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METHODOLOGICAL NOTES

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Nonrelativistic quantum theory of stimulated Cherenkov radiation and Compton scattering in a plasma

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Contents

Abstract. **The role of quantum effects in the theory of Langmuir waves in a collisionless plasma is shown to be important for flows of quantum particles resonantly interacting with plasma oscillations and for plasma electromagnetic waves undergoing resonant scattering.**

1. Introduction. High-frequency waves in a quantum plasma

The applicability of kinetic equations with a self-consistent field (the Vlasov equation in the classical case, and the Wigner equation in the quantum case) for describing collisionless plasma was substantiated by N N Bogoliubov in his famous study [1]. The applicability condition in the case of electron gas is reduced to the following:

$$e^2 n^{1/3} \ll \varepsilon_0 \,, \tag{1.1}$$

where e is the electron charge, n is the electron concentration (number density), $\varepsilon_0 = mV_0^2/2$ is the average energy of electron chaotic motion, m is the electron mass, and V_0 is the average electron velocity (for nondegenerate electron Maxwellian gas, $V_0 = V_T$, where $V_T = \sqrt{k_B T/m}$ is the thermal velocity; in the presence of degeneracy, $V_0 = V_F$, where $V_F = (3\pi^2)^{1/3}\hbar n^{1/3}/m$ is the Fermi velocity, and T is the temperature). For further convenience, we rewrite inequality (1.1) using the gas parameter η as

$$\eta = \frac{\hbar\omega_L}{\varepsilon_0} \leqslant \frac{\hbar\omega_L}{\varepsilon_F} \approx \left(\frac{e^2}{\langle r\rangle\,\varepsilon_F}\right)^{1/2} \ll 1 \,, \tag{1.2}$$

where $\varepsilon_F = mV_F^2/2$ is the Fermi energy, $\omega_L = \sqrt{4\pi e^2 n/m}$ is the electron Langmuir frequency, and $\langle r\rangle = n^{-1/3}$ is the average distance between electrons. The smallness of parameter (1.2) must necessarily be taken into account when considering quantum effects in a plasma.

The electromagnetic processes considered in this paper are determined by the high-frequency dielectric response of a plasma to an electromagnetic field. Therefore, we write out here the known expressions for transverse and longitudinal permittivities of electron quantum plasma [2] (see also [3]):

$$\begin{aligned} \varepsilon^{\mathrm{tr}}(\omega,\mathbf{k}) &= 1 - \frac{\omega_L^2}{\omega^2}\left(1 + \int \frac{[\mathbf{kv}]^2 f(\mathbf{p})}{(\omega-\mathbf{kv})^2 - \omega_\hbar^2}\,\mathrm{d}\mathbf{p}\right), \\ \varepsilon^{\mathrm{l}}(\omega,\mathbf{k}) &= 1 - \omega_L^2 \int \frac{f(\mathbf{p})}{(\omega-\mathbf{kv})^2 - \omega_\hbar^2}\,\mathrm{d}\mathbf{p}\,. \end{aligned} \tag{1.3}$$

Here, $f(\mathbf{p})$ is the distribution function of plasma electrons over their momenta $\mathbf{p} = m\mathbf{v}$, ω and $\mathbf{k}$ are the frequency and wave vector of electromagnetic perturbations in a plasma, and $\omega_\hbar$ is the quantum frequency that determines the one-particle oscillation frequency spectrum (the electron de Broglie wave spectrum)[1] of an electron with the momentum $\hbar\mathbf{k}$:

$$\omega_\hbar = \frac{\hbar k^2}{2m} \,. \tag{1.4}$$

For nondegenerate equilibrium (Maxwellian) plasma, $f(\mathbf{p})$ is the Maxwell distribution function, and for degenerate plasma it is the Fermi distribution. Anyway, the characteristic 'width' of the distribution function $f(\mathbf{p})$ is determined by mV_0.

[1] It makes sense to talk about such a wave here since the plasma electron momentum changes by a multiple of $\hbar\mathbf{k}$ in interactions with an electromagnetic field.

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For the case of a cold plasma considered here, we have $f(\mathbf{p}) = \delta(\mathbf{p})$; therefore, formulas (1.3) are reduced to

$$\varepsilon^{\text{tr}}(\omega, \mathbf{k}) = 1 - \frac{\omega_{\text{L}}^2}{\omega^2}\,, \qquad \varepsilon^{\text{l}}(\omega, \mathbf{k}) = 1 - \frac{\omega_{\text{L}}^2}{\omega^2 - \omega_\hbar^2}\,. \tag{1.5}$$

The structure of expressions (1.3) (the presence of differences of squares in the denominators) suggests that formulas (1.5) are correct only in the high-frequency limit when conditions

$$|\omega \pm \omega_\hbar| \gg k V_0 \tag{1.6}$$

are satisfied; these conditions should necessarily be taken into account when analyzing the oscillation spectra of a quantum plasma.

