exp\,(4k^2 \sigma^2 \tau_0 t)\,.$$

In this case, for the inverse of the density field,

$$\frac{\rho_0}{\rho(x, t)} = \exp\big(T(t)\big) \cos^2(kx) + \exp\big(-T(t)\big) \sin^2(kx)\,,$$

after spatial averaging, we obtain the expressions

$$\overline{\left(\frac{\rho_0}{\rho(x, t)}\right)} = \frac{1}{2}\left[\exp\big(T(t)\big) + \exp\big(-T(t)\big)\right],$$

$$\overline{\left(\frac{\rho_0^2}{\rho^2(x, t)}\right)} = \frac{1}{4} + \frac{3}{8}\left[\exp\big(2T(t)\big) + \exp\big(-2T(t)\big)\right],$$

which also almost always grow with time. After averaging over an ensemble of realizations of the Wiener process $T(t)$, we obtain expressions exponentially growing with time.

2.1.2 Clustering of the magnetic field energy. The same model of the velocity field was used in Ref. [36] to illustrate the clustering of the magnetic field energy in the framework of Eqn (16). In this case, explicit expressions for the magnetic field can also be obtained that resemble those for the tracer density. For a homogeneous initial condition $\mathbf{H}(\mathbf{r}, 0) = \mathbf{H}_0$, the magnetic field, just like the tracer field, depends on a single coordinate x, i.e., $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(x, t)$, and induction

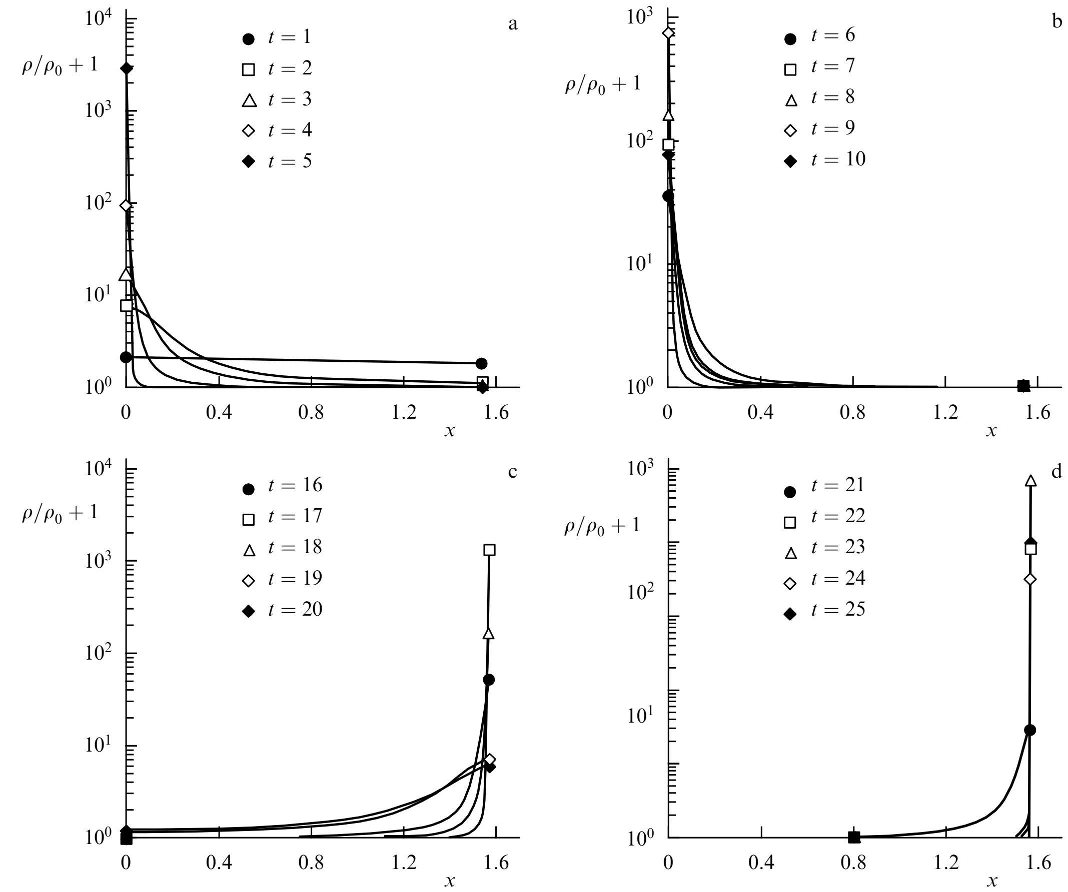

Figure 8. Spatio-temporal evolution of the Eulerian density field.

equation (16) then becomes

$$\left(\frac{\partial}{\partial t} + v_x(t)\sin 2(kx)\,\frac{\partial}{\partial x}\right)\mathbf{H}(x,t) = 2k\cos 2(kx)\left[\mathbf{v}(t)\,H_x(x,t) - v_x(t)\,\mathbf{H}(x,t)\right].$$

It follows from this equation that the x-component of the magnetic field is conserved [i.e. $H_x(x,t) = H_{x0}$], and that an additional source of the magnetic field (generation) appears in the transverse plane (y,z) because of the presence of H_{x0}:

$$\left(\frac{\partial}{\partial t} + v_x(t)\sin 2(kx)\,\frac{\partial}{\partial x}\right)\mathbf{H}_\perp(\mathbf{r},t) = 2k\cos 2(kx)\left(\mathbf{v}_\perp(t)H_{x0} - v_x(t)\,\mathbf{H}_\perp(x,t)\right), \qquad (41)$$
$$\mathbf{H}_\perp(x,0) = \mathbf{H}_{\perp 0}.$$

Solving Eqn (41) by the method of characteristics, it is possible to show that its solution is *statistically equivalent* to the expression [36]

$$\mathbf{H}_\perp(x,t) = \frac{\rho(x,t)}{\rho_0}\,\mathbf{H}_{\perp 0} + 2kH_{x0}\int_0^t \mathrm{d}\tau\,\frac{\exp\left(T(\tau)\right)\cos^2(kx) - \exp\left(-T(\tau)\right)\sin^2(kx)}{\left[\exp\left(T(\tau)\right)\cos^2(kx) + \exp\left(-T(\tau)\right)\sin^2(kx)\right]^2} \times v_x(\tau)\,\mathbf{v}_\perp(\tau), \qquad (42)$$

where the density field is described by formula (37). Statistical equivalence implies that all one-point (in time and space) characteristics of the solution of Eqn (41) coincide with the statistical characteristics of (42).

The first term in Eqn (42) describes magnetic field clustering, resembling that of the density field if $H_{\perp 0} \neq 0$. The second term describes the generation of a magnetic field in the transverse (y,z) plane due to the presence of the initial field H_{x0}. When $H_{\perp 0} = 0$, this term, proportional to the squared random velocity field, yields the answer. The structure of this field, like that of the density field, also undergoes clustering.

Figure 9 displays simulation results pertaining to the spatio-temporal evolution of a realization of the generated magnetic field energy in the transverse plane $E(x,t) = \mathbf{H}_\perp^2(x,t)$ in dimensionless variables (40) for $H_{\perp 0} = 0$, and the dynamics of the transfer of magnetic energy perturbations from one domain boundary to another for the same realization of the random process $T(t)$ (see Fig. 7).

We note that the total energy of the generated magnetic field over the interval $[0, \pi/2]$ rapidly grows with time.

Therefore, using the simple model of a divergent random velocity field, which yields to complete analysis, we clearly see processes of density field clustering and magnetic field energy generation (magnetic dynamo). The energy is confined to narrow spatial structures, or in other words, clustering occurs here.

2.2 Statistical model description

We now consider a general model of velocity field (33), where $f(kx)$ is a periodic function. In this case, as previously, for homogeneous initial conditions $\rho(\mathbf{r},0) = \rho_0$

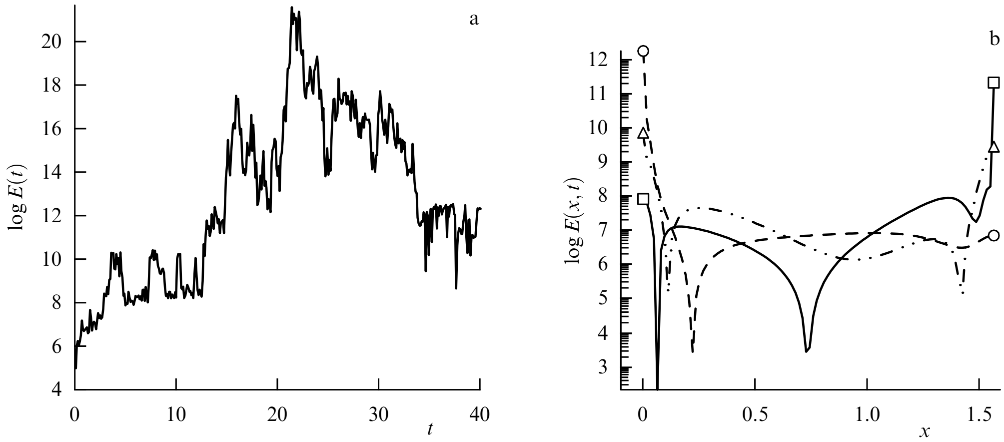

Figure 9. (a) Time evolution of the total magnetic field energy in the interval $[0, \pi/2]$ and (b) the dynamics of cluster disappearance at point 0 and emergence at $\pi/2$. The circles, triangles, and squares respectively correspond to time instants $t = 10.4$, 10.8, and 11.8.

and $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0$, all functions depend on a single spatial coordinate x: $\rho(\mathbf{r}, t) = \rho(x, t)$ and $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(x, t)$.

We note that when determining various statistical averages connected with equations, we need in the general case to split correlations of a random Gaussian field $\mathbf{u}(\mathbf{r}, t)$ with functionals of this field. For a Gaussian field that is spatially homogeneous and stationary, with the correlation function $B_{kl}(\mathbf{r} - \mathbf{r}', t - t') = \langle u_k(\mathbf{r}, t)\, u_l(\mathbf{r}', t')\rangle$, such splitting is based on the Furutsu–Novikov formula (see, e.g., Refs [2–6])

$$\big\langle u_k(\mathbf{r}, t)\, R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big]\big\rangle = \int_0^t \mathrm{d}t' \int \mathrm{d}\mathbf{r}'\, B_{kl}(\mathbf{r} - \mathbf{r}', t - t') \left\langle \frac{\delta R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big]}{\delta u_l(\mathbf{r}', t')} \right\rangle, \quad (43)$$

where $0 \leqslant \tau \leqslant t$. In the approximation of the Gaussian field delta correlated in time,

$$B_{kl}(\mathbf{r}, t) = 2B_{kl}(\mathbf{r})\,\delta(t)\,, \qquad B_{kl}(\mathbf{r}) = \int_0^\infty \mathrm{d}t\, B_{kl}(\mathbf{r}, t)\,, \quad (44)$$

the general formula is simplified and takes the form

$$\big\langle u_k(\mathbf{r}, t)\, R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big]\big\rangle = \int \mathrm{d}\mathbf{r}'\, B_{kl}(\mathbf{r} - \mathbf{r}') \left\langle \frac{\delta R\big[t; \mathbf{u}(\mathbf{r}, \tau)\big]}{\delta u_l(\mathbf{r}', t - 0)} \right\rangle, \quad (45)$$

where $0 \leqslant \tau \leqslant t$.

2.2.1 Clustering of the density field. We first note that for the simple model of velocity field (33), the one-dimensional passive tracer diffusion problem in (36) can be solved for any statistics of the random process $v_x(t)$, without assuming it to be Gaussian or delta correlated. This is because the functional dependence of the problem solution $\rho(x, t)$ on the random process $v_x(t)$ has the structure [2–6]

$$\rho(x, t) = \rho\big(x, t; v(\tau)\big) = \rho\big(x, T(t)\big)\,,$$

where the new 'random' time $T(t) = \int_0^t \mathrm{d}\tau\, v_x(\tau)$ is introduced. Equation (36) in this case acquires a deterministic form:

$$\frac{\partial}{\partial T}\,\rho(x, T) + \frac{\partial}{\partial x}\, f(kx)\,\rho(x, T) = 0\,.$$

Consequently, for the variational derivative of the field $\rho(x, t)$, we obtain

$$\frac{\delta\rho\big[x, t; v(\tau)\big]}{\delta v(t')} = \frac{\partial\rho(x, T)}{\partial T}\,\theta(t - t') = -\theta(t - t')\,\frac{\partial}{\partial x}\, f(kx)\,\rho(x, t)\,,$$

where $\theta(t)$ is the Heaviside function. The variational derivative of the random field $\rho(x, t)$ is thus expressed in terms of an ordinary spatial derivative of this field.

To obtain the one-time probability density of the random field $\rho(x, t)$, we introduce its indicator function $\varphi(x, t; \rho) = \delta(\rho(x, t) - \rho)$. This function, according to the general method described in [2–6], satisfies the Liouville equation

$$\left(\frac{\partial}{\partial t} + v_x(t)\, f(kx)\,\frac{\partial}{\partial x}\right)\varphi(x, t; \rho) = v_x(t)\,\frac{\partial f(kx)}{\partial x}\,\frac{\partial}{\partial\rho}\big[\rho\varphi(x, t; \rho)\big] \quad (46)$$

with the initial condition $\varphi(x, 0; \rho) = \delta(\rho_0 - \rho)$. For simplicity, we assume, for example, that $v_x(t)$ is a Gaussian random stationary process with the parameters

$$\big\langle v(t)\big\rangle = 0\,,$$
$$B_v(t - t') = \big\langle v_x(t)\, v_x(t')\big\rangle \quad \big(\sigma^2 = B_v(0) = \big\langle v_x^2(t)\big\rangle\big)\,.$$

Then averaging Liouville equation (46) over an ensemble of realizations of the random process $v_x(t)$ and using Eqn (43), we obtain an equation for the probability density of the density field $P(x, t; \rho) = \langle\varphi(x, t; \rho)\rangle$, written as

$$\frac{\partial}{\partial t}\, P(x, t;\, \rho) = \int_0^t \mathrm{d}t'\, B(t') \bigg[\frac{\partial}{\partial x}\, f^2(x)\,\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\, f(x)\,\frac{\mathrm{d}f(x)}{\mathrm{d}x}\,\frac{\partial}{\partial\rho}\,\rho - \frac{\mathrm{d}f(x)}{\mathrm{d}x}\, f(x)\,\frac{\partial}{\partial x}\left(1 + \frac{\partial}{\partial\rho}\,\rho\right) + \left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2 \frac{\partial}{\partial\rho}\,\rho\left(1 + \frac{\partial}{\partial\rho}\,\rho\right)\bigg] P(x, t;\, \rho)\,,$$

with the initial condition $P(0, x; \rho) = \delta(\rho_0 - \rho)$.

