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Multipole expansions in magnetostatics

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Contents

Abstract. **Multipole expansions of the magnetic field of a spatially restricted system of stationary currents and those for the potential function of such currents in an external magnetic field are studied using angular momentum algebraic techniques. It is found that the expansion for the magnetic induction vector is made identical to that for the electric field strength of a neutral system of charges by substituting electric for magnetic multipole moments. The toroidal part of the multipole expansion for the magnetic field vector potential can, due to its potential nature, be omitted in the static case. Also, the potential function of a system of currents in an external magnetic field and the potential energy of a neutral system of charges in an external electric field have identical multipole expansions. For axisymmetric systems, the expressions for the field and those for the potential energy of electric and magnetic multipoles are reduced to simple forms, with symmetry axis orientation dependence separated out.**

1. Introduction

A simple way of obtaining the multipole expansion of the electrostatic field potential φ of a system of charges distributed over a finite spatial domain with a volume density $\rho(\mathbf{r})$ is well known (see, for instance, Refs [1, 2]). To do this requires substituting into the expression for the field potential

$$\varphi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')\,\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

the expansion of the kernel of integral operator (1) in a series of Legendre polynomials $P_l(x)$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{l=0}^{\infty} \frac{r'^{\,l}}{r^{l+1}}\, P_l(\mathbf{n}\,\mathbf{n}')\,, \tag{2}$$

where $\mathbf{r} = r\mathbf{n}$, $\mathbf{r}' = r'\mathbf{n}'$, $|\mathbf{n}| = |\mathbf{n}'| = 1$, and employing the summation theorem for spherical functions $Y_{lm}(\mathbf{n})$:

$$P_l(\mathbf{n}\,\mathbf{n}') = 4\pi \sum_m \frac{1}{2l+1}\, Y^*_{lm}(\mathbf{n}')\, Y_{lm}(\mathbf{n})\,. \tag{3}$$

Expansion (2) is valid for $r' < r$ (otherwise the positions of r' and r should be interchanged on the right-hand side), so that sufficiently far away from the field sources, where this condition is already fulfilled, we obtain as a result of this substitution the expansion of the potential (1) in a multipole series (the multipole expansion):

$$\varphi(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}}\, Q_{lm}\, \frac{Y_{lm}(\mathbf{n})}{r^{l+1}} \equiv \sum_{l=0}^{\infty} \varphi_l(\mathbf{r})\,. \tag{4}$$

Here, the electric 2^l-pole moment of the system of charges Q_{lm} is defined in the following way:

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}} \int r^l \rho(\mathbf{r})\, Y^*_{lm}(\mathbf{n})\,\mathrm{d}\mathbf{r}\,. \tag{5}$$

For a given l, the set of $(2l+1)$ quantities Q^*_{lm}, which are transformed as spherical functions Y_{lm} under rotations of the coordinate system, makes up, according to the conventional

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terminology [3, 4], an irreducible l-order tensor, with the scalar Q_{00} coinciding with the charge of the system, Q^*_{1m} coinciding with the spherical components of its dipole moment

$$\mathbf{d} = \int \mathbf{r}\rho(\mathbf{r})\,\mathrm{d}\mathbf{r}\,, \tag{6}$$

and Q^*_{2m} being expressed in terms of the tensor components of the system's quadrupole moment [1].

The first terms in multipole expansion (4) (the field of a point charge, the fields of a dipole and a quadrupole) are easy to obtain without using expansion (2), which is based on the notion of a generating function for Legendre polynomials. All one has to do is to expand $|\mathbf{r}-\mathbf{r}'|^{-1}$ in a Taylor series in terms of the components of vector $\mathbf{r}'$ and substitute the result in the integrand of expression (1) [1, 2]. In higher orders in r'/r, however, it is difficult to separate out irreducible tensors (5) in expansion coefficients in this way. We are also reminded that the multipole expansion of the potential is especially useful away from the system of charges constituting the field sources, where series (4) converges rapidly and the field of the system is primarily determined by its first nonvanishing term.

In magnetostatics, it is usual practice to restrict oneself to the first $\sim r^{-2}$ term of the expansion of the vector potential in considering a field at large distances from the system of currents — that is, to the field of the magnetic dipole, which is expressed in terms of the magnetic (magnetic dipole) moment of the system (see, for instance, Refs [1, 2]). And although it is implied that the multipole expansion, similar to expansion (4), of the vector potential of the magnetic field does exist, to our knowledge it has not been adduced in the literature. To avoid misunderstanding, we emphasize that the expansion of the vector potential given, for instance, in Ref. [5] may not be termed a multipole. It is obtained simply by substituting the expansion of $|\mathbf{r}-\mathbf{r}'|^{-1}$ in a Taylor series in terms of the components of $\mathbf{r}'$ into the expression for the vector potential of a magnetic field, which differs from expression (1) in that ρ is replaced by $\mathbf{j}/c$, where $\mathbf{j}$ is the current density, and c is the speed of light in vacuum. This gives rise to a series for the vector potential whose coefficients (termed the magnetic multipole moments by the authors of book [5]) are not irreducible tensors. Furthermore, as will be evident from our subsequent discussion, this series contains an infinite number of terms (they are expressed in terms of the so-called toroidal moments of the system) which are potential and may therefore be omitted.

The aim of the present work is to show that multipole expansions in magnetostatics do exist, i.e., the magnetic field of a system of currents, as well as the potential energy of this system in an external magnetic field, may be represented in the form of series whose coefficients are irreducible tensors dependent on the distribution of the currents. The well-known results of electrostatics would thereby be generalized to the case of magnetostatics.

The author was impelled to write these notes upon attentively reading review [6], which analyzes at length the multipole expansion of field potentials in the radiation theory and introduces toroidal multipole moments of a system, which play, along with electric and magnetic multipole moments, an important part in higher orders of the long-wavelength approximation. In the limiting case of zero radiation frequency, it is possible to obtain from formulas in Ref. [6] the multipole expansion of the vector potential of the magnetic field of stationary currents expressed in terms of spherical vectors. In doing so, the toroidal part of the series resulting in the static case turns out to be potential (see Section 2.2 below) and does not make a contribution to the multipole expansion of magnetic induction vector $\mathbf{B}$. The latter expansion may be derived by a similar passage to the limit from the expression for $\mathbf{B}$ adduced in the well-known book by M Rose [7], although in the application of his results to the radiation theory Rose, owing to certain inaccuracies, lost the contribution from toroidal moments, as rightly noted in Ref. [6]. Our further analysis showed that the multipole expansion in magnetostatics may be derived in a simpler way — by direct generalization of the method employed in electrostatics (see Section 2.1). In this case, advantage is taken of the mathematical apparatus of the algebra of angular momentum, and in particular of irreducible tensors, which is widely applied in atomic and nuclear physics [3, 4], with the fields of magnetic multipoles being automatically expressed not in terms of spherical vectors but — equivalently — in terms of tensor products of irreducible tensors, which turns out to be more convenient in some cases.

We also considered the multipole expansions of the potential energy of a system of charges or currents in an external field. The expansion of the potential energy of a system of charges in a series in terms of multipole moments is given in Ref. [1], so that it only remained for us carry this result to completion by expressing the coefficients of this series in terms of the derivatives of the potential of the external electric field. In the case of the potential energy of a system of currents in an external magnetic field, all the work had to be done from the beginning to the end. It turned out in doing so that the resultant multipole expansion of the potential function of currents is identical to the multipole expansion of the energy of an electrically neutral system of charges in an external electric field (like the multipole expansions of the electric field strength of a system of charges and of the vector $\mathbf{B}$ of a system of currents) and is obtained from it by replacing the electric multipole moments with the magnetic ones and replacing the electric field strength with the magnetic induction vector $\mathbf{B}$.

Lastly, considered in Section 4 are axisymmetric systems. The apparatus of irreducible tensors turns out to be highly effective here and permits representing the resultant multipole expansions in a compact invariant form with an explicit dependence on all vector quantities which define the field of the multipole or its energy in an external field.

2. Series expansion of the field of a system of currents

2.1. Multipole series for the vector potential. Magnetic multipole moment

To find the multipole expansion of the vector potential $\mathbf{A}(\mathbf{r})$ of a system of stationary currents distributed over a finite spatial domain with density $\mathbf{j}(\mathbf{r})$, we substitute expansions (2), (3) into the solution of the vector Poisson equation written in the form of a volume potential [compare with expression (1)] [1, 2]

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{j}(\mathbf{r}')\,\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,.$$

As a result of this substitution, we obtain the following series for the vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{1}{c}\sum_{l,m} \frac{4\pi}{2l+1}\left(\int r'^{l}\,\mathbf{j}(\mathbf{r}')\,Y^*_{lm}(\mathbf{n}')\,\mathrm{d}\mathbf{r}'\right)\frac{Y_{lm}(\mathbf{n})}{r^{l+1}}$$
$$\equiv \sum_{l=0}^{\infty}\mathbf{A}_l(\mathbf{r})\,. \qquad (7)$$

In the case of constant currents, one has

$$\int \mathbf{j}(\mathbf{r})\,\mathrm{d}\mathbf{r} = 0\,; \qquad (8)$$

this result, in view of the current stationarity condition and zero normal component of vector $\mathbf{j}$ on the surface limiting the domain where $\mathbf{j} \neq 0$, namely

$$\operatorname{div}\mathbf{j} = 0\,, \qquad j_n\Big|_S = 0\,, \qquad (9)$$

is easily verified by substituting the vector $(\mathbf{ar})\,\mathbf{j}$, $\mathbf{a} = \text{const}$, into the integral identity of the Gauss–Ostrogradsky theorem. Expansion (7) therefore begins with the $l = 1$ term, i.e., with the field of the magnetic dipole. The main problem consists in the fact that the coefficients (integral expressions) of series (7) defined by the distribution of currents are not irreducible tensors.