Expression (1.5) for high-frequency transverse permittivity does not contain a quantum term, and consequently the spectrum of transverse electromagnetic waves is the same as in the classical limit [2, 4]. The spectrum of longitudinal waves, determined by zeros of the longitudinal permittivity $\varepsilon^{\text{l}}(\omega, \mathbf{k}) = 0$, is given by

$$\omega = \pm\sqrt{\omega_{\text{L}}^2 + \omega_\hbar^2}\,. \tag{1.7}$$

It is exactly this formula that was given in Ref. [2] as a spectrum of longitudinal quantum waves in a cold plasma. However, the applicability conditions for formula (1.7) were not specified in Ref. [2]. Let us find these conditions.

Substituting formula (1.7) into inequality (1.6), we reduce it to

$$\sqrt{1 + \left(\frac{1}{4}\,\eta\kappa^2\right)^2} - \frac{1}{4}\,\eta\kappa^2 \gg \kappa\,, \tag{1.8}$$

where $\kappa = k V_0/\omega_{\text{L}}$ is the dimensionless wavenumber, and η is the small gas parameter (1.2). Since function $\sqrt{1 + x^2} - x$ monotonically decreases from one to zero, inequality (1.8) can be satisfied only for $\kappa \ll 1$, i.e., when

$$k V_0 \ll \omega_{\text{L}}\,. \tag{1.9}$$

Inequality (1.9), as an applicability condition for formulas (1.5) and therefore for spectrum (1.7), holds for any value of the gas parameter η.[2]

Let us estimate, taking into account inequality (1.9), the value of quantum correction to the spectrum (1.7). From formula (1.9), we obtain the maximum wavenumber $k_{\max} = \omega_{\text{L}}/V_0$ that determines the maximum acceptable value of quantum frequency (1.4): $\omega_{\hbar,\max} = \hbar k_{\max}^2/(2m)$. Therefore, since $k \ll k_{\max}$ and inequality (1.2) is satisfied, the following inequalities are true:

$$\frac{\omega_\hbar}{\omega_{\text{L}}} \ll \frac{\omega_{\hbar,\max}}{\omega_{\text{L}}} = \frac{1}{4}\,\eta \ll 1\,, \qquad \frac{\omega_\hbar}{k V_0} \ll \frac{\hbar k_{\max}}{2m V_0} = \frac{1}{4}\,\eta \ll 1\,. \tag{1.10}$$

Thus, the quantum correction to spectrum (1.7) is small within the applicability limits of the collisionless plasma approximation. Moreover, owing to the second inequality in Eqn (1.10), the quantum correction in formula (1.7) is smaller than the classical thermal correction disregarded in this formula that de facto is an excess of precision.[3]

It should be noted that inequality (1.6) leading to inequality (1.9) is not a principal one since it merely signifies an applicability condition for the cold plasma approximation in the quantum theory, i.e., for formulas (1.5). Meanwhile, for the applicability of general formulas (1.3), we need only satisfy inequalities (1.2). From the structure of expressions (1.3) we can see that the quantum contribution to the plasma permittivities is always determined by frequency $\omega_\hbar$, and the thermal effects are determined by the value of $k V_0$. Therefore, it makes sense to estimate quantities entering Eqn (1.10) without assuming that inequality (1.6) is satisfied. Lifting the restriction on the value of wavenumber k, we arrive at

$$\frac{\omega_\hbar}{k V_0} \leqslant \frac{\omega_\hbar}{k V_{\text{F}}} \approx \frac{\langle r \rangle}{\lambda}\,, \qquad \frac{\omega_\hbar}{\omega_{\text{L}}} \approx \frac{\langle r \rangle^2}{\eta \lambda^2}\,, \tag{1.11}$$

where $\lambda = 2\pi/k$ is the wavelength of a Langmuir wave. For Langmuir waves, due to quasineutrality violation in volumes with large number of particles, the ratio $\langle r \rangle/\lambda$ is small by definition. Therefore, the quantum correction to the Langmuir wave spectrum is smaller than the thermal correction due to chaotic motions of plasma electrons, regardless of inequality (1.6).

From the second estimate in Eqn (1.11) it follows that the quantum frequency can, in principle, be larger than the electron Langmuir frequency (because $\eta \ll 1$). However, this is not exactly so in a Maxwellian plasma. Indeed, owing to the smallness of the ratio $\langle r \rangle/\lambda$, inequality $\omega_\hbar < k V_T$ holds but, simultaneously, $k V_T < \omega_{\text{L}}$ should be satisfied, because otherwise Langmuir waves would not exist in a plasma due to the strong Landau damping. Therefore, weakly damped Langmuir waves are possible in a Maxwellian plasma only for $\omega_\hbar < k V_T < \omega_{\text{L}}$ when the quantum effects are small. However, in a degenerate plasma where the Langmuir wave damping is absent according to the classical theory also for $k V_{\text{F}} > \omega_{\text{L}}$ (the zero sound) there are possible cases where the second ratio in Eqn (1.11) is not small. The quantum kinetic theory of Langmuir waves in an electron plasma taking into account electron thermal motions was developed in paper [5]; it was demonstrated there that quantum corrections to the frequency spectra are indeed always small although they can lead to qualitatively new effects (e.g., to the collisionless damping of zero sound in a degenerate plasma).