We note that if the function $f(x)$ has a characteristic variability scale k^{-1} along x and is a periodic function ('fast variability'), then additionally averaging over this scale (with respect to x), we obtain an equation for 'slow' spatial variations of the probability density that is independent of x:

$$\frac{\partial}{\partial t} P(t;\rho) = \overline{\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2} \int_0^t \mathrm{d}t'\, B(t')\, \frac{\partial^2}{\partial \rho^2}\, \rho^2 P(t;\rho)\,,$$
$$P(0;\rho) = \delta(\rho_0 - \rho)\,.$$

This equation corresponds to the probability density of a log-normal random process. For $t \gg \tau_0$, this equation is simplified to

$$\frac{\partial}{\partial t} P(t;\rho) = D_\rho \frac{\partial^2}{\partial \rho^2}\, \rho^2 P(t;\rho)\,, \qquad P(0;\rho) = \delta(\rho_0 - \rho)\,, \quad (47)$$

where the diffusion coefficient D_ρ in the ρ-space is

$$D_\rho = \overline{\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2} \int_0^\infty \mathrm{d}t'\, B(t') = \sigma^2 \tau_0 \overline{\left(\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right)^2}\,,$$

and τ_0 is the correlation radius of the random process $v(t)$ in time.

Equation (47) with the diffusion coefficient D_ρ coincides with Eqn (22) for $\alpha = D = D_\rho$, and just the solution of that equation is plotted in Figs 5 and 6. The normalized random density field $\rho(x,t)/\rho_0$ in this case for one-point (in time and space) statistical characteristics coincides with the statistical characteristics of the random process $y(t;\alpha)$, i.e., the spatially homogeneous field $\rho(x,t)/\rho_0$ is *statistically equivalent* to the random process $y(t;\alpha)$.

In this case, the density moment functions take the form $(\tau = D_\rho t)$

$$\left\langle \rho^n(x,t) \right\rangle = \rho_0^n \exp\big(n(n-1)\tau\big)\,,$$
$$\left\langle \frac{1}{\rho^n(x,t)} \right\rangle = \frac{1}{\rho_0^n} \exp\big(n(n+1)\tau\big)\,, \quad n = 1, 2, \ldots,$$

and grow exponentially with time according to virtually the same laws.

For the density field $\rho(x,t)$ and its inverse $1/\rho(x,t)$, we have the respective typical realization curves $(\tau = D_\rho t)$

$$\rho^*(x,t) = \rho_0 \exp(-\tau)\,, \qquad \frac{\rho_0}{\rho^*(x,t)} = \exp\tau \qquad (48)$$

for an arbitrary point x. The first equality points to clustering of the field $\rho(x,t)$ with time and the second indicates that the field $1/\rho(x,t)$ grows with time at any spatial location.

Exponential growth of moments of the random processes $\rho(x,t)$ and $1/\rho(x,t)$ is maintained by fluctuations of these processes relative to typical realization curves $\rho^*(x,t)$ and $1/\rho^*(x,t)$ toward large values of $\rho(x,t)$ and $1/\rho(x,t)$.

2.2.2 Clustering of the magnetic field energy. Passing to dimensionless variables (40), we rewrite Eqn (41) for the transverse component of the magnetic field induction as

$$\left(\frac{\partial}{\partial t} + v_x(t)\frac{\partial}{\partial x} f(x)\right)\mathbf{H}(x,t) = \mathbf{v}(t) f'(x) H_{x0}\,, \qquad (49)$$
$$\mathbf{H}(x,0) = \mathbf{H}_0$$

(the label indicating that the field is transverse to the x-axis is omitted here and hereafter).

We introduce the indicator function of the magnetic field

$$\varphi(x,t;\mathbf{H}) = \delta\big(\mathbf{H}(x,t) - \mathbf{H}\big)\,.$$

Using the standard procedure for first-order partial differential equations [2–6], we can derive the Liouville equation, which is equivalent to the original problem:

$$\frac{\partial}{\partial t}\varphi(x,t;\mathbf{H})$$
$$= \left[-\frac{\partial}{\partial x} f(x) + f'(x)\left(1 + \frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right)\right] v_x(t)\,\varphi(x,t;\mathbf{H})$$
$$- f'(x) H_{x0}\, v_i(t)\frac{\partial}{\partial H_i}\varphi(x,t;\mathbf{H})\,, \qquad (50)$$

subject to the initial conditions $\varphi(x,0;\mathbf{H}) = \delta(\mathbf{H}_0 - \mathbf{H})$.

To obtain the magnetic field probability density $P(x,t;\mathbf{H}) = \langle \varphi(\mathbf{x},\mathbf{t};\mathbf{H})\rangle$, we average Eqn (50) over an ensemble of random vector functions $\mathbf{v}(t)$, which is supposed to be delta-correlated in time in what follows. As a result, using expression (45), we obtain the equation for the probability density:

$$\frac{\partial}{\partial t} P(x,t;\mathbf{H})$$
$$= \frac{\partial}{\partial x}\left(f^2(x)\frac{\partial}{\partial x} - f(x) f'(x)\frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right) P(x,t;\mathbf{H})$$
$$+ \left(1 + \frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right)\left(-f(x)f'(x)\frac{\partial}{\partial x} + \big[f'(x)\big]^2\frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right) P(x,t;\mathbf{H})$$
$$+ \big[f'(x)\big]^2 E_{x0}\frac{\partial}{\partial H_i}\frac{\partial}{\partial H_i} P(x,t;\mathbf{H})\,.$$

If the function $f(x)$ has a characteristic variability scale k^{-1} in x and is periodic ('fast variability'), then, additionally averaging this equation over this scale (with respect to x), we obtain the equation for 'slow' spatial variability:

$$\frac{\partial}{\partial t} P(x,t;\mathbf{H}) = \frac{\partial}{\partial x}\left(\overline{f^2(x)}\frac{\partial}{\partial x}\right) P(x,t;\mathbf{H})$$
$$+ \overline{\big[f'(x)\big]^2}\left(1 + \frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right)\frac{\partial}{\partial \mathbf{H}}\mathbf{H} P(x,t;\mathbf{H})$$
$$+ \overline{\big[f'(x)\big]^2} E_{x0}\frac{\partial^2}{\partial \mathbf{H}^2} P(x,t;\mathbf{H})\,.$$

Recalling now that the function $P(x,t;\mathbf{H})$ is independent of x by virtue of the initial conditions, we obtain

$$\frac{\partial}{\partial t} P(t;\mathbf{H})$$
$$= D\left(\frac{\partial}{\partial \mathbf{H}}\mathbf{H} + \frac{\partial}{\partial \mathbf{H}}\mathbf{H}\frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right) P(t;\mathbf{H}) + D E_{x0}\frac{\partial^2}{\partial \mathbf{H}^2} P(t;\mathbf{H})\,,$$

where $D = \overline{\big[f'(x)\big]^2}$. Normalizing the time as $t \to D k^2 \sigma^2 \tau_0 t$, we finally arrive at the equation

$$\frac{\partial}{\partial t} P(t;\mathbf{H}) = \left(\frac{\partial}{\partial \mathbf{H}}\mathbf{H} + \frac{\partial}{\partial \mathbf{H}}\mathbf{H}\frac{\partial}{\partial \mathbf{H}}\mathbf{H}\right) P(t;\mathbf{H})$$
$$+ E_{x0}\frac{\partial^2}{\partial \mathbf{H}^2} P(t;\mathbf{H})\,. \qquad (51)$$

We now derive an equation for the probability density of the generated energy of the transverse magnetic field $E(t) = \mathbf{H}^2(t)$. For this, we multiply both sides of Eqn (51) by $\delta(\mathbf{H}^2 - E)$ and integrate it over $\mathbf{H}$. As a result, we obtain the equation

$$\frac{\partial}{\partial t} P(t;E) = \left(2\frac{\partial}{\partial E}E + 4\frac{\partial}{\partial E}E\frac{\partial}{\partial E}E\right)P(t;E) + 4E_{x0}\frac{\partial}{\partial E}E\frac{\partial}{\partial E}P(t;E)\,. \tag{52}$$

Multiplying Eqn (52) by E and integrating over E, we obtain the equation for the mean generated part of the transverse magnetic field energy:

$$\frac{\partial}{\partial t}\langle E(t)\rangle = 2\langle E(t)\rangle + 4E_{x0}\,, \qquad \langle E(0)\rangle = 0\,. \tag{53}$$

Its solution $\langle E(t)\rangle = 2E_{x0}[\exp(2t) - 1]$ demonstrates exponential growth of the mean energy.

We normalize the energy with E_{x0}. As a result, the equation becomes

$$\frac{\partial}{\partial t} P(t;E) = \left(2\frac{\partial}{\partial E}E + 4\frac{\partial}{\partial E}E\frac{\partial}{\partial E}E\right)P(t;E) + 4\frac{\partial}{\partial E}E\frac{\partial}{\partial E}P(t;E) \tag{54}$$

with the initial condition

$$P(0;E) = \delta(E-\beta)\,,$$

where we introduce the parameter $\beta = E_{\perp 0}/E_{x0}$.

It should be borne in mind that (54) is a boundary value problem and its boundary conditions for E are that the probability density is zero at $E = 0$ and $E = \infty$ for any $t > 0$.

In the general case, for the nth moment of the energy of the transverse magnetic field, we obtain the equation

$$\frac{\partial}{\partial t}\langle E^n(t)\rangle = 2n(2n-1)\langle E^n(t)\rangle + 4n^2\langle E^{n-1}(t)\rangle\,,$$

$$\langle E^n(0)\rangle = \beta^n\,.$$

Its solution for $\beta = 0$ and $t \gg 1$ behaves as

$$\langle E^n(t)\rangle = \frac{2^n(n!)^2}{(4n-3)!!}\exp\left[2n(2n-1)t\right]\,, \quad n = 1, 2, \ldots\,.$$

Notably, in the general case of Eqn (54) for the probability distribution, there exists a stationary distribution with infinite moments that is independent of the parameter β:

$$P(\infty;E) = \frac{1}{2(E+1)^{3/2}}\,. \tag{55}$$

It should be remembered that this distribution is a generalized function and is valid on the interval $(0,\infty)$. The probability density must be equal to zero at the point $E = 0$.

We note that for $\beta = 1$ and $E_{x0} = 0$, the probability density for the energy normalized by $E_{\perp 0}$ is described by Eqn (54) in the absence of the last term and, consequently, the probability distribution is log-normal:

$$P(t;E) = \frac{1}{4\sqrt{\pi t}\,E}\exp\left\{-\frac{1}{16t}\ln^2\left[E\exp(2t)\right]\right\}. \tag{56}$$

This structure of the probability density apparently dominates only for very small times when the generating term is still unessential. Figure 10 displays solutions of Eqn (54) (solid curves) obtained by numerical integration for $\beta = 1$ and $t = 0.05$ and $t = 0.25$. The dashed curves show log-normal distribution (56). It follows from Fig. 10 that already at small times, the term in Eqn (54) providing energy generation prevails. Figure 11 presents the solution of Eqn (54) (i.e., the probability density of the generated magnetic energy) with the parameter $\beta = 0$ for $t = 10$ and asymptotic form (55) (solid and dashed curves, respectively). It is seen that only the 'tail' of the solution of Eqn (54) tends to asymptotic solution (55) (Fig. 11a), but it does so very slowly (Fig. 11b). The intersection of these curves occurs because of the normalization of the probability densities considered here. It is likely that the probability density $P(t;E)$ further approaches the tail of the stationary distribution.

We also mention that in the general case, $\langle \ln E(t)\rangle$ is independent of the last (diffusion) term in Eqn (54) and, the same as for the log-normal distribution, the Lyapunov exponent becomes (see, e.g., Refs [14, 33, 34])

$$\langle \ln E(t)\rangle = \ln\beta - 2t\,.$$

As $\beta \to +0$, this exponent tends to minus infinity, which indicates the absence of a magnetic field at practically any location in space, i.e., shows the clustering. In this case, the typical realization curve cannot be calculated.

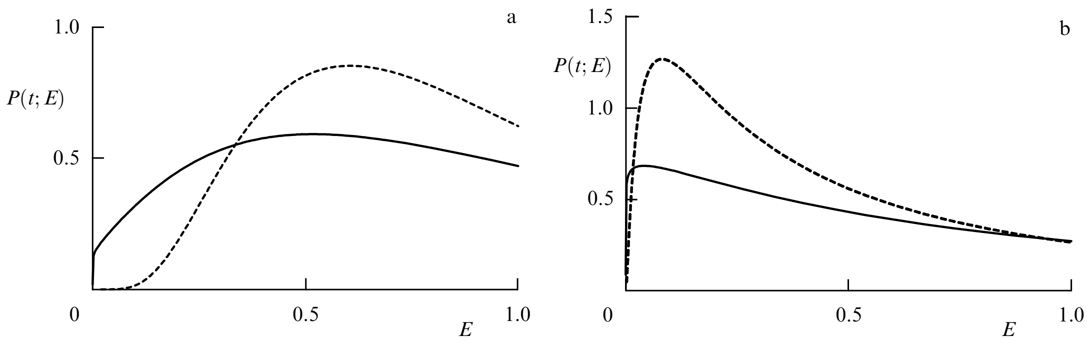

Figure 10. Probability density distribution for $\beta = 1$ at time instants (a) $t = 0.05$ and (b) $t = 0.25$.