To separate out irreducible tensors in series (7), which are defined by the current distribution of the system, i.e., to transform this expansion of the vector potential to the multipole expansion, we take advantage of the apparatus of angular momentum algebra. In what follows we shall use several conventional definitions and designations utilized in this apparatus [3], which are given here for the convenience of the reader. The spherical components of arbitrary vector $\mathbf{a}$, which form an irreducible tensor of rank one, are expressed in terms of its Cartesian components in the following way:

$$a_0 = a_z\,, \qquad a_{\pm 1} = \mp\frac{1}{\sqrt{2}}(a_x \pm \mathrm{i}a_y)\,.$$

Irreducible tensors (tensor products of irreducible tensors) of ranks $L = |l - l'|, |l - l'| + 1, \ldots, l + l'$ are formed with the use of Clebsch–Gordan coefficients $C^{LM}_{lml'm'}$ from two irreducible tensors A_{lm} and $B_{l'm'}$ of ranks l and l', respectively, according to the rule

$$\{A_l \otimes B_{l'}\}_{LM} = \sum_{m,m'} C^{LM}_{lml'm'} A_{lm} B_{l'm'}\,. \qquad (10)$$

From two tensors it is possible at $l = l'$ to form a scalar (an irreducible tensor of rank zero) proportional to the scalar product of the two tensors, which will be designated by parentheses:

$$(A_l\,B_l) = \sum_m (-1)^m A_{l,-m} B_{l,m} = (-1)^l\sqrt{2l+1}\,\{A_l \otimes B_l\}_{00}\,. \qquad (11)$$

Notice also that the irreducible zero- and first-rank tensors composed of two vectors are proportional to their scalar and vector products, respectively:

$$\{\mathbf{a}\otimes\mathbf{b}\}_0 = -\frac{1}{\sqrt{3}}\,\mathbf{ab}\,, \qquad \{\mathbf{a}\otimes\mathbf{b}\}_1 = \frac{\mathrm{i}}{\sqrt{2}}\,\mathbf{a}\times\mathbf{b}\,. \qquad (12)$$

Using designation (11), we write out the expression for the spherical components of vectors $\mathbf{A}_l$, which enter in expansion (7), in the following way:

$$(A_l)_m = \frac{4\pi(-1)^l}{c\sqrt{2l+1}}\,\frac{1}{r^{l+1}}\int r'^{l} j_m\{Y_l(\mathbf{n}') \otimes Y_l(\mathbf{n})\}_{00}\,\mathrm{d}\mathbf{r}'\,. \qquad (13)$$

To rearrange expression (13) to the desired form, one might expand with the help of Clebsch–Gordan coefficients the direct product of irreducible tensors $j_m Y_{lm'}(\mathbf{n}')$ in terms of irreducible tensors $\{\mathbf{j}\otimes Y_l(\mathbf{n}')\}_{JM}$, and then represent the integrand of expression (13) as the sum of irreducible tensors composed of these tensors and $Y_{lm'}(\mathbf{n})$. The same result will be obtained if the coupling scheme of angular momenta in the integrand of expression (13) is changed with a standard technique [3, 4]. Then, upon calculation of the 6j-symbols emerging when changing the coupling scheme, one finds that

$$j_m\{Y_l(\mathbf{n}') \otimes Y_l(\mathbf{n})\}_{00} = \Big\{\mathbf{j}\otimes\{Y_l(\mathbf{n}')\otimes Y_l(\mathbf{n})\}_0\Big\}_{1m}$$
$$= \sum_{J=l,l\pm 1}(-1)^{1+l+J}\sqrt{\frac{2J+1}{3(2l+1)}}\Big\{\{\mathbf{j}\otimes Y_l(\mathbf{n}')\}_J \otimes Y_l(\mathbf{n})\Big\}_{1m}\,.$$

In view of this identity, we obtain the desired representation for vector $\mathbf{A}_l$ (13):

$$(A_l)_m = \frac{4\pi}{c(2l+1)\sqrt{3}}\,\frac{1}{r^{l+1}}\sum_{J=l,l\pm 1}(-1)^{J+1}\sqrt{2J+1}$$
$$\times\left\{\int r'^{l}\{\mathbf{j}\otimes Y_l(\mathbf{n}')\}_J\,\mathrm{d}\mathbf{r}' \otimes Y_l(\mathbf{n})\right\}_{1m}\,. \qquad (14)$$

The current distribution-dependent parameters which define the potential $\mathbf{A}_l(\mathbf{r})$ (14) are irreducible tensors of ranks $J = l, l \pm 1$. In particular, $\mathbf{A}_1(\mathbf{r})$ is expressed in terms of irreducible tensors

$$\sqrt{\frac{4\pi}{3}}\int r\{\mathbf{j}\otimes Y_1(\mathbf{n})\}_{Jm}\,\mathrm{d}\mathbf{r} = \int\{\mathbf{j}\otimes\mathbf{r}\}_{Jm}\,\mathrm{d}\mathbf{r}\,, \qquad (15)$$

with $J = 0, 1, 2$. Using the identity

$$\int j_{m1} r_{m_2}\,\mathrm{d}\mathbf{r} = -\int j_{m_2} r_{m_1}\,\mathrm{d}\mathbf{r}\,,$$

which is easily substantiated, in view of conditions (9), with the help of the Gauss–Ostrogradsky theorem, and the symmetry property of the Clebsch–Gordan coefficients, it is easy to verify [see definition (10)] that tensors (15) turn to zero for $J = 0$ and 2. Furthermore, taking into account the second of formulas (12), one finds that

$$\mathbf{A}_1(\mathbf{r}) = \frac{4\pi}{3cr^2}\left\{\int r'\{\mathbf{j}\otimes Y_1(\mathbf{n}')\}_1\,\mathrm{d}\mathbf{r}' \otimes Y_1(\mathbf{n})\right\}_1$$
$$= \frac{1}{cr^3}\left\{\int\{\mathbf{j}\otimes\mathbf{r}'\}_1\,\mathrm{d}\mathbf{r}' \otimes \mathbf{r}\right\}_1$$

is written in the well-known form of the potential of a magnetic dipole field [1, 2]:

$$\mathbf{A}_1(\mathbf{r}) = \frac{\boldsymbol{\mu}\times\mathbf{r}}{r^3}\,, \qquad (16)$$

where

$$\boldsymbol{\mu} = \frac{1}{2c}\int (\mathbf{r}\times\mathbf{j})\,\mathrm{d}\mathbf{r} \tag{17}$$

is the magnetic dipole moment of the currents.

The potentials of the fields of higher-order multipoles ($l > 1$) are defined, according to representation (14), by the irreducible tensors

$$\int r^l \{\mathbf{j}\otimes Y_l(\mathbf{n})\}_{Jm}\,\mathrm{d}\mathbf{r}\,,$$

where $J = l, l\pm 1$. We shall show that this tensor vanishes for $J = l+1$. This is most easily verified in the following way. Let us introduce spherical vectors [7, 8]

$$\mathbf{Y}_{Llm}(\mathbf{n}) = \sum_{m_1 m_2} C^{Lm}_{lm_1 1m_2} Y_{lm_1}(\mathbf{n})\mathbf{e}_{m_2}\,, \tag{18}$$

where the unit vectors $\mathbf{e}_m$, $m = 0, \pm 1$, which form the so-called helical basis, are expressed in terms of the unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ of the Cartesian coordinate system:

$$\mathbf{e}_0 = \mathbf{e}_z\,, \qquad \mathbf{e}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\mathbf{e}_x \pm \mathrm{i}\mathbf{e}_y)\,.$$

In view of definitions (10), (18) and considering that $j_m = \mathbf{j}\mathbf{e}_m$, the desired tensor may be written in terms of the spherical vector:

$$\int r^l \{\mathbf{j}\otimes Y_l(\mathbf{n})\}_{l+1,m}\,\mathrm{d}\mathbf{r} = \int r^l \mathbf{j}\mathbf{Y}_{l+1,l,m}(\mathbf{n})\,\mathrm{d}\mathbf{r}\,.$$

We next make use of the identity (see, for instance, Ref. [7])

$$\begin{aligned}\nabla\big(\phi(\mathbf{r})Y_{Lm}(\mathbf{n})\big) &= -\sqrt{\frac{L+1}{2L+1}}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r} - \frac{L}{r}\,\phi\right)\mathbf{Y}_{L,L+1,m}(\mathbf{n}) \\ &\quad + \sqrt{\frac{L}{2L+1}}\left(\frac{\mathrm{d}\phi}{\mathrm{d}r} + \frac{L+1}{r}\,\phi\right)\mathbf{Y}_{L,L-1,m}(\mathbf{n})\,.\end{aligned} \tag{19}$$

By putting $L = l+1$, $\phi(r) = r^{l+1}$ in it, we find that

$$r^l \mathbf{Y}_{l+1,l,m}(\mathbf{n}) = \nabla f\,, \qquad f(\mathbf{r}) = \frac{r^{l+1}}{\sqrt{(l+1)(2l+3)}}\,Y_{l+1,m}(\mathbf{n})\,.$$

After this, with the aid of the Gauss–Ostrogradsky theorem and conditions (9) obeyed by stationary currents, we arrive at the conclusion that

$$\int r^l \{\mathbf{j}\otimes Y_l(\mathbf{n})\}_{l+1,m}\,\mathrm{d}\mathbf{r} = \int \mathbf{j}\nabla f\,\mathrm{d}\mathbf{r} = \oint_S f j_n\,\mathrm{d}S = 0\,.$$

Thus, only two terms corresponding to $J = l$ and $l-1$ remain in sum (14) which defines the potential $\mathbf{A}_l$ for $l > 1$.

Let us introduce the 2^l-pole magnetic moment M_{lm}, $l = 1, 2, 3, \ldots$ of the system of currents, which is defined in such a way that

$$\begin{aligned}M^*_{lm} &= \frac{\mathrm{i}}{c}\sqrt{\frac{4\pi l}{(l+1)(2l+1)}}\int r^l \{\mathbf{j}\otimes Y_l(\mathbf{n})\}_{lm}\,\mathrm{d}\mathbf{r} \\ &= -\frac{\mathrm{i}}{c}\sqrt{\frac{4\pi l}{(l+1)(2l+1)}}\int r^l \mathbf{j}\mathbf{Y}_{llm}(\mathbf{n})\,\mathrm{d}\mathbf{r}\,,\end{aligned} \tag{20}$$

and the toroidal 2^l-pole moment T_{lm}, $l = 1, 2, 3, \ldots$:

$$\begin{aligned}T^*_{lm} &= -\frac{\sqrt{4\pi/(l+1)}}{c(2l+3)}\int r^{l+1}\{\mathbf{j}\otimes Y_{l+1}(\mathbf{n})\}_{lm}\,\mathrm{d}\mathbf{r} \\ &= -\frac{\sqrt{4\pi/(l+1)}}{c(2l+3)}\int r^{l+1}\mathbf{j}\mathbf{Y}_{l,l+1,m}(\mathbf{n})\,\mathrm{d}\mathbf{r}\,.\end{aligned} \tag{21}$$

From definitions (20), (21) it follows that

$$M^*_{lm} = (-1)^m M_{l,-m}\,, \qquad T^*_{lm} = (-1)^m T_{l,-m}$$

are irreducible lth-rank tensors, with [see expressions (12)] $M^*_{1m} = \mu_m$, where $\boldsymbol{\mu}$ is the magnetic dipole moment (17), and the vector potential $\mathbf{A}_l$ (14) for $l > 1$ is represented in the form

$$\mathbf{A}_l(\mathbf{r}) = \mathbf{A}^{\mathrm{M}}_l(\mathbf{r}) + \mathbf{A}^{\mathrm{T}}_{l-1}(\mathbf{r})\,. \tag{22}$$

Here,

$$\mathbf{A}^{\mathrm{M}}_l(\mathbf{r}) = \frac{(-1)^l \mathrm{i}}{r^{l+1}}\sqrt{\frac{4\pi(l+1)}{3l}}\{M^*_l \otimes Y_l(\mathbf{n})\}_1 \tag{23}$$

is the vector potential of the magnetic 2^l-field, and

$$\mathbf{A}^{\mathrm{T}}_{l-1}(\mathbf{r}) = \frac{(-1)^{l+1}}{r^{l+1}}\sqrt{\frac{4\pi l(2l-1)}{3}}\{T^*_{l-1}\otimes Y_l(\mathbf{n})\}_1 \tag{24}$$

defines the potential of the field of the corresponding toroidal moment. Notice that expression (22) also gives the correct expression for vector $\mathbf{A}_1$ because in this case T^*_{00} (21) turns to zero together with tensor (15) at $J = 0$, the second term in expression (22) vanishes, and the first one [see expression (23)] gives the well-known expression (16) for the potential of a magnetic dipole field.