We should note that oscillation spectra of electron plasma were investigated in review paper [6] by using both the quantum kinetic equation [2] and the quantum hydrodynamic model [3]. However, the applicability conditions were not specified for the results obtained in Ref. [6]. It seems to us that many results of this work are beyond the applicability limits. The first work where these conditions were specified and that demonstrated when quantum effects can show themselves in a cold plasma in the high-frequency range are studies [7–9]. There, the quantum stimulated Cherenkov emission of longitudinal and transverse electromagnetic waves by electron beams in media and the quantum stimulated scattering of electromagnetic waves on a beam were investigated. Below, we discuss the results of this work.

[2] Under some assumptions, it can be inferred that kinetic Vlasov and Wigner equations may be applied even if inequality (1.2) does not hold. In particular, this is done when considering electron gas in metals [2].

[3] Taking into account chaotic electron motion gives, instead of expression (1.7), spectrum $\omega^2 = \omega_{\text{L}}^2 + \alpha k^2 V_0^2 + \beta k^4 V_0^4/\omega_{\text{L}}^2 + \omega_\hbar^2$, where $\alpha, \beta \sim 1$ [2]. Owing to inequality (1.2), the quantum term here is small, even in comparison with the term $\beta k^4 V_0^4/\omega_{\text{L}}^2$.

A conjecture can be made on the structure of frequency spectra for quantum Langmuir waves in the shorter wavelength range where ratio $\langle r\rangle/\lambda$ is large and the self-consistent field representation becomes incorrect. In this case, the interaction of plasma electrons occurs only via collisions. If the collisions are absent [because inequality (1.2) is fulfilled], then the frequency of longitudinal quantum plasma oscillations is determined by the relationship

$$\omega = \omega_{\hbar}\,. \tag{1.12}$$

In contrast to collective Langmuir oscillations, the waves specified by Eqn (1.12) are one-particle waves [2, 5].

2. Stimulated Cherenkov emission of a nonrelativistic electron beam in a plasma. Three-wave process involving quantum waves

It would seem to follow from an analysis of inequalities (1.2) and (1.6) that quantum effects appear as small insignificant corrections in the plasma theory (at least for a cold electron plasma). In reality, that is not exactly so. For example, there is known a well-observed quantum effect [2] in an electron plasma — the diamagnetism of free electron gas, determined by the quantity

$$\frac{\omega^2}{k^2c^2}\left(\varepsilon^{1} - \varepsilon^{\mathrm{tr}}\right)_{\omega/k\to 0} \tag{2.1}$$

that describes the electron contribution to the static (magnetic) permeability. When calculating Eqn (2.1), the large terms of classical origin are cancelled, and the remaining quantum terms become appreciable. An analogous situation takes place in the processes of stimulated Cherenkov emission of longitudinal plasma oscillations by an electron beam and stimulated resonant scattering of a transverse electromagnetic wave on longitudinal oscillations of electron density in a plasma. In the dispersion equations describing these processes, large classical terms are cancelled when the resonant conditions are satisfied, and quantum effects can become substantial in the remaining small terms, thus contributing significantly to the process development times (increments or growth rates).

Let us consider the excitation of longitudinal oscillations in a dense cold electron plasma by a nonrelativistic monoenergetic beam. We shall describe the dense plasma as a classical one, and the electron beam, taking into account its low density, as a quantum one. In these conditions, the dispersion relation for the beam–plasma interaction or, equally, that for the stimulated Cherenkov emission of longitudinal plasma waves by an electron beam is written out as [7]

$$\varepsilon^{1}(\omega,\mathbf{k}) \equiv 1 - \frac{\omega_{\mathrm{Lp}}^2}{\omega^2} - \frac{\omega_{\mathrm{Lb}}^2}{(\omega - \mathbf{ku})^2 - \omega_{\hbar}^2} = 0\,. \tag{2.2}$$

Here, $\mathbf{u}$ is the velocity of beam electrons, and ω_{Lb} and ω_{Lp} are the Langmuir frequencies of beam and plasma electrons, respectively. Let us clarify that the left-hand side of Eqn (2.2) stands for the longitudinal permittivity of the beam–plasma system. In particular, the beam contribution to the permittivity is obtained from the second formula in Eqn (1.5) by substituting $\omega_{\mathrm{L}} \to \omega_{\mathrm{Lb}}$ and $\omega \to \omega - \mathbf{ku}$. The latter substitution takes into account the Doppler frequency shift owing to electron motions. To simplify concrete formulas, we write them out for the case of perturbations propagating along the beam direction, i.e., at $\mathbf{ku} = ku$; we shall not do that in general formulas.