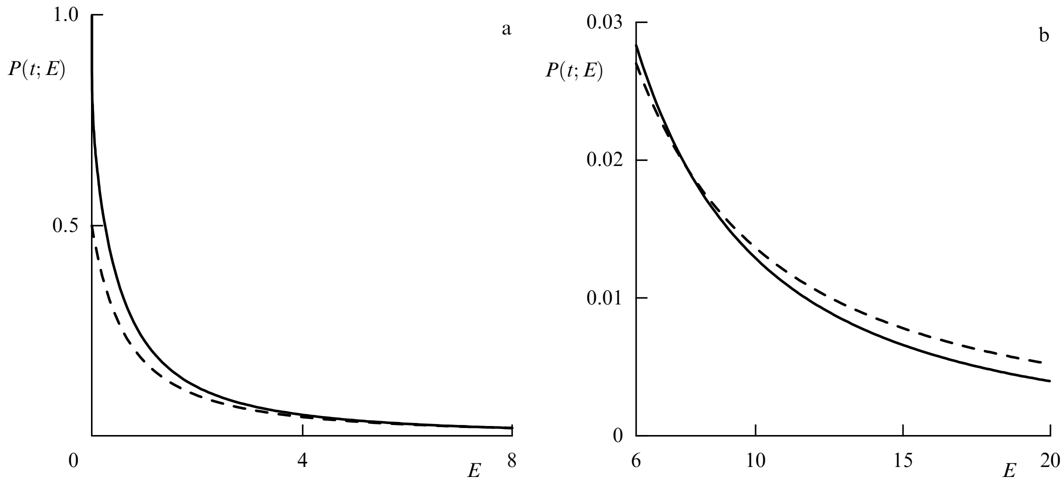

Figure 11. Probability density distribution for $\beta = 0$ and $t = 10$ in the intervals $(0, 8)$ (a) and $(6, 20)$ (b) of energy values.

3. Isotropic model of the random magnetohydrodynamic velocity field

3.1 Gaussian vector random field

We now consider the basic spatio-temporal statistical characteristics of the Gaussian vector random field $\mathbf{u}(\mathbf{r}, t)$ with zero mean and the correlation tensor

$$B_{ij}(\mathbf{r}, t; \mathbf{r}', t') = \left\langle u_i(\mathbf{r}, t)\, u_j(\mathbf{r}', t') \right\rangle .$$

For completeness, the random field $\mathbf{u}(\mathbf{r}, t)$ is in general assumed divergent ($\operatorname{div} \mathbf{u}(\mathbf{r}, t) \neq 0$), Gaussian, and statistically homogeneous with spherical symmetry, but not mirror symmetry in space, being stationary and having the correlation and spectral tensors ($\tau = t - t_1$)

$$B_{ij}(\mathbf{r} - \mathbf{r}_1, \tau) = \left\langle u_i(\mathbf{r}, t)\, u_j(\mathbf{r}_1, t_1) \right\rangle$$

$$= \int \mathrm{d}\mathbf{k}\, E_{ij}(\mathbf{k}, \tau) \exp\left[\mathrm{i}\mathbf{k}(\mathbf{r} - \mathbf{r}_1)\right] ,$$

$$E_{ij}(\mathbf{k}, \tau) = \frac{1}{(2\pi)^d} \int \mathrm{d}\mathbf{r}\, B_{ij}(\mathbf{r}, \tau) \exp(-\mathrm{i}\mathbf{k}\mathbf{r}) ,$$

where d is the space dimension. By virtue of symmetry assumptions, the correlation tensor $B_{ij}(\mathbf{r} - \mathbf{r}_1, \tau)$ has the vector structure ($\mathbf{r} - \mathbf{r}_1 \to \mathbf{r}$) [38]

$$B_{ij}(\mathbf{r}, \tau) = B_{ij}^{\mathrm{iso}}(\mathbf{r}, \tau) + C(r, \tau)\varepsilon_{ijk}\, r_k , \tag{57}$$

where

$$B_{ij}^{\mathrm{iso}}(\mathbf{r}, \tau) = A(r, \tau)\, r_i r_j + B(r, \tau)\, \delta_{ij}$$

is the isotropic part of the correlation tensor and ε_{ijk} is the pseudotensor taking the values $\varepsilon_{ijk} = 0$ if some of the indices i, j, and k coincide, and $\varepsilon_{ijk} = \pm 1$ if all the indices i, j, and k are different and are arranged in cyclic or anticyclic order (see, e.g., Ref. [38]). The product of two pseudotensors is already a tensor, and the following identity holds:

$$\varepsilon_{ilm}\varepsilon_{jpq} = \delta_{ij}\delta_{lp}\delta_{mq} + \delta_{ip}\delta_{lq}\delta_{mj} + \delta_{iq}\delta_{lj}\delta_{mp} - \delta_{ij}\delta_{lq}\delta_{mp} - \delta_{ip}\delta_{lj}\delta_{mq} - \delta_{iq}\delta_{lp}\delta_{mj} ; \tag{58}$$

hence, for $j = m$ (with summation implied over repeating indices),

$$\varepsilon_{ilm}\varepsilon_{mpq} = (d - 2)(\delta_{ip}\delta_{lq} - \delta_{iq}\delta_{lp}) , \tag{59}$$

and therefore the contraction vanishes in two dimensions.

The isotropic part of the correlation tensor is associated with the spatial spectral tensor having the form

$$E_{ij}^{\mathrm{iso}}(\mathbf{k}, \tau) = E_{ij}^{\mathrm{s}}(\mathbf{k}, \tau) + E_{ij}^{\mathrm{p}}(\mathbf{k}, \tau) ,$$

where the spectral components of the velocity field tensor have the structure

$$E_{ij}^{\mathrm{s}}(\mathbf{k}, \tau) = E^{\mathrm{s}}(k, \tau)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) , \quad E_{ij}^{\mathrm{p}}(\mathbf{k}, \tau) = E^{\mathrm{p}}(k, \tau)\, \frac{k_i k_j}{k^2} .$$

Here, $E^{\mathrm{s}}(k, \tau)$ and $E^{\mathrm{p}}(k, \tau)$ denote the solenoidal and potential components of the spectral density of the velocity field.

We now define the function $B_{ij}(\mathbf{r})$ as the time integral of correlation function (57):

$$B_{ij}(\mathbf{r}) = \int_0^{\infty} \mathrm{d}\tau\, B_{ij}(\mathbf{r}, \tau) = B_{ij}^{\mathrm{iso}}(\mathbf{r}) + C(r)\varepsilon_{ijk}\, r_k . \tag{60}$$

Then $B_{ij}(\mathbf{0}) = D_0 \delta_{ij}$, and therefore the quantity

$$B_{ii}(\mathbf{0}) = D_0 d = \tau_0 \sigma_{\mathbf{u}}^2 = \int \mathrm{d}\mathbf{k} \left[(d - 1)\, E^{\mathrm{s}}(k) + E^{\mathrm{p}}(k)\right] \tag{61}$$

defines the time correlation radius of the velocity field τ_0, where $\sigma_{\mathbf{u}}^2 = B_{ii}(\mathbf{0}, 0) = \langle \mathbf{u}^2(\mathbf{r}, t) \rangle$ is its variance and

$$E^{\mathrm{s}}(k) = \int_0^{\infty} \mathrm{d}\tau\, E^{\mathrm{s}}(k, \tau) , \qquad E^{\mathrm{p}}(k) = \int_0^{\infty} \mathrm{d}\tau\, E^{\mathrm{p}}(k, \tau) . \tag{62}$$

For correlation function (60), the computations involving spatial derivatives of the velocity field become essentially simpler. We have

$$\frac{\partial B_{ij}(\mathbf{0})}{\partial r_k} = C(0)\varepsilon_{ijk} , \tag{63}$$

$$-\frac{\partial^2 B_{ij}(\mathbf{0})}{\partial r_k\, \partial r_l} = \frac{D^{\mathrm{s}}}{d(d+2)}\left[(d+1)\delta_{kl}\delta_{ij} - \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\right] + \frac{D^{\mathrm{p}}}{d(d+2)}\left[\delta_{kl}\delta_{ij} + \delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}\right] , \tag{64}$$

$$\frac{\partial^3 B_{kp}(\mathbf{0})}{\partial r_n\, \partial r_m\, \partial r_j} = -2\alpha(\varepsilon_{kpj}\delta_{nm} + \varepsilon_{kpm}\delta_{nj} + \varepsilon_{kpn}\delta_{mj}) , \tag{65}$$

whence

$$\frac{\partial^3 B_{kp}(\mathbf{0})}{\partial \mathbf{r}^2 \, \partial r_j} = -2\alpha(d+2)\varepsilon_{kpj} \,,$$

where

$$D^{\rm s} = \int \mathrm{d}\mathbf{k}\, k^2 E^{\rm s}(k) = \frac{1}{d-1}\int_0^\infty \mathrm{d}\tau \left\langle \boldsymbol{\omega}(\mathbf{r}, t+\tau)\, \boldsymbol{\omega}(\mathbf{r}, t)\right\rangle ,$$

$$D^{\rm p} = \int \mathrm{d}\mathbf{k}\, k^2 E^{\rm p}(k) = \int_0^\infty \mathrm{d}\tau \left\langle \frac{\partial \mathbf{u}(\mathbf{r}, t+\tau)}{\partial \mathbf{r}} \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial \mathbf{r}} \right\rangle , \quad (66)$$

$$C(r) = C(0) - \alpha \mathbf{r}^2 \,,$$

$\boldsymbol{\omega}(\mathbf{r}, t) = \mathrm{rot}\, \mathbf{u}(\mathbf{r}, t)$ is the curl of the velocity field, and the function $C(r)$ describes statistical characteristics of the helicity of the velocity field $\mathbf{u}(\mathbf{r}, t)$.

In what follows, in computing the statistical characteristics of particles and the density field, we use the approximation of a velocity field $\mathbf{u}(\mathbf{r}, t)$ delta-correlated in time. In that framework, the correlation tensor $B_{ij}(\mathbf{r}, t)$ is approximated by expression (44).

3.2 Probabilistic description of the density field in a random velocity field

To describe the local behavior of tracer realizations in a random velocity field, we must know the probability distribution for its density, which can be obtained only in the absence of the dynamic diffusion effect.

To describe the density field in the Eulerian representation, we introduce the indicator function $\varphi(\mathbf{r}, t; \rho) = \delta(\rho(\mathbf{r}, t) - \rho)$ restricting to the surface $\rho(\mathbf{r}, t) = \rho = \mathrm{const}$ in the three-dimensional case, or to a contour in two dimensions. The Liouville equation for it in the absence of mean flow becomes [2–6]

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{r}, t)\frac{\partial}{\partial \mathbf{r}}\right)\varphi(\mathbf{r}, t; \rho) = \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial \mathbf{r}} \frac{\partial}{\partial \rho}\left[\rho \varphi(\mathbf{r}, t; \rho)\right] , \quad (67)$$

and the one-point probability density for solutions of dynamical equation (15) then coincides with the indicator function averaged over an ensemble of realizations of the random velocity field, $P(\mathbf{r}, t; \rho) = \langle \delta(\rho(\mathbf{r}, t) - \rho)\rangle$.

Averaging Eqn (67) over an ensemble of realizations of the field $\mathbf{u}(\mathbf{r}, t)$, using Furutsu–Novikov formula (45) and formulas (63) and (64), we obtain the equation for the probability density of the density field in the form

$$\left(\frac{\partial}{\partial t} - D_0 \Delta\right) P\big(\mathbf{r}, t; \rho|\rho_0(\mathbf{r})\big) = D_\rho \frac{\partial^2}{\partial \rho^2}\, \rho^2 P\big(\mathbf{r}, t; \rho|\rho_0(\mathbf{r})\big) ,$$
$$P\big(\mathbf{r}, 0; \rho|\rho_0(\mathbf{r})\big) = \delta\big(\rho - \rho_0(\mathbf{r})\big) , \quad (68)$$

where the diffusion coefficients in the $\mathbf{r}$-space, D_0, and in ρ-space, $D_\rho = D^{\rm p}$, are given in Eqns (61) and (66), and the vertical bar indicates the dependence of the probability density of the Eulerian field $\rho(\mathbf{r}, t)$ on the initial distribution $\rho_0(\mathbf{r})$.

The solution of Eqn (68) is given by

$$P\big(\mathbf{r}, t; \rho|\rho_0(\mathbf{r})\big) = \frac{1}{2\rho\sqrt{\pi D_\rho t}} \exp\left(D_0 t \frac{\partial^2}{\partial \mathbf{r}^2}\right)$$
$$\times \exp\left\{-\frac{\ln^2\left[\rho \exp(D_\rho t)/\rho_0(\mathbf{r})\right]}{4 D_\rho t}\right\} . \quad (69)$$

If the initial tracer density is homogeneous, $\rho_0(\mathbf{r}) = \rho_0 = \mathrm{const}$, the probability distribution of density is independent of $\mathbf{r}$ and in this case the Eulerian density field is lognormal with the probability density and the corresponding probability distribution function

$$P(t; \rho|\rho_0) = \frac{1}{2\rho\sqrt{\pi\tau}} \exp\left\{-\frac{\ln^2\left[\rho \exp(\tau)/\rho_0\right]}{4\tau}\right\} ,$$
$$F(t; \rho|\rho_0) = \Pr\left(\frac{\ln\left[\rho \exp(\tau)/\rho_0\right]}{2\sqrt{\tau}}\right) , \quad (70)$$

where $\tau = D_\rho t$ and $\Pr(z)$ is probability integral (26).

In considering one-point characteristics of the density field $\rho(\mathbf{r}, t)$, the problem, as mentioned above, is statistically equivalent to the analysis of a random process, in which case all moment functions starting with the second grow exponentially with time:

$$\big\langle \rho(\mathbf{r}, t)\big\rangle = \rho_0 \,, \quad \big\langle \rho^n(\mathbf{r}, t)\big\rangle = \rho_0^n \exp\big(n(n-1)\tau\big) \,, \quad (71)$$

and the typical realization curve for the density field decays exponentially with time at any fixed spatial location:

$$\rho^*(t) = \rho_0 \exp(-\tau) \,,$$

which tells us that the medium density fluctuations have a cluster character in arbitrary divergent flows. The formation of the Eulerian density field statistics at any fixed spatial location is maintained by density fluctuations around this curve.