Potentials (23) and (24) are also written out in an equivalent form in terms of spherical vectors (18). In view of definitions (10), (18) and the symmetry properties of Clebsch–Gordan coefficients, one may verify by comparing the spherical components of the vectors that

$$\mathbf{A}^{\mathrm{M}}_l(\mathbf{r}) = -\frac{\mathrm{i}}{r^{l+1}}\sqrt{\frac{4\pi(l+1)}{l(2l+1)}}\sum_m M_{lm}\mathbf{Y}_{llm}(\mathbf{n})\,, \tag{25}$$

$$\mathbf{A}^{\mathrm{T}}_{l-1}(\mathbf{r}) = -\frac{\sqrt{4\pi l}}{r^{l+1}}\sum_m T_{l-1,m}\mathbf{Y}_{l-1,l,m}(\mathbf{n})\,. \tag{26}$$

It is in this form that these fields will enter into the multipole expansion, when one passes to the static limit in the expression for the vector potential of the field of a radiating system [6], i.e., to the zero radiation frequency. In this case, expression (20) for the magnetic multipole moment, written in terms of the spherical vector, coincides with the definition of the magnetic multipole moment given in Ref. [6], upon correction of an evident misprint in the multiplier. The expression for the toroidal multipole moment in Ref. [6] consists of two terms and, on the face of it, does not coincide with expression (21). However, by invoking the technique of angular momentum algebra and the properties of spherical functions, it is possible to show that these expressions are equivalent under current stationarity conditions (9). Here, we do not adduce the corresponding substantiation, because in Section 2.2 we shall show that the field (24), (26), being potential, does not make a contribution to the magnetic induction vector and, accordingly, the toroidal part of the vector potential in the static case may be omitted from the multipole expansion.

So, our resultant multipole expansion of the vector potential has the following form

$$\mathbf{A}(\mathbf{r}) = \sum_{l=1}^{\infty} \mathbf{A}_l(\mathbf{r}) \,,$$

where vectors $\mathbf{A}_l$ are defined by expressions (22)–(24) or by their equivalent expressions (22), (25), and (26).

2.2 Magnetic induction vector of a magnetic multipole

To find the multipole expansion of magnetic induction vector $\mathbf{B} = \operatorname{rot} \mathbf{A}$, we calculate the rotor of either of the two terms which form the vector $\mathbf{A}_l$ (22).

First of all, let us show that

$$\operatorname{rot} \mathbf{A}^{\mathrm{T}}_{l-1}(\mathbf{r}) = 0 \,. \tag{27}$$

This is most easily verified (see also Appendix 1) proceeding from the representation of $\mathbf{A}^{T}_{l-1}$ in the form (26) and using identity (19). We substitute $L = l - 1$ and $\phi(r) = r^{-l}$ into this identity to find that

$$\frac{1}{r^{l+1}} \, \mathbf{Y}_{l-1,l,m}(\mathbf{n}) = \frac{1}{\sqrt{l(2l-1)}} \, \nabla\left(\frac{1}{r^l} \, Y_{l-1,m}(\mathbf{n})\right),$$

whence there follows expression (27). Therefore, toroidal moments and the corresponding fields of toroidal multipoles do not make a contribution to the static magnetic field. This result is discussed in Section 2.3 from an informal point of view.

To calculate the rotor of vector $\mathbf{A}^{\mathrm{M}}_l$ written in the form (25), it is possible to make use of the well-known representation (see, for instance, Refs [7, 8]) of a spherical vector

$$\mathbf{Y}_{llm}(\mathbf{n}) = -\frac{\mathrm{i}\,\mathbf{r} \times \boldsymbol{\nabla}\, Y_{lm}(\mathbf{n})}{\sqrt{l(l+1)}} \,. \tag{28}$$

Then, by applying identity (19) and considering that the function $Y_{lm}(\mathbf{r})/r^{l+1}$ obeys the Laplace equation, it is easily shown (see also Appendix 1) that

$$\operatorname{rot}\left(\frac{1}{r^{l+1}} \, \mathbf{Y}_{llm}(\mathbf{n})\right) = -\mathrm{i}\, \frac{\sqrt{l(2l+1)}}{r^{l+2}} \, \mathbf{Y}_{l,l+1,m}(\mathbf{n}) \,. \tag{29}$$

With the aid of expressions (29) and (25), we find the expression for the magnetic induction vector of the magnetic 2^l-field, written in terms of the spherical vectors:

$$\mathbf{B}_l(\mathbf{r}) = \operatorname{rot} \mathbf{A}^{\mathrm{M}}_l(\mathbf{r}) = -\frac{\sqrt{4\pi(l+1)}}{r^{l+2}} \sum_m M_{lm} \mathbf{Y}_{l,l+1,m}(\mathbf{n}) \,. \tag{30}$$

This result is also obtained in the correct passage to the static limit in the corresponding formulas [7].

When definitions (18), (10) and the symmetry properties of Clebsch–Gordan coefficients are taken into account, it may be shown that field $\mathbf{B}_l$ (30) is also written out in the equivalent form in terms of the irreducible tensor product of two irreducible tensors:

$$\mathbf{B}_l(\mathbf{n}) = \frac{(-1)^l}{r^{l+2}} \sqrt{\frac{4\pi(l+1)(2l+1)}{3}} \left\{ M^*_l \otimes Y_{l+1} \right\}_1 . \tag{31}$$

It is shown in Appendix 1 how expression (31) may be derived by direct calculation of the rotor of the vector potential of the magnetic multipole field, represented in the form of expression (23).

In the case of magnetic dipole field $\mathbf{B}_1$, expression (31) is easily reduced to the well-known form [1, 2]

$$\mathbf{B}_1 = \frac{3\mathbf{n}(\mathbf{n}\,\boldsymbol{\mu}) - \boldsymbol{\mu}}{r^3} \,, \tag{32}$$

where $\boldsymbol{\mu}$ is the system's magnetic moment (17). For this purpose, it suffices to substitute into expression (31) the spherical function in the form (see, for instance, Ref. [3])

$$Y_{2m}(\mathbf{n}) = \sqrt{\frac{15}{8\pi}} \{\mathbf{n} \otimes \mathbf{n}\}_{2m}$$

and $M^*_1 = \boldsymbol{\mu}$, with $\mathbf{B}_1$ written out as

$$\mathbf{B}_1 = -\frac{\sqrt{15}}{r^3} \left\{ \boldsymbol{\mu} \otimes \{\mathbf{n} \otimes \mathbf{n}\}_2 \right\}_1 ,$$

and then apply the identity

$$\left\{ \boldsymbol{\mu} \otimes \{\mathbf{n} \otimes \mathbf{n}\}_2 \right\}_1 = \sqrt{\frac{3}{5}} \left[\frac{1}{3}\, \mathbf{n}^2 \boldsymbol{\mu} - \mathbf{n}(\mathbf{n}\,\boldsymbol{\mu}) \right],$$

which is obtained by changing the coupling scheme of angular momenta in the irreducible tensor $\{\{\boldsymbol{\mu} \otimes \mathbf{n}\}_0 \otimes \mathbf{n}\}_1$:

$$\left\{ \{\boldsymbol{\mu} \otimes \mathbf{n}\}_0 \otimes \mathbf{n} \right\}_1 = \frac{1}{3} \left\{ \boldsymbol{\mu} \otimes \{\mathbf{n} \otimes \mathbf{n}\}_0 \right\}_1 + \frac{\sqrt{5}}{3} \left\{ \boldsymbol{\mu} \otimes \{\mathbf{n} \otimes \mathbf{n}\}_2 \right\}_1 ,$$

and employ relations (12).

Thus, the multipole expansion of the magnetic induction vector has the form

$$\mathbf{B} = \sum_{l=1}^{\infty} \mathbf{B}_l \,,$$

where the magnetic 2^l-field $\mathbf{B}_l$ is defined by expressions (30) or (31), and the 2^l-pole magnetic moment M_{lm} is defined by formula (20).

2.3 Magnetoelectrostatic analogies

It is well known that the expression for the field strength of an electric dipole is similar in form to expression (32) for the magnetic induction vector of a magnetic dipole and is obtained from it by replacing the magnetic moment $\boldsymbol{\mu}$ (17) with the dipole moment $\mathbf{d}$ (6) of the system of charges (see, for instance, Refs [1, 2]). We will show that this analogy persists in all the following terms of the multipole expansion.

The multipole expansion of the electrostatic field strength of a system of charges is obtained from the corresponding expansion of the field potential (4):

$$\mathbf{E} = -\nabla\varphi = \sum_{l=0}^{\infty} \mathbf{E}_l \,, \tag{33}$$

where $\mathbf{E}_l = -\nabla\varphi_l$, with

$$\varphi_l = \frac{1}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}} \sum_m Q_{lm} Y_{lm}(\mathbf{n}) \tag{34}$$

being the electric 2^l-field potential. By calculating $\nabla(Y_{lm}(\mathbf{n})/r^{l+1})$ with the aid of identity (19), we find the electric 2^l-field strength

$$\mathbf{E}_l = -\frac{\sqrt{4\pi(l+1)}}{r^{l+2}} \sum_m Q_{lm} \mathbf{Y}_{l,l+1,m}(\mathbf{n}) \,. \qquad (35)$$

For $l \geqslant 1$, $\mathbf{E}_l$ (35) is indeed obtained from expression (30) for the magnetic induction vector of the magnetic 2^l-field by way of formal replacement of the 2^l-pole magnetic moment M_{lm} (20) with the electric one Q_{lm} (5). Clearly, the vector $\mathbf{E}_l$ may also be written in terms of the irreducible tensor product of two irreducible tensors [compare with expression (31)]:

$$\mathbf{E}_l = \frac{(-1)^l}{r^{l+2}} \sqrt{\frac{4\pi(l+1)(2l+1)}{3}} \left\{ Q_l^* \otimes Y_{l+1}(\mathbf{n}) \right\}_1 . \qquad (36)$$

The first term $\mathbf{E}_0$ of multipole expansion (33) coincides with the field of a point charge Q_{00}, which is easily seen by inserting $Y_{1m}(\mathbf{n}) = \sqrt{3/(4\pi)}\, n_m$ into expression (36) for $l = 0$. The remaining part of this series is identical to the multipole expansion of vector $\mathbf{B}$ and passes into it by way of formal change $Q_{lm} \to M_{lm}$. The identity of the structures of the multipole expansions of the electric field strength $\mathbf{E}$ in electrostatics and of the vector $\mathbf{B}$ in magnetostatics is not accidental. Indeed, beyond the system of currents — the field sources, only where the multipole expansion is applicable — the magnetic field obeys the system of equations

$$\operatorname{div} \mathbf{B} = 0 \,, \qquad \operatorname{rot} \mathbf{B} = 0 \,,$$

which permit introducing the scalar field potential ψ, such that $\mathbf{B} = -\nabla\psi$, for which we obtain the Laplace equation, as in electrostatics beyond the system of charges. The solution of this equation decreasing for $r \to \infty$ may be expanded in a series in terms of spherical functions:

$$\psi(\mathbf{r}) = \sum_{l,m} a_{lm} \frac{Y_{lm}(\mathbf{n})}{r^{l+1}} \,.$$

We next apply formula (19) to obtain

$$\mathbf{B} = -\nabla\psi = -\sum_{l,m} \frac{\sqrt{(l+1)(2l+1)}}{r^{l+2}}\, a_{lm} \mathbf{Y}_{l,l+1,m}(\mathbf{n}) \,. \qquad (37)$$

The structural similarity between the series (37) and the multipole expansion (33), (35) in electrostatics is evident. In magnetostatics, only the term $\sim 1/r^2$ is missing from the expansion of $\mathbf{B}$ on the strength of condition (8), and the corresponding term $\sim 1/r$ is missing from expansion (7) of the vector potential $\mathbf{A}$, so that the multipole expansions begin with the $l = 1$ term. A comparison of the terms of series (37) with $\mathbf{B}_l$ (30) shows that

$$a_{lm} = \sqrt{\frac{4\pi}{2l+1}}\, M_{lm} \,.$$

In summary, it should be noted that the considerations outlined above provide one more informal way, which is based on electromagnetostatic analogies, of ascertaining that toroidal moments in magnetostatics do not make a contribution to the multipole expansion of the magnetic induction vector. Indeed, the term $\sim 1/r^{l+1}$ in the multipole expansion of the vector potential is defined by two irreducible tensors characterizing the distribution of currents — the field sources: by a tensor of rank l — the magnetic multipole moment — and a tensor of rank $(l-1)$ — the toroidal multipole moment [see expressions (22)–(26)]. Meanwhile, from expression (37) it follows that the term corresponding to $\sim 1/r^{l+2}$ in the multipole expansion of $\mathbf{B}$ is defined by only one tensor — the tensor a_{lm} of rank l, i.e., toroidal moments may not enter into the expansion of $\mathbf{B}$.

