

# Universal randomness

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## Contents

<b>1. Introduction</b>	<b>259</b>
1.1 Combinatorics; 1.2 Polynuclear crystal growth; 1.3 Directed polymers; 1.4 Replica method; 1.5 Tracy–Widom distribution function	
<b>2. Directed polymers and one-dimensional quantum bosons</b>	<b>265</b>
<b>3. Solution of the one-dimensional problem of directed polymers</b>	<b>266</b>
<b>4. Conclusions</b>	<b>268</b>
<b>Appendix A. Quantum bosons with repulsive interactions</b>	<b>269</b>
A.1 Eigenfunctions; A.2 Orthonormality of the wave functions	
<b>Appendix B. Quantum bosons with attractive interaction</b>	<b>270</b>
B.1 Ground state; B.2 Eigenfunctions; B.3 Orthonormality of the eigenfunctions; B.4 Propagator	
<b>Appendix C. Integral relations for the Airy function</b>	<b>277</b>
<b>Appendix D. Fredholm determinant with the Airy kernel and the Tracy–Widom distribution</b>	<b>278</b>
<b>References</b>	<b>279</b>

**Abstract.** In the last two decades, it has been established that a single universal probability distribution function, known as the Tracy–Widom (TW) distribution, in many cases provides a macroscopic-level description of the statistical properties of microscopically different systems, including both purely mathematical ones, such as increasing subsequences in random permutations, and quite physical ones, such as directed polymers in random media or polynuclear crystal growth. In the first part of this review, we use a number of models to examine this phenomenon at a simple qualitative level and then consider the exact solution for one-dimensional directed polymers in a random environment, showing that free energy fluctuations in such a system are described by the universal TW distribution. The second part provides detailed appendix material containing the necessary mathematical background for the first part.

## 1. Introduction

Everyone is familiar with the Gaussian distribution function. Whenever we are dealing with a system containing *independent* random parameters, its macroscopic characteristics (according to the central limit theorem) are described by the Gaussian distribution. This kind of universal behavior is trivial, and not very interesting. On the other hand, each nontrivial system usually requires individual consideration, and although there are many universal macroscopic proper-

ties among microscopically different systems (e.g., critical indices of second-order phase transitions), it has never been observed until very recently that a *universal function* (different from the Gaussian one) would describe the macroscopic statistical properties of a whole class of nontrivial random systems.

Originally, Tracy and Widom [1] considered a rather specific mathematical problem, the distribution function of the largest eigenvalue of  $N \times N$  Hermitian matrices (the Gaussian unitary ensemble, GUE) in the limit  $N \rightarrow \infty$ . Presently, many years after that paper, we have a rather comprehensive list of various systems (both purely mathematical and physical) whose macroscopic statistical properties are described by the same universal Tracy–Widom (TW) distribution function. These systems are: the model of the longest increasing subsequences (LIS) [2], zero-temperature lattice directed polymers with geometric disorder [3], the polynuclear growth (PNG) system [4], the oriented digital boiling model [5], the ballistic decomposition model [6], the longest common subsequences (LCS) [7], the one-point distribution of solutions of the KPZ equation [8] (which describes the motion of an interface separating two homogeneous bulk phases) in the long-time limit [9, 10], and finite-temperature directed polymers in random potentials with short-range correlations [11–14]. We note that directed polymers in a quenched random potential have been the subject of intense investigations during the past three decades (see, e.g., [15]). Diverse physical systems such as domain walls in magnetic films [16], vortices in superconductors [17], wetting fronts on planar systems [18], or Burgers turbulence [19] can be mapped to this model, which exhibits numerous nontrivial features deriving from the interplay between elasticity and disorder.

In the introductory part of this review, we discuss three random statistical systems at a simple qualitative level: the combinatorial model of the longest increasing subsequences

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(Section 1.1), the polynuclear crystal growth model (Section 1.2), and one-dimensional directed polymers in a random potential (Section 1.3). In addition, in Section 1.4, the main ideas of the replica method (to be used in solving the problem of directed polymers) are described, and in Section 1.5, we finally define the TW distribution function and study its main properties.

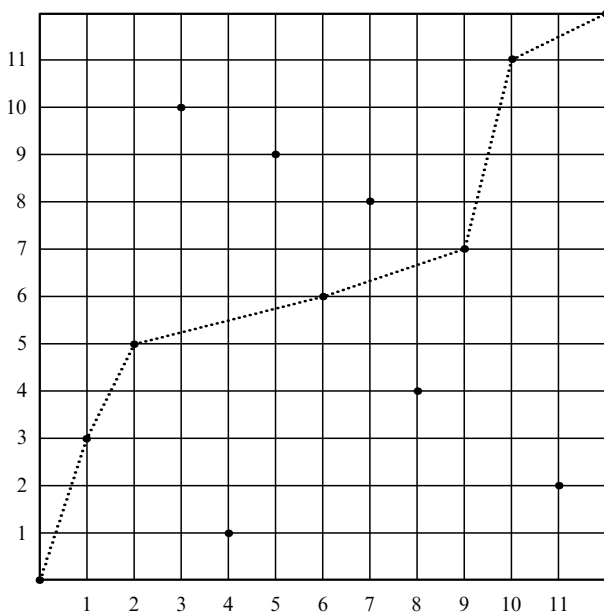
Sections 2 and 3 are devoted to the exact solution of the one-dimensional directed polymer problem. In particular, in Section 2, the main ideas of this solution and its methodological tools are described. Section 3 contains a formal derivation of the TW distribution function for free energy fluctuations in one-dimensional directed polymers with the  $\delta$ -correlated random potential. The second part of this review contains several technical appendices where all the necessary mathematical machinery used in the previous sections is collected.

**1.1 Combinatorics**

We start with a purely mathematical ‘toy’ model, which, as we see in what follows, is closely related to the physical problems of polynuclear crystal growth and one-dimensional directed polymers. This combinatorial problem deals with statistical properties of the longest increasing subsequences under random permutations of  $N$  positive integers  $\{1, 2, \dots, N\}$  [20]. For an arbitrary permutation of these integers, we find all possible *increasing* subsequences, and let  $l_N$  denote the length of the longest one. As an example, we consider the case  $N = 11$  and take an arbitrary permutation,

$$\{3, 5, 10, 1, 9, 6, 8, 4, 7, 11, 2\}. \tag{1.1}$$

This permutation contains several different increasing subsequences  $\{3, 5, 10, 11\}$ ,  $\{1, 9, 11\}$ , and so on); the longest among them are  $\{3, 5, 6, 7, 11\}$  and  $\{3, 5, 6, 8, 11\}$ . In other words,  $l_N = 5$  for this particular permutation. A simple graphical representation of this permutation problem is shown in Fig. 1. There, the set of 11 dots inside the  $12 \times 12$



**Figure 1.** Geometrical representation of permutation (1) for  $N = 11$ . The dotted line corresponds to the longest increasing subsequence  $\{3, 5, 6, 7, 11\}$ .

square represents permutation (1.1): to each integer on the  $x$  axis we associate one and only one integer on the  $y$  axis (for  $x = 1$ , this is  $y = 3$ , for  $x = 2$ , this is  $y = 5$ , and so on). Various increasing subsequences are represented by all possible *directed paths* connecting the origin  $(0, 0)$  with the upper-right corner  $(12, 12)$  of this square and passing over the dots inside the square. A directed path means that only ‘right-and-up’ moves are allowed in going from one dot to another. For example, from the point  $(6, 6)$ , we can jump only to  $(7, 8)$ ,  $(9, 7)$ , and  $(10, 11)$ . In other words, when going from the origin to the upper-right corner  $(12, 12)$ , both  $x$  and  $y$  coordinates can only increase at each step. In terms of these rules, the longest increasing subsequence is represented by a directed path that goes over the maximum possible number of dots. We note that the longest increasing subsequence for a given permutation is not necessarily unique. In the considered example, there are two longest subsequences:  $\{3, 5, 6, 7, 11\}$  and  $\{3, 5, 6, 8, 11\}$ .

Considering all the  $N!$  possible permutations of  $N$  numbers, we find that the length  $l_N$  is a random variable. Assuming that all permutations are equiprobable, the problem is to find the distribution function of the random quantity  $l_N$ .

It turns out that for large  $N$ , the *average size* of the longest increasing subsequence,  $\bar{l}_N$ , is proportional to  $\sqrt{N}$ , namely,  $\bar{l}_N \sim 2\sqrt{N}$  [21]. Moreover, in the limit  $N \rightarrow \infty$ , the quantity  $l_N$  is *self-averaging*:  $\lim_{N \rightarrow \infty} l_N/\sqrt{N} = 2$  (in other words, the distribution function of the random variable  $l_N/\sqrt{N}$  shrinks to the  $\delta$ -function in the limit  $N \rightarrow \infty$ ).

At the qualitative level, it is easy to understand why the typical value of  $l_N$  must be proportional to  $\sqrt{N}$ . Indeed, for large  $N$ , a generic permutation of the numbers  $\{1, 2, \dots, N\}$  in terms of the matrix shown in Fig. 1 is represented by a *uniform* distribution of  $N$  dots inside the  $N \times N$  square. Therefore, the density of the dots is proportional to  $1/N$  and, accordingly, the typical distance between neighboring dots must be proportional to  $\sqrt{N}$ . This means that the typical number of dots on a path running from the  $(0, 0)$  corner to  $(N + 1, N + 1)$  and having the length of the order of  $N$  must be of the order of  $N/\sqrt{N} = \sqrt{N}$ .

However, the main interest in this system is not the average value  $\bar{l}_N$  itself but the statistics of the *fluctuations* of  $l_N$  around its mean  $2\sqrt{N}$ . This problem turned out to be quite nontrivial, and it was shown only recently that in the limit of large  $N$ , the fluctuations of  $l_N$  scale as  $N^{1/6}$  (which is much less than the mean  $2\sqrt{N}$ ). In other words, the random variable  $l_N$  can be represented for large  $N$  as [2, 22]

$$l_N \simeq 2\sqrt{N} + N^{1/6} s, \tag{1.2}$$

where the random quantity  $s \sim 1$  is introduced that is described as  $N \rightarrow \infty$  by a universal  $N$ -independent distribution function  $P_{TW}(s)$ , called the Tracy–Widom distribution:

$$\lim_{N \rightarrow \infty} \text{Prob} \left( \frac{l_N - 2\sqrt{N}}{N^{1/6}} = s \right) = P_{TW}(s). \tag{1.3}$$

**1.2 Polynuclear crystal growth**

The above mathematical toy model is actually equivalent to the physical model of crystal growth occurring due to randomly located nucleation centers. This is the so-called model of *polynuclear crystal growth*, which describes the growth of two-dimensional crystal monolayers in  $2 + 1$  dimensions.

We return to Fig. 1, where the randomly distributed  $N$  dots are now assumed to represent the nucleation centers. In this model, the crystal monolayers grow in the vertical direction (toward the reader) in accordance with the following rules. From each nucleation center, we draw a monolayer level step as a straight line in the horizontal direction to the right and in the vertical direction up, until this line meets with another line starting from another center. In this way, we obtain monolayer ‘terraces,’ which mount from the lower-left to the upper-right corner of the square (see Fig. 2).

It is easy to understand that for a given (random) configuration of the nucleation centers, the number of terraces  $h_N$  is equal to the size of the longest increasing permutation  $l_N$  in the combinatorial problem considered in Section 1.1. Depending on the random configurations of the nucleation centers inside the  $N \times N$  square, the number  $h_N$  of monolayer terraces is a random quantity, and in the limit  $N \rightarrow \infty$ , its statistics is described by the TW distribution, Eqn (1.3) [5]. Presently, the study of various modifications of the above PNG-type model of crystal growth is a vast field of research (see, e.g., [23]). It is also interesting to note that the existence of the TW distribution in PNG-like systems was quite recently confirmed experimentally [24].

**1.3 Directed polymers**

It is clear that when the system size in the above examples is large, then the presence of the underlying ‘microscopic’ lattice is not essential. Therefore, instead of considering the problem in terms of permutations, as in Section 1.1, we can introduce a homogeneous distribution of dots inside a continuous square (Fig. 3). We introduce the ‘time’ axis that goes from the origin (the lower-left corner) and runs along the diagonal to the upper-right corner of the square. A directed polymer is then a line that starts at the origin and arrives at the upper-right corner by hopping over the dots randomly scattered over the square so as to follow the ‘time’ direction. Only those hops are allowed under which the time coordinate of the path increases. For a given random configuration of the dots, we

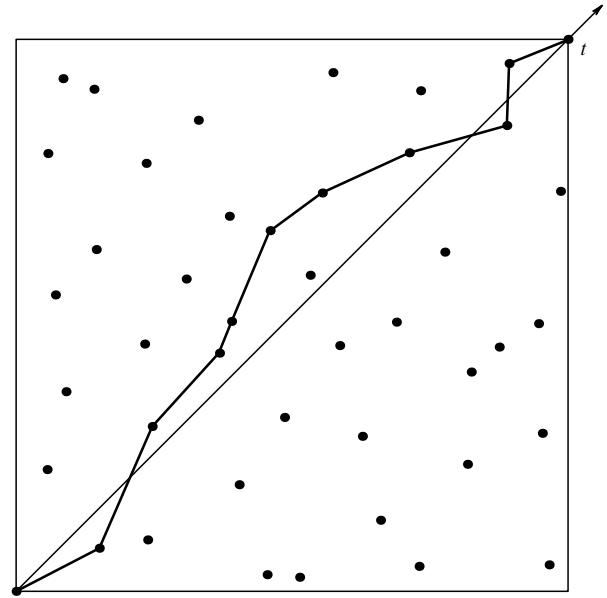


Figure 3. Directed path running over randomly distributed dots.

have to choose the path that contains the maximum number of dots, which we call the polymer ‘length.’ In this formulation, the problem looks somewhat different from the problem in Section 1.1. It can be shown, however, that in the limit of large time  $t$  and large  $N$ , these two problems become equivalent [22].

Let the length of the diagonal of the square be equal to  $t$ . Instead of the total number  $N$  of dots, we can take the independent parameter of the problem to be equal to the density of dots  $\rho$ , assuming that it remains constant as  $t \rightarrow \infty$ . We then express the length  $l(t)$  of an ordered polymer (i.e., the number of dots through which it passes) as a function of time (the square size)  $t$ . By definition,  $\rho = 2N/t^2$ . Because we consider the problem with a constant density  $\rho \sim 1$ , it follows that  $t \propto \sqrt{N}$ . In this setting, the mean of  $l(t)$  is proportional to  $t$  (instead of  $\sqrt{N}$  in the combinatorial problem), and the typical fluctuations around this mean scales as  $t^{1/3}$  instead of  $N^{1/6}$ , and instead of Eqn (1.3), the statistics of fluctuations of the directed polymer length is given by

$$\lim_{t \rightarrow \infty} \text{Prob} \left( \frac{l(t) - \sqrt{2\rho} t}{t^{1/3}} = s \right) = P_{\text{TW}}(s). \tag{1.4}$$

In other words, in the *thermodynamic limit*  $t \rightarrow \infty$ , all the three systems considered above are equivalent, which does not look very surprising if we compare Figs 1–3.

In statistical physics, the problem of directed polymers is set up somewhat differently. We introduce a square lattice in which the discrete ‘time’ axis,  $t = 1, 2, \dots, L$ , is now horizontal (Fig. 4). The vertical direction is described by a discrete parameter  $\phi = 0, \pm 1, \pm 2, \dots, \pm M$ . At each lattice site  $(\phi, t)$ , we place *independent* random quantities  $V(\phi, t)$  described by the Gaussian distribution

$$\mathcal{P}[V] = \prod_{\phi, t} \sqrt{\frac{1}{2\pi u}} \exp \left( -\frac{1}{2u} V^2(\phi, t) \right). \tag{1.5}$$

The parameter  $u$  defines the typical strength of the random potentials  $V(\phi, t)$ . According to Eqn (1.5),

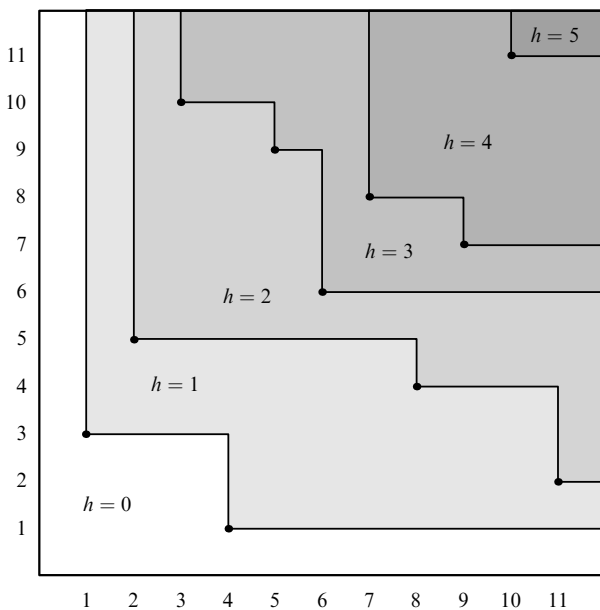


Figure 2. (2+1)-dimensional crystal terraces with ‘nucleation centers’ corresponding to the permutation model in Fig. 1.