We have discussed the one-point probability distribution of a tracer density in the Eulerian representation, which has already allowed us to draw a set of conclusions on the behavior of density field realizations in time at fixed spatial locations. This same distribution also allows clarifying certain characteristic features of the spatio-temporal structure of density field realizations.

For clarity, we limit ourselves to the two-dimensional case. As discussed in Section 1.1, important knowledge on the spatial behavior of realizations is provided by an analysis of isolines defined by the equality $\rho(\mathbf{r}, t) = \rho = \mathrm{const}$. In particular, functionals of the density field such as the total area $S(t, \rho)$ of the domain within which $\rho(\mathbf{r}, t) > \rho$ and the total mass of the tracer $M(t, \rho)$ confined inside this domain, whose means are defined by the one-point probability density, are described by the expressions

$$\big\langle S(t, \rho)\big\rangle = \int_\rho^\infty \mathrm{d}\tilde\rho \int \mathrm{d}\mathbf{r}\, P\big(\mathbf{r}, t; \tilde\rho|\rho_0(\mathbf{r})\big) ,$$
$$\big\langle M(t, \rho)\big\rangle = \int_\rho^\infty \tilde\rho\, \mathrm{d}\tilde\rho \int \mathrm{d}\mathbf{r}\, P\big(\mathbf{r}, t; \tilde\rho|\rho_0(\mathbf{r})\big) . \quad (72)$$

Inserting solution (69) into Eqn (72) and performing straightforward manipulations, we find the explicit expressions

$$\big\langle S(t, \rho)\big\rangle = \int \mathrm{d}\mathbf{r}\, \Pr\left(\frac{1}{\sqrt{2\tau}} \ln \frac{\rho_0(\mathbf{r}) \exp(-\tau)}{\rho}\right) ,$$
$$\big\langle M(t, \rho)\big\rangle = \int \mathrm{d}\mathbf{r}\, \rho_0(\mathbf{r})\, \Pr\left(\frac{1}{\sqrt{2\tau}} \ln \frac{\rho_0(\mathbf{r}) \exp(\tau)}{\rho}\right) , \quad (73)$$

where the probability integral $\Pr(z)$ is given by Eqn (26). Now taking the asymptotic form of $\Pr(z)$ in (28) into account, we find that for $\tau \gg 1$, the mean area of domains where the density exceeds a level ρ decreases with time according to the law

$$\big\langle S(t,\rho)\big\rangle \approx \frac{1}{\sqrt{\pi\tau\rho}} \exp\left(-\frac{\tau}{4}\right) \int \mathrm{d}\mathbf{r} \sqrt{\rho_0(\mathbf{r})} \,, \tag{74}$$

whereas the mean tracer mass confined there,

$$\big\langle M(t,\rho)\big\rangle \approx M - \sqrt{\frac{\rho}{\pi\tau}} \exp\left(-\frac{\tau}{4}\right) \int \mathrm{d}\mathbf{r} \sqrt{\rho_0(\mathbf{r})} \,, \tag{75}$$

tends monotonically to the total mass $M = \int \mathrm{d}\mathbf{r}\, \rho_0(\mathbf{r})$. This once again lends support to the conclusion drawn earlier that tracer particles tend with time to gather in clusters — compact areas of augmented density surrounded by areas of reduced density.

The dynamics of cluster formation can be illustrated with an example where the tracer is initially uniformly distributed over the plane, $\rho_0(\mathbf{r}) = \rho_0 = \text{const}$. In this case, the mean specific (per unit area) area of the domain within which $\rho(\mathbf{r},t) > \rho$ is

$$s(t,\rho|\rho_0) = \int_\rho^\infty \mathrm{d}\tilde{\rho}\, P(t;\tilde{\rho}|\rho_0) = \Pr\left(\frac{\ln\left[\rho_0 \exp(-\tau)/\rho\right]}{\sqrt{2\tau}}\right), \tag{76}$$

where $P(t;\rho|\rho_0)$ is the solution of Eqn (68) independent of $\mathbf{r}$ [i.e., function (70)], and the specific (per unit area) mean tracer mass confined within this domain is described by the expression

$$\frac{m(t,\rho|\rho_0)}{\rho_0} = \frac{1}{\rho_0}\int_\rho^\infty \tilde{\rho}\, \mathrm{d}\tilde{\rho}\, P(t;\tilde{\rho}|\rho_0) = \Pr\left(\frac{\ln\left[\rho_0 \exp(\tau)/\rho\right]}{\sqrt{2\tau}}\right). \tag{77}$$

It follows from Eqns (76) and (77) that at large time ($\tau \gg 1$), the mean specific area decreases according to the law

$$s(t,\rho|\rho_0) \approx \sqrt{\frac{\rho_0}{\pi\rho\tau}} \exp\left(-\frac{\tau}{4}\right), \tag{78}$$

while practically all mass becomes confined within this domain,

$$\frac{m(t,\rho|\rho_0)}{\rho_0} \approx 1 - \sqrt{\frac{\rho}{\pi\rho_0\tau}} \exp\left(-\frac{\tau}{4}\right). \tag{79}$$

The character of the evolution of clustering with time essentially depends on the ratio ρ/ρ_0. If $\rho/\rho_0 < 1$, then $s(0,\rho) = 1$ and $m(0,\rho) = 1$ at the initial instant. Next, because the tracer particles tend to run apart early in their evolution, they form small regions where $\rho(\mathbf{r},t) < \rho$, containing a negligible part of the total mass. As time progresses, these regions rapidly grow, whereas their mass flows into cluster regions, reaching asymptotic behavior (78) and (79) fairly fast (dashed curves in Fig. 12).

In the opposite, more interesting case $\rho/\rho_0 > 1$, initially $s(0,\rho) = 0$ and $m(0,\rho) = 0$. Because of the initial running apart, small cluster regions are formed where $\rho(\mathbf{r},t) > \rho$; they practically persist in time and rapidly pull in a substantial part of the total mass. Their area decreases with time, but the mass

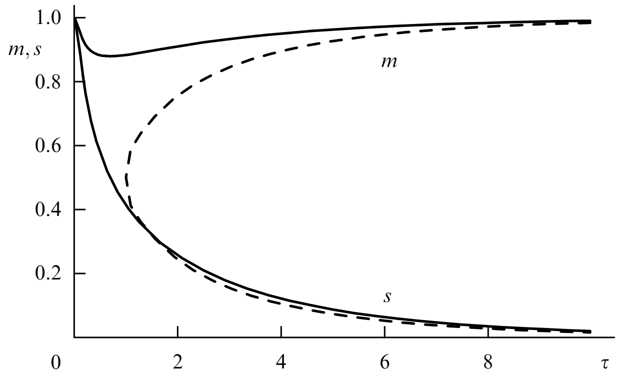

Figure 12. Dynamics of cluster formation for $\rho/\rho_0 = 0.5$.

increases according to asymptotic dependences (78) and (79) (dashed curves in Fig. 13).

We have considered the simplest statistical problem of the diffusion of a scalar tracer in a random velocity field in the absence of regular flow and the dynamic diffusion effect. Also, the approximation of a random field delta-correlated in time was invoked for the statistical description. All factors that have not been accounted for start to act sooner or later, and therefore the results obtained above are valid only at the initial stage of diffusion. In addition, these factors may lead to new physical effects [2–6].

3.3 Probabilistic description of a magnetic field and its energy in a random velocity field

3.3.1 Probabilistic description of a magnetic field. We consider the probabilistic description of a magnetic field on the basis of dynamical equation (16). As in the case of the density field, we assume the random component of the velocity field $\mathbf{u}(\mathbf{r},t)$ to be a divergent ($\operatorname{div}\mathbf{u}(\mathbf{r},t) \neq 0$) random Gaussian field, homogeneous and isotropic in space, and stationary and delta correlated in time.

We introduce the indicator function of the magnetic field $\varphi(\mathbf{r},t;\mathbf{H}) = \delta(\mathbf{H}(\mathbf{r},t) - \mathbf{H})$ which, obviously, satisfies the Liouville equation [36]

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\varphi(\mathbf{r},t;\mathbf{H}) = -\frac{\partial}{\partial H_i}\left[\mathbf{H}\frac{\partial u_i(\mathbf{r},t)}{\partial \mathbf{r}} - H_i\frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}}\right]\varphi(\mathbf{r},t;\mathbf{H}) \,, \tag{80}$$

with the initial condition $\varphi(\mathbf{r},0;\mathbf{H}) = \delta(\mathbf{H}_0(\mathbf{r}) - \mathbf{H})$.

We average Eqn (80) over an ensemble of realizations of the field $\{\mathbf{u}(\mathbf{r},t)\}$ and use Furutsu–Novikov formula (45) to split the resulting correlations. Taking expressions (61), (63), and (64) with parameters (66) into account, we obtain the sought equation for the one-point probability density of the magnetic field $P(\mathbf{r},t;\mathbf{H}) = \langle\varphi(\mathbf{r},t;\mathbf{H})\rangle_{\mathbf{u}}$ [36]:

$$\left(\frac{\partial}{\partial t} - D_0\frac{\partial^2}{\partial \mathbf{r}^2}\right)P(\mathbf{r},t;\mathbf{H}) = \left\{D_1\frac{\partial}{\partial H_l}\frac{\partial}{\partial H_k}H_lH_k + D_2\frac{\partial}{\partial H_l}\frac{\partial}{\partial H_l}H_k^2\right\}P(\mathbf{r},t;\mathbf{H}) \,, \tag{81}$$

where

$$D_1 = \frac{(d^2-2)D^{\mathrm{p}} - 2D^{\mathrm{s}}}{d(d+2)} \,, \qquad D_2 = \frac{(d+1)D^{\mathrm{s}} + D^{\mathrm{p}}}{d(d+2)}$$

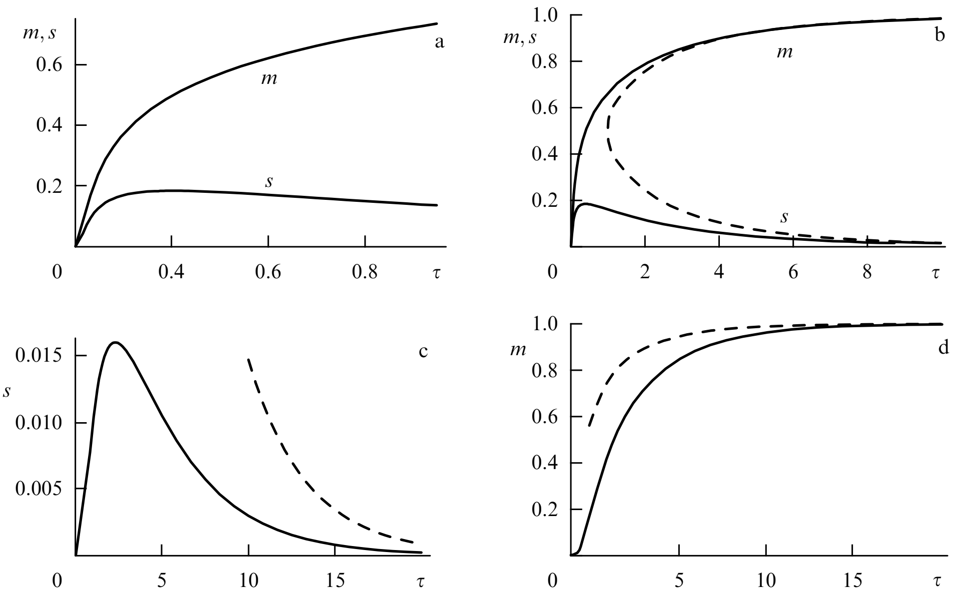

Figure 13. Dynamics of cluster formation for $\rho/\rho_0 = 1.5$ (a, b) and $\rho/\rho_0 = 10$ (c, d).

are the diffusion coefficients and d is the space dimension. In the derivation of Eqn (81), the identity $\partial H_i/\partial H_i = d$ was used. For a three-dimensional problem, $d = 3$, and for a plane-parallel fluid flow, $d = 2$ [see the remarks on Eqn (12)].

We note that for homogeneous initial conditions, the one-point probability density is independent of $\mathbf{r}$, and Eqn (81) becomes

$$\frac{\partial}{\partial t} P(t;\mathbf{H}) = \left\{ D_1 \frac{\partial}{\partial H_l}\frac{\partial}{\partial H_k} H_l H_k + D_2 \frac{\partial}{\partial H_l}\frac{\partial}{\partial H_l} H_k^2 \right\} P(t;\mathbf{H}) \,. \tag{82}$$

We find the one-point correlation of the magnetic field $\langle W_{ij}(t)\rangle = \langle H_i(\mathbf{r},t)\, H_j(\mathbf{r},t)\rangle$ in this case. Multiplying Eqn (81) by H_i and H_j and integrating over $\mathbf{H}$, we obtain the equation

$$\frac{\partial}{\partial t}\langle W_{ij}(t)\rangle = 2D_1 \langle W_{ij}(t)\rangle + 2D_2 \delta_{ij}\langle E(t)\rangle \,,$$

which yields the solution for the mean energy

$$\langle E(t)\rangle = E_0 \exp\left[2\,\frac{d-1}{d}\,(D^{\mathrm{s}} + D^{\mathrm{p}})t\right] \tag{83}$$

and the correlation of the magnetic field

$$\frac{\langle W_{ij}(t)\rangle}{\langle E(t)\rangle} = \frac{1}{d}\, d_{ij} + \left(\frac{W_{ij}(0)}{E_0} - \frac{1}{d}\, d_{ij}\right) \times \exp\left[-2\,\frac{(d+1)D^{\mathrm{s}} + D^{\mathrm{p}}}{d+2}\, t\right] . \tag{84}$$

It follows that the mean energy grows exponentially with time. This growth is accompanied by isotropization of the magnetic field, in accordance with an exponential law. We note that the components of the velocity field enter the respective exponents in an additive way. Obviously, this feature is preserved for any other correlations of the magnetic field and its energy.