3. Multipole expansions of the energy of a system in an external field

In the derivation of the multipole expansion of the energy of a system, it is assumed that there are no sources of an external field (no external charges in the consideration of an electrostatic problem or, accordingly, no external currents in magnetostatics) in the spatial domain where the system is located. If the external field is, in addition, quasiuniform, which is possible only under the assumption made here, the multipole expansion represents, as is well known, a rapidly converging series, and the energy of the system is sufficiently accurately defined by its first nonvanishing term.

3.1 System of charges in an external electric field

The multipole expansion of the energy of a system of charges in an external electric field might be obtained by substituting into expression

$$U = \int \rho(\mathbf{r})\, \varphi(\mathbf{R} + \mathbf{r})\, \mathrm{d}\mathbf{r} \,, \qquad (38)$$

which defines this energy, the Taylor series expansion of the potential φ with the origin of expansion at some point $\mathbf{R}$ inside the system:

$$\varphi(\mathbf{R} + \mathbf{r}) = \sum_{l=0}^{\infty} \frac{1}{l!}\, x_{i_1} x_{i_2} \ldots x_{i_l} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_l} \varphi(\mathbf{R}) \,. \qquad (39)$$

The first terms in the multipole expansion (the energy of a point charge, the energy of a dipole, and the energy of a quadrupole) are indeed easily found by way of this substitution [1, 2]. However, separating out irreducible tensors — the electric multipole moments — becomes a difficult task in the next terms of the series under this procedure. That is why use is made of the following trick [1]. In the domain of the system location, where there are no charges — the sources of the external field — the potential φ satisfies the Laplace equation, and as the solution of this equation is regular for $r \to 0$, it may be expanded in a series in terms of spherical functions:

$$\varphi(\mathbf{R} + \mathbf{r}) = \sum_{l,m} a_{lm}\, r^l\, Y_{lm}(\mathbf{n}) \,. \qquad (40)$$

We next substitute expansion (40) in expression (38), take into account that $Y_{l,-m} = (-1)^m Y^*_{lm}$, and introduce electric 2^l-pole moments in accordance with definition (5) to represent the energy of the system in the form of a series:

$$U = \sum_{l,m} \sqrt{\frac{2l+1}{4\pi}}\, Q_{lm} (-1)^m a_{l,-m} \,. \qquad (41)$$

Expansion (41) was obtained in a book by Landau and Lifshitz [1], and it was therefore shown that the energy of a system of charges is represented in the form of a series in terms of its multipole moments. To bring the multipole expansion of the energy to its final form requires determining the coefficients of this series, which will be done below.

Let us express the coefficients a_{lm} of series (40) in terms of the derivatives of the potential φ at point $\mathbf{R}$. For this purpose, we note that this series may be considered as an equivalent representation of series (39). Writing out the spherical function $Y_{lm}(\mathbf{n})$ in terms of the irreducible tensor of rank l composed of the unit vector $\mathbf{n}$ [3], we may represent the product of r^l and Y_{lm}, which enters expression (40), as

$$r^l Y_{lm}(\mathbf{n}) = \sqrt{\frac{(2l+1)!!}{4\pi l!}} \Big\{ \ldots \{ \{ \mathbf{r} \otimes \mathbf{r} \}_2 \otimes \mathbf{r} \}_3 \ldots \otimes \mathbf{r} \Big\}_{lm} . \quad (42)$$

We also take into consideration that, as shown in Appendix 2, the action of the differential operators which appear in series (39) on the harmonic, i.e., obeying the Laplace equation, function φ may be represented in an equivalent form

$$\begin{aligned} x_{i_1} x_{i_2} \ldots x_{i_l} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_l} \varphi(\mathbf{R}) &= \Big(\Big\{ \ldots \{ \{ \mathbf{r} \otimes \mathbf{r} \}_2 \otimes \mathbf{r} \}_3 \ldots \otimes \mathbf{r} \Big\}_l \\ &\times \Big\{ \ldots \{ \{ \nabla \otimes \nabla \}_2 \otimes \nabla \}_3 \ldots \otimes \nabla \Big\}_l \Big) \varphi(\mathbf{R}) \\ &= \sum_m (-1)^m \Big\{ \ldots \{ \{ \mathbf{r} \otimes \mathbf{r} \}_2 \otimes \mathbf{r} \}_3 \ldots \otimes \mathbf{r} \Big\}_{lm} \\ &\times \Big\{ \ldots \{ \{ \nabla \otimes \nabla \}_2 \otimes \nabla \}_3 \ldots \otimes \nabla \Big\}_{l,-m} \varphi(\mathbf{R}) . \end{aligned} \quad (43)$$

Substituting next expression (42) into expansion (40), substituting expression (43) into expansion (39), and equating in the resultant expressions the coefficients of like components of the irreducible tensors composed of vector $\mathbf{r}$, one obtains the sought representation for coefficients a_{lm}:

$$(-1)^m a_{l,-m} = \sqrt{\frac{4\pi}{(2l+1)!! l!}} \Big\{ \ldots \{ \{ \nabla \otimes \nabla \}_2 \otimes \nabla \}_3 \ldots \otimes \nabla \Big\}_{lm} \varphi(\mathbf{R}) . \quad (44)$$

The multipole expansion of the energy of a system of charges in an external field is found by substituting expression (44) into series (41):

$$U = \sum_{l=0}^{\infty} U_l , \qquad U_l = \frac{1}{\sqrt{l!(2l-1)!!}} \times \Big(Q_l^* \Big\{ \ldots \{ \{ \nabla \otimes \nabla \}_2 \otimes \nabla \}_3 \ldots \otimes \nabla \Big\}_l \Big) \varphi(\mathbf{R}) . \quad (45)$$

The first term in expansion (45) (here, we must put $(2l-1)!! = 1$ at $l = 0$), namely

$$U_0 = Q_{00} \varphi(\mathbf{R}) ,$$

defines the energy of a point charge in an external field. For $l \geqslant 1$, the energy of the corresponding 2^l-field may also be written out in terms of the strength $\mathbf{E} = -\nabla\varphi$ of the external field:

$$U_{l \geqslant 1} = -\frac{1}{\sqrt{l!(2l-1)!!}} \times \Big(Q_l^* \Big\{ \{ \ldots \{ \{ \nabla \otimes \nabla \}_2 \otimes \nabla \}_3 \ldots \otimes \nabla \}_{l-1} \otimes \mathbf{E} \Big\}_l \Big) . \quad (46)$$

The second term U_1 (46) of multipole expansion (45) is the well-known expression for the energy of a dipole with a dipole moment $\mathbf{d}$ (6) in the external field [1, 2]:

$$U_1 = -(Q_1^* \mathbf{E}) = -\mathbf{dE} . \quad (47)$$

To bring the third term U_2 to the well-known form of the energy of a quadrupole in an external field, we notice that the 2^2-pole moment Q_{2m} (5) is related to the quadrupole moment tensor whose Cartesian components are [1, 2]

$$D_{ij} = \int 3(x_i x_j - r^2 \delta_{ij}) \, \rho(\mathbf{r}) \, \mathrm{d}\mathbf{r} \quad (48)$$

in the following way:

$$Q_{2m}^* = \frac{1}{\sqrt{6}} \sum_{m_1, m_2} C_{1m_1 1m_2}^{2m} D_{m_1 m_2} . \quad (49)$$

Relation (49) is most easily obtained by substituting $r^2 Y_{2m}(\mathbf{n})$ in the form (42) into the integral of expression (5), which defines the quadrupole moment Q_{2m}^*. In doing so, it should be taken into account that on the strength of invariance of the unit tensor δ_{ij} the irreducible tensor of rank two composed of its components evidently turns to zero. Tensor (48) is symmetrical and has a zero trace, so that only one nonzero irreducible tensor may be composed of it — second-rank tensor (49). One may therefore introduce auxiliary tensors

$$q_{lm} = \frac{1}{\sqrt{6}} \sum_{m_1, m_2} C_{1m_1 1m_2}^{lm} D_{m_1 m_2} , \qquad q_{2m} = Q_{2m}^* ,$$

$$q_{1m} = 0 , \quad q_{00} = 0$$

and represent U_2 (45) in the form

$$U_2 = \frac{1}{\sqrt{6}} \sum_{l,m} (-1)^{l-m} q_{l,-m} \{ \nabla \otimes \nabla \}_{lm} \, \varphi .$$

Next, using the symmetry property of the Clebsch–Gordan coefficient and the relation inverse to that given by formula (10), it is easy to bring U_2 to the standard form [1, 2]:

$$\begin{aligned} U_2 &= \frac{1}{6} \sum_{m_1, m_2} (-1)^{m_1+m_2} D_{m_1 m_2} \nabla_{-m_1} \nabla_{-m_2} \, \varphi \\ &= \frac{1}{6} D_{ik} \nabla_i \nabla_k \, \varphi = -\frac{1}{6} D_{ik} \nabla_i E_k . \end{aligned} \quad (50)$$

3.2 System of currents in an external magnetic field

The role of the potential energy of a system of currents $\mathbf{j}$ in an external magnetic field defined by the vector potential $\mathbf{A}$ is played, as is well known, by the so-called potential function of the currents [2], which is defined by the expression

$$V = -\frac{1}{c} \int \mathbf{j}(\mathbf{r}) \, \mathbf{A}(\mathbf{R} + \mathbf{r}) \, \mathrm{d}\mathbf{r} , \quad (51)$$

differing in sign from the energy of current interaction with the external field. The multipole expansion for the potential function V will be obtained in this section.

When the Taylor series expansion of the vector potential with the origin at some point $\mathbf{R}$ inside the system, namely

$$\mathbf{A}(\mathbf{R} + \mathbf{r}) = \sum_{l=0}^{\infty} \frac{1}{l!} \, x_{i_1} x_{i_2} \ldots x_{i_l} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_l} \mathbf{A}(\mathbf{R}) , \quad (52)$$

is substituted into expression (51), it is easy to find the first term, corresponding to $l = 1$, of the multipole expansion of the potential function, which is expressed in terms of the magnetic dipole moment (17) of the system of currents and the magnetic induction vector of the external magnetic field [2]:

$$V_1 = -\boldsymbol{\mu}\mathbf{B}\,. \tag{53}$$

The term corresponding to $l = 0$ in expression (52) does not make a contribution to this expansion on the strength of condition (8). It is difficult to separate out in such a direct manner the irreducible tensors — magnetic 2^l-pole moments — in the following terms of the series. That is why we generalize the method employed in the previous section for finding the multipole expansion of energy in electrostatics to the case of vector field **A**.