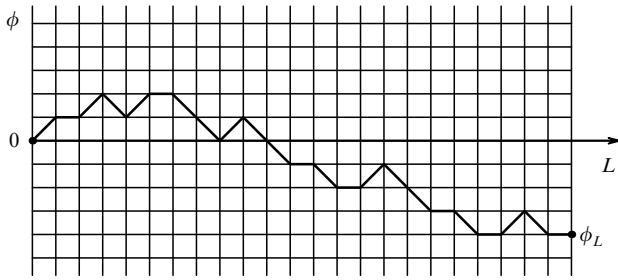


Figure 4. Directed polymer on a square lattice.

$$\overline{V(\phi, t)} = 0 \text{ and}$$

$$\overline{V(\phi, t) V(\phi', t')} = u \delta_{\phi, \phi'} \delta_{t, t'}. \quad (1.6)$$

A directed polymer is here a path that starts at the origin ( $t = 0, \phi = 0$ ) and goes over the lattice sites to the right end of the system,  $t = L$ . At each time step  $t \rightarrow t + 1$ , the polymer trajectory  $\phi(t)$  can deviate up or down by one step or may not deviate at all:  $\phi(t + 1) = \phi(t) + \sigma(t + 1)$ , where  $\sigma = \pm 1, 0$ . Assuming that the polymer is a kind of elastic string, we can introduce the ‘elastic’ (positive) energy  $\propto [\phi(t + 1) - \phi(t)]^2$  for every polymer deviation. In this way, for a given trajectory  $\phi(t)$ , we can introduce the notion of its energy

$$H[\phi(t)] = \sum_{t=1}^{L-1} \left[ \frac{1}{2} [\phi(t+1) - \phi(t)]^2 + V(\phi(t), t) \right]. \quad (1.7)$$

For given random values of the potential  $V(\phi, t)$ , this expression contains two competing contributions: the elastic energy terms try to make the trajectory as horizontal as possible, while the second (random) term forces the trajectory to deviate in the search for the most negative values of the random potential  $V(\phi, t)$ . The optimal trajectory  $\phi_*(t)$  is determined by a minimum of the energy, Eqn (1.7):

$$E(L) = \min_{\phi(t)} \left\{ \sum_{t=1}^{L-1} \left[ \frac{1}{2} [\phi(t+1) - \phi(t)]^2 + V(\phi(t), t) \right] \right\}. \quad (1.8)$$

Being a function of the random potential  $V(\phi, t)$ , this quantity is also random, and can be considered an analog of the random length of the directed polymers in the previous example shown in Fig. 3. An essential difference between the two quantities is that in this last case, the elastic terms are absent altogether, while the (negative) contribution of the potential energy is associated with a fixed energy  $V_0$  carried by the dots, which [unlike the Gaussian random potentials  $V(\phi, t)$ ] are *geometrically* random. Another important difference is that, unlike the directed polymer in Fig. 4, which is defined by *local* wandering over the lattice (with one-step deviations only), the trajectory in Fig. 3 can jump *any distance* from dot to dot, not necessarily between the nearest neighbors. Therefore, *a priori*, there is no reason to expect that these two quantities,  $E(L)$  in Eqn (1.8) and  $l(t)$  in the example in Fig. 3, would have the same statistical properties. But if we *suppose* that in the thermodynamic limit  $t \sim L \rightarrow \infty$ , these two types of models become equivalent due to some profound reasons that are not yet fully understood, then it follows that the energy in (1.8) must depend on the system size as  $E(L) \simeq f_0 L + L^{1/3} s$ , where  $f_0$  is the usual *bulk* (self-averaging) energy density and the random quantity

$s \sim 1$  must be described by the  $L$ -independent universal TW distribution function  $P_{TW}(s)$ .

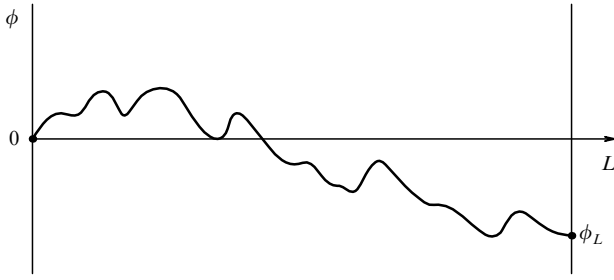
The system described by Hamiltonian (1.7) is the usual one-dimensional statistical model with quenched disorder. That its ground state energy  $E(L)$  is proportional to the system size  $L$  can be explained in a very simple way. Indeed, to find the global minimum of energy (1.7), we can in the first approximation use an algorithm that is local in time. At each time step, among the three possible options (‘up,’ ‘horizontal,’ and ‘down’), we always choose the one yielding the smallest value of the random potential  $V(\phi, t)$  (in this approximation, we neglect the presence of the trajectories that choose a locally unfavorable option to gain a more favorable energy globally). In this way, the second term in Eqn (1.7) provides the contribution that is proportional to  $-\sqrt{u} L$  [and not  $\sqrt{uL}$ , as it would be for an *arbitrary* trajectory  $\phi(t)$ ]. Because the contribution of the first (elastic) term in Eqn (1.7) is also proportional to  $L$ , we find that the leading contribution to the energy of the optimal trajectory is  $E(L) \simeq -(\text{const}) L$ , and, moreover, we can be sure that this contribution is negative, because the energy of the optimal trajectory in the absence of a random potential is zero (it is just a straight horizontal line), while the presence of a random potential can only decrease the energy. On the other hand, the fact that *finite-size corrections* in this system are of the order  $L^{1/3}$  (and *not*  $L^{1/2}$ , as is usually the case) is a highly nontrivial phenomenon, which is very difficult to explain in simple terms.

In terms of the standard statistical mechanics, the lattice model in Eqns (1.7) and (1.8) is essentially a zero-temperature system, because for a given realization of the random potential, we only account for the trajectory that is the global minimum. It is clear that the search for the global minimum is a highly nontrivial task because the algorithms that are local in time are insufficient. On the other hand, as it often happens, life can be made easier by considering a more general (i.e., more complicated) problem. We introduce a *temperature*  $T$  into the system, and hence, in addition to the fluctuations due to the random potential, we also consider thermal fluctuations produced by the trajectories away from the global energy minimum. In other words, instead of the global minimum energy  $E(L)$  in Eqn (1.8), we now consider the *free energy*:

$$F(L, T) = -T \ln \left[ \sum_{\phi(t)} \exp \left( -\frac{1}{T} H[\phi(t)] \right) \right], \quad (1.9)$$

where the expression under the logarithm is the partition function in which the summation ranges over *all* trajectories starting at the origin. In the case where the global minimum trajectory is unique (which is usually the case in a large system), the energy defined in Eqn (1.8) is obtained by taking the zero temperature limit of the free energy:  $E(L) = \lim_{T \rightarrow 0} F(L, T)$ .

The lattice model defined by Eqns (1.7) and (1.9) is sufficiently simple for numerical investigations (see, e.g., [25]) which, in particular, provide conclusive evidence that the free energy fluctuations do indeed scale as  $L^{1/3}$ . On the other hand, a problem defined on a lattice is quite difficult for analytic studies. Because we are mainly interested in the thermodynamic limit  $L \rightarrow \infty$ , it is natural to expect that the lattice becomes irrelevant at large scales (in the continuum limit). Passing to the continuum limit in Hamiltonian (1.7) is straightforward. Assuming that the lattice spacing tends to zero and replacing the finite differences in the elastic term



**Figure 5.** Continuous elastic string in a random potential, Eqn (1.10).

with the gradient, we obtain

$$H[\phi(t)] = \int_0^L d\tau \left\{ \frac{1}{2} \left[ \frac{d\phi(\tau)}{d\tau} \right]^2 + V(\phi(\tau), \tau) \right\}, \quad (1.10)$$

where, as before, the random potential  $V(\phi, \tau)$  is described by the Gaussian distribution. In the continuous limit, instead of discrete relation (1.6), we obtain

$$\overline{V(\phi, \tau) V(\phi', \tau')} = u \delta(\phi - \phi') \delta(\tau - \tau'). \quad (1.11)$$

The partition function of this system is now expressed as a path integral:

$$Z = \int_{-\infty}^{+\infty} d\phi_L \int_{\phi(0)=0}^{\phi(L)=\phi_L} \mathcal{D}[\phi(\tau)] \exp(-\beta H[\phi]), \quad (1.12)$$

where  $\beta = 1/T$  is the inverse temperature and the integration is over all trajectories starting at the origin  $\phi(\tau = 0) = 0$  and having free boundary conditions at  $\tau = L$ . In this way, instead of the lattice trajectories in Fig. 4, we have continuous trajectories, as in Fig. 5. The continuous model thus defined eventually yields to theoretical analysis.

First of all, we can note that in the absence of the random potential in Hamiltonian (1.10), the system describes simple thermal diffusion. Indeed, the probability that the trajectory arrives at the point  $\phi(L) = \phi_L$  at the time  $\tau = L$  is given by the partition function

$$Z_0(\phi_L) = \int_{\phi(0)=0}^{\phi(L)=\phi_L} \mathcal{D}[\phi(\tau)] \exp \left\{ -\frac{1}{2T} \int_0^L d\tau \left[ \frac{d\phi(\tau)}{d\tau} \right]^2 \right\}. \quad (1.13)$$

Simple Gaussian integration [with the proper choice of the *integration measure*, which is an arbitrary normalization factor for the partition function in (1.13) or (1.12)] leads to

$$Z_0(\phi_L) = \frac{1}{\sqrt{2\pi TL}} \exp \left( -\frac{\phi_L^2}{2TL} \right). \quad (1.14)$$

It follows that the typical deviation  $\langle \phi_L^2 \rangle^{1/2}$  of the trajectory due to thermal fluctuations increases as  $\sqrt{T} L^{1/2}$  (and vanishes in the zero-temperature limit).

The picture becomes much more complicated in the presence of a random potential. It is natural to assume that the elastic energy and the energy due to the random potential must be of the same order. For a given value of the typical deviation  $\phi_L$ , the contribution of the elastic term, and hence the total energy of the system, must be of the order of  $\phi_L^2/L$ . Thus, if we assume that the free energy fluctuations of this

system scale as  $L^{1/3}$  in the presence of disorder (which is now established without a doubt; see, e.g., [26–29]), then we conclude that the typical value of trajectory deviations due to the action of the random potential must increase as  $\phi_L \sim L^{2/3}$ , which is much faster than the pure thermal diffusion scaling  $L^{1/2}$ .

Our central problem in what follows is to study statistical properties of the model defined by Eqns (1.10) and (1.11). It turns out that despite essential differences in the definition of this system and the models discussed in Sections 1.1–1.3, all these models not only exhibit the same scaling with the parameter  $L$  but are actually *equivalent* in the thermodynamic limit  $L \rightarrow \infty$ . The central result proved in the next sections is that in the limit  $L \rightarrow \infty$ , the free energy of the system can be represented as

$$F = f_0 L + c L^{1/3} f, \quad (1.15)$$

where  $f_0$  is the linear free energy density and  $c = (1/2)(\beta^2 u^2)^{1/3}$ . The random quantity  $f \sim 1$  [just like the quantity  $s$  in Eqns (1.2)–(1.4)] is described by the universal TW distribution function (see Section 1.5).

### 1.4 Replica method

The replica method is widely used in studies of systems with quenched disorder (see [30, 31]). For simplicity, we consider the model of directed polymers described by Hamiltonian (1.10) and restrict ourselves to zero boundary conditions:  $\phi(0) = \phi(L) = 0$ . The partition function of a given sample described by Hamiltonian (1.10) is

$$Z[V] = \int_{\phi(0)=0}^{\phi(L)=0} \mathcal{D}[\phi(\tau)] \exp(-\beta H[\phi, V]). \quad (1.16)$$

On the other hand, the partition function is related to the total free energy  $F[V]$  as

$$Z[V] = \exp(-\beta F[V]). \quad (1.17)$$

The free energy  $F[V]$  depends on a specific realization of the random potential and is therefore also a random variable. Taking the  $N$ th power of both sides of (1.17) and averaging over realizations of the random potential, we obtain

$$\overline{Z^N[V]} \equiv Z[N, L] = \overline{\exp(-\beta N F[V])}, \quad (1.18)$$

where the quantity in the left-hand side is called the *replica partition function*. Substituting the free energy in the form  $F = f_0 L + c L^{1/3} f$  from Eqn (1.15) and redefining the replica partition function,

$$Z[N, L] = \tilde{Z}[N, L] \exp(-\beta N f_0 L), \quad (1.19)$$

we obtain

$$\tilde{Z}[N, L] = \overline{\exp(-\lambda N f)}, \quad (1.20)$$

where we introduce the new parameter  $\lambda = \beta c L^{1/3}$ . The averaging in the right-hand side of (1.20), which involves the random quantity  $f$ , can be represented in terms of its distribution function  $P_L(f)$  (which depends on the system size  $L$  in general). In this way, we arrive at the following general relation between the replica partition function  $\tilde{Z}[N, L]$  and the distribution function of the free energy

fluctuations  $P_L(f)$ :

$$\tilde{Z}[N, L] = \int_{-\infty}^{+\infty} df P_L(f) \exp(-\lambda N f). \tag{1.21}$$

Of course, the most interesting object is the thermodynamic limit distribution function  $P_*(f) = \lim_{L \rightarrow \infty} P_L(f)$ , which is expected to be a universal quantity free of any parameters. The above equation is readily seen to be a bilateral Laplace transform of  $P_L(f)$ , and, formally at least, it allows restoring this function via the inverse Laplace transform of the replica partition function  $\tilde{Z}[N, L]$ . For this, we have to first compute  $\tilde{Z}[N, L]$  for an arbitrary integer  $N$  and then, if possible, perform an analytic continuation of this function from integer to arbitrary complex values of  $N$ . In this approach, unfortunately, the analytic continuation is often a rather dubious operation [32, 33]. The classic example of this situation is provided by the Derrida random energy model [34], in which  $Z(N)$  increases as  $\exp(N^2)$  at large  $N$ , and it turns out that there are many different distributions yielding the same values of  $Z(N)$  but providing different values for the average free energy of the system. In our present system of directed polymers with Hamiltonian (1.10), the situation is even worse because, as we see in what follows, the replica partition function increases here as  $\exp(N^3)$  at large  $N$ , and in this situation its analytic continuation from integer to noninteger  $N$  is ambiguous. (In order that the analytic continuation of a function  $Z(N)$  from integer to arbitrary values of  $N$  be unambiguous,  $Z(N)$  must increase not faster than exponentially as  $N \rightarrow \infty$ .) It turns out, however, that in our present case, it is possible to bypass the problem of the analytic continuation if, instead of the distribution function  $P_*(f)$ , we study its integral representation

$$W(x) = \int_x^\infty df P_*(f), \tag{1.22}$$

which expresses the probability of finding a value of a random variable  $f$  greater than a given number  $x$ . Formally, the function  $W(x)$  can be defined as

$$\begin{aligned} W(x) &= \lim_{\lambda \rightarrow \infty} \sum_{N=0}^\infty \frac{(-1)^N}{N!} \exp(\lambda N x) \overline{\tilde{Z}^N} \\ &= \lim_{\lambda \rightarrow \infty} \sum_{N=0}^\infty \frac{(-1)^N}{N!} \overline{\exp(\lambda N x - \lambda N f)} \\ &= \lim_{\lambda \rightarrow \infty} \overline{\exp[-\exp(\lambda(x-f))]} = \overline{\theta(f-x)}. \end{aligned} \tag{1.23}$$

Thus, the probability function  $W(x)$  can be computed in terms of the replica partition function  $\tilde{Z}[N, L]$  by summing over all positive integer values of the replica parameter  $N$ :

$$W(x) = \lim_{\lambda \rightarrow \infty} \sum_{N=0}^\infty \frac{(-1)^N}{N!} \exp(\lambda N x) \tilde{Z}[N, L]. \tag{1.24}$$

Of course, keeping in mind that  $\tilde{Z}[N, L]$  increases as  $\exp(N^3)$  for large  $N$ , we see that the above series is not entirely innocuous. Here, in accordance with the *troubles conservation law*, instead of an analytic continuation problem, we are facing a formally divergent series. Nevertheless, it is shown in Section 3 that this *sign alternating* divergent series can be regularized in the standard way (similarly to the formally divergent sign alternating series  $\sum_{k=0}^\infty (-1)^k a^k = (1+a)^{(-1)}$  which at  $|a| > 1$  is defined as the analytic continuation from

the region  $|a| < 1$ ). This eventually allows us to prove that the thermodynamic limit function  $W(x)$  in Eqn (1.24) can be expressed in terms of the universal Tracy–Widom distribution function, which is considered in the next subsection.