3.3.2 Probabilistic description of the magnetic field energy. We now introduce the indicator function for the magnetic field energy $E(\mathbf{r},t) = \mathbf{H}^2(\mathbf{r},t)$ — $\varphi(\mathbf{r},t;E) = \delta(E(\mathbf{r},t) - E)$, in terms of which the probability density $P(\mathbf{r},t;E)$ is expressed as

$$P(\mathbf{r},t;E) = \big\langle \delta\big(E(\mathbf{r},t) - E\big)\big\rangle_{\mathbf{u}} = \big\langle \delta\big(\mathbf{H}^2(\mathbf{r},t) - E\big)\big\rangle_{\mathbf{H}} \,.$$

To derive the equation for this function, we multiply Eqn (81) by $\delta(\mathbf{H}^2 - E)$ and integrate it over $\mathbf{H}$. As a result, we obtain the equation [36]

$$\left(\frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial \mathbf{r}^2}\right) P(\mathbf{r},t;E) = \left\{ \alpha \frac{\partial}{\partial E} E + D \frac{\partial}{\partial E} E \frac{\partial}{\partial E} E \right\} P(\mathbf{r},t;E) \,, \tag{85}$$
$$P(\mathbf{r},0;E) = \delta\big(E - E_0(\mathbf{r})\big) \,,$$

with the parameters

$$\alpha = 2\,\frac{d-1}{d+2}\,(D^{\mathrm{p}} - D^{\mathrm{s}}) \,, \qquad D = 4(d-1)\,\frac{(d+1)D^{\mathrm{p}} + D^{\mathrm{s}}}{d(d+2)} \,.$$

In this case, α can be both positive and negative. For the one-point characteristics, changing the sign of α implies passing from the field $E(\mathbf{r},t)$ to $\tilde{E}(\mathbf{r},t) = 1/E(\mathbf{r},t)$. The case $\alpha = 0$ ($D^{\mathrm{p}} = D^{\mathrm{s}}$) deserves special treatment.

The solution of Eqn (85) is given by

$$P(\mathbf{r},t;E) = \frac{1}{2E\sqrt{\pi D t}} \exp\left(D_0 t \frac{\partial^2}{\partial \mathbf{r}^2}\right) \times \exp\left\{-\frac{\ln^2\big[E\exp(\alpha t)/E_0(\mathbf{r})\big]}{4Dt}\right\} . \tag{86}$$

For a spatially homogeneous initial energy distribution $E_0(\mathbf{r}) = E_0$, probability density (86) is independent of $\mathbf{r}$ and is described by the formula

$$P(t;E) = \frac{1}{2E\sqrt{\pi D t}} \exp\left\{-\frac{\ln^2\big[E\exp(\alpha t)/E_0\big]}{4Dt}\right\} . \tag{87}$$

In this case, the one-point statistical characteristics of the energy $E(\mathbf{r},t)/E_0$ are statistically equivalent to statistical characteristics of the random process $E(t) = y(t;\alpha)$ in Eqn (2).

A characteristic feature of distribution (87) is the appearance of long, gently sloping tail for $Dt \gg 1$, which indicates the increased role of strong peaks of the process $E(t;\alpha)$ in the formation of one-point statistics in time. For this distribution, all moments of the magnetic field energy grow exponentially with time (for positive as well as negative n):

$$\left\langle E^n(t)\right\rangle = E_0^n \exp\left[-2n\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})t + 4n^2(d-1)\,\frac{(d+1)D^{\mathrm{p}}+D^{\mathrm{s}}}{d(d+2)}\,t\right].$$

In particular, for $n=1$, the mean specific energy is

$$\left\langle E(t)\right\rangle = E_0\exp(\gamma t)\,,\qquad \gamma = \frac{2(d-1)}{d}(D^{\mathrm{p}}+D^{\mathrm{s}})\,, \qquad (88)$$

while

$$\left\langle \ln\frac{E(t)}{E_0}\right\rangle = -\alpha t = -2\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})t\,,$$

and, consequently, the parameter α is the *Lyapunov characteristic exponent*. In this case, the *typical realization curve* of the random process $E(t)$, which determines the behavior of the magnetic field energy in concrete realizations, is exponential at any fixed spatial location,

$$E^*(t) = E_0\exp(-\alpha t) = E_0\exp\left[-2\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})t\right],$$

and either grows or decays with time. Hence, for $\alpha>0$ ($D^{\mathrm{p}}>D^{\mathrm{s}}$), the typical realization curve decays exponentially at each spatial location, which bears witness to the cluster structure of the magnetic field, and the growth of the magnetic field energy moments in this case is due to rare but strong fluctuations of energy relative to the typical realization curve, as is common for log-normal processes. In the other case, $\alpha<0$ ($D^{\mathrm{p}}<D^{\mathrm{s}}$), the typical realization curve grows exponentially with time, which indicates an increase in the magnetic energy at each spatial location. Figure 14 schematically shows realizations of the magnetic field energy in a random velocity field for different signs of the parameter α.

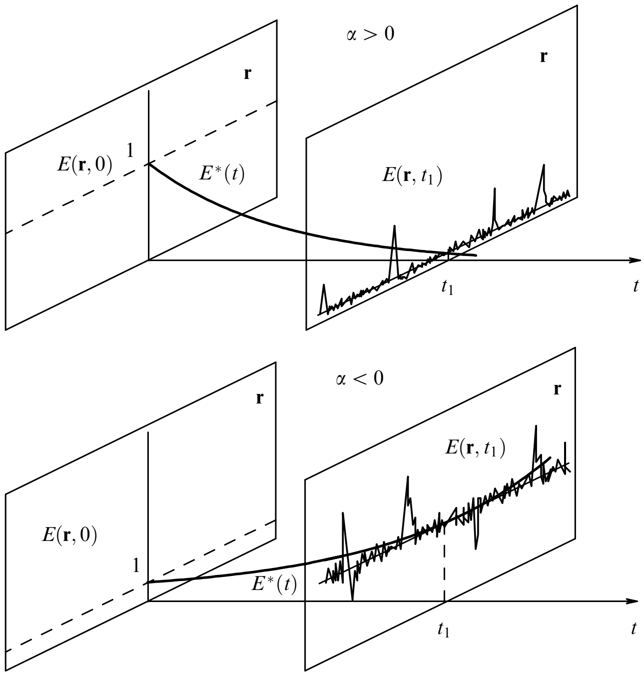

Figure 14. Schematic behavior of random realizations of the field energy for $\alpha>0$ and $\alpha<0$.

The indicator function of the magnetic field energy also enables obtaining general information on the spatial structure of the energy field. In particular, functionals of the magnetic field energy such as the total volume (in three dimensions) or the area (in two dimensions) of the domain where $E(\mathbf{r},t)>E$,

$$V(t,E) = \int \mathrm{d}\mathbf{r}\,\theta\big(E(\mathbf{r},t)-E\big) = \int \mathrm{d}\mathbf{r}\int_E^\infty \mathrm{d}\tilde E\,\delta\big(E(\mathbf{r},t)-\tilde E\big)\,,$$

and the total magnetic field energy contained there,

$$\mathcal{E}(t,E) = \int \mathrm{d}\mathbf{r}\,E(\mathbf{r},t)\,\theta\big(E(\mathbf{r},t)-E\big) = \int \mathrm{d}\mathbf{r}\int_E^\infty \tilde E\,\mathrm{d}\tilde E\,\delta\big(E(\mathbf{r},t)-\tilde E\big)\,,$$

whose means are defined by one-point probability density (86), are described, in the general case, by the equalities

$$\left\langle V(t,E)\right\rangle = \int_E^\infty \mathrm{d}\tilde E \int \mathrm{d}\mathbf{r}\,P(\mathbf{r},t;\tilde E)\,,$$

$$\left\langle \mathcal{E}(t,E)\right\rangle = \int_E^\infty \tilde E\,\mathrm{d}\tilde E \int \mathrm{d}\mathbf{r}\,P(\mathbf{r},t;\tilde E)\,.$$

The mean values of these functionals are independent of diffusion in $\mathbf{r}$-space (the coefficient D_0) and for the probability density (86), we find that as $t\to\infty$ (for $\alpha>0$), the mean volume decays asymptotically with time according to the law

$$\left\langle V(t,E)\right\rangle \approx \frac{1}{\alpha}\sqrt{\frac{D}{\pi E^{\alpha/D}t}}\exp\left(-\frac{\alpha^2 t}{4D}\right)\int \mathrm{d}\mathbf{r}\,\sqrt{E_0^{\alpha/D}(\mathbf{r})}\,.$$

For $\alpha<0$, the mean volume occupies the whole space in the limit as $t\to\infty$.

In both cases, the following asymptotic expressions are obtained for the total energy as $t\to\infty$ (since $\alpha<2D$):

$$\left\langle \mathcal{E}(t,E)\right\rangle \times \exp(\gamma t)\int \mathrm{d}\mathbf{r}\,E_0(\mathbf{r})\left\{1-\frac{1}{(2D-\alpha)}\sqrt{\frac{D}{\pi t}\left(\frac{E}{E_0(\mathbf{r})}\right)^{(2D-\alpha)/D}} \times \exp\left[-\frac{(2D-\alpha)^2 t}{4}\right]\right\},$$

where the parameter γ is given by (88). Hence, for $\alpha>0$, 100% of the total mean energy is confined to the clusters.

For homogeneous initial conditions, the corresponding asymptotic expressions for the specific values of the volume of large fluctuations and their total energy become

$$\left\langle \mathcal{V}_{\text{hom}}(t,E)\right\rangle \approx \begin{cases} \dfrac{1}{\alpha}\sqrt{\dfrac{D}{\pi t}\left(\dfrac{E_0}{E}\right)^{\alpha/D}}\, \exp\left(-\dfrac{\alpha^2 t}{4D}\right), & \alpha > 0\,, \\ 1-\dfrac{1}{|\alpha|}\sqrt{\dfrac{D}{\pi t}\left(\dfrac{E}{E_0}\right)^{|\alpha|/D}}\, \exp\left(-\dfrac{|\alpha|^2 t}{4D}\right), & \alpha < 0\,, \end{cases}$$

and, because $2D-\alpha>0$,

$$\left\langle \mathcal{E}_{\text{hom}}(t,E)\right\rangle \approx E_0 \exp(\gamma t)\left\{1-\frac{1}{2D-\alpha}\sqrt{\frac{D}{\pi t}\left(\frac{E}{E_0}\right)^{(2D-\alpha)/D}} \times \exp\left[-\frac{(2D-\alpha)^2 t}{4D}\right]\right\}.$$

Accordingly, for $\alpha>0$ ($D^{\text{p}}>D^{\text{s}}$), the specific total volume tends to zero, while the specific total energy confined within it coincides with that in the entire space. This indicates clustering of the magnetic field energy.

In the case $\alpha<0$ ($D^{\text{p}}<D^{\text{s}}$), clustering is absent, the magnetic field energy increases, and the specific volume where the specific mean energy grows with time occupies the entire space. We specially note that the exponential growth of the magnetic energy E everywhere (small-scale dynamo) implies the clustering of the inverse quantity $1/E$. In other words, clusters of compact regions with expelled magnetic field (magnetic zeros) then appear.

3.3.3 The case $\alpha=0$ ($D^{\text{p}}=D^{\text{s}}$). In this case, the energy of the magnetic field with a homogeneous initial condition is statistically equivalent to random process (10). All one-point characteristics are independent of $\mathbf{r}$ and the one-point probability density assumes the form

$$P(t;E)=\frac{1}{2E\sqrt{\pi D t}}\exp\left[-\frac{\ln^2(E/E_0)}{4Dt}\right].$$

The random processes $E(t)$ and $1/E(t)$ are then also statistically equivalent, and all magnetic field energy moments grow exponentially with time: $\langle E^n(t)\rangle = E_0^n \exp(n^2\gamma t)$, where $\gamma=4(d-1)D^{\text{p}}/d$; in particular, for $n=1$, the mean specific energy is $\langle E(t)\rangle = E_0\exp(\gamma t)$.

It follows that the specific mean volume in the limit $t\to\infty$ asymptotically tends to half the total volume, whereas the specific mean energy tends to the total mean energy.

Therefore, in the case $\alpha=0$ ($D^{\text{p}}=D^{\text{s}}$), we see that clustering is absent, but intermittency is naturally preserved. The same conclusion also applies to the field $f(\mathbf{r},t)$ (10).

4. Integral one-point characteristics of passive vector fields

In Section 3, which dealt with one-point characteristics of the density field and magnetic field in the absence of dynamic diffusion effects, the equations were given for the probability densities of these fields. This allowed us to derive conditions on cluster structure formation based on the ideas of statistical topography. Obtaining more detailed information on clusters requires the knowledge of statistical quantities related to one-point characteristics of derivatives of scalar fields and their correlation with the fields. Unfortunately, considering these quantities involves rather cumbersome manipulations, and gaining insight from the outcome of such a description becomes very difficult. Besides, such a probabilistic description does not allow the inclusion of the effects of dynamical diffusion in the analysis.

We note, however, that in the case of a delta-correlated random velocity field, it is easy to pass from linear equations (11) and (12) to closed equations for the means of these fields, as well as for their higher many-point correlation functions. Accordingly, to analyze the problems formulated above, the only remaining option is to study the two-point correlation function of the density field $R(\mathbf{r},\mathbf{r}_1,t)=\langle\rho(\mathbf{r},t)\,\rho(\mathbf{r}_1,t)\rangle$ and the magnetic field $W_{ij}(\mathbf{r},\mathbf{r}_1,t)=\langle H_i(\mathbf{r},t)\,H_j(\mathbf{r}_1,t)\rangle$. They depend on the difference $\mathbf{r}-\mathbf{r}_1$ for homogeneous initial conditions $\rho(\mathbf{r},0)=\rho_0$ and $\mathbf{H}(\mathbf{r},0)=\mathbf{H}_0$, which substantially simplifies the treatment [37]. As pointed out in Section 1, all statistical means in this case are integral or specific quantities related to a unit volume.

Various correlations of the spatial derivatives of fields considered here can be obtained by subsequent differentiation of these functions with respect to spatial variables. The original equation in this case should contain dissipative terms.

