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METHODOLOGICAL NOTES

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The Kepler problem and collisions of negative masses

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Abstract. **Processes of mechanical interaction between systems containing negative-mass bodies are considered, showing that the laws of physics lead to no inconsistencies when applied to such systems.**

1. Introduction

Recently, an increasing number of papers have appeared that explore phantom matter and its applications [1–9]. In this connection, many misconceptions, sometimes called paradoxes, have emerged, which are related to an inadequate interpretation of the properties of such matter in the approximation of the Newtonian mechanics. The appearance of the perpetual motion machine of the third kind is one such paradox (see, e.g., [10, 11]). This paradox is related to the hypothetical possibility of an unlimited acceleration of two gravitating bodies, one of which consists of phantom matter and has a negative mass.

Here, we analyze the mechanics of systems containing negative-mass bodies. The aim of this paper is to resolve inconsistencies in this issue and to show that at least the Newtonian mechanics of phantom matter is in agreement with the common knowledge and laws of physics. Possible applications of the considered problem are pointed out.

For positive masses, all calculations are carried out in [12]. Here, we make calculations accounting for negative masses.

In this paper, as in the classical Kepler problem, the inertial mass is assumed to be equal to the gravitational mass, in accordance with the equivalence principle.

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2. Classical two-body Kepler problem

In the classical Kepler problem, the laws of motion of each of two gravitating bodies with masses m_1 and m_2 are determined in the center-of-mass system (CMS) by radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{r}_1 = \frac{m_2 \mathbf{r}}{m_1 + m_2}\,, \quad \mathbf{r}_2 = -\frac{m_1 \mathbf{r}}{m_1 + m_2}\,, \quad m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 = 0\,, \tag{1}$$

where $\mathbf{r}$ is the radius vector from the first body to the second body, and the coordinate origin is in the center of inertia.

We assume that the bodies interact gravitationally with the interaction energy

$$U(r) \equiv -\frac{G m_1 m_2}{r}\,, \tag{2}$$

where G is the gravitational constant.

The complete solution is determined by four parameters (see [12]):

1) the reduced mass $\mu \equiv m_1 m_2/(m_1 + m_2)$;

2) the value α of the gravitational interaction of the bodies, $\alpha \equiv G m_1 m_2$;

3) the conserved angular momentum of the moving bodies

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad (\mathbf{p} \equiv \mu \dot{\mathbf{r}})\,; \tag{3}$$

4) the conserved total energy of the bodies

$$E = \frac{\mathbf{p}^2}{2\mu} - \frac{\alpha}{r}\,. \tag{4}$$

The energy parameter E determines the trajectory type: an ellipse (for $E < 0$) or a hyperbola (for $E > 0$), and the value of L (together with E) determines the eccentricity of the orbit e.

From the methodical standpoint, care should be taken to avoid errors when inserting negative values (which are always positive in the classical case) into a radicand or taking them out of a radicand.

The effective potential has the form

$$U_{\text{eff}}(r) = \frac{L^2}{2\mu r^2} - \frac{\alpha}{r} \,. \tag{5}$$

Because the results are independent of the sign of the angular momentum L, we hereafter assume a positive value of the ratio $l \equiv L/\mu$.

The integral of the trajectory is written as

$$\varphi = \int \frac{l}{r^2} \left[\frac{2E}{\mu} + \frac{2\alpha}{\mu r} - \frac{l^2}{r^2} \right]^{-1/2} \mathrm{d}r \,. \tag{6}$$

We introduce the eccentricity squared

$$e^2 \equiv 1 + \frac{2E\mu l^2}{\alpha^2} \,. \tag{7}$$

The different cases are shown in Figs 1–3. We note that for $m_1 + m_2 < 0$, the eccentricity can only be larger than unity ($e^2 > 1$) [see (4)]. This case corresponds to Fig. 3a, b.

Setting $\xi \equiv l/r > 0$, we rewrite (6) in the form

$$\varphi = -\int \frac{\mathrm{d}\xi}{\sqrt{2E/\mu + 2\alpha\xi/(\mu l) - \xi^2}} = -\int \left[\left(\frac{e\alpha}{\mu l} \right)^2 - \left(\xi - \frac{\alpha}{\mu l} \right)^2 \right]^{-1/2} \mathrm{d}\xi \,. \tag{8}$$

It follows that a solution exists only for $e^2 > 0$.