### 1.5 Tracy–Widom distribution function

Originally, the Tracy–Widom distribution function was derived in studying statistical properties of the Gaussian unitary ensemble (GUE) of random Hermitian matrices [1]. Let  $G_{ij}$  be an  $N \times N$  complex Hermitian matrix whose elements are independent random quantities with the Gaussian distribution

$$\mathcal{P}[\mathbf{G}] = B_N \exp\left[-\frac{1}{2} \text{Tr}(\mathbf{G}^2)\right], \tag{1.25}$$

where  $B_N$  is a normalization constant. The joint probability density of  $N$  eigenvalues  $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$  of such matrices is well known to be given by [35]

$$\mathcal{P}[\lambda_1, \lambda_2, \dots, \lambda_N] = C_N \prod_{i \neq j}^N |\lambda_i - \lambda_j|^2 \exp\left(-\sum_{i=1}^N \lambda_i^2\right), \tag{1.26}$$

where  $C_N$  is a normalization constant. Using this result, we can calculate various averaged characteristics of the eigenvalue statistics. The simplest is the average density of the eigenvalues  $\rho(\lambda, N) = (1/N) \sum_{i=1}^N \langle \delta(\lambda - \lambda_i) \rangle$ , where the averaging  $\langle \dots \rangle$  is performed with probability distribution (1.26). Using the symmetry of this distribution under permutations of the eigenvalues, we represent it as

$$\rho(\lambda, N) = \left[ \prod_{i=2}^N \int_{-\infty}^{+\infty} d\lambda_i \right] \mathcal{P}[\lambda, \lambda_2, \dots, \lambda_N]. \tag{1.27}$$

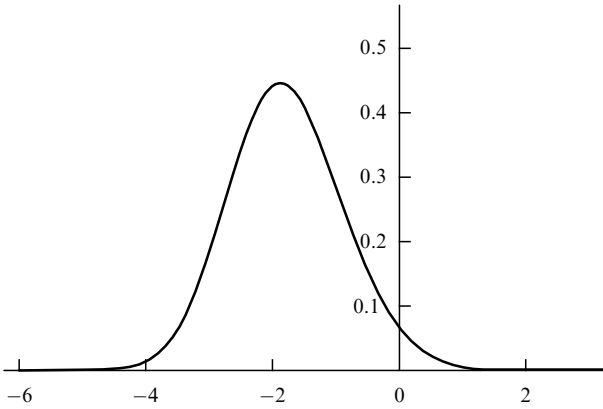
It can be shown [35] that in the limit  $N \rightarrow \infty$ ,  $\rho(\lambda, N)$  is the so-called Wigner semicircle:

$$\rho(\lambda, N) = \sqrt{\frac{2}{N\pi^2} \left(1 - \frac{\lambda^2}{2N}\right)}. \tag{1.28}$$

In other words, in the limit of large  $N$ , the eigenvalues lie within the interval  $[-\sqrt{2N} < \lambda < \sqrt{2N}]$ . This is one of the central results of the random matrix theory. In particular, this result reveals that on average, the maximum eigenvalue  $\lambda_{\max}$  is equal to  $\sqrt{2N}$ . But this eigenvalue is a random quantity in and of itself, and we can address the question of the full probability distribution of the largest eigenvalue  $\lambda_{\max}$ . This distribution can be computed in terms of the joint probability density, Eqn (1.26). Introducing the standard notation of the random matrix theory, we define the function  $F_2(s) \equiv \text{Prob}[\lambda_{\max} < s]$  that gives the probability that  $\lambda_{\max}$  is less than a given value  $s$  (in this notation, the functions  $F_1(s)$ ,  $F_2(s)$ , and  $F_4(s)$  respectively denote the probability distributions of the largest eigenvalues in the Gaussian orthogonal ensemble (GOE), Gaussian unitary ensemble (GUE), and Gaussian symplectic ensemble (GSE) [36]). By definition, therefore,

$$F_2(s) = \left[ \prod_{i=1}^N \int_{-\infty}^s d\lambda_i \right] \mathcal{P}[\lambda_1, \lambda_2, \dots, \lambda_N] \equiv \int_{-\infty}^s d\lambda P_{\text{TW}}(\lambda). \tag{1.29}$$

Just this problem was solved by Tracy and Widom in 1994 [1]. They showed that for large  $N$ , the typical fluctuations of  $\lambda_{\max}$



**Figure 6.** Tracy–Widom distribution function  $P_{TW}(x)$ , Eqn (1.32).

around its mean  $\sqrt{2N}$  scale as  $N^{-1/6}$ , namely [cf. Eqn (1.2)],

$$\lambda_{\max} = \sqrt{2N} + \frac{1}{\sqrt{2} N^{1/6}} s, \tag{1.30}$$

where the random quantity  $s \sim 1$  is described by the  $N$ -independent distribution  $P_{TW}(s) = dF_2(s)/ds$ . The function  $F_2(x)$  has the explicit form

$$F_2(s) = \exp\left(-\int_s^\infty dt (t-s) q^2(t)\right), \tag{1.31}$$

and, accordingly,

$$P_{TW}(s) = \frac{d}{ds} F_2(s) = \exp\left[-\int_s^\infty dt (t-s) q^2(t)\right] \int_s^\infty dt q^2(t), \tag{1.32}$$

where  $q(t)$  is the solution of the Painlevé II equation,<sup>1</sup>

$$q'' = tq + 2q^3, \tag{1.33}$$

with the boundary condition  $q(t \rightarrow +\infty) \sim \text{Ai}(t)$ . The function  $P_{TW}(s)$  is plotted in Fig. 6. We note that the asymptotic tails of this function are strongly asymmetric. While its right tail coincides with the Airy function asymptotic form  $P_{TW}(s \rightarrow +\infty) \sim \exp[-(4/3)s^{3/2}]$ , the left tail exhibits a much faster decay,  $P_{TW}(s \rightarrow -\infty) \sim \exp[-(1/12)|s|^3]$ .

## 2. Directed polymers and one-dimensional quantum bosons

Explicitly, the replica partition function in Eqn (1.18) of the system described by Hamiltonian (1.10) is given by

$$Z(N, L) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau) \times \exp\left[-\beta \int_0^L d\tau \sum_{a=1}^N \left\{ \frac{1}{2} [\partial_\tau \phi_a(\tau)]^2 + V[\phi_a(\tau), \tau] \right\}\right]. \tag{2.1}$$

<sup>1</sup> There exist six Painlevé differential equations, discovered about a hundred years ago [37] (see, e.g., [38]). It is proved that the general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of any of the previously known functions, including all classical special functions. At present, the Painlevé equations have many applications in various parts of modern physics, including statistical mechanics, plasma physics, nonlinear waves, quantum field theory, and general relativity.

Because the random potential  $V[\phi, \tau]$  has a Gaussian distribution, the disorder averaging (...) in the above equation is very simple:

$$\exp\left[-\beta \int_0^L d\tau \sum_{a=1}^N V[\phi_a(\tau), \tau]\right] = \exp\left[\frac{\beta^2}{2} \iint_0^L d\tau d\tau' \sum_{a,b=1}^N \overline{V[\phi_a(\tau), \tau] V[\phi_b(\tau'), \tau']}\right]. \tag{2.2}$$

Using definition (1.11), we find

$$Z(N, L) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau) \times \exp\left[-\frac{1}{2} \beta \int_0^L d\tau \left\{ \sum_{a=1}^N [\partial_\tau \phi_a(\tau)]^2 - \beta u \sum_{a,b=1}^N \delta[\phi_a(\tau) - \phi_b(\tau)] \right\}\right]. \tag{2.3}$$

We note that the second term in (2.3) contains formally divergent contributions proportional to  $\delta(0)$  (due to the terms with  $a = b$ ). In fact, this is just a manifestation of the fact that the model in Eqns (1.10) and (1.11) is ill-defined at short distances and requires a proper lattice regularization. Of course, the corresponding lattice model in (1.6) and (1.7) contains no divergences, and the above terms with  $a = b$  produce the irrelevant constant  $(1/2)L\beta^2 u N \delta(0)$  (where the lattice version of the delta-function is the Kronecker symbol, which has a finite value). Because the lattice regularization has no impact on the continuous long-distance properties of the considered system, this term is omitted in what follows.

Introducing the *replica Hamiltonian*

$$H_N[\Phi] = \frac{1}{2} \int_0^L d\tau \left( \sum_{a=1}^N [\partial_\tau \phi_a(\tau)]^2 - \beta u \sum_{a \neq b}^N \delta[\phi_a(\tau) - \phi_b(\tau)] \right), \tag{2.4}$$

which describes  $N$  scalar fields  $\Phi(\tau) \equiv \{\phi_1(\tau), \dots, \phi_N(\tau)\}$ , we obtain the standard expression for replica partition function (2.3):

$$Z(N, L) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(L)=0} \mathcal{D}\phi_a(\tau) \exp(-\beta H_N[\Phi]), \tag{2.5}$$

According to the above definition, this partition function describes the statistics of  $N$  elastic strings  $\phi_a(\tau)$  with a two-body attracting potential and with zero boundary conditions. To reformulate the above problem in terms of one-dimensional quantum bosons, instead of the above replica partition function (2.5), we introduce a more general object,

$$\Psi(\mathbf{x}; t) = \prod_{a=1}^N \int_{\phi_a(0)=0}^{\phi_a(t)=x_a} \mathcal{D}\phi_a(\tau) \exp(-\beta H_N[\Phi]), \tag{2.6}$$

which describes  $N$  trajectories  $\phi_a(\tau)$  all of which start at zero ( $\phi_a(0) = 0$ ) but end at  $\tau = t$  at arbitrary points  $\{x_1, \dots, x_N\}$ . It is easy to show that instead of calculating  $\Psi(\mathbf{x}; t)$  as a path integral, as in (2.6), we can obtain it as a solution of the linear differential equation

$$\partial_t \Psi(\mathbf{x}; t) = \frac{1}{2\beta} \sum_{a=1}^N \partial_{x_a}^2 \Psi(\mathbf{x}; t) + \frac{1}{2} \beta^2 u \sum_{a \neq b}^N \delta(x_a - x_b) \Psi(\mathbf{x}; t) \tag{2.7}$$

with the initial condition

$$\Psi(\mathbf{x}; 0) = \prod_{a=1}^N \delta(x_a). \quad (2.8)$$

It can easily be seen that Eqn (2.7) is the imaginary-time Schrödinger equation

$$-\partial_t \Psi(\mathbf{x}; t) = \hat{H} \Psi(\mathbf{x}; t) \quad (2.9)$$

with the Hamiltonian

$$\hat{H} = -\frac{1}{2\beta} \sum_{a=1}^N \partial_{x_a}^2 - \frac{1}{2} \beta^2 u \sum_{a \neq b}^N \delta(x_a - x_b). \quad (2.10)$$

We here have the system of  $N$  Bose particles of mass  $\beta$  interacting via the *attractive* two-body potential  $-\beta^2 u \delta(x)$ . The original replica partition function, Eqn (2.5), is then obtained from the time-dependent wave function  $\Psi(\mathbf{x}; t)$  of the system of bosons at the instant  $t = L$  at the point  $x_1 = x_2 = \dots = x_N = 0$ :

$$Z(N, L) = \Psi(\mathbf{0}; L). \quad (2.11)$$

Historically, the main interest in studies of such systems was devoted to quantum bosons with *repulsion* [which corresponds to  $u < 0$  in Hamiltonian (2.10)]. This was largely because in the case of repulsive interactions, the free energy of the system of  $N$  particles reveals the ‘correct’ extensive free energy  $F \propto N$ . The eigenfunctions of  $N$ -boson Hamiltonian (2.10) with repulsion were derived by Lieb and Liniger in 1963 [39] (for the details, see Appendix A and Refs [40, 41]). The system of *attractive* bosons ( $u > 0$ ) remained much less studied. This was largely because this system has a ‘wrong’ thermodynamic limit behavior,  $F \propto -N^3$ , and hence ‘collapses’ as  $N \rightarrow \infty$ . Moreover, as we see in what follows, the structure of the eigenstates of such a system with arbitrary  $N$  is much more complicated compared to the case of repulsion. The spectrum and some properties of the eigenfunctions for attractive bosons were derived by McGuire [42] and Yang [43] in the 1960s (see also Ref. [44, 45]). The detailed structure of these wave functions, the proof of their orthogonality, and the calculation of their normalization are described in Appendix B.

The generic eigenstate of the system of  $N$  attracting bosons consists of  $M$  ( $1 \leq M \leq N$ ) ‘clusters’  $\{\Omega_x\}$  ( $x = 1, \dots, M$ ) of bound particles. Each cluster is characterized by the momentum  $q_x$  of its center-of-mass motion and by the number  $n_x$  of particles contained in it (the integer parameters  $n_x$  are constrained by  $\sum_{x=1}^M n_x = N$ ). Correspondingly, the eigenfunction  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(x_1, \dots, x_N)$  of such a state is characterized by  $M$  continuous parameters  $\mathbf{q} = (q_1, \dots, q_M)$  and  $M$  integer parameters  $\mathbf{n} = (n_1, \dots, n_M)$  [see Appendix B.2, Eqn (B.27)]. The energy spectrum of such states is

$$E_M(\mathbf{q}, \mathbf{n}) = \frac{1}{2\beta} \sum_{x=1}^M n_x q_x^2 - \frac{\kappa^2}{24\beta} \sum_{x=1}^M (n_x^3 - n_x), \quad (2.12)$$

where

$$\kappa = \beta^3 u. \quad (2.13)$$

Thus, a time-dependent solution  $\Psi(\mathbf{x}, t)$  of Schrödinger equation (2.7) with initial conditions (2.8) can be represented

in the form of a linear combination of the eigenfunctions  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$  [see Appendix B.4, Eqn (B.50)]:

$$\Psi(\mathbf{x}, t) = \sum_{M=1}^N \sum_{\mathbf{n}} \int' \mathcal{D}\mathbf{q} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}) \Psi_{\mathbf{q}, \mathbf{n}}^{(M)*}(\mathbf{0}) \exp[-E_M(\mathbf{q}, \mathbf{n}) t], \quad (2.14)$$

where summations over the integer parameters  $n_x$  and integrations over the momenta  $q_x$  are performed in a restricted subspace [see Eqns (B.42)–(B.45) and (B.51)], which reflects the symmetry of the eigenfunctions  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$  under permutations of  $q_1, \dots, q_M$  and  $n_1, \dots, n_M$ . Correspondingly, according to Eqn (2.11), we have the replica partition function of the original directed polymer problem [see Appendix B.4, Eqn (B.59)]

$$Z(N, L) = \sum_{M=1}^N \frac{1}{M!} \left[ \prod_{x=1}^M \int_{-\infty}^{+\infty} \frac{dq_x}{2\pi} \sum_{n_x=1}^{\infty} \right] \delta\left(\sum_{x=1}^M n_x, N\right) \times |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 \exp[-E_M(\mathbf{q}, \mathbf{n}) L], \quad (2.15)$$

where, due to the symmetry of the function  $f(\mathbf{q}, \mathbf{n}) = |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 \exp[-E_M(\mathbf{q}, \mathbf{n}) L]$  under arbitrary permutations of all its  $M$  pairs of arguments  $(q_x, n_x)$ , integrations over the  $M$  momenta  $q_x$  can be extended to the whole space  $R_M$ , and summations over the  $n_x$  are bounded by the only constraint  $\sum_{x=1}^M n_x = N$  (for simplicity, due to the presence of the Kronecker symbol  $\delta(\sum_x n_x, N)$ , summations over the  $n_x$  are extended to infinity).

### 3. Solution of the one-dimensional problem of directed polymers

Using the explicit form of the wave functions  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$ , Eqn (B.27), the expression in Eqn (2.15) for the replica partition function can be represented as [see Appendix B.4, Eqns (B.61) and (B.62)]

$$Z(N, L) = \exp(-\beta N L f_0) \tilde{Z}(N, \lambda), \quad (3.1)$$

where  $f_0 = (1/24)\beta^4 u^2 - (1/\beta L) \ln(\beta^3 u)$  is the linear (self-averaging) free energy density [cf. Eqn (1.19)] and

$$\begin{aligned} \tilde{Z}(N, L) = & N! \int_{-\infty}^{+\infty} \frac{dq}{2\pi\kappa N} \exp\left[-\frac{L}{2\beta} N q^2 + \frac{\kappa^2 L}{24\beta} N^3\right] \\ & + N! \sum_{M=2}^N \frac{1}{M!} \left[ \prod_{x=1}^M \sum_{n_x=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_x}{2\pi\kappa n_x} \right] \\ & \times \delta\left(\sum_{x=1}^M n_x, N\right) \prod_{x < \beta} \frac{|q_x - q_\beta - (i\kappa/2)(n_x - n_\beta)|^2}{|q_x - q_\beta - (i\kappa/2)(n_x + n_\beta)|^2} \\ & \times \exp\left[-\frac{L}{2\beta} \sum_{x=1}^M n_x q_x^2 + \frac{\kappa^2 L}{24\beta} \sum_{x=1}^M n_x^3\right]. \end{aligned} \quad (3.2)$$

The first term in the above expression is the contribution of the ground state ( $M = 1$ ) and the other terms ( $M \geq 2$ ) are the contributions of excited states.

The terms cubic in  $n_x$  in the exponential of Eqn (3.2) can be linearized with the help of the Airy function, using the standard relation (see Appendix C)

$$\exp\left(\frac{1}{3} \lambda^3 n^3\right) = \int_{-\infty}^{+\infty} dy \text{Ai}(y) \exp(\lambda n). \quad (3.3)$$



Redefining the momentum as  $q_x = (\beta\kappa/L)^{1/3} p_x$  and introducing the new parameter

$$\lambda(L) = \frac{1}{2} \left( \frac{L}{\beta} \kappa^2 \right)^{1/3} = \frac{1}{2} (\beta^5 u^2 L)^{1/3}, \tag{3.4}$$

we obtain

$$\begin{aligned} \tilde{Z}(N, \lambda) &= N! \iint_{-\infty}^{+\infty} \frac{dy dp}{4\pi\lambda N} \text{Ai}(y) \exp[\lambda N(y - p^2)] \\ &+ N! \sum_{M=2}^N \frac{1}{M!} \left\{ \prod_{\alpha=1}^M \sum_{n_\alpha=1}^{\infty} \iint_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{4\pi\lambda n_\alpha} \text{Ai}(y_\alpha) \right. \\ &\times \exp[\lambda n_\alpha(y_\alpha - p_\alpha^2)] \left. \right\} \\ &\times \prod_{\alpha < \beta} \frac{|\lambda(n_\alpha - n_\beta) - i(p_\alpha - p_\beta)|^2}{|\lambda(n_\alpha + n_\beta) - i(p_\alpha - p_\beta)|^2} \delta\left(\sum_{\alpha=1}^M n_\alpha, N\right). \end{aligned} \tag{3.5}$$

After shifting the integration variables as  $y_\alpha \rightarrow y_\alpha + p_\alpha^2$ , we obtain a sufficiently compact expression,

$$\begin{aligned} \tilde{Z}(N, \lambda) &= N! \iint_{-\infty}^{+\infty} \frac{dy dp}{4\pi\lambda N} \text{Ai}(y + p^2) \exp(\lambda Ny) \\ &+ N! \sum_{M=2}^N \frac{1}{M!} \left[ \prod_{\alpha=1}^M \sum_{n_\alpha=1}^{\infty} \iint_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{4\pi\lambda n_\alpha} \right. \\ &\times \text{Ai}(y_\alpha + p_\alpha^2) \exp(\lambda n_\alpha y_\alpha) \left. \right] \\ &\times \prod_{\alpha < \beta} \frac{|\lambda(n_\alpha - n_\beta) - i(p_\alpha - p_\beta)|^2}{|\lambda(n_\alpha + n_\beta) - i(p_\alpha - p_\beta)|^2} \delta\left(\sum_{\alpha=1}^M n_\alpha, N\right). \end{aligned} \tag{3.6}$$

Now, using the Cauchy double alternant identity

$$\begin{aligned} &\frac{\prod_{\alpha < \beta} (a_\alpha - a_\beta)(b_\alpha - b_\beta)}{\prod_{\alpha, \beta=1}^M (a_\alpha - b_\beta)} \\ &= (-1)^{M(M-1)/2} \det \left[ \frac{1}{a_\alpha - b_\beta} \right]_{\alpha, \beta=1, \dots, M}, \end{aligned} \tag{3.7}$$

where  $a_\alpha = p_\alpha - i\lambda n_\alpha$  and  $b_\alpha = p_\alpha + i\lambda n_\alpha$  in our case, we express the product in Eqn (3.6) in the determinant form:

$$\begin{aligned} &\prod_{\alpha < \beta} \frac{|\lambda(n_\alpha - n_\beta) - i(p_\alpha - p_\beta)|^2}{|\lambda(n_\alpha + n_\beta) - i(p_\alpha - p_\beta)|^2} \\ &= \left[ \prod_{\alpha=1}^M (2\lambda n_\alpha) \right] \det \left[ \frac{1}{\lambda n_\alpha - ip_\alpha + \lambda n_\beta + ip_\beta} \right]_{\alpha, \beta=1, \dots, M}. \end{aligned} \tag{3.8}$$