4.1 Spatial correlation function for the density field

Starting with Eqn (11), we first write the stochastic dynamical equation for the function $R(\mathbf{r},\mathbf{r}_1,t)=\rho(\mathbf{r},t)\,\rho(\mathbf{r}_1,t)$ and average it over an ensemble of realizations of the random velocity field. Then, using Furutsu–Novikov formula (45), we obtain the partial differential equation for the spatial correlation function $\langle R(\mathbf{r}-\mathbf{r}_1,t)\rangle=\langle\rho(\mathbf{r},t)\,\rho(\mathbf{r}_1,t)\rangle$ ($\mathbf{r}-\mathbf{r}_1\to\mathbf{r}$)

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle R(\mathbf{r},t)\right\rangle &= -\frac{\partial^2\left[B_{ij}(\mathbf{r})+B_{ji}(\mathbf{r})\right]}{\partial r_j\,\partial r_i}\left\langle R(\mathbf{r},t)\right\rangle \\ &-\frac{\partial\left[B_{ij}(\mathbf{r})+B_{ji}(\mathbf{r})\right]}{\partial r_i}\left\langle R_j(\mathbf{r},t)\right\rangle-\frac{\partial\left[B_{ij}(\mathbf{r})+B_{ji}(\mathbf{r})\right]}{\partial r_j}\left\langle R_i(\mathbf{r},t)\right\rangle \\ &+\left[2B_{ij}(\mathbf{0})-B_{ij}(\mathbf{r})-B_{ji}(\mathbf{r})\right]\left\langle R_{ji}(\mathbf{r},t)\right\rangle+2\mu_\rho\left\langle R_{kk}(\mathbf{r},t)\right\rangle,\end{aligned} \tag{89}$$

where $\langle R_{kk}(\mathbf{r},t)\rangle=\partial^2/\partial\mathbf{r}^2\langle R(\mathbf{r})\rangle$, i.e., subscripts used with the function $\langle R(\mathbf{r},t)\rangle$ denote spatial derivatives, and $\langle R_{kk}(\mathbf{0},t)\rangle<0$ in this case.

We note that in the case of an isotropic random velocity field, the random density field $\rho(\mathbf{r},t)$ is a homogeneous and isotropic random field. Then the equation for the correlation function is simplified to

$$\frac{\partial}{\partial t}\left\langle R(\mathbf{r},t)\right\rangle = 2\mu_\rho\,\Delta\left\langle R(\mathbf{r},t)\right\rangle+\frac{\partial^2}{\partial r_i\,\partial r_j}D_{ij}(\mathbf{r})\left\langle R(\mathbf{r},t)\right\rangle,$$

where $D_{ij}(\mathbf{r})$ is the structure matrix of the vector field $\mathbf{u}(\mathbf{r},t)$,

$$D_{ij}(\mathbf{r})=2\left[B_{ij}(0)-B_{ij}(\mathbf{r})\right].$$

The correlation function $\langle R(\mathbf{r},t)\rangle$ is now dependent on the modulus of $\mathbf{r}$, i.e., $\langle R(\mathbf{r},t)\rangle=\langle R(r,t)\rangle$, and as a function of the variables r and t, satisfies the equation

$$\frac{\partial}{\partial t}\left\langle R(r,t)\right\rangle=\frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1} \times\left[\frac{\partial D_{ii}(r)}{\partial r}+\left(2\mu+\frac{r_i r_j}{r^2}D_{ij}(\mathbf{r})\right)\frac{\partial}{\partial r}\right]\left\langle R(r,t)\right\rangle,$$

where, as before, d is the space dimension. This equation can have a stationary solution $\langle R(r,t)\rangle = \langle R(r)\rangle$ as $t\to\infty$, which corresponds to the boundary condition $\langle R(\infty)\rangle = \rho_0^2$, of the form [39]

$$\big\langle R(r)\big\rangle = \rho_0^2 \exp\left(\int_r^\infty \mathrm{d}r' \; \frac{\partial D_{ii}(r')/\partial r'}{2\mu + r_i' r_j' D_{ij}(\mathbf{r}')/r'^2}\right).$$

We note that this quantity at $r=0$ provides a stationary value of the second moment of the density field $\langle R(0)\rangle = \langle \rho^2(\mathbf{r},t)\rangle_{t\to\infty}$; hence,

$$\big\langle \rho^2(\mathbf{r},t)\big\rangle_{t\to\infty} = \rho_0^2 \exp\left(\int_0^\infty \mathrm{d}r' \; \frac{\partial D_{ii}(r')/\partial r'}{2\mu + r_i' r_j' D_{ij}(\mathbf{r}')/r'^2}\right) > \rho_0^2 \,. \tag{90}$$

We return to the general case of a random velocity field without mirror symmetry (i.e., possessing helicity). Setting $\mathbf{r}=\mathbf{0}$ in Eqn (89) and taking Eqn (64) into account, we arrive at the nonclosed equation

$$\left(\frac{\partial}{\partial t} - 2D^{\mathrm{p}}\right)\big\langle R(\mathbf{0},t)\big\rangle = 2\mu_\rho \big\langle R_{kk}(\mathbf{0},t)\big\rangle \,, \qquad \big\langle R(\mathbf{0},0)\big\rangle = \rho_0^2 \,. \tag{91}$$

To derive an approximate solution of Eqn (91), we can resort to the approximate procedure of expanding its right-hand side in a series in the small parameter μ_ρ. For this, we label the quantities $\langle R(\mathbf{0},t)\rangle_\mu$ and $\langle R_{kk}(\mathbf{0},t)\rangle_\mu$ with the index μ and rewrite this equation in the form of an integral equation:

$$\big\langle R(\mathbf{0},t)\big\rangle_\mu = \rho_0^2 \exp\left\{2D^{\mathrm{p}} t + \int_0^t \mathrm{d}\tau \; \frac{2\mu_\rho}{\big\langle R(\mathbf{0},\tau)\big\rangle_\mu}\big\langle R_{kk}(\mathbf{0},\tau)\big\rangle_\mu\right\}. \tag{92}$$

The solution at the initial stage, when dissipation can be neglected, is written as

$$\big\langle R(\mathbf{0},t)\big\rangle_0 = \big\langle \rho^2(\mathbf{r},t)\big\rangle_0 = \rho_0^2 \exp\left(2D^{\mathrm{p}} t\right), \tag{93}$$

and, expectedly, coincides with formula (71) for $n=2$ and is defined by only the potential component of the spectral function of the velocity field.

To proceed, we represent the right-hand side as a series in the parameter μ. In the first approximation, the problem solution becomes

$$\big\langle R(\mathbf{0},t)\big\rangle_1 = \rho_0^2 \exp\left\{2D^{\mathrm{p}} t + \int_0^t \mathrm{d}\tau \; \frac{2\mu_\rho}{\big\langle R(\mathbf{0},\tau)\big\rangle_0}\big\langle R_{kk}(\mathbf{0},\tau)\big\rangle_0\right\}. \tag{94}$$

Clearly, solution (94) exponentially grows at small times and, having reached its maximum value at a time instant $t_{\max}$, decreases with time until it attains its stationary value, in accordance with the physical meaning of the problem considered here.

We note that the structure of Eqn (89) implies that the presence of helicity has no impact on the statistics of the density field gradient because only the symmetric matrix $B_{ij}(\mathbf{r}) + B_{ji}(\mathbf{r})$ enters the coefficients of this equation. For such a matrix, all odd derivatives with respect to $\mathbf{r}$ vanish at $\mathbf{r}=\mathbf{0}$ [see Eqn (65)]. Odd derivatives with respect to $\mathbf{r}$ of the function $\langle R(\mathbf{r},t)\rangle$ also vanish at $\mathbf{r}=\mathbf{0}$. Taking these circumstances into account, it can be argued that the helicity of the tracer density gradient vanishes because it is related to the quantity

$$\left\langle \frac{\partial \rho(\mathbf{r},t)}{\partial r_i} \frac{\partial^2 \rho(\mathbf{r},t)}{\partial r_j \, \partial r_k}\right\rangle = \big\langle R_{ijk}(\mathbf{0},t)\big\rangle \,.$$

4.1.1 One-point statistical characteristics of the density field gradient. We now introduce the vector of the density field gradient $p_k(\mathbf{r},t) = \partial\rho(\mathbf{r},t)/\partial r_k$. The spatial correlation tensor of the density gradient and its variance are given by

$$P_{kl}(\mathbf{r}-\mathbf{r}_1,t) \equiv -\big\langle R_{kl}(\mathbf{r}-\mathbf{r}_1,t)\big\rangle \,, \qquad \sigma_{\mathbf{p}}^2(t) \equiv -\big\langle R_{kk}(\mathbf{0},t)\big\rangle \,. \tag{95}$$

We note that as $t\to\infty$, Eqn (91) gives the expression

$$\sigma_{\mathbf{p}}^2(\infty) = \frac{D^{\mathrm{p}}}{\mu_\rho}\big\langle R(\mathbf{0},\infty)\big\rangle$$

for the stationary value of the variance of the density field gradient.

To find the temporal evolution of the one-point statistical characteristics of the density field gradient, we differentiate Eqn (89) with respect to r_k and r_l and set $\mathbf{r}=0$. As a result, using equality (64), we obtain the equation

$$\begin{aligned}\frac{\partial}{\partial t}\big\langle R_{kl}(\mathbf{0},t)\big\rangle_0 &= -2\,\frac{\partial^4 B_{ij}(\mathbf{0})}{\partial r_i\,\partial r_j\,\partial r_k\,\partial r_l}\big\langle R(\mathbf{0},t)\big\rangle_0 \\ &+ 2\,\frac{(4+d)D^{\mathrm{p}}}{d}\big\langle R_{kl}(\mathbf{0},t)\big\rangle_0 \\ &+ 2\,\frac{D^{\mathrm{s}}}{d(d+2)}\Big[(d+1)\,\delta_{kl}\big\langle R_{ii}(\mathbf{0},t)\big\rangle_0 - 2\big\langle R_{kl}(\mathbf{0},t)\big\rangle_0\Big] \\ &+ 2\,\frac{D^{\mathrm{p}}}{d(d+2)}\Big(\delta_{kl}\big\langle R_{ii}(\mathbf{0},t)\big\rangle_0 + 2\big\langle R_{kl}(\mathbf{0},t)\big\rangle_0\Big).\end{aligned}$$

We also take into account that because the generation source for the gradient field is isotropic,

$$\begin{aligned}2\,\frac{\partial^4 B_{ij}(\mathbf{0})}{\partial r_i\,\partial r_j\,\partial r_k\,\partial r_l} &= -\frac{2}{d}\int_0^\infty \mathrm{d}\tau \left\langle \frac{\partial \mathbf{u}(\mathbf{r},t)}{\partial \mathbf{r}}\,\Delta\,\frac{\partial \mathbf{u}(\mathbf{r},t-\tau)}{\partial \mathbf{r}}\right\rangle \delta_{kl} \\ &= \frac{1}{d}\,D_\rho^{(4)}\delta_{kl} \quad \big(D_\rho^{(4)} > 0\big)\,,\end{aligned}$$

the tensor $\langle R_{kl}(\mathbf{0},t)\rangle$ is also isotropic, and hence

$$\big\langle R_{kl}(\mathbf{0},t)\big\rangle_0 = \frac{1}{d}\big\langle R_{ii}(\mathbf{0},t)\big\rangle_0\,\delta_{kl}\,.$$

Consequently, the variance of density field gradient (95) satisfies the equation

$$\begin{aligned}\frac{\partial}{\partial t}\big\langle R_{kk}(\mathbf{0},t)\big\rangle_0 &= -D_\rho^{(4)}\big\langle R(\mathbf{0},t)\big\rangle_0 \\ &+ 2\,\frac{D^{\mathrm{s}}(d-1) + (d+5)D^{\mathrm{p}}}{d}\big\langle R_{kk}(\mathbf{0},t)\big\rangle_0\,,\end{aligned} \tag{96}$$

where the second moment of the density field $\langle R(\mathbf{0},t)\rangle_0$ is given by Eqn (93).

The solution of Eqn (96) is

$$\sigma_{\mathbf{p}}^2(t) = \frac{D_\rho^{(4)}\big\langle R(\mathbf{0},t)\big\rangle_0}{A}\big[\exp(At) - 1\big]\,, \tag{97}$$

where $A = 2[D^{\mathrm{s}}(d-1) + (d+5)D^{\mathrm{p}}]/d$.

We note that in the case of weak compressibility ($D^{\mathrm{p}} \ll D^{\mathrm{s}}$), although this solution is generated by the potential component of the spectral density of the random velocity field, the increment of exponential growth coincides with that in the incompressible fluid flow with an inhomogeneous initial density distribution [2–5], because $A = 2D^{\mathrm{s}}(d-1)/d$ in that case.

We return to Eqn (94), which can be rewritten, with the help of Eqn (97), in the final form

$$\left\langle \rho^2(t)\right\rangle_1 = \rho_0^2 \exp\left\{2D^{\mathrm{p}}t - 2\,\frac{\mu_\rho D_\rho^{(4)}}{A^2}\left[\exp(At) - 1 - At\right]\right\}. \tag{98}$$

This solution reaches a maximum at

$$At_{\max} \approx \ln \frac{AD^{\mathrm{p}}}{\mu_\rho D_\rho^{(4)}},$$

with $\langle R(\mathbf{0}, t)\rangle_1$ attaining the value

$$\left\langle R(\mathbf{0}, t)\right\rangle_{1\max} \approx \rho_0^2 \left(\frac{AD^{\mathrm{p}}}{\mu_\rho D_\rho^{(4)}}\right)^{2D^{\mathrm{p}}/A} \exp\left(-\frac{2D^{\mathrm{p}}}{A}\right).$$

The condition that the action of dynamic diffusion can be neglected is $t \ll t_{\max}$. For $t > t_{\max}$, the function $\langle R(\mathbf{0}, t)\rangle_1$ rapidly decreases with time. As mentioned above, in the general case, $\langle R(\mathbf{0}, t)\rangle$ approaches the stationary value (90) as $t \to \infty$.

4.2 Spatial correlation function of the magnetic field

As mentioned in Section 3.3, in the general case of a divergent random velocity field, the probability densities for the magnetic field and its energy can be calculated. In principle, this technique can be further used to compute various quantities related to spatial derivatives of the magnetic field in the absence of dynamic diffusion. But the resultant equations are unwieldy and it would be difficult to draw any conclusions based on them.