The trajectory equation is obtained in the form

$$\varphi(\xi) - \text{const} = -\arcsin \left[\frac{\xi - \alpha/(\mu l)}{\left| e\alpha/(\mu l) \right|} \right], \tag{9}$$

or, by introducing the specific radius as $r_0 \equiv l^2 |\mu/\alpha|$, we obtain

$$\frac{r_0}{r} = \operatorname{sign}\left(\frac{\alpha}{\mu} \right) - e \sin\varphi = \operatorname{sign}(m_1 + m_2) - e \sin\varphi \,. \tag{10}$$

It then follows that for $m_1 + m_2 < 0$, the eccentricity must be $e > 1$.

Instead of (5), we introduce the normalized effective potential $\tilde{U}_{\text{eff}}$:

$$\tilde{U}_{\text{eff}}(r) = \frac{r_0^2}{2r^2} - \operatorname{sign}(m_1 + m_2) \frac{r_0}{r} \,, \quad U_{\text{eff}}(r) \equiv \frac{\alpha^2}{\mu l^2} \tilde{U}_{\text{eff}}(r) \,. \tag{11}$$

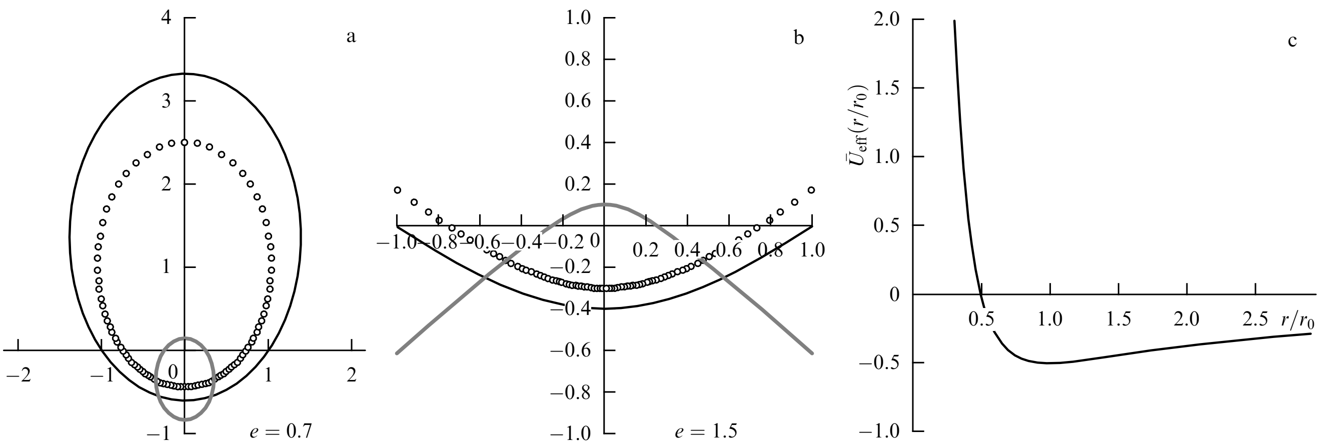

Figure 1. Trajectories (a, b) of the reduced mass $\mu > 0$ (thin line), the mass $m_1 = 3$ (thick line), and $m_2 = 1$ (dots); (c) plot of $\tilde{U}_{\text{eff}}(r/r_0)$.