Substituting the expression for the replica partition function  $\tilde{Z}(N, \lambda)$  in the definition of the probability function (1.24), we can sum over  $N$  using the Kronecker symbol, which ensures that only the value  $N = \sum_{\alpha=1}^M n_\alpha$  contributes. The result is

$$\begin{aligned} W(x) &= \lim_{\lambda \rightarrow \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \left[ \prod_{\alpha=1}^M \iint_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \text{Ai}(y_\alpha + p_\alpha^2) \right. \right. \\ &\times \sum_{n_\alpha=1}^{\infty} (-1)^{n_\alpha-1} \exp(\lambda n_\alpha(y_\alpha + x)) \left. \right] \\ &\times \det \left[ \frac{1}{\lambda n_\alpha - ip_\alpha + \lambda n_\beta + ip_\beta} \right] \left. \right\}. \end{aligned} \tag{3.9}$$

It is easy to see that this expression is an expansion of the Fredholm determinant  $\det(1 - \hat{K})$  (see [46] and Appendix D) with the kernel

$$\begin{aligned} \hat{K} &\equiv K[(n, p); (n', p')] \\ &= \left[ \int_{-\infty}^{+\infty} dy \text{Ai}(y + p^2) (-1)^{n-1} \exp(\lambda n(y + x)) \right] \\ &\times \frac{1}{\lambda n - ip + \lambda n' + ip'}. \end{aligned} \tag{3.10}$$

Using the exponential representation of this determinant, we obtain

$$W(x) = \lim_{\lambda \rightarrow \infty} \exp \left[ - \sum_{M=1}^{\infty} \frac{1}{M} \text{Tr} \hat{K}^M \right], \tag{3.11}$$

where

$$\begin{aligned} \text{Tr} \hat{K}^M &= \left[ \prod_{\alpha=1}^M \iint_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \text{Ai}(y_\alpha + p_\alpha^2) \right. \\ &\times \sum_{n_\alpha=1}^{\infty} (-1)^{n_\alpha-1} \exp(\lambda n_\alpha(y_\alpha + x)) \left. \right] \\ &\times [(\lambda n_1 - ip_1 + \lambda n_2 + ip_2)(\lambda n_2 - ip_2 + \lambda n_3 + ip_3) \dots \\ &\dots (\lambda n_M - ip_M + \lambda n_1 + ip_1)]^{-1}. \end{aligned} \tag{3.12}$$

Substituting

$$\begin{aligned} &\frac{1}{\lambda n_\alpha - ip_\alpha + \lambda n_{\alpha+1} + ip_{\alpha+1}} \\ &= \int_0^\infty d\omega_\alpha \exp[-(\lambda n_\alpha - ip_\alpha + \lambda n_{\alpha+1} + ip_{\alpha+1})\omega_\alpha], \end{aligned} \tag{3.13}$$

in the above expression, we can easily perform the summation over the integer  $n_\alpha$ . Taking into account that

$$\lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} (-1)^{n-1} \exp(\lambda n z) = \lim_{\lambda \rightarrow \infty} \frac{\exp(\lambda z)}{1 + \exp(\lambda z)} = \theta(z), \tag{3.14}$$

we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \text{Tr} \hat{K}^M &= \prod_{\alpha=1}^M \iint_{-\infty}^{+\infty} \frac{dy_\alpha dp_\alpha}{2\pi} \int_0^\infty d\omega_\alpha \text{Ai}(y_\alpha + p_\alpha^2) \\ &\times \theta(y_\alpha + x - \omega_\alpha - \omega_{\alpha-1}) \exp[ip_\alpha(\omega_\alpha - \omega_{\alpha-1})], \end{aligned} \tag{3.15}$$

where  $\omega_0 \equiv \omega_M$  by definition. Shifting the integration variables as  $y_\alpha \rightarrow y_\alpha - x + \omega_\alpha + \omega_{\alpha-1}$  and  $\omega_\alpha \rightarrow \omega_\alpha + x/2$ , we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \text{Tr} \hat{K}^M &= \prod_{\alpha=1}^M \int_0^\infty dy_\alpha \int_{-\infty}^{+\infty} \frac{dp_\alpha}{2\pi} \\ &\times \int_{-x/2}^\infty d\omega_\alpha \text{Ai}(y_\alpha + p_\alpha^2 + \omega_\alpha + \omega_{\alpha-1}) \exp[ip_\alpha(\omega_\alpha - \omega_{\alpha-1})]. \end{aligned} \tag{3.16}$$

Using the Airy function integral representation and taking into account that it satisfies the differential equation  $\text{Ai}''(t) = t \text{Ai}(t)$ , we can easily prove the following integral

relations (see Appendix C):

$$\begin{aligned} & \int_0^\infty dy \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \text{Ai}(y + p^2 + \omega + \omega') \exp[ip(\omega - \omega')] \\ &= 2^{-1/3} \int_0^\infty dy \text{Ai}(2^{1/3}\omega + y) \text{Ai}(2^{1/3}\omega' + y) \\ &= \frac{\text{Ai}(2^{1/3}\omega) \text{Ai}'(2^{1/3}\omega') - \text{Ai}'(2^{1/3}\omega) \text{Ai}(2^{1/3}\omega')}{\omega - \omega'}. \end{aligned} \quad (3.17)$$

Redefining  $\omega_\alpha \rightarrow \omega_\alpha 2^{-1/3}$ , we find

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \text{Tr} \hat{K}^M &= \iint \dots \int_{-x/2^{2/3}}^\infty d\omega_1 d\omega_2 \dots d\omega_M \\ &\times K_A(\omega_1, \omega_2) K_A(\omega_2, \omega_3) \dots K_A(\omega_M, \omega_1), \end{aligned} \quad (3.18)$$

where

$$K_A(\omega, \omega') = \frac{\text{Ai}(\omega) \text{Ai}'(\omega') - \text{Ai}'(\omega) \text{Ai}(\omega')}{\omega - \omega'} \quad (3.19)$$

is the so-called Airy kernel. This proves that in the thermodynamic limit, the probability function  $W(x)$  in (1.22) is given by the Fredholm determinant

$$W(x) = \det[1 - \hat{K}_A] \equiv F_2\left(-\frac{x}{2^{2/3}}\right), \quad (3.20)$$

where  $\hat{K}_A$  is the integral operator defined on the semi-infinite line  $[-x/2^{2/3}, \infty)$  with Airy kernel (3.19). The function commonly denoted as  $F_2(s)$  is just the Tracy–Widom distribution function (see Appendix D). It can be shown to admit the explicit representation

$$F_2(s) = \exp\left(-\int_s^\infty dt (t-s) q^2(t)\right), \quad (3.21)$$

where the function  $q(t)$  is the solution of the Painlevé II equation  $q'' = tq + 2q^3$  with the boundary condition  $q(t \rightarrow +\infty) \sim \text{Ai}(t)$ . We note that in accordance with Eqns (1.22), (1.32), and (3.20),  $P_*(x) = 2^{-2/3} P_{\text{TW}}(-2^{-2/3}x)$ .

## 4. Conclusions

The first breakthrough in the studies of one-dimensional directed polymers in a random potential was due to the work of Kardar [29], in which the problem was reduced to an  $N$ -particle system of quantum bosons with an attractive two-body interaction. Because of the focus on the properties of the system in the thermodynamic limit, it seemed initially that considering only the contribution of the ground state, whose energy was well known, would suffice.

Indeed, for any number of particles  $N > 1$ , the contribution of the excited states in the limit  $L \rightarrow \infty$  is exponentially small compared to that of the ground state. In the framework of this approach, it has been demonstrated that the free energy fluctuations increase as  $L^{1/3}$ , while the typical value of the polymer deviation scale increases as  $L^{2/3}$  which, notably, was in perfect agreement with numerical studies.

But more detailed investigations demonstrated that the above approach contains serious pathologies. In particular, it turned out that the *second cumulant* of free energy ( $\overline{F^2} - \overline{F}^2$ ) is identically zero! This is possible only in two cases: either the quantity  $F$  is not random (which contradicts the fact that its

fluctuations scale as  $L^{1/3}$ ) or the distribution function of  $F$  is not positive definite (which, of course, makes no physical sense). A simple mathematical analysis demonstrated that the origin of this pathology is hidden in the replica's ‘magic operations’: on the one hand, all the calculations are performed under the assumption that the parameter  $N$  (the number of particles) is integer, but on the other hand, in the thermodynamic limit  $L \rightarrow \infty$ , the relevant values of  $N$  that define the physical properties of the original random system are in the region  $N \rightarrow 0$ . In other words, the replica method assumes an *analytic continuation* of the result obtained for arbitrary integers  $N$  to the region  $N \rightarrow 0$ . The problem is that, first, such an analytic continuation is not always unambiguous (see [33, 34]) and, second, any approximations in the calculations of the replica partition function are quite risky as regards the region  $N \rightarrow 0$ .

In the problem under consideration, the neglected contributions that are exponentially small at integers  $N > 1$  turn out to be essential in the region  $N \rightarrow 0$ , which defines the properties of the free energy distribution function  $P_*(f) = \lim_{L \rightarrow \infty} P_L(f)$ . In other words, the problem is that the two limits,  $L \rightarrow \infty$  and  $N \rightarrow 0$ , do not commute [47, 48].

Nevertheless, the approximation in which the excited states of the  $N$ -body bosonic system are neglected turned out to be applicable to the calculation of the *left tail* of  $P_*(f)$  (i.e., the asymptotic behavior of this function as  $f \rightarrow -\infty$ ). Assuming the universal scaling  $L^{1/3}$  of the free energy fluctuations, it was possible to show that the left tail of  $P_*(f)$  is given by the asymptotic form of the Airy function,  $P_*(f \rightarrow -\infty) \sim \exp(-(2/3)|f|^{3/2})$  [49].

The form of the *right tail* of this distribution function,  $P_*(f \rightarrow +\infty) \sim \exp[-(\text{const})f^3]$ , was first derived in terms of the optimal fluctuation approach in [50–52], where it was demonstrated that both (left and right) asymptotic forms of the free energy distribution function  $P_*(f)$  for directed polymers are consistent with the Tracy–Widom distribution [1], which was known by that time to describe statistical properties of many other systems [2–7].

The TW distribution for directed polymers in a random medium with local correlations was first derived in terms of the distribution of solutions of the KPZ equation [9, 10], which, in particular, describes the domain wall growth and which is equivalent to the present system. Almost simultaneously, the exact solution of the one-dimensional directed polymer problem was also found in terms of the replica method, which involved summation over the whole spectrum of excited states in the corresponding  $N$ -particle quantum boson system [11–14]. That calculation allowed deriving the full free energy distributions and proving that it coincides with the TW distribution function. Tracy–Widom function (3.21) (see also Fig. 6) was originally derived in a purely mathematical problem, as the probability distribution of the largest eigenvalue of an  $N \times N$  random Hermitian matrix in the limit  $N \rightarrow \infty$  [1].

Amazingly, it gradually turned out that this function describes statistical properties of numerous random systems in physics, which may or may not be similar to one another. Here, we have not some particular instances of coincident properties but a universal distribution function free of any parameters. It is therefore valid to say that we here deal with a kind of ‘superuniversality’ applicable to the entire class of random systems.

In this review, we have described the exact solution of the problem that had remained unsolved for almost thirty years.

We stress that this solution was obtained in terms of the *replica method*. This is a very rare case where the solution of a nontrivial problem is found without using the heuristic ‘replica magic’ operations quite typical for this method, which usually leads one to think that the ‘replica method’ and the ‘exact solution’ are two absolutely incompatible things. Having found the exact solution may cause the feeling of regret because we must stop here and abandon the subject, which does take courage and resoluteness. But on the other hand, the methodology and the created mathematical technique could be used for solving numerous other problems still awaiting their solutions.

## Appendix A Quantum bosons with repulsive interactions

### A.1 Eigenfunctions

The eigenstate equation for an  $N$ -particle system of  $\delta$ -interacting one-dimensional quantum bosons is

$$\frac{1}{2} \sum_{a=1}^N \partial_{x_a}^2 \Psi(\mathbf{x}) + \frac{1}{2} \kappa \sum_{a \neq b}^N \delta(x_a - x_b) \Psi(\mathbf{x}) = -\beta E \Psi(\mathbf{x}) \quad (\text{A.1})$$

(where  $\kappa = \beta^3 u$ ). Due to the symmetry of the wave function under permutations of its arguments, it is sufficient to consider it in the sector

$$x_1 < x_2 < \dots < x_N \quad (\text{A.2})$$

and at its boundary. Inside this sector, the wave function  $\Psi(\mathbf{x})$  satisfies the equation

$$\frac{1}{2} \sum_{a=1}^N \partial_{x_a}^2 \Psi(\mathbf{x}) = -\beta E \Psi(\mathbf{x}), \quad (\text{A.3})$$

which describes  $N$  free particles, and its general solution is a linear combination of  $N$  plane waves characterized by  $N$  momenta  $\{q_1, q_2, \dots, q_N\} \equiv \mathbf{q}$ . Integrating Eqn (A.1) over the variable  $(x_{i+1} - x_i)$  in a small interval around zero (such that  $|x_{i+1} - x_i| < \epsilon \rightarrow 0$ ) and assuming that the other variables  $\{x_j\}$  (with  $j \neq i, i + 1$ ) belong to sector (A.2), it is easy to show that the wave function  $\Psi(\mathbf{x})$  must satisfy the boundary conditions

$$(\partial_{x_{i+1}} - \partial_{x_i} + \kappa) \Psi(\mathbf{x}) \Big|_{x_{i+1}=x_i+0} = 0. \quad (\text{A.4})$$

The general expression for a function satisfying both Eqn (A.3) and boundary conditions Eqn (A.4) can be written as

$$\begin{aligned} \Psi_{q_1 \dots q_N}(x_1, \dots, x_N) &\equiv \Psi_{\mathbf{q}}^{(N)}(\mathbf{x}) \\ &= C \left( \prod_{a < b}^N [\partial_{x_a} - \partial_{x_b} + \kappa] \right) \det [\exp(iq_c x_d)]_{(c,d)=1, \dots, N}, \end{aligned} \quad (\text{A.5})$$

where  $C$  is a normalization constant to be fixed later. Because the wave function is a linear combination of plane waves, it clearly satisfies Eqn (A.3). To demonstrate that it also satisfies boundary conditions (A.4), we verify this for  $i = 1$  for simplicity. According to Eqn (A.5), the wave function  $\Psi_{\mathbf{q}}^{(N)}(\mathbf{x})$  can be represented in the form

$$\Psi_{\mathbf{q}}^{(N)}(\mathbf{x}) = -(\partial_{x_2} - \partial_{x_1} - \kappa) \tilde{\Psi}_{\mathbf{q}}^{(N)}(\mathbf{x}), \quad (\text{A.6})$$

where

$$\begin{aligned} \tilde{\Psi}_{\mathbf{q}}^{(N)}(\mathbf{x}) &= C \left( \prod_{a=3}^N [\partial_{x_1} - \partial_{x_a} + \kappa] [\partial_{x_2} - \partial_{x_a} + \kappa] \right) \\ &\times \left( \prod_{3 \leq a < b}^N [\partial_{x_a} - \partial_{x_b} + \kappa] \right) \det [\exp(iq_c x_d)]_{(c,d)=1, \dots, N}. \end{aligned} \quad (\text{A.7})$$

It can be easily seen that this function is *antisymmetric* with respect to the permutation of  $x_1$  and  $x_2$ . Substituting Eqn (A.6) in Eqn (A.4) (with  $i = 1$ ) yields

$$-[(\partial_{x_2} - \partial_{x_1})^2 - \kappa^2] \tilde{\Psi}_{\mathbf{q}}^{(N)}(\mathbf{x}) \Big|_{x_2=x_1} = 0. \quad (\text{A.8})$$

Given the antisymmetry of the left-hand side of (A.8) under the permutation of  $x_1$  and  $x_2$ , the above condition is indeed satisfied at the boundary  $x_1 = x_2$ .