On imposition of the gauge condition

$$\operatorname{div}\mathbf{A} = 0\,, \tag{54}$$

the potential of the external magnetic field in the domain of currents **j**, which is devoid of the sources of the external field according to the foregoing assumption, obeys the Laplace equation. The solutions of the vector Laplace equation, regular as $r \to 0$, are the vectors

$$r^l\mathbf{Y}_{llm}(\mathbf{n})\,, \quad r^{l-1}\mathbf{Y}_{l,l-1,m}(\mathbf{n})\,, \quad \text{and} \quad r^{l+1}\mathbf{Y}_{l,l+1,m} \tag{55}$$

proportional to the spherical vectors (18). The spherical vectors $\mathbf{Y}_{l\lambda m}(\mathbf{n})$, $l = 0, 1, 2, \ldots$, $m = 0, \pm 1, \pm 2, \ldots, \pm l$, $\lambda = l, l \pm 1$ for $l \neq 0$, and $\lambda = 1$ at $l = 0$ form, as is well known [7], a complete orthonormal set on a unit sphere. That is why the solution of the vector Laplace equation, regular for $r \to 0$, may be expanded in a series in terms of vectors (55). In our case, this expansion is simplified. Indeed, it is easy to show (see also Ref. [7]) by using the definition of spherical vectors (18) and identity (19) that the divergence of the first two vectors (55) turns to zero, while

$$\operatorname{div}\left(r^{l+1}\mathbf{Y}_{l,l+1,m}(\mathbf{n})\right) = -\sqrt{\frac{l+1}{2l+1}}(2l+3)\,r^l Y_{lm}(\mathbf{n})\,.$$

Consequently, the third of vectors in Eqn (55) does not satisfy the gauge condition (54) and may not enter into the expansion of the vector potential **A**:

$$\mathbf{A}(\mathbf{R}+\mathbf{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}r^l\mathbf{Y}_{llm}(\mathbf{n}) + b_{lm}r^{l-1}\mathbf{Y}_{l,l-1,m}(\mathbf{n})\right]. \tag{56}$$

The second term under the summation symbol in expression (56) is potential and, therefore, does not make a contribution to either the magnetic field $\mathbf{B} = \operatorname{rot}\mathbf{A}$ or the potential function (51) of currents. Its potentiality may be verified by way of the direct calculation:

$$\operatorname{rot}\left(r^{l-1}\mathbf{Y}_{l,l-1,m}(\mathbf{n})\right) = 0$$

by applying the method described in Appendix 1, or on the strength of identity (19), whence it follows that

$$r^{l-1}\mathbf{Y}_{l,l-1,m}(\mathbf{n}) = \frac{\nabla\left(r^l Y_{lm}(\mathbf{n})\right)}{\sqrt{l(2l+1)}}\,.$$

After that, it is easy to show with the aid of the Gauss–Ostrogradsky theorem and conditions (9), which are obeyed by a stationary current, that the indicated terms of series (56) do not make a contribution to the potential function (51). Next, substituting expansion (56) into integral (51), introducing magnetic 2^l-pole moments defined in accordance with expression (20), and considering the identity

$$\mathbf{Y}_{llm}(\mathbf{n}) = (-1)^{1-m}\mathbf{Y}^*_{l,l,-m}(\mathbf{n})$$

obvious for spherical vectors (18), we arrive at the following series for the potential function of currents:

$$V = -\sum_{l,m}\mathrm{i}\sqrt{\frac{(l+1)(2l+1)}{4\pi l}}\,M_{lm}(-1)^m a_{l,-m}\,. \tag{57}$$

Therefore, we have shown that the potential function (51) is expanded in a series in terms of magnetic multipole moments; to bring this expansion to its final form requires defining the coefficients a_{lm} of series (57).

Let us express a_{lm}, which also are coefficients of series (56) for the vector potential, in terms of the derivatives of the components of the magnetic induction vector

$$\mathbf{B} = \operatorname{rot}\mathbf{A} = \sum_{l,m} a_{lm}\operatorname{rot}\left(r^l\mathbf{Y}_{llm}(\mathbf{n})\right). \tag{58}$$

The rotor of the first of vectors (55), which appears in expression (58), may be calculated by a standard procedure when the spherical vector is represented in the form (28), or by applying the technique described in Appendix 1. The result takes the form

$$\operatorname{rot}\left(r^l\mathbf{Y}_{llm}(\mathbf{n})\right) = \mathrm{i}\sqrt{(l+1)(2l+1)}\,r^{l-1}\,\mathbf{Y}_{l,l-1,m}(\mathbf{n})\,,$$

so that expansion of vector **B** (58) will contain only vectors of the second type in Eqn (55), which are, like the field **B** itself in the domain of currents **j**, potential and solenoidal:

$$\mathbf{B}(\mathbf{R}+\mathbf{r}) = \sum_{l,m} a_{lm}\mathrm{i}\sqrt{(l+1)(2l+1)}\,r^{l-1}\,\mathbf{Y}_{l,l-1,m}(\mathbf{n})\,. \tag{59}$$

Let us compare the expression for the spherical component B_M of vector (59) with the Taylor series expansion of this component:

$$\begin{aligned}B_M(\mathbf{R}+\mathbf{r}) &= \sum_{l=0}^{\infty}\sum_{m=-(l+1)}^{l+1} a_{l+1,m}\mathrm{i}\sqrt{(l+2)(2l+3)}\,r^l\left(Y_{l+1,l,m}(\mathbf{n})\right)_M \\ &= \sum_{l=0}^{\infty}\frac{1}{l!}\,x_{i_1}x_{i_2}\ldots x_{i_l}\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_l}B_M(\mathbf{R})\,.\end{aligned}$$

In making this comparison, one should take into account that the expression for the spherical component of a spherical vector has, in accordance with its definition (18), the form

$$\left(Y_{l+1,l,m}(\mathbf{n})\right)_M = \sum_{m_1} C^{l+1,m}_{lm_1 1,-M}(-1)^M Y_{lm_1}(\mathbf{n})\,,$$

and use representation (42) for $r^l Y_{lm_1}(\mathbf{n})$ and the following identity (A.5) for the harmonic function B_M:

$$\begin{aligned}&x_{i_1}x_{i_2}\ldots x_{i_l}\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_l}B_M(\mathbf{R}) \\ &\quad= \sum_{m_1}(-1)^{m_1}\Big\{\ldots\{\{\mathbf{r}\otimes\mathbf{r}\}_2\otimes\mathbf{r}\}_3\ldots\otimes\mathbf{r}\Big\}_{lm_1} \\ &\quad\times\Big\{\ldots\{\{\nabla\otimes\nabla\}_2\otimes\nabla\}_3\ldots\nabla\Big\}_{l,-m_1}B_M(\mathbf{R})\,.\end{aligned}$$

As a result, we arrive at the system of equations for the coefficients a_{lm}:

$$\frac{1}{l!}\Big\{\ldots\{\{\nabla\otimes\nabla\}_2\otimes\nabla\}_3\ldots\otimes\nabla\Big\}_{l,-m_1} B_M(\mathbf{R})$$
$$= \mathrm{i}\sqrt{\frac{(l+2)(2l+3)!!}{4\pi l!}}\sum_m a_{l+1,m}(-1)^{M-m_1} C^{l+1,m}_{lm_1 1,-M}\,. \quad (60)$$

To find these coefficients, one should multiply equations (60) by

$$C^{l+1,m'}_{l,-m_1 1M} = C^{l+1,-m'}_{lm_1 1,-M}$$

and, on summation over m_1 and M, form an irreducible tensor on the left-hand side, and also make use of the orthogonality condition for Clebsch–Gordan coefficients on the right-hand side. As a result, the desired coefficients turn out to be expressed in terms of the derivatives of the magnetic field $\mathbf{B}$:

$$(-1)^m a_{l,-m} = -\mathrm{i}\sqrt{\frac{4\pi}{(l+1)(2l+1)!!(l-1)!}}$$
$$\times\Big\{\Big\{\ldots\{\{\nabla\otimes\nabla\}_2\otimes\nabla\}_3\ldots\otimes\nabla\Big\}_{l-1}\otimes\mathbf{B}\Big\}_{lm}\,, \quad (61)$$

where $l = 1, 2, 3, \ldots$.

By substituting expression (61) into series (57), we find the final expression for the multipole expansion of the potential function of a system of currents in an external magnetic field:

$$V = \sum_{l=1}^{\infty} V_l\,, \qquad V_l = -\frac{1}{\sqrt{l!(2l-1)!!}}$$
$$\times\Big(M_l^*\Big\{\Big\{\ldots\{\{\nabla\otimes\nabla\}_2\otimes\nabla\}_3\ldots\otimes\nabla\Big\}_{l-1}\otimes\mathbf{B}\Big\}_l\Big)\,. \quad (62)$$

Here, it was taken into account that $M_{lm} = (-1)^m M^*_{l,-m}$, and the scalar product of two irreducible tensors was introduced in accordance with definition (11). Clearly, expansion (62) is analogous to the multipole expansion of the energy of an electrically neutral system of charges in an external electric field, with the potential function V_l of a magnetic 2^l-field resulting from the expression for the energy of electric 2^l-field (46) by way of the substitutions $Q_{lm}\to M_{lm}$, $\mathbf{E}\to\mathbf{B}$. We emphasize that this result is well known for the potential function (53) of a magnetic dipole.

The potential function V_2 (62) of a magnetic quadrupole, namely

$$V_2 = -\frac{1}{\sqrt{6}}\big(M_2^*\{\nabla\otimes\mathbf{B}\}_1\big)\,,$$

may also be represented in a form similar to expression (50): [1]

$$V_2 = -\frac{1}{6}\,\mu_{ik}\nabla_i B_k\,. \quad (63)$$

[1] Given in Ref. [6, Appendix 1] is the expression for V_2 obtained by direct expansion of the vector potential of an external field in a Taylor series:

$$V_2 = -m_{ik}\nabla_i B_k\,.$$

The magnetic quadrupole moment tensor m_{ik} introduced by the authors of Ref. [6] differs from μ_{ik} (65) by the following factor:

$$m_{ik} = \frac{1}{3}\,\mu_{ik}\,.$$

A comparison of these expressions with formula (63) shows that a factor of $1/2$ was lost in Ref. [6].

In complete analogy with the electric quadrupole moment tensor D_{ik} (48), the magnetic quadrupole moment tensor μ_{ik}, whose spherical components are related to M^*_{2m} by the expression of the form (49)

$$M^*_{2m} = \frac{1}{\sqrt{6}}\sum_{m_1,m_2} C^{2m}_{1m_1 1m_2}\mu_{m_1 m_2}\,, \quad (64)$$

is introduced into expression (63). An explicit expression for the components of the symmetric tensor μ_{ik} having a zero trace is easy to find by taking advantage of the definition of M^*_{2m} (20). By substituting $r^2 Y_{2m}(\mathbf{n})$ in the form (42) into expression (20), changing the coupling scheme of angular momenta, and employing the second of identities (12), one brings the expression for M^*_{2m} to the form (64), where

$$\mu_{ik} = \frac{1}{c}\int\big([\mathbf{r}\times\mathbf{j}]_i\,x_k + [\mathbf{r}\times\mathbf{j}]_k\,x_i\big)\,\mathrm{d}\mathbf{r}\,. \quad (65)$$

In concluding this section, it should be noted that the coincidence of multipole energy expansions in electrostatics and magnetostatics is not accidental. Indeed, the scalar construction defining U_l (46) and V_l (62) is the sole scalar which may be composed of an irreducible tensor of rank l (2^l-pole moment) and the $(l-1)$th-order derivatives of the components of vector $\mathbf{E}$ or $\mathbf{B}$, respectively. The coincidence of the coefficients of the scalar products of the irreducible tensors in expressions (46) and (62) is related to the definition of electric and magnetic multipole moments. The latters are so defined that the multipole expansions of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ look similar (see Section 2.3). At the same time, the energy (electric or magnetic) of interaction of the two systems is, as is well known, symmetrical in the sense that it may be calculated as the energy of system 1 in the field of system 2 or, conversely, as the energy of system 2 in the field of system 1 [2] (in particular, the energy of the interaction of the 2^{l_1}-pole of system 1 with the external 2^{l_2}-field of system 2 must coincide with the energy of the 2^{l_2}-pole of system 2 in the external 2^{l_1}-field of system 1). Therefore, the coefficient of scalar products in expressions (46) and (62), which ensures this symmetry, might only differ from $[l!(2l-1)!!]^{-1/2}$ by a factor equal for all l.