By analyzing different moment functions, we can already account for the coefficient of dynamic diffusion. But, once again, all equations are very cumbersome in the general case of a divergent velocity field. Therefore, in the subsequent analysis of the statistical characteristics of spatial derivatives of a magnetic field, we limit ourselves to the solenoidal velocity field ($\operatorname{div}\mathbf{u}(\mathbf{r}, t) = 0$), i.e., consider the turbulent fluid flow incompressible. Taking the compressibility into account changes only the coefficients in the resultant equation in an additive way, but does not change the basic tendency in the behavior of moment functions.

We introduce the function $W_{ij}(\mathbf{r}, \mathbf{r}_1; t) = H_i(\mathbf{r}, t)H_k(\mathbf{r}_1, t)$. Then for the correlation function of the magnetic field, similarly to how it was done for the density correlation function, we obtain the partial differential equation ($\mathbf{r} - \mathbf{r}_1 \to \mathbf{r}$)

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle W_{ij}(\mathbf{r}; t)\right\rangle &= -2\,\frac{\partial^2 B_{ij}(\mathbf{r})}{\partial r_k\,\partial r_m}\left\langle W_{km}(\mathbf{r}; t)\right\rangle \\ &\quad - 2\,\frac{\partial\left[B_{ik}(\mathbf{0}) - B_{ik}(\mathbf{r})\right]}{\partial r_s}\,\frac{\partial}{\partial r_k}\left\langle W_{sj}(\mathbf{r}; t)\right\rangle \\ &\quad - 2\,\frac{\partial\left[B_{kj}(\mathbf{0}) - B_{kj}(\mathbf{r})\right]}{\partial r_s}\,\frac{\partial}{\partial r_k}\left\langle W_{is}(\mathbf{r}; t)\right\rangle \\ &\quad + \left[2B_{kq}(\mathbf{0}) - B_{kq}(\mathbf{r}) - B_{qk}(\mathbf{r})\right]\frac{\partial^2}{\partial r_q\,\partial r_k}\left\langle W_{ij}(\mathbf{r}; t)\right\rangle \\ &\quad + 2\mu_H\left\langle W_{ij;\,ss}(\mathbf{r}; t)\right\rangle, \end{aligned} \tag{99}$$

where

$$\left\langle W_{ij;\,ss}(\mathbf{r}; t)\right\rangle = \frac{\partial^2}{\partial \mathbf{r}^2}\left\langle W_{ij}(\mathbf{r}; t)\right\rangle$$

is the dissipation tensor and the derivatives of $W_{ik}(\mathbf{r}, t)$ with respect to $\mathbf{r}$ are denoted by additional indices after the semicolon.

We now derive the equation for the one-point correlation function assuming that $\mathbf{r} = 0$ in Eqn (99):

$$\frac{\partial}{\partial t}\left\langle W_{ij}(t)\right\rangle = -2\,\frac{\partial^2 B_{ij}(\mathbf{0})}{\partial r_k\,\partial r_m}\left\langle W_{km}(t)\right\rangle + 2\mu_H\left\langle W_{ij;\,ss}(\mathbf{0}; t)\right\rangle.$$

Taking the solenoidal part of Eqn (64) ($D^{\mathrm{p}} = 0$) into account, we then obtain the nonclosed equation for the correlation:

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle W_{ij}(t)\right\rangle &= \frac{2(d+1)D^{\mathrm{s}}}{d(d+2)}\,\delta_{ij}\left\langle E(t)\right\rangle \\ &\quad - \frac{4D^{\mathrm{s}}}{d(d+2)}\left\langle W_{ij}(t)\right\rangle + 2\mu_H\left\langle W_{ij;\,ss}(\mathbf{0}; t)\right\rangle,\end{aligned}$$

where $\langle E(t)\rangle = \langle \mathbf{H}^2(\mathbf{r}, t)\rangle$ is the mean magnetic field energy. Setting $i = j$ in this equation, we obtain the equation for the mean energy

$$\frac{\partial}{\partial t}\left\langle E(t)\right\rangle = \frac{2(d-1)D^{\mathrm{s}}}{d}\left\langle E(t)\right\rangle - 2\mu_H D(t),$$

where

$$D(t) = \left\langle\left[\operatorname{rot}\mathbf{H}(\mathbf{r}, t)\right]^2\right\rangle = -\left\langle W_{ii;\,jj}(\mathbf{0}, t)\right\rangle \tag{100}$$

describes the dissipation and is related to the variance of the current strength in the magnetohydrodynamic flow. We note that as $t \to \infty$, the mean energy reaches a stationary level $\langle E(\infty)\rangle$. Because $\langle E(t)\rangle \geqslant E_0$, the dissipation also reaches the stationary level

$$D(0) = \frac{(d-1)D^{\mathrm{s}}}{d\mu_H}\left\langle E(\infty)\right\rangle.$$

As in the case of a density field, we seek the solution of this equation as a series in the parameter μ. In the first approximation, we obtain the solution

$$\left\langle E(t)\right\rangle_1 = E_0 \exp\left(\frac{2(d-1)D^{\mathrm{s}}t}{d} - \int_0^t \mathrm{d}\tau\,\frac{2\mu_H}{\left\langle E(\tau)\right\rangle_0}\,D_0(\tau)\right), \tag{101}$$

where $D_0(t)$ corresponds to dissipation in the absence of the dynamic diffusion effect. In this case, the problem solution at the initial evolution stage, when the dissipation can be neglected, grows exponentially:

$$\left\langle E(t)\right\rangle_0 = E_0 \exp\left[\frac{2(d-1)}{d}\,D^{\mathrm{s}}t\right]. \tag{102}$$

4.2.1 The magnetic field helicity. Hereafter, we neglect the dynamic diffusion and suppress the zero index. We derive an equation for the quantity

$$\left\langle W_{kp;\,j}(\mathbf{r}; t)\right\rangle = \frac{\partial}{\partial r_j}\left\langle W_{kp}(\mathbf{r}; t)\right\rangle = \left\langle\frac{\partial H_k(\mathbf{r}, t)}{\partial r_j}\,H_p(\mathbf{r}_1, t)\right\rangle,$$

which facilitates computation of the magnetic field helicity

$$H(t) = \varepsilon_{ijk}\langle W_{ki;j}(\mathbf{0};t)\rangle = \langle \mathbf{H}\,\mathrm{rot}\,\mathbf{H}\rangle\,.$$

Differentiating Eqn (99) with respect to r_j, we arrive at the equation

$$\frac{\partial}{\partial t}\langle W_{kp;j}(\mathbf{r};t)\rangle = -2\,\frac{\partial^3 B_{kp}(\mathbf{r})}{\partial r_n\,\partial r_m\,\partial r_j}\langle W_{nm}(\mathbf{r};t)\rangle$$
$$-2\,\frac{\partial^2 B_{kp}(\mathbf{r})}{\partial r_n\,\partial r_m}\langle W_{nm;j}(\mathbf{r};t)\rangle + 2\,\frac{\partial^2 B_{kn}(\mathbf{r})}{\partial r_m\,\partial r_j}\langle W_{mp;n}(\mathbf{r};t)\rangle$$
$$-2\,\frac{\partial\big(B_{kn}(\mathbf{0})-B_{kn}(\mathbf{r})\big)}{\partial r_m}\,\frac{\partial}{\partial r_n}\langle W_{mp;j}(\mathbf{r};t)\rangle$$
$$+2\,\frac{\partial^2 B_{np}(\mathbf{r})}{\partial r_m\,\partial r_j}\langle W_{km;n}(\mathbf{r};t)\rangle$$
$$-2\,\frac{\partial\big(B_{np}(\mathbf{0})-B_{np}(\mathbf{r})\big)}{\partial r_m}\,\frac{\partial}{\partial r_n}\langle W_{km;j}(\mathbf{r};t)\rangle$$
$$-\frac{\partial\big(B_{nq}(\mathbf{r})+B_{qn}(\mathbf{r})\big)}{\partial r_j}\,\frac{\partial^2}{\partial r_q\,\partial r_n}\langle W_{kp}(\mathbf{r};t)\rangle$$
$$+\big(2B_{nq}(\mathbf{0})-B_{nq}(\mathbf{r})-B_{qn}(\mathbf{r})\big)\,\frac{\partial^2}{\partial r_q\,\partial r_n}\langle W_{kp;j}(\mathbf{r};t)\rangle\,. \quad (103)$$

At $\mathbf{r}=0$, it becomes an equation for the one-point correlation:

$$\frac{\partial}{\partial t}\langle W_{kp;j}(\mathbf{0};t)\rangle = -2\,\frac{\partial^3 B_{kp}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_j}\langle W_{nm}(\mathbf{0};t)\rangle$$
$$-2\,\frac{\partial^2 B_{kp}(\mathbf{0})}{\partial r_n\,\partial r_m}\langle W_{nm;j}(\mathbf{0};t)\rangle + 2\,\frac{\partial^2 B_{kn}(\mathbf{0})}{\partial r_m\,\partial r_j}\langle W_{mp;n}(\mathbf{0};t)\rangle$$
$$+2\,\frac{\partial^2 B_{np}(\mathbf{0})}{\partial r_m\,\partial r_j}\langle W_{km;n}(\mathbf{0};t)\rangle\,.$$

Using (64), we obtain the following second-order equation for the solenoidal part of the correlation function:

$$\left(\frac{\partial^2}{\partial t^2}-\frac{2(d+5)D^{\mathrm{s}}}{d(d+2)}\,\frac{\partial}{\partial t}-8\,\frac{d-1}{d^2(d+2)}\,[D^{\mathrm{s}}]^2\right)\langle W_{kp;j}(\mathbf{0};t)\rangle$$
$$=-2\,\frac{\partial^3 B_{kp}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_j}\left(\frac{\partial}{\partial t}-\frac{2(d+3)D^{\mathrm{s}}}{d(d+2)}\right)\langle W_{nm}(\mathbf{0};t)\rangle$$
$$+\frac{4(d+1)D^{\mathrm{s}}}{d(d+2)}\left(\frac{\partial^3 B_{jp}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_k}+\frac{\partial^3 B_{kj}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_p}\right)\langle W_{nm}(\mathbf{0};t)\rangle\,, \quad (104)$$

with the source in the right-hand side related to the absence of mirror symmetry.

We expand the function $C(r)$ in the correlation function of velocity field (66) as $C(r)=C(0)-\alpha\mathbf{r}^2+\ldots$ and use Eqn (65). We solve Eqn (104) bearing in mind the growing exponents of one-point magnetic field correlation function (102):

$$\langle W_{nm}(\mathbf{0};t)\rangle = \frac{1}{d}\,\delta_{nm}\langle E(t)\rangle_0\,.$$

Equation (104) is then simplified and becomes

$$\left(\frac{\partial}{\partial t}-\frac{4D^{\mathrm{s}}}{d}\right)\left(\frac{\partial}{\partial t}+\frac{2(d-1)D^{\mathrm{s}}}{d(d+2)}\right)\langle W_{kp;j}(\mathbf{0};t)\rangle$$
$$=8\alpha D^{\mathrm{s}}\,\frac{(d+3)(d-1)}{d}\,\varepsilon_{kpj}\langle E(t)\rangle_0\,. \quad (105)$$

Equation (105) has two characteristic exponents, one positive, corresponding to growth, and the other negative, corresponding to a decaying solution. The solution of Eqn (105) that grows with time is sought in the form

$$\langle W_{kp;j}(\mathbf{0};t)\rangle = U_{kp;j}(t)\exp\left(\frac{4D^{\mathrm{s}}}{d}\,t\right).$$

The equation for $U_{kp;j}(t)$ is then simplified and takes the form of a 'reduced' equation:

$$\frac{\partial U_{kp;j}(t)}{\partial t} = 4\alpha\varepsilon_{kpj}\,\frac{(d+2)(d-1)(d+3)}{3(d+1)}\,E_0\exp\left[\frac{2(d-3)D^{\mathrm{s}}}{d}\,t\right].$$

Because we know that the helicity is zero in two dimensions, we consider the case of three dimensions and obtain

$$U_{kp;j}(t) = 20\alpha\varepsilon_{kpj}E_0 t\,.$$

Accordingly, the basic exponentially growing solution becomes

$$\langle W_{kp;j}(\mathbf{0};t)\rangle = U_{kp;j}(t)\exp\left(\frac{4D^{\mathrm{s}}}{3}\,t\right) = 20\alpha\varepsilon_{kpj}\langle E(t)\rangle_0 t\,. \quad (106)$$

By virtue of Eqn (59) ($\varepsilon_{ijk}\varepsilon_{kij}=6$), we obtain the following final expression for the magnetic field helicity in three dimensions:

$$H_0(t) = 120\alpha\langle E(t)\rangle_0\,t\,. \quad (107)$$

4.2.2 Magnetic field energy dissipation (variance of the current strength in a flow). The magnetic field energy dissipation is defined by Eqn (100), and to estimate it, we need to derive an equation for the quantity

$$\langle W_{kp;js}(\mathbf{0};t)\rangle = \frac{\partial}{\partial r_s}\langle W_{kp;j}(\mathbf{0};t)\rangle = \left\langle\frac{\partial H_k(\mathbf{r},t)}{\partial r_s\,\partial r_j}\,H_p(\mathbf{r}_1,t)\right\rangle_{\mathbf{r}=\mathbf{r}_1}.$$

Differentiating Eqn (103) with respect to r_s, setting $\mathbf{r}=0$, and contracting all functions over the indices $k=p$ and $j=s$ using formula (64), we obtain the equation

$$\frac{\partial}{\partial t}\langle W_{kk;ss}(\mathbf{0};t)\rangle = -2\,\frac{\partial^4 B_{kk}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_s\,\partial r_s}\langle W_{nm}(\mathbf{0};t)\rangle$$
$$-4\,\frac{\partial^3 B_{kk}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_s}\langle W_{nm;s}(\mathbf{0};t)\rangle + 2\,\frac{\partial^3 B_{kn}(\mathbf{0})}{\partial r_m\,\partial r_s\,\partial r_s}\langle W_{mk;n}(\mathbf{0};t)\rangle$$
$$+2\,\frac{\partial^3 B_{nk}(\mathbf{0})}{\partial r_m\,\partial r_s\,\partial r_s}\langle W_{km;n}(\mathbf{0};t)\rangle + \frac{4(d+1)D^{\mathrm{s}}}{d+2}\langle W_{mm;ss}(\mathbf{0};t)\rangle$$

in the case of an incompressible fluid flow.