Because the eigenfunction  $\Psi_{\mathbf{q}}^{(N)}(\mathbf{x})$  satisfying Eqn (A.1) must be *symmetric* under the permutations of its arguments, function (A.5) can be easily continued outside the sector in (A.2), to the entire space of variables  $\{x_1, x_2, \dots, x_N\} \in R_N$ ,

$$\begin{aligned} \Psi_{\mathbf{q}}^{(N)}(\mathbf{x}) &= C \left( \prod_{a < b}^N [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b)] \right) \\ &\times \det [\exp(iq_c x_d)]_{(c,d)=1, \dots, N}, \end{aligned} \quad (\text{A.9})$$

where, by definition, the differential operator  $\partial_{x_a}$  acts only on the exponential terms and not on the  $\operatorname{sgn}(x)$  functions in the prefactor and we have redefined the normalization factor as  $i^{N(N-1)/2} C \rightarrow C$  for further convenience. Explicitly, the determinant in (A.9) can be written as

$$\det [\exp(iq_c x_d)]_{(c,d)=1, \dots, N} = \sum_P (-1)^{[P]} \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right], \quad (\text{A.10})$$

where the summation ranges over the permutations of  $N$  momenta  $\{q_1, q_2, \dots, q_N\}$  over the  $N$  particles  $\{x_1, x_2, \dots, x_N\}$  and  $[P]$  denotes the parity of the permutation. In this way, eigenfunctions (A.9) become

$$\begin{aligned} \Psi_{\mathbf{q}}^{(N)}(\mathbf{x}) &= C \sum_P (-1)^{[P]} \\ &\times \left( \prod_{a < b}^N [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b)] \right) \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right]. \end{aligned} \quad (\text{A.11})$$

Taking the derivatives, we obtain

$$\begin{aligned} \Psi_{\mathbf{q}}^{(N)}(\mathbf{x}) &= C \sum_P (-1)^{[P]} \left( \prod_{a < b}^N [q_{p_a} - q_{p_b} + i\kappa \operatorname{sgn}(x_a - x_b)] \right) \\ &\times \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right]. \end{aligned} \quad (\text{A.12})$$

It is evident that the wave functions  $\Psi_{\mathbf{q}}^{(N)}(\mathbf{x})$  are *antisymmetric* with respect to permutations of the momenta  $q_1, \dots, q_N$ . Finally, substituting the expression for eigenfunctions (A.5) [which is valid in sector (A.2)] in Eqn (A.3), we find the energy spectrum

$$E(\mathbf{q}) = \frac{1}{2\beta} \sum_{a=1}^N q_a^2. \quad (\text{A.13})$$

## A.2 Orthonormality of the wave functions

We now show that the above eigenfunctions with different momenta are orthogonal to each other. We consider two functions  $\Psi_{\mathbf{q}}^{(N)}(\mathbf{x})$  and  $\Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x})$  assuming that

$$\begin{aligned} q_1 < q_2 < \dots < q_N, \\ q'_1 < q'_2 < \dots < q'_N. \end{aligned} \quad (\text{A.14})$$

Using representation (A.11), we find the overlap of these two functions as

$$\begin{aligned} \overline{\Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x}) \Psi_{\mathbf{q}}^{(N)}(\mathbf{x})} &\equiv \int_{-\infty}^{+\infty} d^N \mathbf{x} \Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x}) \Psi_{\mathbf{q}}^{(N)}(\mathbf{x}) \\ &= |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N \mathbf{x} \left\{ \left( \prod_{a < b}^N [i(\partial_{x_a} - \partial_{x_b}) \right. \right. \\ &\quad \left. \left. - i\kappa \operatorname{sgn}(x_a - x_b)] \right) \exp \left[ -i \sum_{a=1}^N q'_{p'_a} x_a \right] \right\} \\ &\quad \times \left\{ \left( \prod_{a < b}^N [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b)] \right) \right. \\ &\quad \left. \times \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right] \right\}. \end{aligned} \quad (\text{A.15})$$

Integrating by parts leads to

$$\begin{aligned} \overline{\Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x}) \Psi_{\mathbf{q}}^{(N)}(\mathbf{x})} &= |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N \mathbf{x} \exp \left[ -i \sum_{a=1}^N q'_{p'_a} x_a \right] \\ &\quad \times \left( \prod_{a < b}^N [-i(\partial_{x_a} - \partial_{x_b}) - i\kappa \operatorname{sgn}(x_a - x_b)] \right) \\ &\quad \times \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b) \right] \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right] \end{aligned} \quad (\text{A.16})$$

or

$$\begin{aligned} \overline{\Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x}) \Psi_{\mathbf{q}}^{(N)}(\mathbf{x})} &= |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|} \int_{-\infty}^{+\infty} d^N \mathbf{x} \exp \left[ -i \sum_{a=1}^N q'_{p'_a} x_a \right] \\ &\quad \times \left( \prod_{a < b}^N [-(\partial_{x_a} - \partial_{x_b})^2 + \kappa^2] \right) \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right]. \end{aligned} \quad (\text{A.17})$$

Taking the derivatives and integrating, we find

$$\begin{aligned} \overline{\Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x}) \Psi_{\mathbf{q}}^{(N)}(\mathbf{x})} &= |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|} \left( \prod_{a < b}^N [(q_{p_a} - q_{p_b})^2 + \kappa^2] \right) \\ &\quad \times \int_{-\infty}^{+\infty} d^N \mathbf{x} \exp \left[ i \sum_{a=1}^N (q_{p_a} - q'_{p'_a}) x_a \right] \\ &= |C|^2 \sum_{P, P'} (-1)^{|P|+|P'|} \left( \prod_{a < b}^N [(q_{p_a} - q_{p_b})^2 + \kappa^2] \right) \\ &\quad \times \left[ \prod_{a=1}^N (2\pi) \delta(q_{p_a} - q'_{p'_a}) \right]. \end{aligned} \quad (\text{A.18})$$

Because the momenta are confined to sectors (A.14), it can be easily seen that the only nonzero contribution to this expression is given by terms with coincident permutations of the momenta,  $P = P'$ , and all these contributions are equal. Because the total number of permutations is  $N!$ , we obtain the orthogonality condition of the eigenfunctions in the form

$$\begin{aligned} \overline{\Psi_{\mathbf{q}'}^{(N)'}(\mathbf{x}) \Psi_{\mathbf{q}}^{(N)}(\mathbf{x})} &= |C|^2 N! \left( \prod_{a < b}^N [(q_a - q_b)^2 + \kappa^2] \right) \left[ \prod_{a=1}^N (2\pi) \delta(q_a - q'_a) \right]. \end{aligned} \quad (\text{A.19})$$

Hence, for this set of functions to be orthonormal, the normalization constant must be given by

$$|C(\mathbf{q})|^2 = \frac{1}{N! \prod_{a < b}^N [(q_a - q_b)^2 + \kappa^2]}. \quad (\text{A.20})$$

The proof of the completeness of this set is given in Ref. [41]. We note that the above eigenfunctions of problem (A.1) are an orthonormal set for any sign of the coupling parameter  $\kappa$ , i.e., for both the repulsive,  $\kappa < 0$ , and attractive,  $\kappa > 0$ , cases. But this set is complete only in the case of repulsion, while in the case of attractive interactions,  $\kappa > 0$ , in addition to solutions (A.11), which describe the continuous free particle spectrum, there also exists a family of totally different discrete solutions describing bound states of the particles. A detailed mathematical description of these states is given in Appendix B.

## Appendix B Quantum bosons with attractive interaction

### B.1 Ground state

The simplest example of a bound eigenstate defined by Eqn (A.1) (with  $\kappa > 0$ ) is the wave function with all the  $N$  particles bound into a single ‘cluster’:

$$\Psi_q^{(1)}(\mathbf{x}) = C \exp \left[ iq \sum_{a=1}^N x_a - \frac{1}{4} \kappa \sum_{a, b=1}^N |x_a - x_b| \right], \quad (\text{B.1})$$

where  $C$  is a normalization constant (to be defined below) and  $q$  is the momentum of the free center-of-mass motion. Substituting this function in Eqn (A.1), it is easy to verify that this is indeed an eigenfunction of our problem, and the energy of this state is given by

$$E = -\frac{1}{2\beta} \sum_{a=1}^N \left[ iq - \frac{1}{2} \kappa \sum_{b=1}^N \operatorname{sgn}(x_a - x_b) \right]^2. \quad (\text{B.2})$$

Because the value of  $E$  in (B.2) is independent of the order of particles, we assume for simplicity that they are ordered as indicated in Eqn (A.2). Then, substituting the well-known relations

$$\sum_{b=1}^N \operatorname{sgn}(x_a - x_b) = -(N + 1 - 2a), \quad (\text{B.3})$$

$$\sum_{a=1}^N a = \frac{1}{2} N(N + 1), \quad (\text{B.4})$$

$$\sum_{a=1}^N a^2 = \frac{N}{6} (N + 1)(2N + 1) \quad (\text{B.5})$$

in Eqn (B.2), we obtain

$$E = \frac{N}{2\beta} q^2 - \frac{\kappa^2}{24\beta} (N^3 - N) \equiv E_1(q, N). \tag{B.6}$$

The normalization constant  $C$  is defined by the orthonormality condition

$$\begin{aligned} \overline{\Psi_{q'}^{(1)*}(\mathbf{x}) \Psi_q^{(1)}(\mathbf{x})} &\equiv \int_{-\infty}^{+\infty} dx_1 \dots dx_N \Psi_{q'}^{(1)*}(\mathbf{x}) \Psi_q^{(1)}(\mathbf{x}) \\ &= (2\pi) \delta(q - q'). \end{aligned} \tag{B.7}$$

Substituting Eqn (B.1) here yields

$$\begin{aligned} \overline{\Psi_{q'}^{(1)*}(\mathbf{x}) \Psi_q^{(1)}(\mathbf{x})} &= |C|^2 \int_{-\infty}^{+\infty} dx_1 \dots dx_N \\ &\times \exp \left[ i(q - q') \sum_{a=1}^N x_a - \frac{1}{2} \kappa \sum_{a,b=1}^N |x_a - x_b| \right] \\ &= |C|^2 N! \int_{-\infty}^{+\infty} dx_1 \int_{x_1}^{+\infty} dx_2 \dots \\ &\dots \int_{x_{N-1}}^{+\infty} dx_N \exp \left[ i(q - q') \sum_{a=1}^N x_a + \kappa \sum_{a=1}^N (N + 1 - 2a)x_a \right], \end{aligned} \tag{B.8}$$

where we used the relation

$$\frac{1}{2} \sum_{a,b=1}^N |x_a - x_b| = - \sum_{a=1}^N (N + 1 - 2a)x_a \tag{B.9}$$

for the particles ordered in accordance with (A.2). Integrating first over  $x_N$  and then over  $x_{N-1}$ , and proceeding until  $x_1$ , we find

$$\begin{aligned} \overline{\Psi_{q'}^{(1)*}(\mathbf{x}) \Psi_q^{(1)}(\mathbf{x})} &= |C|^2 N! \left( \prod_{r=1}^{N-1} \frac{1}{r[(N-r)\kappa - i(q-q')]} \right) \\ &\times \int_{-\infty}^{+\infty} dx_1 \exp [iN(q - q')x_1] \\ &= |C|^2 N! \left( \prod_{r=1}^{N-1} \frac{1}{r(N-r)\kappa} \right) (2\pi) \delta(N(q - q')) \\ &= |C|^2 \frac{N\kappa}{N! \kappa^N} (2\pi) \delta(q - q'). \end{aligned} \tag{B.10}$$

According to Eqn (B.7), this defines the normalization constant

$$C = \sqrt{\frac{\kappa^N N!}{\kappa N}} \equiv C^{(1)}(q). \tag{B.11}$$

We note that the eigenstate described by wave function (B.1) exists only in the case of attraction,  $\kappa > 0$ ; otherwise, this function is divergent at infinity and is therefore not normalizable.

We note that wave function (B.1) can also be derived from the general expression for the eigenfunctions, Eqn (A.12), by introducing (discrete) imaginary parts for the momenta  $q_a$ . We assume again that the particle positions are ordered according to Eqn (A.2), and define the particle momenta as

$$q_a = q - \frac{i}{2} \kappa(N + 1 - 2a). \tag{B.12}$$

Substituting this in Eqn (A.12), we obtain

$$\begin{aligned} \Psi_q^{(1)}(x_1 < x_2 < \dots < x_N) &\propto \sum_P (-1)^{|P|} \left( \prod_{a < b}^N \left[ \left( q - \frac{i}{2} \kappa(N + 1 - 2P_a) \right) \right. \right. \\ &\quad \left. \left. - \left( q - \frac{i}{2} \kappa(N + 1 - 2P_b) \right) - i\kappa \right] \right) \\ &\times \exp \left[ iq \sum_{a=1}^N x_a + \frac{\kappa}{2} \sum_{a=1}^N (N + 1 - 2P_a)x_a \right] \\ &\propto \sum_P (-1)^{|P|} \left( \prod_{a < b}^N [P_b - P_a + 1] \right) \\ &\times \exp \left[ iq \sum_{a=1}^N x_a + \frac{\kappa}{2} \sum_{a=1}^N (N + 1 - 2P_a)x_a \right]. \end{aligned} \tag{B.13}$$

Here, it can be easily seen that due to the presence of the product  $\prod_{a < b}^N [P_b - P_a + 1]$ , only the trivial permutation  $P_a = a$  makes a nonzero contribution to the sum over permutations (if we permute any two numbers in the sequence  $1, 2, \dots, N$ , we can always find two numbers  $a < b$  such that  $P_b = P_a - 1$ ). Therefore,

$$\begin{aligned} \Psi_q^{(1)}(x_1 < x_2 < \dots < x_N) &\propto \exp \left[ iq \sum_{a=1}^N x_a + \frac{\kappa}{2} \sum_{a=1}^N (N + 1 - 2a)x_a \right]. \end{aligned} \tag{B.14}$$

Taking relation (B.9) into account, we recover wave function (B.1), which is symmetric with respect to its  $N$  arguments and can therefore be easily extended outside sector (A.2). Finally, substituting (B.12) in the general expression for the energy spectrum, Eqn (A.13), we obtain

$$E = \frac{1}{2\beta} \sum_{a=1}^N \left[ q - \frac{i}{2} \kappa(N + 1 - 2a) \right]^2. \tag{B.15}$$

By simple summation [using Eqns (B.4) and (B.5)] we recover Eqn (B.6).

### B.2 Eigenfunctions

The general eigenfunction of the system of  $N$  attractive bosons is characterized by  $N$  parameters  $\{q_a\}$  ( $a = 1, 2, \dots, N$ ), which may have imaginary parts. To describe the general structure of these parameters, it is convenient to group them into  $M$  ( $1 \leq M \leq N$ ) ‘vector’ momenta,

$$q_r^\alpha = q_\alpha - \frac{i}{2} \kappa (n_\alpha + 1 - 2r), \tag{B.16}$$

where  $q_\alpha$  ( $\alpha = 1, 2, \dots, M$ ) are the continuous (real) parameters and the discrete imaginary components of each ‘vector’ are labeled by  $r = 1, 2, \dots, n_\alpha$ . The integers  $n_\alpha$  must satisfy the constraint

$$\sum_{\alpha=1}^M n_\alpha = N. \tag{B.17}$$

Therefore, an eigenstate is characterized in general by parameters of three types: the discrete number  $M$  ( $1 \leq M \leq N$ ) that defines the number of ‘vector’ momenta, the set of  $M$  integer parameters  $\{n_1, n_2, \dots, n_M\} \equiv \mathbf{n}$  (which are the numbers of imaginary components of each

‘vector’), and the set of  $M$  continuous real momenta  $\{q_1, q_2, \dots, q_M\} \equiv \mathbf{q}$ .

The general expression for the eigenfunction is given in Eqns (A.9)–(A.12). To understand the structure of the determinant of the  $N \times N$  matrix  $\exp(iq_a x_b)$ , which defines this wave function, the  $N$  momenta  $q_a$  in (B.16) can be ordered as follows:

$$\begin{aligned} \{q_a\} &\equiv \{q_r^\alpha\} \\ &= \{q_1^1, q_2^1, \dots, q_{n_1}^1; q_1^2, q_2^2, \dots, q_{n_2}^2; \dots; q_1^M, q_2^M, \dots, q_{n_M}^M\}. \end{aligned} \quad (\text{B.18})$$

By definition,

$$\det [\exp(iq_a x_c)]_{(c,d)=1,\dots,N} = \sum_P (-1)^{|P|} \exp \left[ i \sum_{a=1}^N q_{p_a} x_a \right], \quad (\text{B.19})$$

where the summation ranges over all permutations of  $N$  complex momenta  $\{q_a\}$ , Eqn (B.18), over the  $N$  particles  $\{x_1, x_2, \dots, x_N\}$ , and  $[P]$  denotes the parity of the permutation. For a given permutation  $P$ , a number- $a$  particle is assigned the momentum component  $q_{r(a)}^{\alpha(a)}$ . The particles assigned the momenta with the same  $\alpha$  (having the same real part  $q_\alpha$ ) are said to belong to a *cluster*  $\Omega_\alpha$ . For a given permutation  $P$ , the particles belonging to the same cluster are therefore numbered by the ‘internal’ index  $r = 1, \dots, n_\alpha$ . Hence, according to Eqn (A.11),

$$\begin{aligned} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}) &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_P (-1)^{|P|} \left( \prod_{a < b}^N [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b)] \right) \\ &\times \exp \left[ i \sum_{c=1}^N q_{r(c)}^{\alpha(c)} x_c \right], \end{aligned} \quad (\text{B.20})$$

where  $C_{\mathbf{q}, \mathbf{n}}^{(M)}$  is the normalization constant to be defined later. Substituting Eqn (B.16) here and taking the derivatives, we obtain

$$\begin{aligned} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}) &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_P (-1)^{|P|} \prod_{a < b}^N \left[ \left( q_{\alpha(a)} - \frac{i\kappa}{2} [n_{\alpha(a)} + 1 - 2r(a)] \right) \right. \\ &- \left. \left( q_{\alpha(b)} - \frac{i\kappa}{2} [n_{\alpha(b)} + 1 - 2r(b)] \right) + i\kappa \operatorname{sgn}(x_a - x_b) \right] \\ &\times \exp \left[ i \sum_{c=1}^N q_{\alpha(c)} x_c + \frac{\kappa}{2} \sum_{c=1}^N (n_{\alpha(c)} + 1 - 2r(c)) x_c \right]. \end{aligned} \quad (\text{B.21})$$

The pre-exponential product in the above equation contains two types of factors: pairs of points  $(a, b)$  that belong to different clusters  $[\alpha(a) \neq \alpha(b)]$  and pairs of points that belong to the same cluster  $[\alpha(a) = \alpha(b)]$ . In the second case, the product  $\Pi_\alpha$  over the pairs of points belonging to a cluster  $\Omega_\alpha$  reduces to

$$\Pi_\alpha \propto \prod_{a < b \in \Omega_\alpha} [r(b) - r(a) - \operatorname{sgn}(x_a - x_b)]. \quad (\text{B.22})$$

Similarly to the ground-state wave function in (B.13) and (B.14), it is easy to see that due to the presence of this product in the summations over the ‘internal’ permutations  $r(a)$  (inside the cluster  $\Omega_\alpha$ ), only one permutation makes a nonzero contribution. To prove this statement, we first note that the wave function  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$  is symmetric under permuta-

tions of its  $N$  arguments  $\{x_a\}$ , and it therefore suffices to consider the case where the positions of the particles are ordered in the simplest way,  $x_1 < x_2 < \dots < x_N$ . The particles  $\{x_{a_k}\}$  ( $k = 1, 2, \dots, n_\alpha$ ) belonging to the same cluster  $\Omega_\alpha$  are then also ordered,  $x_{a_1} < x_{a_2} < \dots < x_{a_{n_\alpha}}$  (where  $a_1 < a_2 < \dots < a_{n_\alpha}$ ). In this case,

$$\Pi_\alpha \propto \prod_{k < l}^{n_\alpha} [r(l) - r(k) + 1]. \quad (\text{B.23})$$

It is evident that the above product is nonzero only for the trivial permutation  $r(k) = k$  [indeed, if we permute any two numbers in the sequence  $1, 2, \dots, n_\alpha$ , we can always find two numbers  $k < l$  such that  $r(l) = r(k) - 1$ ]. In the case of the trivial permutation,

$$\Pi_\alpha \propto \prod_{k < l}^{n_\alpha} [l - k + 1]. \quad (\text{B.24})$$

Including the values of all these ‘internal’ products of clusters into the redefined normalization constant  $C_{\mathbf{q}, \mathbf{n}}^{(M)}$ , we obtain the wave function (B.21) in the form

$$\begin{aligned} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}) &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_P' (-1)^{|P|} \prod_{\substack{a < b \\ \alpha(a) \neq \alpha(b)}}^N \left[ \left( q_{\alpha(a)} - \frac{i\kappa}{2} n_{\alpha(a)} \right) \right. \\ &- \left. \left( q_{\alpha(b)} - \frac{i\kappa}{2} n_{\alpha(b)} \right) + i\kappa (r(a) - r(b) - 1) \right] \\ &\times \exp \left[ i \sum_{c=1}^N q_{\alpha(c)} x_c + \frac{\kappa}{2} \sum_{c=1}^N (n_{\alpha(c)} + 1 - 2r(c)) x_c \right], \end{aligned} \quad (\text{B.25})$$

where the product now ranges only over the pairs of particles belonging to *different* clusters, and the symbol  $\sum_P'$  means that the summation is only over the permutations  $P$  in which the ‘internal’ indices  $r(a)$  are taken in the ascending order.