4. Axisymmetric systems

Let us assume that the distribution of charges or currents is axisymmetric with the symmetry axis Z, whose direction is defined by a unit vector $\mathbf{k}$. In this case, both the multipole expansions of the field of a system and the expansion of the energy of the system in an external field are significantly simplified. The irreducible tensor apparatus proves to be highly effective for these systems and permits, in particular, explicitly separating out the dependences on the symmetry $\mathbf{k}$-axis direction in all formulas.

4.1 Multipole moments of axisymmetric systems

From the definition of 2^l-pole electric (magnetic) moment Q_{lm} (M_{lm}) it follows that the set of its complex conjugate quantities is transformed according to the irreducible representation of the group of rotations having the dimension $2l+1$ and forms an irreducible tensor of rank l. Therefore, the multipole moment (for definiteness we shall write out the corresponding formulas for the electric moment Q_{lm}) is transformed under rotations of the coordinate system (CS) according to the law

$$Q_{lm} = \sum_M Q_{lM} D^{(l)*}_{Mm}(\gamma, \beta, \alpha)\,, \tag{66}$$

where Q_{lM} are the components of the 2^l-pole moment in the CS XYZ, and Q_{lm} in the CS xyz, (γ, β, α) are Euler's angles specifying the orientation of the XYZ CS relative to the xyz CS, and $D^{(l)}_{Mm}$ is the matrix of finite rotations (Wigner's D function) [9]. Let XYZ in expression (66) be the intrinsic CS of our axisymmetric system, whose Z-axis coincides with the symmetry axis, and let β and α — the polar and azimuthal angles of the Z-axis in the xyz CS — give the **k** direction. We shall also require expressions for the D function in some special cases:

$$D^{(l)}_{Mm}(\gamma, 0, 0) = \exp(\mathrm{i}M\gamma)\, \delta_{Mm}\,, \tag{67}$$

$$D^{(l)}_{0m}(\gamma, \beta, \alpha) = D^{(l)}_{0m}(0, \beta, \alpha) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\mathbf{k})\,. \tag{68}$$

The axial symmetry of the system signifies that the multipole moment components Q_{lM} remain invariable under CS rotation through an arbitrary angle γ about the Z-axis, and on the strength of transformation law (66) and formula (67) we therefore obtain

$$Q_{lM} = Q_{lM} \exp(\mathrm{i}\gamma M)\,. \tag{69}$$

The last relation shows that only the zero component of the 2^l-pole moment $Q_{l0} \equiv Q^{(k)}_l$ is nonzero in the intrinsic CS:

$$Q_{lM} = Q^{(k)}_l \delta_{M0}\,. \tag{70}$$

We substitute expression (70) into expression (66) and take into consideration relationship (68) which links the D function to the spherical function to determine the dependence of Q_{lm} on the orientation of the symmetry axis of the system:

$$Q_{lm} = \sqrt{\frac{4\pi}{2l+1}}\, Q^{(k)}_l Y^*_{lm}(\mathbf{k})\,. \tag{71}$$

From formulas (5) and (20), which define electric and magnetic multipole moments, it follows that

$$Q^*_{lm} = (-1)^m Q_{l,-m}\,, \qquad M^*_{lm} = (-1)^m M_{l,-m}\,.$$

The zero components of these tensors are therefore real, and in accordance with formula (71) one finds

$$Q^*_{lm} = \sqrt{\frac{4\pi}{2l+1}}\, Q^{(k)}_l Y_{lm}(\mathbf{k})\,, \qquad M^*_{lm} = \sqrt{\frac{4\pi}{2l+1}}\, M^{(k)}_l Y_{lm}(\mathbf{k})\,. \tag{72}$$

Formulas (72) define the multipole moments of an axisymmetric system in an arbitrarily oriented CS whose origin is on the system's symmetry axis. It can also be said that these formulas establish the dependence of the multipole moments, which are defined relative to a point lying on the symmetry axis, on the orientation of this axis. In this case, $Q^{(k)}_l$ and $M^{(k)}_l$ depend, generally speaking, on the position of the point relative to which the multipole moments are defined.

Let us also discuss the case where the **k**-axis is no more than a symmetry axis of order n, i.e., the charge (current) distribution and with it the multipole moments remain invariable under rotations through the angle $\gamma_n = 2\pi/n$ about this axis. In this case, according to expression (69), we have

$$Q_{lM} = Q_{lM} \exp\left(-\frac{\mathrm{i}2\pi M}{n}\right),$$

so that from $Q_{lM} \neq 0$ it follows that $M = 0, \pm n, \pm 2n, \ldots$. For a given l, it will be recalled, $M = 0, \pm 1, \pm 2, \ldots, \pm l$, making it evident that for $l < n$ only the zero component of the 2^l-pole moment is nonzero in the intrinsic CS. As for an arbitrarily oriented CS whose origin is located on the system's symmetry axis, Q_{lm} and M_{lm} are defined for $l < n$ by expressions (72). This result is evident for an electric or magnetic dipole ($l = 1$) defined by vector **d** or **μ**, respectively: the vector which characterizes the system at a point lying on the symmetry axis of even the lowest second order should be aligned with this axis, i.e., it has only one nonzero component in the intrinsic CS. Thus, when the system possesses a symmetry axis of order n, the fields of the 2^l-pole of this and of an axisymmetric system coincide for $l < n$; the same is true of the interaction of a 2^l-pole with an external field.

By way of example, let us consider a symmetric toroidal distribution of currents. The system comprises n similar equidistant turns wound on a torus, which carry equal currents (an n-fold symmetric toroid). In the limiting case of $n \to \infty$, we obtain a toroidal solenoid. For a finite n, the system evidently possesses a symmetry axis of order n, which coincides with the torus axis, so that $M_{lM} = M^{(k)}_l \delta_{M0}$ for $l < n$, as shown above. In view of expression (28), meanwhile, for a spherical vector it follows from definition (20) that

$$M^{(k)}_l = M_{l0} = \frac{\sqrt{4\pi/(2l+1)}}{c(l+1)} \int r^l (\mathbf{r} \times \mathbf{j})\, \nabla Y_{l0}(\mathbf{n})\, \mathrm{d}\mathbf{r} = 0\,,$$

because $Y_{l0}(\theta, \varphi)$ is independent of the φ angle, with the result that vectors **r**, **j**, and ∇Y_{l0} turn out to be coplanar. For a toroid with a symmetry axis of order n, therefore, $M_{lm} = 0$ for $l < n$. For a magnetic dipole moment this result is evident: the magnetic moment of each of the turns is perpendicular to the plane of the turn, whereas the total magnetic moment of the toroid should be directed along its axis. It should be noted that the system under discussion possesses nonzero toroidal multipole moments [6] [in the static case they may be calculated by formula (21)]. However, they play no part in magnetostatics, making no contribution to the multipole expansion of the magnetic induction vector **B** (see Section 2.2). Hence, the multipole expansion of the n-fold symmetric toroid field begins with a magnetic 2^n-field, and **B** therefore decreases as $1/r^{n+2}$ for long distances [see expressions (30), (31)]. For an axisymmetric toroidal current ($n \to \infty$), all magnetic multipole moments turn to zero and, accordingly, there is no magnetic field outside of the system. The absence of a magnetic field beyond a toroidal solenoid with an infinitely dense winding is a well-known result, which is given in many textbooks on general physics and substantiated with the aid of Ampère's circuital law (the theorem on the circulation of a magnetic induction vector).

4.2 Electric field of an axisymmetric system of charges

The potential of an electric multipole field is defined by formula (34). Substituting into it the electric 2^l-pole moment in the form of expression (71) and applying the summation theorem for spherical functions (3), one finds the potential for the field of an axisymmetric 2^l-pole:

$$\varphi_l(\mathbf{r}) = \frac{Q_l^{(k)}}{r^{l+1}}\, P_l\,(\mathbf{kn})\,, \qquad l = 0, 1, 2, \ldots\,, \tag{73}$$

where

$$P_l(x) = \frac{1}{2^l l!}\,\frac{\mathrm{d}^l}{\mathrm{d}x^l}\,(x^2 - 1)^l$$

is a Legendre polynomial.

The electric field strength may be found from formula (73): $\mathbf{E}_l = -\nabla\varphi_l$. We nevertheless take advantage of the previously derived general expression (36) for $\mathbf{E}_l$ and substitute Q^*_{lm} (72) into it to find for an axisymmetric system:

$$\mathbf{E}_l = \frac{(-1)^l}{r^{l+2}}\, 4\pi \sqrt{\frac{l+1}{3}}\, Q_l^{(k)} \big\{ Y_l(\mathbf{k}) \otimes Y_{l+1}(\mathbf{n}) \big\}_1\,. \tag{74}$$

An irreducible tensor composed of two spherical functions is referred to in the literature as a bipolar harmonic (BH) [3]. A BH of rank one enters into expression (74). As shown in Ref. [10], a BH of a given rank comprising spherical functions of arbitrary ranks may be reduced to the simplest BHs of this rank with spherical functions of the lowest possible rank. In this case, the coefficients of expansion of the initial BH in terms of these simplest BHs are expressed in terms of Legendre polynomials (the summation theorem for spherical functions may be considered as the simplest of reductions of this kind). For the BH of interest, in particular, the reduction formula has the form

$$\big\{ Y_l(\mathbf{k}) \otimes Y_{l'}(\mathbf{n}) \big\}_1 = -\frac{1}{4\pi} \sqrt{\frac{3}{l_{\max}}}\, \big[(-1)^l P_l^{(1)}(x)\mathbf{k} + (-1)^{l'} P_{l'}^{(1)}(x)\mathbf{n} \big]\,, \tag{75}$$

where

$$l' = l \pm 1\,, \quad l_{\max} = \max\,(l, l')\,, \quad P_l^{(1)}(x) = \frac{\mathrm{d}P_l(x)}{\mathrm{d}x}\,, \quad x = \mathbf{kn}\,.$$

In what follows, we shall require the formula of reduction of yet another BH [10]:

$$\big\{ Y_l(\mathbf{k}) \otimes Y_l(\mathbf{n}) \big\}_1 = \frac{\mathrm{i}(-1)^{l+1}}{4\pi} \sqrt{\frac{3(2l+1)}{l(l+1)}}\, P_l^{(1)}(x)\, \mathbf{k} \times \mathbf{n}\,. \tag{76}$$

Identity (75) permits obtaining a relatively simple expression for the field strength of the axisymmetric 2^l-pole (74):

$$\mathbf{E}_l = \frac{Q_l^{(k)}}{r^{l+2}} \big[P_{l+1}^{(1)}(\mathbf{kn})\mathbf{n} - P_l^{(1)}(\mathbf{kn})\mathbf{k} \big]\,, \quad l = 0, 1, 2, \ldots\,. \tag{77}$$

By way of example we shall give the formulas which expressions (73) and (77) reduce to in the special cases of $l = 1$ and 2. To write down the results, we require explicit expressions for the three simplest Legendre polynomials:

$$P_1(x) = x\,, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)\,, \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)\,.$$

For $l = 1$, we have

$$\varphi_1 = \frac{Q_1^{(k)}}{r^2}\, \mathbf{kn}\,, \qquad \mathbf{E}_1 = \frac{Q_1^{(k)}}{r^3} \big[3(\mathbf{kn})\mathbf{n} - \mathbf{k} \big]\,.$$

In accordance with definition (5), $Q_1^{(k)} = Q_{10}$ coincides with the z-component of the system's dipole moment (6) in the intrinsic CS, and the formulas given here reproduce the well-known expressions for the potential and field strength of a dipole with the dipole moment $\mathbf{d} = Q_1^{(k)}\mathbf{k}$ [1, 2]. Therefore, the axial symmetry of the system in the case of a dipole field does not lead to any simplifications. This is evidently due to the following circumstance: in a CS, whose Z-axis is aligned with the dipole moment vector $\mathbf{d}$, only one of its components, $d_z = Q_{10}$, is nonzero and, consequently, Q_{1M} has the form of expression (70) in the absence of symmetry, as well.