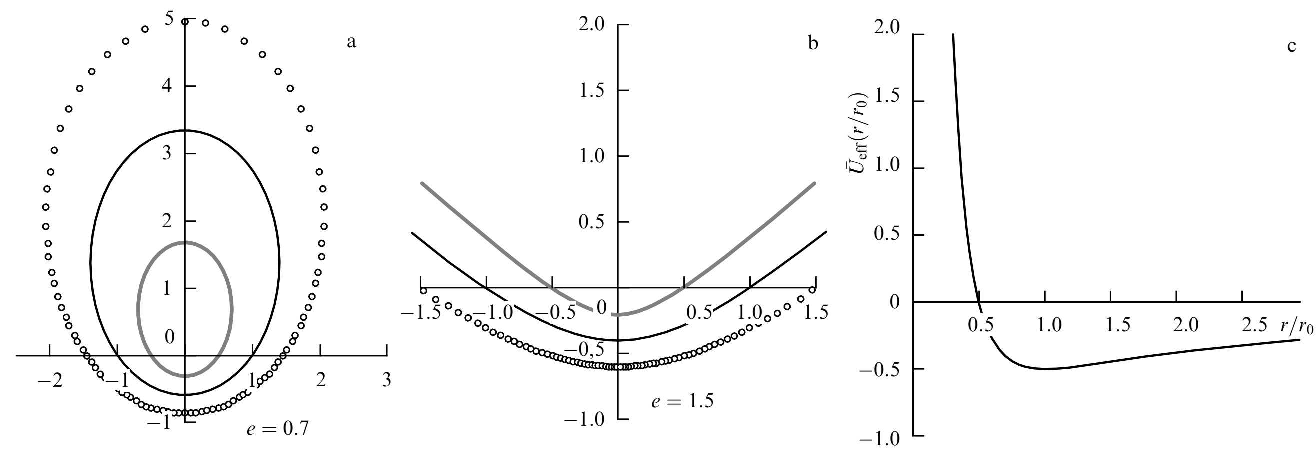

Figure 2. Trajectories (a, b) of the reduced mass $\mu < 0$ (thin line), the mass $m_1 = 3$ (thick line), and $m_2 = -1$ (dots); (c) plot of $\tilde{U}_{\text{eff}}(r/r_0)$.

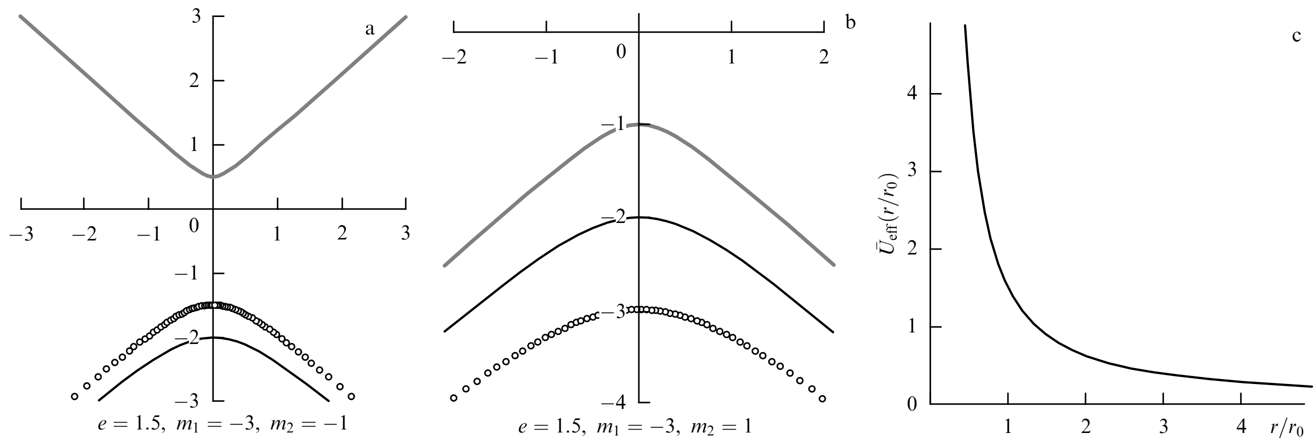

Figure 3. Trajectories (a, b) of the reduced mass m_1 (thin line), the mass m_2 (thick line) and μ (dots); (c) plot of $\tilde{U}_{\rm eff}(r/r_0)$.

For a positive total mass, the dimensionless effective potential $\tilde{U}_{\rm eff}$ has a local minimum and trajectories can be ellipses if $\tilde{U}_{\rm eff} < 0$. *This corresponds to the orbital stability for these masses, although the effective potential* $U_{\rm eff}$ *has a local maximum* (*and has no minimum*) *for* $\mu < 0$. This is a characteristic feature of a negative reduced mass.

3. Classification of interactions with negative masses

Clearly, in contrast to the classical situation (see Fig. 1), there can be only three different cases if negative masses are involved.

(1) The reduced mass is negative, $\mu < 0$, and $\alpha < 0$, i.e., the masses have different signs and their sum is positive (see Fig. 2). Both bodies lie along one radial line on one side of the center of inertia (which is at the coordinate origin). In the limit case $m_2 \to -m_1$, the bodies are removed to infinity from the center of inertia. Both finite ($e < 1$) and infinite ($e > 1$) trajectories are possible.