Using the symmetry of the wave function  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$  under permutations of its arguments, we can easily continue expression (B.25) outside the sector  $x_1 < x_2 < \dots < x_N$ , to the entire space  $R_N$ . Using the relation

$$\begin{aligned} \sum_{a \in \Omega_\alpha} (n_\alpha + 1 - 2r(a)) x_a &= \sum_{k=1}^{n_\alpha} (n_\alpha + 1 - 2k) x_{a_k} \\ &= -\frac{1}{2} \sum_{k, l=1}^{n_\alpha} |x_{a_k} - x_{a_l}| \end{aligned} \quad (\text{B.26})$$

(where  $x_{a_1} < x_{a_2} < \dots < x_{a_{n_\alpha}}$ ), we find the following sufficiently compact representation for the wave function  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$  with *arbitrary* particle positions [cf. Eqn (B.20)]:

$$\begin{aligned} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}) &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_P' (-1)^{|P|} \prod_{\substack{a < b \\ \alpha(a) \neq \alpha(b)}}^N [-i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b)] \\ &\times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{c \in \Omega_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^M \sum_{c, c' \in \Omega_\alpha} |x_c - x_{c'}| \right]. \end{aligned} \quad (\text{B.27})$$

We note that because the mutual positions of particles belonging to different clusters can be arbitrary, the clusters can overlap and ‘penetrate’ one another. In other words, the name ‘cluster’ does not imply isolated, spatially compact particle arrangements.

Finally, to obtain the energy spectrum of the above eigenstates, it suffices to substitute Eqns (B.16) and (B.17) in Eqn (A.13). Simple summation [using (B.4) and (B.5)] then leads to

$$E_M(\mathbf{q}, \mathbf{n}) = \frac{1}{2\beta} \sum_{\alpha=1}^M \sum_{r=1}^{n_\alpha} (q_\alpha^r)^2 = \frac{1}{2\beta} \sum_{\alpha=1}^M n_\alpha q_\alpha^2 - \frac{\kappa^2}{24\beta} \sum_{\alpha=1}^M (n_\alpha^3 - n_\alpha). \quad (\text{B.28})$$

**B.3 Orthonormality of the eigenfunctions**

We define the overlap of two wave functions characterized by two sets of parameters  $(M, \mathbf{n}, \mathbf{q})$  and  $(M', \mathbf{n}', \mathbf{q}')$  as

$$Q_{\mathbf{n}, \mathbf{n}'}^{(M, M')}(\mathbf{q}, \mathbf{q}') \equiv \int_{-\infty}^{+\infty} d^N \mathbf{x} \Psi_{\mathbf{q}', \mathbf{n}'}^{(M')}(\mathbf{x}) \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}). \quad (\text{B.29})$$

Substituting Eqn (B.27) here, we obtain

$$Q_{\mathbf{n}, \mathbf{n}'}^{(M, M')}(\mathbf{q}, \mathbf{q}') = C_{\mathbf{q}, \mathbf{n}}^{(M)} C_{\mathbf{q}', \mathbf{n}'}^{(M')*} \sum_P \sum_{P'} (-1)^{[P]+[P']} \times \int_{-\infty}^{+\infty} d^N \mathbf{x} \left( \prod_{\substack{a < b \\ \alpha'(a) \neq \alpha'(b)}}^N \left[ i(\partial_{x_a} - \partial_{x_b}) - i\kappa \operatorname{sgn}(x_a - x_b) \right] \right) \times \exp \left[ -i \sum_{\alpha=1}^{M'} q'_\alpha \sum_{c \in \Omega'_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^{M'} \sum_{c, c' \in \Omega'_\alpha} |x_c - x_{c'}| \right] \times \left( \prod_{\substack{a < b \\ \alpha(a) \neq \alpha(b)}}^N \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b) \right] \right) \times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{c \in \Omega_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^M \sum_{c, c' \in \Omega_\alpha} |x_c - x_{c'}| \right], \quad (\text{B.30})$$

where  $\{\Omega_\alpha\}$  and  $\{\Omega'_\alpha\}$  denote the clusters formed by the respective permutations  $P$  and  $P'$ . Integrating by parts, we obtain

$$Q_{\mathbf{n}, \mathbf{n}'}^{(M, M')}(\mathbf{q}, \mathbf{q}') = C_{\mathbf{q}, \mathbf{n}}^{(M)} C_{\mathbf{q}', \mathbf{n}'}^{(M')*} \sum_P \sum_{P'} (-1)^{[P]+[P']} \times \int_{-\infty}^{+\infty} d^N \mathbf{x} \exp \left[ -i \sum_{\alpha=1}^{M'} q'_\alpha \sum_{c \in \Omega'_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^{M'} \sum_{c, c' \in \Omega'_\alpha} |x_c - x_{c'}| \right] \times \left( \prod_{\substack{a < b \\ \alpha'(a) \neq \alpha'(b)}}^N \left[ -i(\partial_{x_a} - \partial_{x_b}) - i\kappa \operatorname{sgn}(x_a - x_b) \right] \right) \times \left( \prod_{\substack{a < b \\ \alpha(a) \neq \alpha(b)}}^N \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b) \right] \right) \times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{c \in \Omega_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^M \sum_{c, c' \in \Omega_\alpha} |x_c - x_{c'}| \right]. \quad (\text{B.31})$$

We first consider the case where the integer parameters of the two functions coincide,  $M = M'$ ,  $\mathbf{n} = \mathbf{n}'$ , and assume in addition that all these integer parameters  $\{n_\alpha\}$  are *distinct*, i.e.,  $1 \leq n_1 < n_2 < \dots < n_M$ .

In summing over the permutations in Eqn (B.31), we then find two types of terms:

(a) ‘diagonal’ terms in which the two permutations coincide,  $P = P'$ ;

(b) ‘off-diagonal’ terms in which the two permutations are different,  $P \neq P'$ .

The contribution of the ‘diagonal’ terms is given by

$$Q_{\mathbf{n}, \mathbf{n}}^{(M, M)^{(A)}}(\mathbf{q}, \mathbf{q}') = C_{\mathbf{q}, \mathbf{n}}^{(M)} C_{\mathbf{q}', \mathbf{n}}^{(M)*} \sum_P \int_{-\infty}^{+\infty} d^N \mathbf{x} \times \exp \left[ -i \sum_{\alpha=1}^M q'_\alpha \sum_{c \in \Omega_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^M \sum_{c, c' \in \Omega_\alpha} |x_c - x_{c'}| \right] \times \left( \prod_{\substack{a < b \\ \alpha(a) \neq \alpha(b)}}^N \left[ -(\partial_{x_a} - \partial_{x_b})^2 + \kappa^2 \right] \right) \times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{c \in \Omega_\alpha} x_c - \frac{\kappa}{4} \sum_{\alpha=1}^M \sum_{c, c' \in \Omega_\alpha} |x_c - x_{c'}| \right]. \quad (\text{B.32})$$

It is evident that all permutations  $\alpha(a)$  in the above equation make the same contribution, and it is therefore sufficient to consider only the contribution of the ‘trivial’ permutation in (B.18). We let  $\alpha_0(a)$  denote the clusters corresponding to this permutation. We can now redefine the order of summation over the particles such that instead of the ‘straight’ summation over  $a = 1, 2, \dots, N$ , the particles are summed over two indices  $(\alpha, r)$ :  $\{x_a\} \rightarrow \{x_r^\alpha\}$  ( $\alpha = 1, \dots, M$ ;  $r = 1, \dots, n_\alpha$ ) indicating to which cluster  $\alpha$  a given particle belongs and what its ‘internal’ cluster number  $r$  is. Due to the symmetry of the integrand in Eqn (B.32) under permutations of the particles inside clusters, it suffices to find the contribution of the integration over the sectors  $x_1^\alpha < x_2^\alpha < \dots < x_{n_\alpha}^\alpha$ . In this way, using relation (B.26), we obtain

$$Q_{\mathbf{n}, \mathbf{n}}^{(M, M)^{(A)}}(\mathbf{q}, \mathbf{q}') = C_{\mathbf{q}, \mathbf{n}}^{(M)} C_{\mathbf{q}', \mathbf{n}}^{(M)*} \frac{N!}{n_1! n_2! \dots n_M!} \times \left[ \prod_{\alpha=1}^M \left( n_\alpha! \int_{-\infty}^{+\infty} dx_1^\alpha \int_{x_1^\alpha}^{+\infty} dx_2^\alpha \dots \int_{x_{n_\alpha-1}^\alpha}^{+\infty} dx_{n_\alpha}^\alpha \right) \right] \times \exp \left[ -i \sum_{\alpha=1}^M q'_\alpha \sum_{r=1}^{n_\alpha} x_r^\alpha + \frac{\kappa}{2} \sum_{\alpha=1}^M \sum_{r=1}^{n_\alpha} (n_\alpha + 1 - 2r) x_r^\alpha \right] \times \left( \prod_{\alpha < \beta} \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left[ -(\partial_{x_r^\alpha} - \partial_{x_{r'}^\beta})^2 + \kappa^2 \right] \right) \times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{r=1}^{n_\alpha} x_r^\alpha + \frac{\kappa}{2} \sum_{\alpha=1}^M \sum_{r=1}^{n_\alpha} (n_\alpha + 1 - 2r) x_r^\alpha \right], \quad (\text{B.33})$$

where the factor  $N!/n_1! \dots n_M!$  is the total number of permutations of  $M$  clusters containing  $n_1, n_2, \dots, n_M$  elements. Taking the derivatives, we obtain

$$Q_{\mathbf{n}, \mathbf{n}}^{(M, M)^{(A)}}(\mathbf{q}, \mathbf{q}') = C_{\mathbf{q}, \mathbf{n}}^{(M)} C_{\mathbf{q}', \mathbf{n}}^{(M)*} N! \left( \prod_{\alpha < \beta} \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left| \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa (r - r' - 1) \right|^2 \right) \times \prod_{\alpha=1}^M \left\{ \int_{-\infty}^{+\infty} dx_1^\alpha \int_{x_1^\alpha}^{+\infty} dx_2^\alpha \dots \int_{x_{n_\alpha-1}^\alpha}^{+\infty} dx_{n_\alpha}^\alpha \right\} \times \exp \left[ i(q_\alpha - q'_\alpha) \sum_{r=1}^{n_\alpha} x_r^\alpha + \kappa \sum_{r=1}^{n_\alpha} (n_\alpha + 1 - 2r) x_r^\alpha \right]. \quad (\text{B.34})$$

Simple integrations over the particle coordinates  $x_r^z$  yield [cf. Eqns (B.8)–(B.10)]

$$Q_{\mathbf{n},\mathbf{n}}^{(M,M)^{(A)}}(\mathbf{q},\mathbf{q}') = |C_{\mathbf{q},\mathbf{n}}^{(M)}|^2 N! \left( \prod_{\alpha < \beta}^M \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left| \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa (r - r' - 1) \right|^2 \right) \times \prod_{\alpha=1}^M \left[ \frac{n_\alpha \kappa}{(n_\alpha!)^2 \kappa^{n_\alpha}} (2\pi) \delta(q_\alpha - q'_\alpha) \right]. \tag{B.35}$$

We now prove that the ‘off-diagonal’ terms in Eqn (B.31), in which the permutations  $P$  and  $P'$  are different, make no contribution. Here, we can also choose one of the permutations, say  $P$ , to be the ‘trivial’ one represented by Eqn (B.18), with the clusters corresponding to this permutation again denoted by  $\alpha_0(a)$ . In view of the symmetry of the wave functions, it is sufficient to consider the contribution of the sector  $x_1 < x_2 < \dots < x_N$ . According to Eqn (B.31), we then have

$$Q_{\mathbf{n},\mathbf{n}}^{(M,M)^{(B)}}(\mathbf{q},\mathbf{q}') \propto \sum_{P'} (-1)^{|P'|} \int_{x_1 < \dots < x_N} d^N \mathbf{x} \times \exp \left[ -i \sum_{\alpha=1}^M q'_\alpha \sum_{a \in \Omega'_\alpha} x_a - \frac{\kappa}{4} \sum_{\alpha=1}^M \sum_{a,b \in \Omega'_\alpha} |x_a - x_b| \right] \times \left( \prod_{\substack{a < b \\ \alpha_0(a) \neq \alpha_0(b)}}^N \left[ -i(\partial_{x_a} - \partial_{x_b}) + i\kappa \operatorname{sgn}(x_a - x_b) \right] \right) \times \left( \prod_{\substack{a < b \\ \alpha'(a) \neq \alpha'(b)}}^N \left[ -i(\partial_{x_a} - \partial_{x_b}) - i\kappa \operatorname{sgn}(x_a - x_b) \right] \right) \times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{a \in \Omega_\alpha} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^M \sum_{a \in \Omega_\alpha} (n_\alpha + 1 - 2r(a)) x_a \right]. \tag{B.36}$$

Here, the symbol  $\{\Omega_\alpha^\circ\}$  denotes the clusters of the trivial permutation  $\alpha_0(a)$ . Because  $P' \neq P$ , some of the clusters  $\Omega'_\alpha$  must be different from  $\Omega_\alpha^\circ$ . As an illustration, we consider the particular case of  $N = 10$  particles, with three clusters with the respective numbers of particles  $n_1 = 5$  (denoted by the symbol ‘ $\circ$ ’),  $n_2 = 2$  (denoted by the symbol ‘ $\times$ ’), and  $n_3 = 3$  (denoted by the symbol ‘ $\triangle$ ’):

Particle number $a$	1	2	3	4	5	6	7	8	9	10
Permutation $\alpha_0(a)$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\triangle$	$\triangle$	$\triangle$
Permutation $\alpha'(a)$	$\circ$	$\circ$	$\circ$	$\triangle$	$\circ$	$\times$	$\times$	$\circ$	$\triangle$	$\triangle$

Here, in the permutation  $\alpha'(a)$ , the particle  $a = 4$  belongs to the cluster  $\alpha = 3$  [and not to the cluster  $\alpha = 1$  as in the permutation  $\alpha_0(a)$ ], and the particle  $a = 8$  belongs to the cluster  $\alpha = 1$  [and not to the cluster  $\alpha = 3$  as in the permutation  $\alpha_0(a)$ ]. We now inspect the structure of the products in Eqn (B.36). Unlike the first product, which contains no ‘internal’ products among particles belonging to the cluster  $\Omega_1^\circ$ , the second product does contain such products. Moreover, the signs of the differential operators  $\partial_{x_a} - \partial_{x_b}$  in the second product are *opposite* to the ‘normal’

(original) ones in the first product. Due to these two circumstances (the presence of the ‘internal’ products and the ‘wrong’ signs of the differential operators), ‘off-diagonal’ contributions (B.36) vanish. Indeed, in the above example, the second product contains the term

$$\Pi'_{4,5} \equiv \left[ -i(\partial_{x_4} - \partial_{x_5}) + i\kappa \right] \times \exp \left[ i \sum_{\alpha=1}^3 q_\alpha \sum_{a \in \Omega_\alpha^\circ} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^3 \sum_{a \in \Omega_\alpha^\circ} (n_\alpha + 1 - 2r(a)) x_a \right] \tag{B.37}$$

(we recall that the particles in the clusters  $\Omega_\alpha^\circ$  are ordered and, in particular,  $x_4 < x_5$ ). Taking the derivatives, we obtain

$$\Pi'_{4,5} \propto \left[ -\left( iq_1 + \frac{\kappa}{2} (n_1 + 1 - 2r(4)) \right) - iq_1 - \frac{\kappa}{2} (n_1 + 1 - 2r(5)) \right] + \kappa \propto [r(4) - r(5) + 1] = 0, \tag{B.38}$$

because  $r(a) = a$  in the first cluster.

It is easy to realize that the above example reflects the general situation. Because we have so far assumed all the cluster sizes  $n_\alpha$  to be different for any permutation  $\alpha'(a)$ , we can always find a cluster  $\Omega_\alpha^\circ$  such that some of its particles belong to the same cluster number  $\alpha$  in the permutation  $\alpha'(a)$ , whereas the others do not. It then suffices to consider the contribution of the product of two *neighboring* points

$$\Pi'_{k,k+1} = \left[ -i(\partial_{x_k} - \partial_{x_{k+1}}) + i\kappa \right] \times \exp \left[ i \sum_{\alpha=1}^M q_\alpha \sum_{a \in \Omega_\alpha} x_a + \frac{\kappa}{2} \sum_{\alpha=1}^M \sum_{a \in \Omega_\alpha} (n_\alpha + 1 - 2r(a)) x_a \right], \tag{B.39}$$

where, in the permutation  $\alpha'(a)$ , the particle  $k$  belongs to the cluster number  $\alpha$  and the particle  $k + 1$  belongs to some other cluster. Taking the derivatives, we find

$$\Pi'_{k,k+1} \propto [r(k) - r(k + 1) + 1] = 0, \tag{B.40}$$

because  $r(a)$  is the ‘internal’ particle number in the cluster  $\Omega_\alpha^\circ$ , where  $r(k + 1) = r(k) + 1$  [cf. Eqns (B.22)–(B.24)].