In the single-particle approximation, $\omega_{\mathrm{Lb}} \to 0$, we find from relation (2.2) the known quantum condition for the Cherenkov resonance between an electron and a longitudinal wave [10]:

$$\omega = \mathbf{ku} \mp \omega_{\hbar}\,. \tag{2.3}$$

Condition (2.3) with the 'minus' sign represents the condition for Cherenkov emission, and with the 'plus' sign is the quantum condition for Cherenkov absorption. We note that, in the plasma theory, stimulated emission of a beam is described as some resonant beam instability [10]. For instability to be developed, the dispersion relation describing it with respect to frequency ω has a complex solution with $\operatorname{Im}\omega > 0$.

When analyzing dispersion relation (2.2), we assume that the strong inequality is fulfilled, viz.

$$\omega_{\mathrm{Lp}} \gg \omega_{\mathrm{Lb}}\,. \tag{2.4}$$

Substituting $\omega = \omega_{\mathrm{Lp}}$ into formula (2.3), we obtain the single-particle Cherenkov resonance points for the beam and the plasma wave:

$$k_{1,2} = \frac{mu}{\hbar}\left(1 \mp \sqrt{1-\mu}\,\right),\qquad k_{3,4} = -\frac{mu}{\hbar}\left(1 \pm \sqrt{1+\mu}\,\right), \tag{2.5}$$

where μ is the important (for subsequent consideration) quantum parameter defined as

$$\mu = \frac{\hbar\omega_{\mathrm{Lp}}}{mu^2/2}\,. \tag{2.6}$$

The values of $k_{1,2}$ in Eqn (2.5) determine the wavenumbers of emitted plasma oscillations, and those of $k_{3,4}$ are the wavenumbers of plasma oscillations absorbed by beam electrons. If inequality $\mu > 1$ is satisfied, resonances in the points $k_{1,2}$ are absent. Since at resonances in the points $k_{3,4}$ there is no radiation emission (see below), inequality $\mu > 1$ is the stability condition for a low-density beam in a plasma, which can also be justified by analyzing dispersion relation (2.2) with respect to ω: there are no complex roots for $\mu > 1$. Therefore, we are interested here only in the case of $\mu < 1$. Moreover, we suppose that this inequality is rather strong. The physical meaning of inequality $\mu < 1$ as an instability condition is that an electron loses energy $\hbar\omega_{\mathrm{Lp}}$ during plasmon emission. But this is possible only if the plasmon energy is less than the electron kinetic energy. Thus, inequality $\mu < 1$ is the principal quantum electron energy threshold for the Cherenkov beam instability development in a plasma [7].

In the vicinity of the resonance points $k_{1,2}$, dispersion relation (2.2), when inequalities (2.4) and $\mu \ll 1$ are fulfilled, takes the form

$$\delta\omega^2(\delta\omega - 2\omega_{\hbar 1,2}) = \frac{1}{2}\,\omega_{\mathrm{Lb}}^2\omega_{\mathrm{Lp}}\,, \tag{2.7}$$

where $\delta\omega = \omega - \omega_{\mathrm{Lp}}$ is the complex instability increment, and $\omega_{\hbar 1,2} = \hbar k_{1,2}^2/2m$. If inequality $|\delta\omega| \gg 2\omega_{\hbar 1,2}$ is satisfied, then from Eqn (2.7) follows the growth rate of the standard classical beam instability in a plasma [4, 10]. We are not interested here in this case. However, when the opposite inequality takes place, it follows from Eqn (2.7) for the growth rate of the instability:

$$\delta\omega_{1,2} = \mathrm{i}\left(\frac{\omega_{\mathrm{Lb}}^2\omega_{\mathrm{Lp}}}{4\omega_{\hbar 1,2}}\right)^{1/2} \to \begin{cases} \delta\omega_1 = \mathrm{i}\left(\dfrac{\omega_{\mathrm{Lb}}^2}{\omega_{\mathrm{Lp}}^2}\dfrac{mu^2}{2\hbar\omega_{\mathrm{Lp}}}\right)^{1/2}\omega_{\mathrm{Lp}}\,, \\ \delta\omega_2 = \mathrm{i}\,\dfrac{1}{4}\left(\dfrac{\omega_{\mathrm{Lb}}^2}{\omega_{\mathrm{Lp}}^2}\dfrac{2\hbar\omega_{\mathrm{Lp}}}{mu^2}\right)^{1/2}\omega_{\mathrm{Lp}}\,. \end{cases} \tag{2.8}$$

These instability increments are obviously purely quantum. In the classical limit (when $\hbar \to 0$), the growth rate $\delta\omega_1$ increases and becomes the rate of the classical beam instability in a plasma, while the rate $\delta\omega_2$ vanishes.