The potential and strength of the field of an axisymmetric quadrupole are written out in the following form in view of expressions (73) and (77):

$$\varphi_2 = \frac{Q_2^{(k)}}{2r^3} \big[3(\mathbf{kn})^2 - 1 \big]\,, \tag{78}$$

$$\mathbf{E}_2 = \frac{3Q_2^{(k)}}{2r^4} \Big\{ \big[5(\mathbf{kn})^2 - 1 \big]\mathbf{n} - 2(\mathbf{kn})\mathbf{k} \Big\}\,. \tag{79}$$

The quantity $Q_2^{(k)} = Q_{20}$, which enters into these expressions, is, according to definition (5), proportional to the z-component of the system's quadrupole moment tensor D_{ij} (48) in the intrinsic CS (i.e., to the third principal value of this tensor):

$$Q_2^{(k)} = \frac{1}{2}\, D_{zz}\,. \tag{80}$$

With regard to relationship (80), potential φ_2 (78) coincides with the expression for the potential of the field of an axisymmetric quadrupole, which was given in Refs [1, 2].

4.3 Magnetic field of an axisymmetric system of currents

The similarity between the multipole expansions of the electric field strength and the magnetic induction vector was discussed at length in Section 2.3. A simple comparison of formulas (31) and (36) shows that the magnetic induction vector $\mathbf{B}_l$ ($l = 1, 2, 3, \ldots$) of the 2^l-field of an axisymmetric system of currents should be defined by expression (77) in which the component $Q_l^{(k)}$ of the electric multipole moment should be replaced by the corresponding component $M_l^{(k)}$ of the magnetic multipole moment [see formulas (72)]:

$$\mathbf{B}_l = \frac{M_l^{(k)}}{r^{l+2}} \big[P_{l+1}^{(1)}(\mathbf{kn})\mathbf{n} - P_l^{(1)}(\mathbf{kn})\mathbf{k} \big]\,.$$

In particular, the magnetic quadrupole field $\mathbf{B}_2$ is defined by formula (79) with $Q_2^{(k)}$ replaced by

$$M_2^{(k)} = \frac{1}{2}\, \mu_{zz}\,,$$

where, on the strength of relationships (49) and (64), the relationship between $M_2^{(k)}$ and the principal z-value of the magnetic quadrupole moment tensor μ_{ij} (65) duplicates formula (80).

To find the vector potential of the magnetic multipole field of an axisymmetric system, we substitute the 2^l-pole magnetic moment in the form of formula (72) into the expression for $\mathbf{A}_l^{\mathrm{M}}$ (23):

$$\mathbf{A}_l^{\mathrm{M}} = \frac{(-1)^l \mathrm{i}}{r^{l+1}}\, 4\pi \sqrt{\frac{l+1}{3l(2l+1)}}\, M_l^{(k)} \big\{ Y_l(\mathbf{k}) \otimes Y_l(\mathbf{n}) \big\}_1\,.$$

We next apply the BH reduction formula (76) to arrive at the final result

$$\mathbf{A}_l^{\mathrm{M}} = \frac{M_l^{(k)}}{l r^{l+1}}\, P_l^{(1)}(\mathbf{kn})\, \mathbf{k} \times \mathbf{n}\,, \quad l = 1, 2, 3, \ldots\,. \tag{81}$$

Specifically, at $l = 1$ formula (81) reduces to the well-known expression for the potential (16) of a magnetic dipole field with a magnetic moment $\boldsymbol{\mu} = M_1^{(k)}\mathbf{k}$. The vector potential of an axisymmetric magnetic quadrupole is defined, according to formula (81), by the expression

$$\mathbf{A}_2^{\mathrm{M}} = \frac{3M_2^{(k)}}{2r^3}\,(\mathbf{kn})\, \mathbf{k} \times \mathbf{n}\,.$$

It should be emphasized that the vector potential $\mathbf{A}_l^{\mathrm{M}}$ (81) is collinear with the vector product $\mathbf{k} \times \mathbf{n}$ for any multipolarity l.

4.4 Energy of an axisymmetric system in an external field

First, let us simplify expression (45) for the energy of an electric multipole in an external field. By substituting the electric 2^l-pole moment in the form of formula (72) into expression (45) and, furthermore, expressing the corresponding spherical function in terms of the irreducible tensor composed of the unit vector $\mathbf{k}$ [see expression (42)], we represent the energy of interaction between the axisymmetric multipole and external field in the following form

$$U_l = \frac{1}{l!}\, Q_l^{(k)} \Big(\big\{ \ldots \{ \mathbf{k} \otimes \mathbf{k} \}_2 \ldots \otimes \mathbf{k} \big\}_l \times \big\{ \ldots \{ \nabla \otimes \nabla \}_2 \ldots \nabla \big\}_l \Big)\, \varphi(\mathbf{R})\,. \tag{82}$$

For the differential operator acting on the harmonic function φ (the potential of the external field) in expression (82), it is possible to take advantage of representation (A.5), which permits writing U_l in its relatively simple final form

$$U_l = \frac{1}{l!}\, Q_l^{(k)} k_{i_1} k_{i_2} \ldots k_{i_l} \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_l}\, \varphi(\mathbf{R}) = \frac{1}{l!}\, Q_l^{(k)} (\mathbf{k}\nabla)^l\, \varphi(\mathbf{R})\,. \tag{83}$$

At $l = 0$, expression (83) reduces to the energy of a point charge $Q_0^{(k)}$ which is equal to the total charge of the system in an external field:

$$U_0 = Q_0^{(k)} \varphi\,.$$

For $l \geqslant 1$, U_l may also be written out in terms of the strength of this field, $\mathbf{E} = -\nabla\varphi$:

$$U_l = -\frac{1}{l!}\, Q_l^{(k)} (\mathbf{k}\nabla)^{l-1} (\mathbf{kE})\,. \tag{84}$$

The potential and strength of the external electric field in expressions (83) and (84) are taken at point $\mathbf{R}$, which lies on the system's symmetry axis, because the multipole moments are defined relative to this point. Should the Z-axis be directed along the system's symmetry axis defined by vector $\mathbf{k}$, expressions (83) for $l \geqslant 1$ and (84) would take on the following form

$$U_l = \frac{1}{l!}\, Q_l^{(k)}\, \frac{\partial^l \varphi}{\partial z^l} = -\frac{1}{l!}\, Q_l^{(k)}\, \frac{\partial^{l-1}}{\partial z^{l-1}}\, E_z\,. \tag{85}$$

At $l = 1$, expression (84) passes into the well-known expression (47) for the energy of a dipole with a dipole moment $\mathbf{d} = Q_1^{(k)}\mathbf{k}$ in the field $\mathbf{E}$, and the energy of an axisymmetric quadrupole equals

$$U_2 = \frac{1}{4}\, D_{zz}\, \frac{\partial^2 \varphi}{\partial z^2}\,,$$

as follows from expression (85) in view of relation (80).

To summarize, we give the expression for the potential function of an axisymmetric magnetic multipole in external field $\mathbf{B}$. A comparison of the above-derived general formulas for the energy U_l (46) of an electric multipole and the potential function V_l (62) of a magnetic multipole shows that the desired expression may be obtained from expressions (84) and (85) by a simple change of $\mathbf{E} \to \mathbf{B}$, and $Q_l^{(k)} \to M_l^{(k)}$:

$$V_l = -\frac{1}{l!}\, M_l^{(k)} (\mathbf{k}\nabla)^{l-1} (\mathbf{kB}) = -\frac{1}{l!}\, M_l^{(k)}\, \frac{\partial^{l-1}}{\partial z^{l-1}}\, B_z\,,$$

where $l = 1, 2, 3, \ldots\,.$

5. Conclusions

The main findings of our investigation are represented by formulas (23) or (25) for the vector potential, by formulas (30) or (31) for the magnetic induction vector of a magnetic multipole field, and by expression (62) for the potential energy of a magnetic multipole in an external magnetic field. The magnetic field of an arbitrary stationary system of currents beyond this system is defined by a series, whose individual terms represent the fields of magnetic multipoles, and the potential energy of this system in an external magnetic field is written out in the form of multipole expansion (62). In this case, the magnetic 2^l-pole moment is defined by an irreducible tensor of rank l (20). The multipole expansions in magnetostatics are identical to the multipole expansions for an electrically neutral system of charges in electrostatics.

When estimating these results, there is no escape from asking oneself the question: how could it happen that in this seemingly well-elaborated area of classical electrodynamics — magnetostatics — the problems of multipole expansions remained unexplored down to the smallest details? This is surprising, the more so as these expansions are well known in electrostatics; an exception is provided, perhaps, only by the foregoing expression (45) for the potential energy of an electric multipole of arbitrary rank in an external field. The reason supposedly lies with the following fact: to completely investigate the multipole expansions in magnetostatics calls, as we have seen, systematically use the powerful and at the same time elegant mathematical apparatus based on the results of the theory of representations of the group of rotations — the irreducible tensor apparatus. Although this apparatus has been employed in classical and quantum theory of radiation for a relatively long time (see Refs [7, 8] as well as Ref. [6]), the employment of this mathematical apparatus in so simple a classical electrodynamics domain like magnetostatics has most likely been regarded as inappropriate. Meanwhile, the very notion of electric or magnetic multipole moment, whose components form an irreducible tensor, is an organic part of this apparatus.

It should be noted that finding the multipole expansion of the field potential in electrostatics is a simpler task owing to the scalar nature of this potential. And although the mathematical apparatus of irreducible tensors is implicitly employed in this case also, to obtain the expansion one needs only to take advantage of the well-known properties of spherical functions, as we briefly mentioned in the Introduction. That is why, the multipole expansion (4) of the electrostatic potential is given even in textbooks. True, when writing the general expression for the field strength of an electric multipole, use should be made either of spherical vectors [see expression (35)] or of the tensor product of irreducible tensors [see expression (36)]. And it is precisely these expressions (we have not encountered them in the literature) that are required to make possible the identification of the entire multipole series for the electric field strength of an electrically neutral system of charges with the multipole expansion of the magnetic induction vector of a system of currents, and not to confine oneself to the statement of the coincidence of the electric and magnetic dipole fields.