Therefore, the only nonzero contribution to the overlap, Eqn (B.29), of two wave functions  $\Psi_{\mathbf{q}',\mathbf{n}}^{(M)}(\mathbf{x})$  and  $\Psi_{\mathbf{q},\mathbf{n}}^{(M)}(\mathbf{x})$  (having the same number of clusters  $M$  and characterized by the same set of integer parameters  $1 \leq n_1 < n_2 < \dots < n_M$ ) comes from ‘diagonal’ terms (B.35):

$$Q_{\mathbf{n},\mathbf{n}}^{(M,M)}(\mathbf{q},\mathbf{q}') = |C_{\mathbf{q},\mathbf{n}}^{(M)}|^2 N! \prod_{\alpha=1}^M \left[ \frac{n_\alpha \kappa}{(n_\alpha!)^2 \kappa^{n_\alpha}} \right] \times \left( \prod_{\alpha < \beta}^M \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left| \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa (r - r' - 1) \right|^2 \right) \prod_{\alpha=1}^M [(2\pi) \delta(q_\alpha - q'_\alpha)]. \tag{B.41}$$

The case where some clusters have the same number of particles  $n_\alpha$  is somewhat more complicated. We consider the overlap of two wave function  $\Psi_{\mathbf{q}',\mathbf{n}}^{(M)}(\mathbf{x})$  and  $\Psi_{\mathbf{q},\mathbf{n}}^{(M)}(\mathbf{x})$  (which, as before, have the same integer parameters  $M$  and  $\mathbf{n}$ ) such that among the  $M$  integers  $n_1, n_2, \dots, n_M$ , there are two equal ones, say,  $n_{\alpha_1} = n_{\alpha_2}$  (where  $\alpha_1 \neq \alpha_2$ ). In the eigenstate



$(\mathbf{q}', \mathbf{n})$ , these two clusters have the center of mass momenta  $q'_{\alpha_1}$  and  $q'_{\alpha_2}$ , and in the eigenstate  $(\mathbf{q}, \mathbf{n})$ , they have the momenta  $q_{\alpha_1}$  and  $q_{\alpha_2}$ . We have seen that in summing over the cluster permutations  $\alpha(a)$  and  $\alpha'(a)$  in Eqn (B.31), a nonzero contribution appears only if the clusters  $\{\Omega_{\alpha}\}$  of the permutation  $\alpha(a)$  totally coincide with the clusters  $\{\Omega'_{\alpha}\}$  of the permutation  $\alpha'(a)$ . When all the  $n_{\alpha}$  are different, this is possible only if the permutation  $\alpha(a)$  coincides with the permutation  $\alpha'(a)$ . But if have  $n_{\alpha_1} = n_{\alpha_2}$ , then there are *two* nonzero options. The first, as before, is given by the ‘diagonal’ terms with  $\alpha(a) = \alpha'(a)$ , and this contribution is proportional to  $\delta(q_{\alpha_1} - q'_{\alpha_1}) \delta(q_{\alpha_2} - q'_{\alpha_2})$ . The second contribution is given by such a permutation  $\alpha'(a)$  in which the cluster  $\Omega'_{\alpha_1}$  [of the permutation  $\alpha'(a)$ ] coincides with the cluster  $\Omega_{\alpha_2}$  [of the permutation  $\alpha(a)$ ] and the cluster  $\Omega'_{\alpha_2}$  [of the permutation  $\alpha'(a)$ ] coincides with the cluster  $\Omega_{\alpha_1}$  [of the permutation  $\alpha(a)$ ], while the other clusters of these two permutations are the same,  $\Omega'_{\alpha} = \Omega_{\alpha}$  ( $\alpha \neq \alpha_1, \alpha_2$ ). It is easy to see that this last contribution is proportional to  $\delta(q_{\alpha_1} - q'_{\alpha_2}) \delta(q_{\alpha_2} - q'_{\alpha_1}) (-1)^{n_{\alpha_1}}$ . In fact, this situation with two equivalent contributions is a consequence of the symmetry of the wave function  $\Psi_{\mathbf{q}', \mathbf{n}}^{(M)}(\mathbf{x})$ . This is evident from the general expression for the wave function, Eqn (A.9), where the permutation of any two momenta  $q_{\alpha_1}$  and  $q_{\alpha_2}$  belonging to the clusters of the same size  $n_{\alpha_1} = n_{\alpha_2}$  corresponds to a permutation of  $n$  columns of the matrix  $\exp(iq_a x_b)$  and produces the factor  $(-1)^{n_{\alpha_1}}$ . Therefore, the states that differ only in a permutation of momenta belonging to equal-size clusters can be considered equivalent. But if we deal with wave function overlaps only between nonequivalent states, then we must restrict ourself in the above case to the sector  $q_{\alpha_1} < q_{\alpha_2}$ ;  $q'_{\alpha_1} < q'_{\alpha_2}$ . The second contribution  $\delta(q_{\alpha_1} - q'_{\alpha_2}) \delta(q_{\alpha_2} - q'_{\alpha_1})$  is then equal to zero, and we therefore return to the above result in Eqn (B.41).

In general, therefore, an eigenstate  $(\mathbf{q}, \mathbf{n})$  with  $M$  clusters has to be described in a somewhat greater detail, by singling out clusters of the same size. Namely, we assume that among the  $M$  clusters, there are  $s_1$  clusters that have the same size  $m_1$ ,  $s_2$  clusters that have the same size  $m_2$ , and so on. The eigenstate is then characterized by a number  $k$ ,  $1 \leq k \leq M$ , of *different-size* clusters. Each such cluster size  $m_i$  is characterized by the ‘degeneracy’  $s_i$  indicating the number of clusters of the same size  $m_i$ . This can be represented as

$$(\mathbf{q}, \mathbf{n}) = \left\{ \underbrace{(q_1, m_1), \dots, (q_{s_1}, m_1)}_{s_1}; \underbrace{(q_{s_1+1}, m_2), \dots, (q_{s_1+s_2}, m_2)}_{s_2}; \dots; \dots; \underbrace{(q_{s_1+\dots+s_{k-1}+1}, m_k), \dots, (q_{s_1+\dots+s_k}, m_k)}_{s_k} \right\}, \quad (\text{B.42})$$

where  $s_1 + s_2 + \dots + s_k = M$  and all the  $k$  integers  $\{m_i\}$  are supposed to be *different*:

$$1 \leq m_1 < m_2 < \dots < m_k. \quad (\text{B.43})$$

For a given  $k$ ,

$$\sum_{\alpha=1}^M n_{\alpha} = \sum_{i=1}^k s_i m_i = N. \quad (\text{B.44})$$

The states that differ only in a permutation of momenta between clusters of the same size are considered equivalent. Therefore, the orthonormality only has to be addressed for nonequivalent states, whose momenta are confined to the

sectors

$$\begin{aligned} q_1 < q_2 < \dots < q_{s_1}; \\ q_{s_1+1} < q_{s_1+2} < \dots < q_{s_1+s_2}; \\ \dots \dots \dots \\ q_{s_1+\dots+s_{k-1}+1} < q_{s_1+\dots+s_{k-1}+2} < \dots < q_{s_1+\dots+s_{k-1}+s_k}. \end{aligned} \quad (\text{B.45})$$

In this representation, we again recover the above result in Eqn (B.41).

We finally consider the case where two eigenstates are described by two *different* sets of integer parameters,  $\mathbf{n}' \neq \mathbf{n}$ . This case is quite simple: that the clusters of the two states are different from each other means that in the summation over the pairs of permutations  $P$  and  $P'$  in Eqn (B.31), there exist no two permutations for which these two sets of clusters  $\{\Omega_{\alpha}\}$  and  $\{\Omega'_{\alpha}\}$  coincide. This, according to the above analysis, means that expression (B.31) then vanishes. We note that the condition  $M' \neq M$  automatically implies that  $\mathbf{n}' \neq \mathbf{n}$ .

Thus, we have proved that

$$\begin{aligned} Q_{\mathbf{n}, \mathbf{n}'}^{(M, M')}(\mathbf{q}, \mathbf{q}') &= |C_{\mathbf{q}, \mathbf{n}}^{(M)}|^2 \delta(M, M') \left( \prod_{\alpha=1}^M \delta(n_{\alpha}, n'_{\alpha}) \right) \\ &\times \left( \prod_{\alpha=1}^M (2\pi) \delta(q_{\alpha} - q'_{\alpha}) \right) N! \prod_{\alpha=1}^M \left[ \frac{n_{\alpha} \kappa}{(n_{\alpha}!)^2 \kappa^{n_{\alpha}}} \right] \\ &\times \prod_{\alpha < \beta} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n_{\beta}} \left| \left( q_{\alpha} - \frac{i\kappa}{2} n_{\alpha} \right) - \left( q_{\beta} - \frac{i\kappa}{2} n_{\beta} \right) + i\kappa (r - r' - 1) \right|^2, \end{aligned} \quad (\text{B.46})$$

where the integer parameters  $\{n_{\alpha}\}$  and  $\{n'_{\alpha}\}$  are assumed to have the structure represented in Eqns (B.42)–(B.44) and the momenta  $\{q_{\alpha}\}$  and  $\{q'_{\alpha}\}$  of the clusters with equal numbers of particles are restricted to sectors (B.45). According to Eqn (B.46), the orthonormality condition defines the normalization constant

$$\begin{aligned} |C_{\mathbf{q}, \mathbf{n}}^{(M)}|^2 &= \frac{1}{N!} \left[ \prod_{\alpha=1}^M \frac{(n_{\alpha}!)^2 \kappa^{n_{\alpha}}}{n_{\alpha} \kappa} \right] \\ &\times \prod_{\alpha < \beta} \prod_{r=1}^{n_{\alpha}} \prod_{r'=1}^{n_{\beta}} \left| \left( q_{\alpha} - \frac{i\kappa}{2} n_{\alpha} \right) - \left( q_{\beta} - \frac{i\kappa}{2} n_{\beta} \right) + i\kappa (r - r' - 1) \right|^{-2}. \end{aligned} \quad (\text{B.47})$$

### B.4 Propagator

The time-dependent solution  $\Psi(\mathbf{x}, t)$  of the Schrödinger equation

$$\beta \partial_t \Psi(\mathbf{x}; t) = \frac{1}{2} \sum_{a=1}^N \partial_{x_a}^2 \Psi(\mathbf{x}; t) + \frac{1}{2} \kappa \sum_{a \neq b}^N \delta(x_a - x_b) \Psi(\mathbf{x}; t) \quad (\text{B.48})$$

with the initial condition

$$\Psi(\mathbf{x}; 0) = \prod_{a=1}^N \delta(x_a) \quad (\text{B.49})$$

can be represented as a linear combination of the eigenfunctions  $\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x})$  in (B.27),

$$\Psi(\mathbf{x}, t) = \sum_{M=1}^N \sum_{\mathbf{n}} \int' \mathcal{D}\mathbf{q} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{x}) \Psi_{\mathbf{q}, \mathbf{n}}^{(M)*}(\mathbf{0}) \exp[-E_M(\mathbf{q}, \mathbf{n})t], \quad (\text{B.50})$$

where the energy spectrum is given by Eqn (B.28). The summations over the integer  $n_x$  are performed here in terms of the parameters  $\{s_i, m_i\}$  introduced in (B.42)–(B.44):

$$\sum_{\mathbf{n}}' \equiv \sum_{k=1}^M \sum_{s_1 \dots s_k=1}^{\infty} \sum_{1 \leq m_1 \dots m_k}^{\infty} \delta\left(\sum_{i=1}^k s_i, M\right) \delta\left(\sum_{i=1}^k s_i m_i, N\right), \quad (\text{B.51})$$

where  $\delta(n, l)$  is the (discrete) Kronecker symbol. We note that due to the presence of these Kronecker symbols, the summations over  $m_i$  and  $s_i$  can be extended to infinity. The symbol  $\int' \mathcal{D}\mathbf{q}$  in (B.50) denotes integration over the  $M$  momenta  $q_x$  in the sectors described in (B.45).

The replica partition function  $Z(N, L)$  of the original directed polymer problem is obtained from the wave function  $\Psi(\mathbf{x}; t)$  at zero boundary conditions:

$$\begin{aligned} Z(N, L) &= \Psi(\mathbf{0}; L) \\ &= \sum_{M=1}^N \sum_{\mathbf{n}}' \int' \mathcal{D}\mathbf{q} |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 \exp[-E_M(\mathbf{q}, \mathbf{n})L]. \end{aligned} \quad (\text{B.52})$$

According to Eqn (B.27), for  $M \geq 2$ , we have

$$\begin{aligned} \Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0}) &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \sum_P' (-1)^{|P|} \\ &\times \prod_{\substack{a < b \\ \alpha(a) \neq \alpha(b)}}^N \left[ \left( q_{\alpha(a)} - \frac{i\kappa}{2} [n_{\alpha(a)} + 1 - 2r(a)] \right) \right. \\ &\left. - \left( q_{\alpha(b)} - \frac{i\kappa}{2} [n_{\alpha(b)} + 1 - 2r(b)] \right) \right] \\ &= C_{\mathbf{q}, \mathbf{n}}^{(M)} \frac{N!}{n_1! n_2! \dots n_M!} \prod_{\alpha < \beta}^M \prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} \left[ \left( q_\alpha - \frac{i\kappa}{2} n_\alpha \right) \right. \\ &\left. - \left( q_\beta - \frac{i\kappa}{2} n_\beta \right) + i\kappa(r - r') \right]. \end{aligned} \quad (\text{B.53})$$

Substituting the value of normalization constant (B.47) in this expression, we obtain

$$\begin{aligned} |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 &= \frac{N! \kappa^N}{\left(\prod_{\alpha=1}^M \kappa n_\alpha\right)} \\ &\times \prod_{\alpha < \beta}^M \frac{\prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r - r')|^2}{\prod_{r=1}^{n_\alpha} \prod_{r'=1}^{n_\beta} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r - r' - 1)|^2}. \end{aligned} \quad (\text{B.54})$$

This expression can be somewhat simplified. Shifting the product over  $r'$  in the denominator by 1, we obtain

$$\begin{aligned} |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 &= \frac{N! \kappa^N}{\left(\prod_{\alpha=1}^M \kappa n_\alpha\right)} \\ &\times \prod_{\alpha < \beta}^M \frac{\prod_{r=1}^{n_\alpha} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r - 1)|^2}{\prod_{r=1}^{n_\alpha} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r - n_\beta - 1)|^2}. \end{aligned} \quad (\text{B.55})$$

Redefining the product parameter  $r$  in the denominator as  $r \rightarrow n_x + 1 - r$  and replacing the obtained expression with its

complex conjugate, we obtain

$$\begin{aligned} |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 &= \frac{N! \kappa^N}{\left(\prod_{\alpha=1}^M \kappa n_\alpha\right)} \\ &\times \prod_{\alpha < \beta}^M \frac{\prod_{r=1}^{n_\alpha} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa(r - 1)|^2}{\prod_{r=1}^{n_\alpha} |(q_\alpha - (i\kappa/2)n_\alpha) - (q_\beta - (i\kappa/2)n_\beta) + i\kappa r|^2}. \end{aligned} \quad (\text{B.56})$$

Shifting the product over  $r$  in the numerator by 1, we finally obtain

$$|\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 = \frac{N! \kappa^N}{\left(\prod_{\alpha=1}^M \kappa n_\alpha\right)} \prod_{\alpha < \beta}^M \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2}. \quad (\text{B.57})$$

For  $M = 1$ , according to Eqns (B.1) and (B.11),

$$|\Psi_q^{(1)}(\mathbf{0})|^2 = \frac{\kappa^N N!}{\kappa N}. \quad (\text{B.58})$$

Because the function  $f(\mathbf{q}, \mathbf{n}) = |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 \exp[-E_M(\mathbf{q}, \mathbf{n})L]$  in Eqn (B.52) is symmetric under permutations of all its  $M$  pairs of arguments  $(q_x, n_x)$ , the integrations over the  $M$  momenta  $q_x$  can be extended outside the sector defined in Eqn (B.45), to the whole space  $R_M$ . As a consequence, there is no need to distinguish equal and different values of the  $n_x$  any more, and the only remaining constraint is the one in Eqn (B.17). We note that this kind of simplification holds only for the wave function  $\Psi(\mathbf{0}; t)$  in (B.52), and *not* for the general form of  $\Psi(\mathbf{x}; t)$  in (B.50), which contains  $N$  arbitrary coordinates  $x_1, \dots, x_N$ . Thus, instead of Eqn (B.52), we obtain

$$\begin{aligned} Z(N, L) &= \sum_{M=1}^N \frac{1}{M!} \left[ \prod_{\alpha=1}^M \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right] \delta\left(\sum_{\alpha=1}^M n_\alpha, N\right) \\ &\times |\Psi_{\mathbf{q}, \mathbf{n}}^{(M)}(\mathbf{0})|^2 \exp[-E_M(\mathbf{q}, \mathbf{n})L]. \end{aligned} \quad (\text{B.59})$$

With Eqns (B.28), (B.57), and (B.58), we obtain the following sufficiently compact representation for the replica partition function:

$$\begin{aligned} Z(N, L) &= N! \kappa^N \left\{ \int_{-\infty}^{+\infty} \frac{dq}{2\pi \kappa N} \exp\left[-\frac{L}{2\beta} N q^2 + \frac{\kappa^2 L}{24\beta} (N^3 - N)\right] \right. \\ &+ \sum_{M=2}^N \frac{1}{M!} \left[ \prod_{\alpha=1}^M \sum_{n_\alpha=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi \kappa n_\alpha} \right] \delta\left(\sum_{\alpha=1}^M n_\alpha, N\right) \\ &\times \prod_{\alpha < \beta}^M \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2} \\ &\times \exp\left[-\frac{L}{2\beta} \sum_{\alpha=1}^M n_\alpha q_\alpha^2 + \frac{\kappa^2 L}{24\beta} \sum_{\alpha=1}^M (n_\alpha^3 - n_\alpha)\right] \left. \right\}. \end{aligned} \quad (\text{B.60})$$