Notice that resonance conditions (2.3) are similar in form to those for the anomalous and normal Doppler effects [11, 12]:

$$\omega = \mathbf{ku} \mp \Omega\,, \tag{2.9}$$

where Ω is the eigenfrequency of electron oscillations. In an external magnetic field, for example, Ω is the electron cyclotron frequency, and $\omega_H = eB/mc$, where B is the external longitudinal magnetic field induction.[4] For a high-density beam, we have $\Omega \sim \omega_{\mathrm{Lb}}$. For the case considered here, Ω is the quantum frequency $\omega_\hbar$. Thus, the quantum beam instabilities in a plasma are analogous to beam instabilities under the conditions of the anomalous Doppler effect or instabilities of the collective stimulated Cherenkov effect type [10]. It is to such a case that the instability is realized only at the resonance points $k_{1,2}$, since they are indeed in the anomalous Doppler effect range. There is no instability at resonance in the normal effect range (points $k_{3,4}$).

Consider now the applicability of results obtained in this section. Notice that conditions (1.1) and (1.2) also hold in the case under consideration. As for condition (1.6), it gives, together with the condition for the quantum instability regime, the following:

$$\omega_{\hbar 1,2} \gg |\delta\omega_{1,2}| \gg k_{1,2}V_0\,, \tag{2.10}$$

where V_0 is the thermal spread of the beam electron velocities.[5] For instability at the wavenumber $k_1 \approx \omega_{\mathrm{Lp}}/u$, taking into account increment (2.8), we obtain from inequalities (2.10):

$$\mu^3 \gg \frac{\omega_{\mathrm{Lb}}^2}{\omega_{\mathrm{Lp}}^2} \gg \mu\,\frac{V_0^2}{u^2}\,, \tag{2.11a}$$

and for instability at the wavenumber $k_2 \approx 2mu/\hbar$, inequalities (2.10) are reduced to the following:

$$1 \gg \mu^3\,\frac{\omega_{\mathrm{Lb}}^2}{\omega_{\mathrm{Lp}}^2} \gg \frac{V_0^2}{u^2}\,. \tag{2.11b}$$

Both inequalities (2.11) can be satisfied. Therefore, the quantum Cherenkov beam instability in a plasma can be realized well, albeit for a very small beam density and a large plasma density.

It might seem that the statement on the possibility of the quantum Cherenkov beam instability in a plasma contradicts the above results showing that Langmuir waves, including the electron beam Langmuir waves, can be described without accounting for quantum effects, since their contribution is small. But, for the beam instability considered here, Langmuir waves are not at all excited in the beam; therefore, their properties are not relevant in this case. To clarify what we mean, let us write out the dispersion relation (2.2) as

$$(1 + \delta\varepsilon_{\mathrm{p}}^{\mathrm{l}})(1 + \delta\varepsilon_{\mathrm{b}}^{\mathrm{l}}) = \delta\varepsilon_{\mathrm{p}}^{\mathrm{l}}\delta\varepsilon_{\mathrm{b}}^{\mathrm{l}}\,, \tag{2.12}$$

where $\delta\varepsilon_{\mathrm{p}}^{\mathrm{l}}$ and $\delta\varepsilon_{\mathrm{b}}^{\mathrm{l}}$ are the plasma and beam electron contributions, respectively, to the common longitudinal permittivity, i.e., to the left-hand side of Eqn (2.2).

Being written in such a way, equation (2.12) explicitly describes the interaction of plasma waves with the beam. Equations $1 + \delta\varepsilon_{\mathrm{p}}^{\mathrm{l}} = 0$ and $1 + \delta\varepsilon_{\mathrm{b}}^{\mathrm{l}} = 0$ are, respectively, the dispersion equations for Langmuir waves in a plasma and a beam, which are not interacting with each other. For the Cherenkov beam–plasma instability, we have $\omega \approx \omega_{\mathrm{Lp}}$, therefore $1 + \delta\varepsilon_{\mathrm{p}}^{\mathrm{l}} \approx 0$, and Langmuir waves are indeed excited in the plasma; we describe these waves classically, which fully agrees with the above results. It can be shown that when the left inequalities in formulas (2.11) are satisfied [to be more precise, for $\omega_{\mathrm{Lb}}^2/\omega_{\mathrm{Lp}}^2 \ll \mu$ in the case (2.11a), and for $\mu\omega_{\mathrm{Lb}}^2/\omega_{\mathrm{Lp}}^2 \ll 1$ in the case (2.11b)], inequality $|\delta\varepsilon_{\mathrm{b}}^{\mathrm{l}}| \ll 1$ will be fulfilled and therefore it makes no sense at all to talk about Langmuir waves in the beam. The considered Cherenkov beam instability is a single-particle one, and the manifestation of quantum effects is due to the quantization of energy transferred by a free beam electron to the classical plasma wave, which can be justified by the following simple reasoning.