(2) The reduced mass is negative, $\mu < 0$, and $\alpha > 0$, i.e., both masses are negative (Fig. 3a). This case corresponds to the interaction of two negative masses and infinite motions with $e > 1$.

(3) The reduced mass is positive, $\mu > 0$, and $\alpha < 0$, i.e., the masses have different signs but their sum is positive (Fig. 3b). This case is similar to the dynamics of two charges of the same sign in electrodynamics and corresponds to infinite motions with $e > 1$.

According to formula (9), the first case (see Fig. 2) is equivalent to the classical Kepler problem in the field of attraction, and it can therefore be reduced to the classical case (see Fig. 1). It corresponds to $m_1 > 0$, $m_2 < 0$, and $m_1 > |m_2|$. The option of an elliptic orbit (Fig. 2a) is then the only possible one in the proposed classification (with negative masses).

4. The equivalence of elliptic trajectories for test masses moving in the gravitational field of masses with opposite signs

We consider two pairs of masses: (m, m^+) and (m, m^-). We assume that $e < 1$, $m > 0$, $m^+ > 0$, $m^- < 0$, and $m + m^- > 0$. These cases are shown in Fig. 1a and Fig. 2a.

We address the question: under what conditions does the elliptic trajectory of the mass m in the first case (the interaction of m with m^+) coincide with that of the mass m in the second case (the interaction of m with m^-)?

Clearly, this is possible for equal eccentricities e and parameters r_{0m} (the radii r_0 corresponding to the mass m). According to (1) and (7), two equations must be satisfied:

$$\frac{2E^+\mu^+(l^+)^2}{(\alpha^+)^2} = -\frac{2E^-\mu^-(l^-)^2}{(\alpha^-)^2} \iff \frac{E^+(l^+)^2}{m^+(m+m^+)} = -\frac{E^-(l^-)^2}{m^-(m+m^-)}\,, \tag{12}$$

$$\frac{m^+r_0^+}{m+m^+} = -\frac{m^-r_0^-}{m+m^-} \iff \frac{m^+(l^+)^2}{(m+m^+)^2} = -\frac{m^-(l^-)^2}{(m+m^-)^2}\,, \tag{13}$$

where the superscripts '+' and '−' stand for the corresponding cases. In addition to (12) and (13), we must equate the angular momenta of the mass m on these trajectories: $L_m^+ = L_m^-$ (or $l_m^+ = l_m^-$). Then, taking into account that $p_m^+ + p_{m^+}^+ = 0$ in the CMS and using relations (1), we obtain

$$L^+ = \mu^+ l^+ = r_m^+ p_m^+ + r_{m^+}^+ p_{m^+}^+ = p_m^+(r_m^+ - r_{m^+}^+) = m l_m^+\left(1 + \frac{m}{m^+}\right), \tag{14}$$

$$l^+ = l_m^+\left(\frac{m+m^+}{m^+}\right)^2, \qquad l^- = l_m^-\left(\frac{m+m^-}{m^-}\right)^2. \tag{15}$$

Hence, after taking (12) and (13) into account, we find the necessary condition in the form

$$\frac{(m+m^+)^2}{(m^+)^3} = -\frac{(m+m^-)^2}{(m^-)^3} \tag{16}$$

or

$$\left(\frac{E^+}{E^-}\right)^2 = -\frac{m^+}{m^-}\,. \tag{17}$$

We suppose that we wish to imitate the motion of the mass m around the mass m^+ by the motion around the mass m^-. We assume that the parameters related to the positive mass are known from observations and the mass m^- is unseen. We express all the parameters related to the negative mass in terms of these parameters.

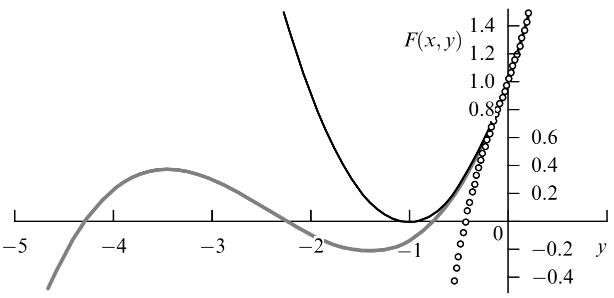

Figure 4. The plot of $F(x, y)$ for $x = 100$ (thin curve), $x = 9$ (thick curve), and $x = 1$ (white dots).