The first term in the above expression is the contribution of the ground state ( $M = 1$ ) and the other terms ( $M \geq 2$ ) are the contributions of excited states. After simple algebra, the above replica partition function can be represented as

$$Z(N, L) = \exp(-\beta N L f_0) \tilde{Z}(N, \lambda), \quad (\text{B.61})$$

where

$$f_0 = \frac{1}{24} \beta^4 u^2 - \frac{1}{\beta L} \ln(\beta^3 u),$$

and

$$\begin{aligned} \tilde{Z}(N, L) = N! & \left\{ \int_{-\infty}^{+\infty} \frac{dq}{2\pi\kappa N} \exp \left[ -\frac{L}{2\beta} Nq^2 + \frac{\kappa^2 L}{24\beta} N^3 \right] \right. \\ & + \sum_{M=2}^N \frac{1}{M!} \left[ \prod_{\alpha=1}^M \sum_{n_\alpha=1}^{\infty} \int_{-\infty}^{+\infty} \frac{dq_\alpha}{2\pi\kappa n_\alpha} \right] \delta \left( \sum_{\alpha=1}^M n_\alpha, N \right) \\ & \times \prod_{\alpha < \beta} \frac{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha - n_\beta)|^2}{|q_\alpha - q_\beta - (i\kappa/2)(n_\alpha + n_\beta)|^2} \\ & \left. \times \exp \left[ -\frac{L}{2\beta} \sum_{\alpha=1}^M n_\alpha q_\alpha^2 + \frac{\kappa^2 L}{24\beta} \sum_{\alpha=1}^M n_\alpha^3 \right] \right\}. \end{aligned} \quad (\text{B.62})$$

### Appendix C Integral relations for the Airy function

The Airy function  $\text{Ai}(x)$  is the solution of the differential equation

$$y''(x) = xy(x) \quad (\text{C.1})$$

with the boundary condition  $y(x \rightarrow +\infty) = 0$ . As  $x \rightarrow +\infty$ , this function tends to zero exponentially fast:

$$\text{Ai}(x \rightarrow +\infty) \simeq \frac{1}{2\sqrt{\pi} x^{1/4}} \exp \left( -\frac{2}{3} x^{3/2} \right), \quad (\text{C.2})$$

and as  $x \rightarrow -\infty$ , it oscillates and tends to zero in accordance with a power law:

$$\text{Ai}(x \rightarrow -\infty) \simeq \frac{1}{\sqrt{\pi} |x|^{1/4}} \sin \left( \frac{2}{3} |x|^{3/2} + \frac{1}{4} \pi \right). \quad (\text{C.3})$$

The Airy function can also be represented in the integral form

$$\text{Ai}(x) = \int_{\mathcal{C}} \frac{dz}{2\pi i} \exp \left( \frac{1}{3} z^3 - zx \right), \quad (\text{C.4})$$

where the integration contour in the complex plane starts at a point at infinity with the argument in the sector  $-\pi/2 < \theta_{(-)} < -\pi/3$  and ends at infinity with the argument in the sector  $\pi/3 < \theta_{(+)} < \pi/2$ . Choosing  $\theta_{(-)} = -\pi/2 + \epsilon$  and  $\theta_{(+)} = \pi/2 - \epsilon$ , where the positive parameter  $\epsilon \rightarrow 0$  is introduced just to ensure the convergence of the integration, we can draw the integration contour in Eqn (C.4) along the imaginary axes  $z = iy$ .

Similarly to the Hubbard–Stratonovich transformation that involves a Gaussian function to linearize quadratic expressions in the exponential,

$$\exp \left( \frac{1}{2} F^2 \right) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} x^2 \right) \exp(Fx), \quad (\text{C.5})$$

the Airy function can be used to linearize *cubic* exponential terms:

$$\exp \left( \frac{1}{3} F^3 \right) = \int_{-\infty}^{+\infty} dx \text{Ai}(x) \exp(Fx), \quad (\text{C.6})$$

where  $F$  is assumed to be a nonnegative quantity. This relation can be easily proved using integral representation (C.4), in which the integration path coincides with the imaginary axis  $z = iy$  and the quantity  $F$  is first taken to be purely imaginary,  $F \rightarrow iF$ . The integration over  $x$  then results in the factor  $\delta(F - y)$ . Further trivial integration over  $y$  yields the result  $\exp(-iF^3/3)$ . Performing the analytic continuation in  $F$  back to real values  $F \rightarrow -iF$ , we arrive at the left-hand side of (C.6).

We prove two other integral relations with the Airy function:

$$\begin{aligned} I_1 & \equiv \int_{-\infty}^{+\infty} dp \text{Ai}(p^2 + \omega_1 + \omega_2) \exp[ip(\omega_1 - \omega_2)] = \\ & = 2^{2/3} \pi \text{Ai}(2^{1/3} \omega_1) \text{Ai}(2^{1/3} \omega_2), \end{aligned} \quad (\text{C.7})$$

$$\begin{aligned} I_2 & \equiv \int_0^{\infty} dy \text{Ai}(y + \omega_1) \text{Ai}(y + \omega_2) \\ & = \frac{\text{Ai}(\omega_1) \text{Ai}'(\omega_2) - \text{Ai}'(\omega_1) \text{Ai}(\omega_2)}{\omega_1 - \omega_2}. \end{aligned} \quad (\text{C.8})$$

Using the integral representation of the Airy function in Eqn (C.4), we obtain

$$\begin{aligned} I_1 & = \int_{-\infty}^{+\infty} dp \\ & \times \int_{\mathcal{C}} \frac{dz}{2\pi i} \exp \left( \frac{1}{3} z^3 - p^2 z - \omega_1 z - \omega_2 z + ip\omega_1 - ip\omega_2 \right). \end{aligned} \quad (\text{C.9})$$

With  $z \equiv z_1$  and  $ip \equiv z_2$ , this can be represented as

$$\begin{aligned} I_1 & = 2\pi \iint_{\mathcal{C}} \frac{dz_1 dz_2}{(2\pi i)^2} \\ & \times \exp \left( \frac{1}{3} z_1^3 + z_1 z_2^2 - \omega_1(z_1 - z_2) - \omega_2(z_1 + z_2) \right), \end{aligned} \quad (\text{C.10})$$

where the integration contour  $\mathcal{C}$  coincides with the imaginary axis. Instead of  $z_1$  and  $z_2$ , we introduce new integration variables

$$\begin{aligned} z_1 - z_2 & = \xi, \\ z_1 + z_2 & = \eta, \end{aligned} \quad (\text{C.11})$$

which yields

$$I_1 = \pi \int_{\mathcal{C}} \frac{d\xi}{2\pi i} \int_{\mathcal{C}} \frac{d\eta}{2\pi i} \exp \left( \frac{1}{6} \xi^3 + \frac{1}{6} \eta^3 - \omega_1 \xi - \omega_2 \eta \right). \quad (\text{C.12})$$

Redefining  $\xi \rightarrow 2^{1/3} \xi$  and  $\eta \rightarrow 2^{1/3} \eta$ , we obtain

$$\begin{aligned} I_1 & = 2^{2/3} \pi \int_{\mathcal{C}} \frac{d\xi}{2\pi i} \exp \left( \frac{1}{3} \xi^3 - 2^{1/3} \omega_1 \xi \right) \\ & \times \int_{\mathcal{C}} \frac{d\eta}{2\pi i} \exp \left( \frac{1}{3} \eta^3 - 2^{1/3} \omega_2 \eta \right) \\ & = 2^{2/3} \pi \text{Ai}(2^{1/3} \omega_1) \text{Ai}(2^{1/3} \omega_2), \end{aligned} \quad (\text{C.13})$$

which proves relation (C.7).

To prove relation (C.8), it suffices to take into account that the Airy function satisfies differential equation (C.1).

Substituting

$$\text{Ai}(y + \omega) = \frac{1}{y + \omega} \text{Ai}''(y + \omega) \tag{C.14}$$

in the left-hand side of Eqn (C.8), we find

$$\begin{aligned} I_2 &= \int_0^\infty dy \frac{1}{(y + \omega_1)(y + \omega_2)} \text{Ai}''(y + \omega_1) \text{Ai}''(y + \omega_2) \\ &= \frac{1}{\omega_1 - \omega_2} \int_0^\infty dy \left[ \frac{1}{y + \omega_2} - \frac{1}{y + \omega_1} \right] \\ &\quad \times \text{Ai}''(y + \omega_1) \text{Ai}''(y + \omega_2) \\ &= \frac{1}{\omega_1 - \omega_2} \int_0^\infty dy \left[ \frac{\text{Ai}''(y + \omega_2)}{y + \omega_2} \text{Ai}''(y + \omega_1) \right. \\ &\quad \left. - \frac{\text{Ai}''(y + \omega_1)}{y + \omega_1} \text{Ai}''(y + \omega_2) \right]. \end{aligned} \tag{C.15}$$

Substituting  $\text{Ai}''(y + \omega_2) = (y + \omega_2) \text{Ai}(y + \omega_2)$  in the first term and  $\text{Ai}''(y + \omega_1) = (y + \omega_1) \text{Ai}(y + \omega_1)$  in the second, we obtain

$$\begin{aligned} I_2 &= \frac{1}{\omega_1 - \omega_2} \left[ \int_0^\infty dy \text{Ai}''(y + \omega_1) \text{Ai}(y + \omega_2) \right. \\ &\quad \left. - \int_0^\infty dy \text{Ai}''(y + \omega_2) \text{Ai}(y + \omega_1) \right]. \end{aligned} \tag{C.16}$$

Simple integration by parts eventually yields

$$I_2 = \frac{1}{\omega_1 - \omega_2} [-\text{Ai}'(\omega_1) \text{Ai}(\omega_2) + \text{Ai}'(\omega_2) \text{Ai}(\omega_1)], \tag{C.17}$$

which proves relation (C.8).

### Appendix D Fredholm determinant with the Airy kernel and the Tracy–Widom distribution

In somewhat simplified terms, the Fredholm determinant  $\det(1 - \hat{K})$  can be defined as follows (see, e.g., [46] for a rigorous mathematical definition):

$$\begin{aligned} \det(1 - \hat{K}) &= 1 + \sum_{n=1}^\infty \frac{(-1)^n}{n!} \iint \dots \int_a^b dt_1 dt_2 \dots dt_n \det [K(t_i, t_j)]_{(i,j)=1,\dots,n}, \end{aligned} \tag{D.1}$$

where the *kernel*  $\hat{K} \equiv K(t, t')$  is a function of two variables defined in the region  $a \leq (t, t') \leq b$ . On the other hand, the Fredholm determinant can also be represented in the exponential form

$$\det(1 - \hat{K}) = \exp \left[ - \sum_{n=1}^\infty \frac{1}{n} \text{Tr} \hat{K}^n \right], \tag{D.2}$$

where

$$\text{Tr} \hat{K}^n \equiv \iint \dots \int_a^b dt_1 dt_2 \dots dt_n K(t_1, t_2) K(t_2, t_3) \dots K(t_n, t_1). \tag{D.3}$$

In this appendix, we reproduce the formal derivation of Tracy and Widom [1] demonstrating that the function  $F_2(s)$  defined as the Fredholm determinant with the Airy kernel can

be expressed in terms of the solution of the Painlevé II differential equation, namely

$$F_2(s) \equiv \det [1 - \hat{K}_A] = \exp \left[ - \int_s^\infty dt (t - s) q^2(t) \right], \tag{D.4}$$

where  $\hat{K}_A$  is the *Airy kernel* defined on the semi-infinite line  $[s, \infty)$

$$K_A(t_1, t_2) = \frac{\text{Ai}(t_1) \text{Ai}'(t_2) - \text{Ai}'(t_1) \text{Ai}(t_2)}{t_1 - t_2} \tag{D.5}$$

and the function  $q(t)$  is the solution of the Painlevé II differential equation

$$q'' = tq + 2q^3 \tag{D.6}$$

with the boundary condition  $q(t \rightarrow +\infty) \sim \text{Ai}(t)$ .

We introduce a new function  $R(t)$  such that

$$F_2(s) = \exp \left[ - \int_s^\infty dt R(t) \right] \tag{D.7}$$

or, according to definition (D.4),

$$R(s) = \frac{d}{ds} \ln [\det(1 - \hat{K}_A)]. \tag{D.8}$$

Here, the logarithm of the determinant can be expressed in terms of the trace:

$$\begin{aligned} \ln [\det(1 - \hat{K}_A)] &= - \sum_{n=1}^\infty \frac{1}{n} \text{Tr} \hat{K}_A^n \\ &\equiv - \sum_{n=1}^\infty \frac{1}{n} \int_s^\infty dt_1 \int_s^\infty dt_2 \\ &\quad \dots \int_s^\infty dt_n K_A(t_1, t_2) K_A(t_2, t_3) \dots K_A(t_n, t_1). \end{aligned} \tag{D.9}$$

Taking the derivative of this expression over  $s$ , we obtain

$$\begin{aligned} R(s) &= - \int_s^\infty dt (1 - \hat{K}_A)^{-1}(s, t) K_A(t, s) \\ &\equiv -K_A(s, s) - \sum_{n=2}^\infty \int_s^\infty dt_1 \int_s^\infty dt_2 \\ &\quad \dots \int_s^\infty dt_{n-1} K_A(s, t_1) K_A(t_1, t_2) \dots K_A(t_{n-1}, s). \end{aligned} \tag{D.10}$$

Substituting the integral representation of the Airy kernel, Eqn (D.5),

$$K_A(t_1, t_2) = \int_0^\infty dz \text{Ai}(t_1 + z) \text{Ai}(t_2 + z), \tag{D.11}$$

after some simple algebraic transformations, we obtain

$$R(s) = \int_s^\infty dt_1 \int_s^\infty dt_2 \text{Ai}(t_1) (1 - \hat{K}_A)^{-1}(t_1, t_2) \text{Ai}(t_2). \tag{D.12}$$

Taking the derivative of this expression with respect to  $s$  and applying the necessary algebraic transformations, we obtain

$$\frac{d}{ds} R(s) = -q^2(s), \tag{D.13}$$

where

$$q(s) = \int_s^\infty dt (1 - \hat{K}_A)^{-1}(s, t) \text{Ai}(t). \tag{D.14}$$

It follows from Eqn (D.13) that

$$R(s) = \int_s^\infty dt q^2(t). \tag{D.15}$$

We introduce two more auxiliary functions,

$$v(s) = \int_s^\infty dt_1 \int_s^\infty dt_2 \text{Ai}(t_1) (1 - \hat{K}_A)^{-1}(t_1, t_2) \text{Ai}'(t_2), \tag{D.16}$$

$$p(s) = \int_s^\infty dt (1 - \hat{K}_A)^{-1}(s, t) \text{Ai}'(t). \tag{D.17}$$

Taking derivatives of the above three functions  $q(s)$ ,  $v(s)$ , and  $p(s)$ , Eqns (D.14), (D.16), and (D.17), after straightforward but cumbersome algebraic transformations, it can be shown that the derivatives of these functions satisfy the following three relations:

$$q' = p - Rq, \tag{D.18}$$

$$p' = sq - pR - 2qv, \tag{D.19}$$

$$v' = -pq. \tag{D.20}$$

Taking the derivative of  $R^2 - 2v$  and using Eqns (D.13) and (D.20), it is easy to see that

$$\frac{d}{ds}(R^2 - 2v) = 2q(p - Rq). \tag{D.21}$$

On the other hand, multiplying both sides of (D.18) by  $2q$  yields

$$\frac{d}{ds} q^2 = 2q(p - Rq). \tag{D.22}$$

Comparing Eqns (D.21) and (D.22) and taking into account that all the above functions tend to zero as  $s \rightarrow \infty$ , we obtain the simple relation

$$R^2 - 2v = q^2. \tag{D.23}$$

Finally, taking the derivative of Eqn (D.18) and using Eqns (D.13), (D.18), (D.19), and (D.23), it is easy to show that the second derivative of  $q(s)$  satisfies the equation

$$q'' = 2q^3 + sq, \tag{D.24}$$

which is a special case of the Painlevé II differential equation [37, 38, 53]. Thus, substituting Eqn (D.15) in Eqn (D.7), we obtain Eqn (D.4) with  $q(s)$  being a solution of differential equation (D.24). In the limit  $s \rightarrow \infty$ , the function  $q(s)$ , according to its definition (D.14), tends to zero, and Eqn (D.24) then turns into the Airy function equation,  $q'' = sq$ . In this limit, therefore,

$$q(s \rightarrow +\infty) \simeq \text{Ai}(s) \sim \exp\left[-\frac{2}{3}s^{3/2}\right]. \tag{D.25}$$

On the other hand, it can be proved [54] that in the opposite limit  $s \rightarrow -\infty$ , the asymptotic form of the solution of the Painlevé equation (D.24) [whose right-tail asymptotic form is

given by the Airy function, Eqn (D.25)] is

$$q(s \rightarrow -\infty) \simeq \sqrt{-\frac{1}{2}s}. \tag{D.26}$$

The Tracy–Widom distribution function  $P_{\text{TW}}(t)$  is related to the above function  $F_2(s)$  as follows. By definition, the function  $F_2(s)$  gives the probability that a random quantity  $t$  described by the probability distribution functions  $P_{\text{TW}}(t)$  has a value less than a given parameter  $s$ :

$$F_2(s) = \int_{-\infty}^s dt P_{\text{TW}}(t). \tag{D.27}$$

Taking the derivative of this relation and using (D.4), we find

$$P_{\text{TW}}(s) = \exp\left[-\int_s^\infty dt (t-s) q^2(t)\right] \int_s^\infty dt q^2(t), \tag{D.28}$$

where the function  $q(s)$  is a solution of differential equation (D.24).

Substituting the two explicit asymptotic forms of  $q(s)$  as  $s \rightarrow +\infty$  and  $s \rightarrow -\infty$  in Eqn (D.28), we can estimate the asymptotic behavior for the right and the left tails of the TW probability distribution function  $P_{\text{TW}}(s)$ :

$$P_{\text{TW}}(s \rightarrow +\infty) \sim \exp\left[-\frac{4}{3}s^{3/2}\right], \tag{D.29}$$

$$P_{\text{TW}}(s \rightarrow -\infty) \sim \exp\left[-\frac{1}{12}|s|^3\right]. \tag{D.30}$$

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