We write out the energy and momentum conservation laws for an electron emitting a wave with the frequency ω and wave vector $\mathbf{k}$:

$$\frac{mu^2}{2} = \frac{mu'^2}{2} + \sigma_0\,, \qquad m\mathbf{u} = m\mathbf{u}' + \mathbf{p}_0\,, \tag{2.13}$$

where $\mathbf{u}'$ is the electron velocity after emission, and σ_0 and $\mathbf{p}_0$ are the energy and momentum of the emitted radiation quantum (in our case, the plasmon with frequency $\omega = \omega_{\mathrm{Lp}}$). From the general wave theory we have $\mathbf{p}_0 = (\mathbf{k}/\omega)\sigma_0$. Then, eliminating $\mathbf{u}'$ from Eqn (2.13), we obtain

$$\omega = \mathbf{ku} - \frac{(\sigma_0/\omega)k^2}{2m}\,. \tag{2.14}$$

If we put $\sigma_0 = \hbar\omega$, we arrive at condition (2.3) with the minus sign, i.e., at the quantum condition of the Cherenkov emission.

The quantum Cherenkov beam instability in a plasma can also be alternatively interpreted [7]. In the electric field of a plasma wave with the potential

$$\varphi(t,\mathbf{r}) = \frac{1}{2}\left[A\exp(-\mathrm{i}\omega t + \mathrm{i}\mathbf{kr}) + A^*\exp(\mathrm{i}\omega t - \mathrm{i}\mathbf{kr})\right], \tag{2.15}$$

[4] Equation (2.2), with the substitution $\omega_\hbar \to \omega_H$, looks like a dispersion relation for the beam–plasma interaction in the classical case when the beam is magnetized, and the plasma is not magnetized [4].

[5] When writing out condition (1.6) for the beam, we should make substitution $\omega \to \omega - \mathbf{ku}$.

the wave function of the beam electron has the following structure:

$$\psi(t,\mathbf{r}) = a_0 \exp(-\mathrm{i}\omega_0 t + \mathbf{k}_0\mathbf{r}) + a_- \exp\left[-\mathrm{i}(\omega_0-\omega)t + \mathrm{i}(\mathbf{k}_0-\mathbf{k})\mathbf{r}\right] + a_+ \exp\left[-\mathrm{i}(\omega_0+\omega)t + \mathrm{i}(\mathbf{k}_0+\mathbf{k})\mathbf{r}\right]. \quad (2.16)$$

Here, $\omega = \omega_{\mathrm{Lp}}$ is the plasma wave frequency, and $\omega_0 = \hbar k_0^2/2m$, $\hbar\mathbf{k}_0 = m\mathbf{u}$. Representation (2.16) in the linear approximation follows directly from the Schrödinger equation for beam electrons. The first term in expression (2.16) describes the de Broglie wave of an unperturbed electron, the second term describes the de Broglie wave of an electron that emitted a plasmon, and the third term stands for the de Broglie wave of an electron that absorbed a plasmon (the plasmon energy and momentum are $\hbar\omega$ and $\hbar\mathbf{k}$, respectively). The quantum Cherenkov emission can be treated as a decay process of the primary de Broglie beam wave into the plasma wave and the secondary de Broglie wave with the frequency $\omega' = \omega_0 - \omega$ and wave vector $\mathbf{k}' = \mathbf{k}_0 - \mathbf{k}$ [the second term in Eqn (2.16)]. The decay conditions are expressed in the form:

$$\omega_0 = \omega + \omega', \qquad \mathbf{k}_0 = \mathbf{k} + \mathbf{k}'. \quad (2.17)$$

However, conditions (2.17) make physical meaning only when every quantity there characterizes a real wave. In particular, the secondary wave should be the real de Broglie wave; therefore, the following dispersion relation should be satisfied:

$$\omega' = \frac{\hbar k'^2}{2m}. \quad (2.18)$$

Now, if we substitute relations (2.17) into formula (2.18) and take into account the definitions of ω_0 and $\mathbf{k}_0$, we obtain condition (2.3) with the minus sign, i.e., the quantum resonance condition for the Cherenkov emission. Thus, the Cherenkov radiation emission, from the quantum point of view, is the resonant interaction process of three waves (three-wave process), namely, two de Broglie waves of a free electron and a plasma wave.[6] It should be noted that there is no real de Broglie wave with $\omega = \omega_\hbar$ (see footnote 1); we are dealing here with some virtual wave determining the energy and momentum portions [σ_0 and $\mathbf{p}_0$ in Eqn (2.13)] lost by a beam electron in the emission process.