We now limit ourselves to excitation sources that grow exponentially with time:

$$-2\,\frac{\partial^4 B_{kk}(\mathbf{0})}{\partial r_n\,\partial r_m\,\partial r_s\,\partial r_s}\langle W_{nm}(\mathbf{0};t)\rangle_0 = -2\,\frac{\partial^4 B_{kk}(\mathbf{0})}{\partial r_n\,\partial r_n\,\partial r_s\,\partial r_s}\langle E(t)\rangle_0$$
$$=-2\int_0^\infty \mathrm{d}\tau\,\Big\langle\big[\Delta\mathbf{u}(\mathbf{r},t+\tau)\big]\big[\Delta\mathbf{u}(\mathbf{r},t)\big]\Big\rangle\langle E(t)\rangle_0$$
$$=-2D_H^{(4)}\langle E(t)\rangle_0\,,$$

where $D_H^{(4)} = \int \mathrm{d}\mathbf{k}\, \mathbf{k}^4 E^{\mathrm{s}}(k)$. As a result, in the three-dimensional case,

$$D_0^{(3)}(t) \approx \left[2D_H^{(4)} + 2571\,\frac{\alpha^2}{D^{\mathrm{s}}}\right] \frac{15\langle E(t)\rangle_0}{28D^{\mathrm{s}}} \exp\left(\frac{28}{15}\, D^{\mathrm{s}} t\right). \tag{108}$$

It follows from the structure of the three-dimensional solution that the dissipation of the magnetic field for large times is determined by the velocity field helicity, and in its absence, by the mean magnetic field energy (102).

In the two-dimensional case ($d = 2$), the helicity is absent, and it follows that

$$D_0^{(2)}(t) \approx \frac{D_H^{(4)}}{D^{\mathrm{s}}} \langle E(t)\rangle_0 \exp(D^{\mathrm{s}} t). \tag{109}$$

In this case, the energy dissipation is only governed by the energy.

Formulas (108) and (109) show that the dissipation grows with time markedly faster than the mean energy does.

4.3 Generalization to the case of inhomogeneous initial conditions

We have considered solutions of a number of problems pertaining to the dynamics of statistical characteristics of the scalar density field and vector magnetic field and their spatial derivatives in the simplest formulation (involving homogeneous initial conditions), with a minimum number of defining parameters related only to the statistical characteristics of the homogeneous velocity field delta correlated in time. In this case, all the studied fields are also spatially homogeneous, but nonstationary random fields. The statistical means like $F_{ij}(\mathbf{r}, \mathbf{r}_1, t) = \langle f_i(\mathbf{r}, t)\, f_j(\mathbf{r}_1, t)\rangle$ depend on the spatial coordinates only in terms of the difference $\mathbf{r} - \mathbf{r}_1$; hence,

$$\frac{\partial}{\partial \mathbf{r}_1} F_{ij}(\mathbf{r}, \mathbf{r}_1, t) = -\frac{\partial}{\partial \mathbf{r}} F_{ij}(\mathbf{r}, \mathbf{r}_1, t),$$

i.e., at $\mathbf{r} = \mathbf{r}_1$, the quantity $\langle f_i(\mathbf{r}, t)\, \partial f_j(\mathbf{r}, t)/\partial \mathbf{r}\rangle$ independent of $\mathbf{r}$ satisfies the identity

$$\left\langle f_i(\mathbf{r}, t)\, \frac{\partial f_j(\mathbf{r}, t)}{\partial \mathbf{r}}\right\rangle = -\left\langle f_j(\mathbf{r}, t)\, \frac{\partial f_i(\mathbf{r}, t)}{\partial \mathbf{r}}\right\rangle. \tag{110}$$

We extensively used it in deriving the equations above, which considerably simplified the analysis of both the system itself and the results obtained because the essential majority of terms vanish at $\mathbf{r} = \mathbf{r}_1$, which implies the absence of advection of statistical characteristics in the problems studied. Specifically, this allowed us to solve the problems considered more completely, while avoiding fairly tedious manipulations.

If the initial conditions are spatially inhomogeneous, the solutions of all problems lose the property of spatial homogeneity, and the equations become very cumbersome. The solutions obtained above, however, carry certain information pertinent to this case, too. Indeed, the integral

$$\int \mathrm{d}\mathbf{r}\, f_i(\mathbf{r}, t)\, \frac{\partial f_j(\mathbf{r}, t)}{\partial \mathbf{r}} = -\int \mathrm{d}\mathbf{r}\, f_j(\mathbf{r}, t)\, \frac{\partial f_i(\mathbf{r}, t)}{\partial \mathbf{r}}$$

also has property (110) (integration by parts). It is therefore understandable that for the density field, in the case of inhomogeneous initial conditions and in the absence of dynamic diffusion, we obtain the solution

$$\int \mathrm{d}\mathbf{r}\, \langle \rho^2(\mathbf{r}, t)\rangle_0 = \int \mathrm{d}\mathbf{r}\, \rho_0^2(\mathbf{r}) \exp(2D^{\mathrm{p}} t) \tag{111}$$

instead of Eqn (93), whereas formula (97) is replaced with

$$\int \mathrm{d}\mathbf{r} \left\langle \left(\boldsymbol{\nabla}\rho(\mathbf{r}, t)\right)^2 \right\rangle_0 = \int \mathrm{d}\mathbf{r}\, \left(\boldsymbol{\nabla}\rho_0(\mathbf{r})\right)^2 \exp(At) + \frac{D_\rho^{(4)}}{A} \int \mathrm{d}\mathbf{r}\, \langle \rho^2(\mathbf{r}, t)\rangle_0 \left[\exp(At) - 1\right], \tag{112}$$

where $A = 2\left[(d-1)D^{\mathrm{s}} + (d+5)D^{\mathrm{p}}\right]/d$.

Analogously, for the mean magnetic field energy in the case of inhomogeneous initial conditions and in the absence of dynamic diffusion, we obtain

$$\int \mathrm{d}\mathbf{r}\, \langle E(\mathbf{r}, t)\rangle_0 = \int \mathrm{d}\mathbf{r}\, E_0(\mathbf{r}) \exp\left(2\,\frac{d-1}{d}\,(D^{\mathrm{s}} + D^{\mathrm{p}})\, t\right)$$

instead of formula (83).

We can therefore argue that the relations and links among the various quantities obtained above are integral from the standpoint of inhomogeneous problems, and serve, speaking metaphorically, as the ‘skeleton’ supporting the dynamics of complex stochastic motions. Notably, the terms that were reduced to zero in our analysis, have the divergence (‘flux’) form in the case of an inhomogeneous problem.

5. Waves in randomly inhomogeneous media

The problem of propagation of monochromatic radiation in random multidimensional media is described by complex-valued parabolic equation (13). In this case, the intensity of the wave field $I(x, \mathbf{R}) = |u(x, \mathbf{R})|^2$ satisfies continuity equation (14), which coincides in form with equation (11) for the tracer density field in a random potential flow, and hence the wave field intensity undergoes clustering, which is manifested through the emerging *caustic field structure*.

For small distances traveled by a wave, the distribution of the wave field intensity has a log-normal character. As the distance increases, the nonlinear character of the equation for the complex-valued phase must be taken into account. This distance range, called the *region of strong focusing*, is notoriously difficult for analytic research. If the distance of wave propagation is increased even further, the statistical characteristics of intensity approach the saturated regime, and this region is called the *region of strong intensity fluctuations*.

In this region, the statistical characteristics of the wave field intensity cease to depend on the propagation path and take the form

$$\langle I^n(x, \mathbf{R})\rangle = n!, \qquad P(x, I) = \exp(-I).$$

These expressions can be derived by representing solutions of Eqn (13) in terms of a path integral [2–6, 18].

In this case, the mean specific area of domains inside which $I(x, \mathbf{R}) > I$ and the mean specific power concentrated in them are constant and do not describe the behavior of the wave field intensity in particular realizations. In addition, even passing to a statistically equivalent random process does not bring new information because the typical realization curve is also a constant. Here, the wave field structure can be explained by resorting to the analysis of quantities such as the

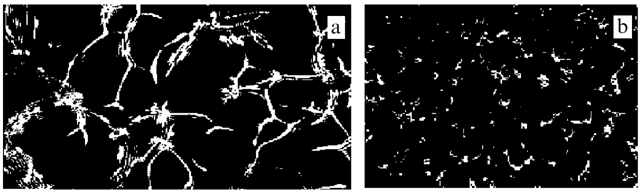

Figure 15. Transverse section of a laser beam propagating through a turbulent medium (under laboratory conditions) in the region of strong focusing (a) and in the region of strong (saturated) fluctuations (b).

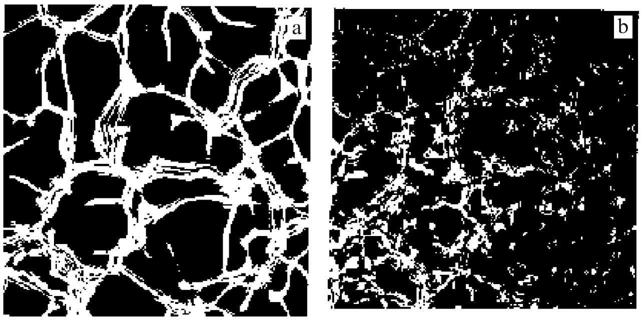

Figure 16. Transverse section of a laser beam propagating through a turbulent medium (numerical simulations) in the region of strong focusing (a) and in the region of strong (saturated) fluctuations (b).

mean specific contour length and mean specific number of wave intensity contours described by functionals like (18) or (20).

The aforementioned quantities continue to grow with distance; consequently, contour subdivision occurs, which was observed in laboratory experiments and in numerical simulations. Figure 15 presents images of the transverse section of a laser beam propagating through a turbulent medium in the laboratory research in Ref. [40], for various amplitudes of dielectric permittivity fluctuations. Analogous images in Ref. [41] are given in Fig. 16. The images were obtained as a result of numerical simulations performed in Refs [42, 43]. They explicitly show the emergence of a caustic structure of the wave field.

A similar situation should also be observed in the case of the monochromatic nonlinear *problem of wave self-interaction* in randomly inhomogeneous media described by the non-linear parabolic equation (nonlinear Schrödinger equation)

$$\frac{\partial}{\partial x}\, u(x,\mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}}\, u(x,\mathbf{R}) + \frac{\mathrm{i}k}{2}\,\varepsilon\big(x,\mathbf{R};I(x,\mathbf{R})\big)\,u(x,\mathbf{R})\,,$$
$$u(0,\mathbf{R}) = u_0(\mathbf{R})\,,$$

because Eqn (14) is independent of the shape of the function $\varepsilon(x,\mathbf{R};I(x,\mathbf{R}))$. The interest in this equation is motivated by the problem of so-called *rogue waves* (see, e.g., Refs [44-49]). These waves, undoubtedly, arise because of water mass clustering, but, in our opinion, they cannot be described on the basis of equations (linear or nonlinear) for separate harmonics.

6. Conclusions

As shown in this paper, the growth with time of quantities such as moments of the density field $\langle\rho^n(\mathbf{r},t)\rangle$ and magnetic field energy $\langle E^n(\mathbf{r},t)\rangle$ for positive and negative values of n (i.e., $\langle 1/\rho^n(\mathbf{r},t)\rangle$ and $\langle 1/E^n(\mathbf{r},t)\rangle$) in accordance with very similar laws has no relation to the clustering of these fields. It is therefore clear that multiple attempts, beginning with classic work [50] and series [51–60], to describe complex dynamical processes in separate realizations of random fields $\rho(\mathbf{r},t)$ and $\mathbf{H}(\mathbf{r},t)$ based on the computation of 'turbulent' diffusion coefficients in the $\mathbf{r}$-space using equations for mean fields and other moments and correlation functions are unfounded, because all these functions are described by the tails of probability distributions, while stochastic structure formation is described by the central part of the distributions.

We studied a set of integral characteristics that describe the dynamics as a whole, which allowed us to separate processes of field generation without diverting our attention to side phenomena rooted in advection of these quantities by a random velocity field. For a compressible flow (in a divergent velocity field), the density field always undergoes clustering with probability one. The characteristic time of cluster formation is estimated as $D^{\mathrm{p}}t \sim 1$, where the quantity D^{p} is determined by a potential constituent of the spectral component of the velocity field. We note that even for an incompressible fluid in hydrodynamical flows, the density field experiences clustering for a 'buoyant' tracer, when a finite inertia of the tracer field is considered, or for multiphase flows, i.e., always when a potential component arises in the spectrum of the tracer velocity field, which is different from the velocity of the fluid itself.

As concerns the magnetic field energy $E(\mathbf{r},t)$, clustering in an isotropic turbulent velocity field develops in the three-dimensional case only if $D^{\mathrm{p}} > D^{\mathrm{s}}$, where D^{s} depends on the solenoidal component of the velocity field. For acoustic turbulence, clustering thus occurs with probability one. For a plane-parallel fluid flow, clustering of the magnetic field energy in the velocity plane also occurs if the condition $D^{\mathrm{p}} > D^{\mathrm{s}}$ is satisfied, while clustering of the magnetic field energy associated with the component perpendicular to the velocity plane always occurs in the presence of a potential component in the velocity field.

An important question is how to estimate characteristic scales of clusters and separations between different clusters. Clearly, they cannot be described by multi-point probability distributions because any mean quantity is determined by the gently sloping tails of these distributions. These quantities can only be estimated from one-point statistical functionals (an integral describing the dynamics of the system as a whole) like the length of contours, their number, etc., related to derivatives of the random field. In this respect, of indubitable interest is the random log-normal field $f(\mathbf{r},t;\alpha)$ in (5). It allows analytic calculations of all joint statistical characteristics of the field and its spatial derivatives, as well as numerical simulation.

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