We set

$$x \equiv \frac{m^+}{m}\,, \quad y \equiv \frac{m^-}{m}\,. \tag{18}$$

It follows from (16) that

$$F(x, y) \equiv \frac{(1+x)^2}{x^3}\, y^3 + y^2 + 2y + 1 = 0\,. \tag{19}$$

The roots of Eqn (19) for a given x give the answer to the question posed.

We find an approximate solution of Eqn (19) for large x. An elementary application of the perturbation theory for roots of Eqn (19) yields

$$x \gg 1\,, \quad y_1 \approx -\frac{x}{(1+x)^{2/3}}\,, \quad y_2 \approx -1 - \frac{1+x}{x^{3/2}}\,,$$

$$y_3 \approx -1 + \frac{1+x}{x^{3/2}}\,. \tag{20}$$

The first two roots do not satisfy the condition for an elliptic orbit: $m + m^- > 0$. The third root, y_3, corresponds to the necessary conditions.

Graphical solutions of Eqn (19) for some x are shown in Fig. 4. For a given value of m^-, we can then obtain expressions for (13) and (17) using equations for i^- and E^-.

But all these considerations can be valid only if we do not see the real trajectories of the second mass. It also should be borne in mind that different trajectories for all possible m correspond to different masses m^-.

These considerations can be interesting for the astrophysics of wormholes or their remnants [13], because wormhole entrances can have negative masses. Possibly, similar ideas can be used in the analysis of the nature of dark matter.

5. Acceleration of conventional bodies by negative-mass bodies

We calculate the possible deviation of the velocity of a body due to its interaction with another body. We consider the case of infinite trajectories ($e > 1$), because in the case of finite (elliptic) trajectories, no infinitely large values can be obtained in Newtonian mechanics (as we see below). Clearly, in this case, the difference between the velocities of the body reaches a maximum between the perihelion and the infinitely remote point. At these points, the corresponding squares of the body momentum are $p_{\rm p}^2$ and p_∞^2.

According to (4), (7), and (10), we have

$$2\mu E = p_\infty^2 = p_{\rm p}^2 - \frac{2\mu\alpha}{r_{\rm p}} = (e^2 - 1)\,\frac{\alpha^2}{l^2}\,, \tag{21}$$

$$\frac{r_0}{r_{\rm p}} = \operatorname{sign}(m_1 + m_2) + e\,, \tag{22}$$

$$r_0 = \frac{l^2\mu}{\alpha}\,\operatorname{sign}(m_1 + m_2)\,. \tag{23}$$

Hence, we obtain the necessary relation

$$\frac{p_{\rm p}^2}{p_\infty^2} = \frac{e + \operatorname{sign}(m_1 + m_2)}{e - \operatorname{sign}(m_1 + m_2)}\,. \tag{24}$$

The case where $\operatorname{sign}(m_1 + m_2) = 1$ corresponds to classical motion (decelerating as the body recedes to infinity), and therefore the analysis gives nothing new.

The case where $\operatorname{sign}(m_1 + m_2) = -1$ corresponds to acceleration as the body recedes to infinity. In this case, $1 < e \approx 1$ corresponds to high accelerations. According to (21) and (24), we obtain

$$p_\infty^2 \approx 4\left(\frac{e-1}{e+1}\right)\frac{\alpha^2}{l^2} = 4\,\frac{p_{\rm p}^2\alpha^2}{p_\infty^2 l^2} \tag{25}$$

as $e \to 1$.

We here consider the nonrelativistic case $v \ll c$. Setting $r_{\rm g} \equiv 2G|m_1 + m_2|/c^2$, we obtain $4\alpha^2/(\mu^2 l^2) = r_{\rm g}^2 c^4/(r_{\rm p}^2 v_{\rm p}^2)$, where using (25) for essentially nonrelativistic velocities, we find

$$\frac{v_{\rm p}^2}{c^2} \ll \frac{v_\infty^2}{c^2} \approx \frac{r_{\rm g}}{r_{\rm p}} \ll 1\,, \quad \frac{v_\infty^2}{v_{\rm p}^2} \approx \frac{c^2 r_{\rm g}}{v_{\rm p}^2 r_{\rm p}} \ll \frac{c^2}{v_{\rm p}^2}\,. \tag{26}$$

Considering the perihelion as the initial point of the trajectory and treating $v_{\rm p}$ as a perturbation of the initial state at rest, we find that the masses are not accelerated endlessly, but up to some limit determined by this perturbation according to formula (26).