The inverse process of Cherenkov absorption can be treated similarly as the merging (inverse decay), viz.

$$\omega_0 + \omega = \omega', \qquad \mathbf{k}_0 + \mathbf{k} = \mathbf{k}', \quad (2.19)$$

of a primary de Broglie wave and a plasma wave into the secondary de Broglie wave described by the third term in expression (2.16). Indeed, if we substitute Eqn (2.19) into formula (2.18), we obtain condition (2.3) with the plus sign, i.e., the resonance condition of the Cherenkov absorption.

Let us draw attention to differences in the approaches. While the quantization of the radiation field is assumed in the derivation of relation (2.14), in the approach based on the wave interaction we have to do with a quantum electron [formula (2.16)] and the field is not quantized. The result is the same.

[6] Instead of the plasma wave, there can be a wave of any nature satisfying the resonance condition (2.3).

3. Stimulated Compton scattering in a plasma. Four-wave process involving quantum waves

Let us now turn to the discussion of the stimulated Compton scattering of a transverse electromagnetic wave on electrons of cold plasma (or an electron beam), accompanied by the generation of quantum waves. In the classical theory, this process is described in the linear approximation by the known dispersion equation for a three-wave decay of an incident electromagnetic wave (with frequency ω_1, and wave vector $\mathbf{k}_1$) into a scattered transverse wave (frequency ω, and wave vector $\mathbf{k}$) and a longitudinal Langmuir wave [4]:

$$\left[1 + \delta\varepsilon^{\mathrm{l}}(\omega_1-\omega, \mathbf{k}_1-\mathbf{k})\right]\left[k^2 - \frac{\omega^2}{c^2}\,\varepsilon^{\mathrm{tr}}(\omega,\mathbf{k})\right] = \frac{1}{4}(\mathbf{k}_1-\mathbf{k})^2\frac{\omega^2}{\omega_1^2}\frac{[\mathbf{k}\times\mathbf{V}_E]^2}{k^2c^2}\,\delta\varepsilon^{\mathrm{l}}(\omega_1-\omega,\mathbf{k}_1-\mathbf{k}). \quad (3.1)$$

Here, $\mathbf{V}_E = e\mathbf{E}_1/m\omega_1$, where $\mathbf{E}_1$ is the incident wave amplitude. According to equation (3.1), the plasma is modulated by the beat wave at frequency $\omega_1 - \omega$ with wave vector $\mathbf{k}_1 - \mathbf{k}$. It turns out that Eqn (3.1) takes place in the quantum case as well, where $\varepsilon^{\mathrm{tr}}$ and $1 + \delta\varepsilon^{\mathrm{l}} \equiv \varepsilon^{\mathrm{l}}$ are defined by formulas (1.5) (for a beam, $\omega - \mathbf{ku}$ in the function ε^{l} should be taken as a variable instead of ω).

Below, we restrict ourselves to the high-frequency case where frequencies ω_1 and ω significantly exceed the electron Langmuir frequency, and the plasma is transparent to electromagnetic waves, i.e., $\varepsilon^{\mathrm{tr}}$ and ε^{l} are close to unity [$|\delta\varepsilon^{\mathrm{l}}| \ll 1$, similar to the case of Eqn (2.12)]. Then dispersion equation (3.1) is reduced to the form [9]

$$\omega^2 - k^2c^2 = \frac{1}{4}\,\frac{\omega_{\mathrm{L}}^2(\mathbf{e}_1\mathbf{e})^2(\mathbf{k}_1-\mathbf{k})^2V_E^2}{\left[(\omega_1-\omega)-(\mathbf{k}_1-\mathbf{k})\mathbf{u}\right]^2 - \hbar^2(\mathbf{k}_1-\mathbf{k})^4/4m^2}, \quad (3.2)$$

where $\mathbf{e}_1$ and $\mathbf{e}$ are the unit polarization vectors for incident and scattered electromagnetic waves.

The condition for the resonant wave scattering, given by the zeros of the denominator on the right-hand side of Eqn (3.2), is given by

$$(\omega_1-\omega) - (\mathbf{k}_1-\mathbf{k})\mathbf{u} = \pm\frac{\hbar(\mathbf{k}_1-\mathbf{k})^2}{2m}. \quad (3.3)$$

Let us analyze formula (3.3) for the particular case of $\mathbf{u} = 0$ — the scattering on electron gas. With an accuracy up to the quantum term (that we assume to be small), we see that $\omega \approx \omega_1$. Therefore, condition (3.3) can be written out as

$$\omega = \omega_1 \mp \omega_1\frac{\hbar\omega_1}{mc^2}(1-\cos\theta), \quad (3.4)$$

where θ is the scattering angle (the angle between wave vectors $\mathbf{k}$ and $\mathbf{k}_1$). Condition (3.4) with the minus sign represents the known condition for the Compton scattering of light.