In conclusion, I would like to emphasize that the investigation performed in this work is not only of academic significance. The multipole expansion of the magnetic field is the solution to a magnetostatic problem, presented in the form of a series. This series converges even at a finite distance from the system of currents — the field sources — so that calculating the field requires, generally speaking, all terms of this series. At long distances from the sources, the multipole expansion is especially effective, because the series converges rapidly, and to find the field it suffices, as a rule, to limit oneself to the first nonvanishing term. The multipole expansion of the potential function of currents embedded in a quasiuniform external magnetic field is equally effective. However, the first term in the multipole expansions is not necessarily the dipole one. A system with a zero magnetic dipole moment or a system with several zero higher-order magnetic multipole moments is by no means exotic and is not confined to the n-fold symmetric toroid discussed at the end of Section 4.1. For example, it suffices to take a system of currents consisting of two subsystems with antiparallel magnetic dipole moments of equal magnitude (in the simplest case, these are two plane-parallel magnet sheets, i.e., two like plane-parallel turns with equal currents flowing in the opposite directions) to obtain a system whose magnetic properties are defined already by its magnetic quadrupole moment. It is easy to conceive a system with zero magnetic dipole and quadrupole moments, etc. And in all such cases, the field of the system and its behavior in an external magnetic field will be defined by the higher-order terms of multipole expansions. Lastly, we note that the magnetic effects of higher-order multipolarity may, in principle, manifest themselves in the interaction of quantum systems with one another, as well as with a nonuniform external magnetic field. However, the relative magnitude of these effects is rather low, and the feasibility of their experimental examination calls for special consideration in each specific case.

6. Appendices

6.1 Appendix 1

In the determination of the magnetic induction vector from the vector potential expressed in terms of the spherical vectors in the form of expressions (25), (26), one has to calculate the rotor of vector $\mathbf{Y}_{Llm}(\mathbf{n})/r^{l+1}$. We briefly explain the technique of this calculation. At first we write, in view of the second of formulas (12), the spherical rotor component in the form

$$\operatorname{rot}_v \left(\frac{1}{r^{l+1}} \mathbf{Y}_{Llm}(\mathbf{n}) \right) = -\mathrm{i}\sqrt{2} \left\{ \nabla \otimes \frac{1}{r^{l+1}} \mathbf{Y}_{Llm}(\mathbf{n}) \right\}_{1v} . \quad \text{(A.1)}$$

We next make use of definitions (10), (18) and of identity (19) for the gradient, and rearrange in a standard way [3, 4] the resultant sum of the products of three Clebsch–Gordan coefficients to obtain the product of a Clebsch–Gordan coefficient and a $6j$-symbol. Finally, expression (A.1) takes on the following form

$$\operatorname{rot} \left(\frac{1}{r^{l+1}} \mathbf{Y}_{Llm}(\mathbf{n}) \right) = -\mathrm{i}\, \frac{(2l+1)\sqrt{6(l+1)}}{r^{l+2}} \begin{Bmatrix} l+1 & L & 1 \\ 1 & 1 & l \end{Bmatrix} \mathbf{Y}_{L,\,l+1,\,m}(\mathbf{n}) . \quad \text{(A.2)}$$

When executing rearrangements it should be remembered that the spherical component a_v of vector $\mathbf{a}$ may be written out in the form of the scalar product $\mathbf{a}\mathbf{e}_v$. For $L = l - 1$, the $6j$-symbol and with it the whole expression (A.2) turn to zero, which corresponds to the potentiality of field (26) (see Section 2.2). At $L = l$, expression (A.2) reduces, upon calculation of the $6j$-symbol, to formula (29) in the main text.

We also show how it is possible to find the rotor of the vectors having the forms (23) and (24). To this end, we introduce the vector

$$\mathbf{R}_L = \operatorname{rot} \left\{ a_L \otimes \frac{1}{r^{l+1}}\, Y_l(\mathbf{n}) \right\}_1 ,$$

where the irreducible tensor a_L of rank L is independent of $\mathbf{r}$. Employing the second of identities (12) and transposing the factors in the irreducible tensor product with the form (10), which is attended with the multiplication by $(-1)^{L+l+l'}$, we represent $\mathbf{R}_L$ as

$$\mathbf{R}_L = -\mathrm{i}\sqrt{2} \left\{ \nabla \otimes \left\{ a_L \otimes \frac{1}{r^{l+1}}\, Y_l(\mathbf{n}) \right\}_1 \right\}_1 = (-1)^{L+l}\mathrm{i}\sqrt{2} \left\{ \nabla \otimes \left\{ \frac{1}{r^{l+1}}\, Y_l(\mathbf{n}) \otimes a_L \right\}_1 \right\}_1 .$$

On changing the coupling scheme of angular momenta in the last expression, we arrive at

$$\mathbf{R}_L = \mathrm{i}\sqrt{6} \sum_J \sqrt{2J+1} \begin{Bmatrix} J & 1 & l \\ 1 & L & 1 \end{Bmatrix} \times \left\{ \left\{ \nabla \otimes \frac{1}{r^{l+1}}\, Y_l(\mathbf{n}) \right\}_J \otimes a_L \right\}_1 . \quad \text{(A.3)}$$

Next, by employing identity (19), the spherical vector definition (18), and the orthogonality condition for

Clebsch–Gordan coefficients, we find the irreducible tensor

$$\left\{\nabla \otimes \frac{1}{r^{l+1}}\, Y_l(\mathbf{n})\right\}_{JM} = -\frac{2l+1}{r^{l+2}}\sqrt{\frac{l+1}{2l+3}}\, Y_{l+1,M}(\mathbf{n})\delta_{J,l+1}\,,$$

and substitute this formula into expression (A.3) to obtain the final result:

$$\mathbf{R}_L = \operatorname{rot}\left\{a_L \otimes \frac{1}{r^{l+1}}\, Y_l(\mathbf{n})\right\}_1 = -\mathrm{i}\,\frac{\sqrt{6(l+1)}\,(2l+1)}{r^{l+2}} \times \begin{Bmatrix} l+1 & 1 & l \\ 1 & L & 1 \end{Bmatrix}\left\{Y_{l+1}(\mathbf{n}) \otimes a_L\right\}_1. \tag{A.4}$$

For $L = l-1$, the $6j$-symbol in expression (A.4), and with it $\mathbf{R}_{L-1}$, turn to zero, which corresponds to the potentiality of vector (24). Substituting $L = l$, $a_l = M_l^*$ into expression (A.4) and calculating the $6j$-symbol, one finds

$$\operatorname{rot}\left\{M_l^* \otimes \frac{1}{r^{l+1}}\, Y_l(\mathbf{n})\right\}_1 = -\mathrm{i}\,\frac{\sqrt{l(2l+1)}}{r^{l+2}}\left\{M_l^* \otimes Y_{l+1}(\mathbf{n}\right\}_1,$$

with the result that the rotor of vector $\mathbf{A}_l^M(\mathbf{r})$ (23) is written out in the form of expression (31).

6.2 Appendix 2

Let us show that the harmonic function φ, i.e., a function which satisfies Laplace's equation, obeys the following identity

$$x_{i_1}x_{i_2}\ldots x_{i_l}\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_l}\varphi = \Big(\Big\{\ldots\big\{\{\mathbf{r}\otimes\mathbf{r}\}_2\otimes\mathbf{r}\big\}_3\ldots\otimes\mathbf{r}\Big\}_l \times \Big\{\ldots\big\{\{\nabla\otimes\nabla\}_2\otimes\nabla\big\}_3\ldots\otimes\nabla\Big\}_l\Big)\varphi\,. \tag{A.5}$$

To prove this, we first of all write out the operator which appears on the left-hand side of expression (A.5) in terms of the spherical components of the vectors:

$$x_{i_1}x_{i_2}\ldots x_{i_l}\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_l} = \sum_{m_1,m_2,\ldots,m_l}(-1)^{m_1+m_2+\ldots+m_l}x_{m_1}x_{m_2}\ldots x_{m_l}\nabla_{-m_1}\nabla_{-m_2}\ldots\nabla_{-m_l}\,. \tag{A.6}$$

We next invert relation (10), which defines the irreducible tensor product of two irreducible tensors, to find that

$$x_{m_1}x_{m_2} = \sum_{l_1,M_1} C^{l_1M_1}_{1m_11m_2}\{\mathbf{r}\otimes\mathbf{r}\}_{l_1M_1}\,,$$

and continue the same procedure to arrive at the relation

$$x_{m_1}x_{m_2}\ldots x_{m_l} = \sum_{l_1,M_1,l_2,M_2,\ldots,l_{l-1},M} C^{l_1M_1}_{1m_11m_2}C^{l_2M_2}_{l_1M_11m_3}\ldots C^{l_{l-1}M}_{l_{l-2}M_{l-2}1m_l} \times \Big\{\ldots\big\{\{\mathbf{r}\otimes\mathbf{r}\}_{l_1}\otimes\mathbf{r}\big\}_{l_2}\ldots\otimes\mathbf{r}\Big\}_{l_{l-1}M}. \tag{A.7}$$

On substituting relation (A.7) into the right-hand side of relation (A.6) and making use of the symmetry properties of Clebsch–Gordan coefficients and of definition (10), we obtain an equivalent representation of operator (A.6):

$$x_{i_1}x_{i_2}\ldots x_{i_l}\nabla_{i_1}\nabla_{i_2}\ldots\nabla_{i_l} = \sum_{l_1,l_2,\ldots,l_{l-1}}(-1)^{l-l_{l-1}} \times \sum_M(-1)^M\Big\{\ldots\big\{\{\mathbf{r}\otimes\mathbf{r}\}_{l_1}\otimes\mathbf{r}\big\}_{l_2}\ldots\otimes\mathbf{r}\Big\}_{l_{l-1},M} \times \Big\{\ldots\big\{\{\nabla\otimes\nabla\}_{l_1}\otimes\nabla\big\}_{l_2}\ldots\otimes\nabla\Big\}_{l-1,-M}$$
$$= \sum_{l_1,l_2,\ldots,l_{l-1}}(-1)^{l-l_{l-1}}\Big(\Big\{\ldots\big\{\{\mathbf{r}\otimes\mathbf{r}\}_{l_1}\otimes\mathbf{r}\big\}_{l_2}\ldots\otimes\mathbf{r}\Big\}_{l_{l-1}} \times \Big\{\ldots\big\{\{\nabla\otimes\nabla\}_{l_1}\otimes\nabla\big\}_{l_2}\otimes\nabla\Big\}_{l_{l-1}}\Big)\,. \tag{A.8}$$

Let us now compare the Taylor series (39) with the expansion of the harmonic function in a series (40). The terms with a given l (the terms $\sim r^l$) in expressions (39) and (40) should evidently coincide. We also take into consideration that, as is easily shown, an irreducible tensor of rank l, which depends only on the unit vector $\mathbf{n}$, reduces, correct to a factor, to the spherical function $Y_{lm}(\mathbf{n})$. Therefore, the irreducible tensors (composed of the vector $\mathbf{r}$) with lower intermediate momenta and final l_{l-1} momentum than in expression (42), which appear in expression (A.8), are proportional to the spherical functions $Y_{l_{l-1},M}(\mathbf{n})$. For a given l, however, in expression (40) there are no spherical functions of lower rank than l (here, l is the maximum possible finite momentum in the irreducible tensors indicated). Consequently, the action of operator (A.8) on the harmonic function does reduce to that described by expression (A.5).

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