Inequality (26) bounds the value of the possible acceleration in Newtonian mechanics with negative masses.

Above, we did not consider the special case where $m_1 = -m_2$, $\mu \to \pm\infty$. This case is degenerate and must be considered as one of the limits

$$m_1 + m_2 > 0\,, \qquad m_1 \to -m_2$$

or

$$m_1 + m_2 < 0\,, \qquad m_1 \to -m_2\,.$$

In both cases, as follows from (26), it is impossible to reach an unlimited acceleration (in principle); furthermore, reaching maximum velocities v_∞ already requires not only equal masses of the bodies (in absolute values) but also a superfine tuning of the initial conditions. These requirements reduce the probability of such a (noncritical) situation to zero.

6. Collision of two bodies

We now consider the collision of two masses with arbitrary (including sign) masses. This problem is solved in the general form in [12]. The final result for an elastic collision has the

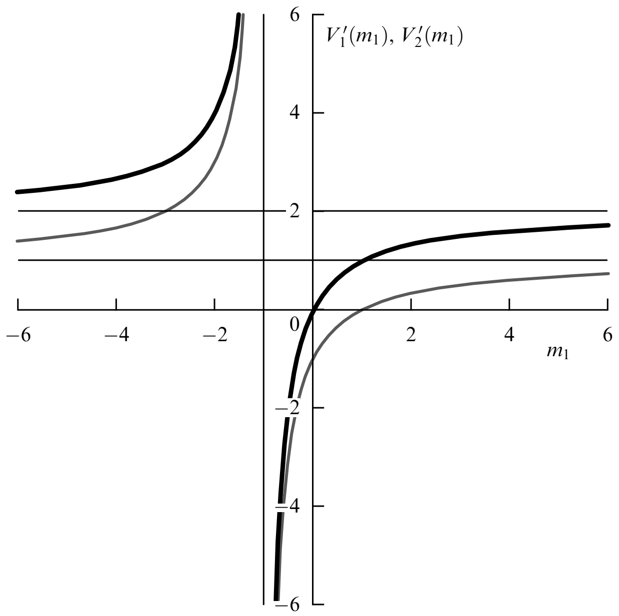

Figure 5. Plots of $V_1'(m_1)$ (thin curve) and $V_2'(m_1)$ (thick curve).

form (in the CMS)

$$\mathbf{v}_1' = \frac{m_2|\mathbf{v}_1 - \mathbf{v}_2|}{m_1 + m_2} \frac{\mathbf{v}_1'}{v_1'}, \quad \mathbf{v}_2' = \frac{-m_1|\mathbf{v}_1 - \mathbf{v}_2|}{m_1 + m_2} \frac{\mathbf{v}_1'}{v_1'}. \tag{27}$$

Here, subscripts '1' and '2' denote the particle numbers and primed quantities correspond to a moment after the collision.

In the laboratory frame, denoting the corresponding velocities by $\mathbf{V}$, we obtain an analog of (27) in the form

$$\mathbf{V}_1' = \frac{m_2|\mathbf{V}_1 - \mathbf{V}_2|}{m_1 + m_2} \frac{\mathbf{v}_1'}{v_1'} + \frac{m_1\mathbf{V}_1 + m_2\mathbf{V}_2}{m_1 + m_2},$$

$$\mathbf{V}_2' = \frac{-m_1|\mathbf{V}_1 - \mathbf{V}_2|}{m_1 + m_2} \frac{\mathbf{v}_1'}{v_1'} + \frac{m_1\mathbf{V}_1 + m_2\mathbf{V}_2}{m_1 + m_2}. \tag{28}$$

Setting $V_1 = 1, m_2 = 1$, and $V_2 = 0$ for clarity, we can express the dependences $V_1'(m_1)$ and $V_2'(m_1)$ for a head-on collision (Fig. 5).[1]

Similarly to the Kepler problem, no contradictions appear here.

7. Conclusion

The study presented in this paper shows that in the Newtonian treatment of the interaction of two bodies, including bodies with negative masses, no contradictions or paradoxes emerge.

The ideas considered here are of interest for the astrophysics of wormholes or their remnants, as well as for possible applications to the analysis of dark matter.

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[1] Assuming the direction of $\mathbf{v}_1$ to be +1 for clarity, we obtain $\mathbf{v}_1'/v_1' = -1$.