At the resonance point, we look for a solution of Eqn (3.2) (at $\mathbf{u} = 0$) in the form

$$\omega = kc + \delta\omega = \omega_1 \mp \Omega_\hbar + \delta\omega, \qquad \Omega_\hbar = \omega_1\frac{\hbar\omega_1}{mc^2}(1-\cos\theta). \quad (3.5)$$

Substituting Eqn (3.5) into Eqn (3.2), we obtain the following equation for the increment $\delta\omega$:

$$\delta\omega^2(\delta\omega \mp 2\Omega_\hbar) = \frac{1}{4}\frac{\tilde{V}_E^2}{c^2}\,\omega_L^2\omega_1(1-\cos\theta)\,. \qquad (3.6)$$

Here, we introduced the notation $\tilde{V}_E^2 = V_E^2(\mathbf{e}_1\mathbf{e}_2)^2$ to shorten subsequent expressions. It is reasonable to compare this equation with analogous equation (2.7). If inequality $|\delta\omega| \gg 2\Omega_\hbar$ is fulfilled, then from equation (3.6) follows the standard classical plasma instability increment due to the stimulated Thomson scattering of light. We are not interested here in that case. But when the opposite inequality is satisfied, we obtain from Eqn (3.6) the following expression for the instability increment [9]:

$$\delta\omega = \mathrm{i}\,\frac{1}{2}\frac{\tilde{V}_E}{c}\,\omega_L\left(\frac{mc^2}{2\hbar\omega_1}\right)^{1/2}. \qquad (3.7)$$

Increment (3.7) is obviously a pure quantum in nature. It characterizes the quantum plasma instability due to the stimulated Compton scattering of light. In the classical limit (for $\hbar \to 0$), increment (3.7) increases and converts into the classical Thomson scattering increment.

When calculating increment (3.7), we took the upper (minus) sign in equation (3.6). If we take the plus sign in the quantum limit in Eqn (3.6) [and, therefore, in formula (3.4)], we obtain $\delta\omega^2 > 0$, which means the absence of instability. This is understandable because $\omega > \omega_1$ in this case, and scattering with increasing frequency on a stationary electron is impossible. It is easy to see that the results obtained also hold for the scattering on a beam, with the only difference being that expression (3.4) becomes more complicated due to the Doppler effect (see Ref. [9] for details).

The applicability conditions for the results obtained in this section follow from inequalities $\Omega_\hbar \gg |\delta\omega| \gg |\mathbf{k}_1 - \mathbf{k}|V_0$ and reduce to the following [see Eqn (2.11a)]:

$$\left(\frac{\hbar\omega_1}{mc^2}\right)^3 \gg \frac{V_E^2}{c^2}\frac{\omega_L^2}{\omega_1^2} \gg \frac{\hbar\omega_1}{mc^2}\frac{V_0^2}{c^2}\,. \qquad (3.8)$$

These conditions are satisfied well for strong fields, when $V_E \gg V_0$, and for sufficiently high frequency ω_1 of the incident wave.

The considered quantum effect of the stimulated Compton scattering can be interpreted as a resonant interaction process involving electromagnetic waves and de Broglie waves of a free electron. For that, we substitute the beat wave frequency $\omega_1 - \omega$ and wave vector $\mathbf{k}_1 - \mathbf{k}$ instead of ω and $\mathbf{k}$ into decay conditions (2.19), since the electron moves in the field of two electromagnetic waves — incident and scattered. Then, we obtain the following resonance conditions:

$$\omega_0 + \omega_1 - \omega = \omega'\,, \qquad \mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k} = \mathbf{k}'\,. \qquad (3.9)$$

By substituting next conditions (3.9) into formula (2.18), we arrive at condition (3.3) with the plus sign, i.e., the quantum Compton scattering condition. Thus, the Compton scattering effect comprises the resonant interaction of four waves — two electromagnetic and two de Broglie.

4. Conclusion

From the above analysis, we can conclude the following:

(1) When describing Langmuir waves in a gas plasma, quantum effects always lead to small corrections, at most comparable to corrections due to the thermal motion of plasma electrons. Quantum effects are important for the resonant interactions of free plasma electrons with electromagnetic waves of various natures. Here, the quantization of electromagnetic energy transferred to electrons in the interaction process is substantial.

(2) For the propagation of a low-density beam in a dense electron plasma, development of the quantum Cherenkov beam instability becomes possible. When increasing the beam density, the quantum instability converts into the standard classical single-particle stimulated Vavilov–Cherenkov effect. The quantum Cherenkov beam instability comprises a three-wave decay process of the de Broglie beam electron wave into a Langmuir plasma wave and another de Broglie wave.

(3) For the propagation of an intense high-frequency electromagnetic wave in a plasma, the quantum effect of its Compton scattering on plasma electrons is realized. Its classical analogue is the stimulated Thomson scattering effect taking place in the lower-frequency range. The quantum Compton effect comprises the resonant four-wave interaction of two electromagnetic waves and two de Broglie waves of free plasma electrons.

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