

METHODOLOGICAL NOTES

PACS numbers: 03.50.De, 41.20.Jb, **52.35.–g**

Impedance technique for modeling electromagnetic wave propagation in anisotropic and gyrotropic media

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DOI: 10.3367/UFNe.0181.201102c.0151

Contents

Abstract. **A convenient and physically unambiguous interpretation of the invariant embedding technique is discussed in application to problems of electromagnetic wave propagation in complex media characterized by anisotropic and gyrotropic dielectric or magnetic response.**

1. Introduction

We here describe a relatively simple general method for addressing problems of electromagnetic wave propagation and linear transformation in complex media characterized by anisotropic and gyrotropic dielectric or magnetic response. Such media are exemplified by magnetized plasma [1–4], optically active crystals [5] (including magnetically ordered [6, 7] and liquid [8–10] ones), magnetic semiconductors [7, 11], artificial meta-materials [12], a magnetized vacuum [13, 14], etc. The main problem to be considered in this study is the reconstruction of the electromagnetic field inside an inhomogeneous layer of a linear medium with a planar boundary onto which external monochromatic radiation is incident from the vacuum or a half-space filled with a homogeneous medium. Related to this problem is that of finding the field in the external medium, e.g., in the wave reflected from the layer and transmitted behind it.

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Received 20 January 2010, revised 29 July 2010
Uspekhi Fizicheskikh Nauk **181** (2) 151 – 172 (2011)
DOI: 10.3367/UFNr.0181.201102c.0151
Translated by Yu V Morozov; edited by A M Semikhatov

To illustrate the essence of the proposed method, we consider the simplest and best-known case of an electromagnetic plane wave incident on an inhomogeneous layer of an isotropic medium. We suppose that the medium is characterized be the dielectric $\varepsilon(z)$ and magnetic $\mu(z)$ permittivities dependent only on a single spatial coordinate z; the layer occupies the region $z \in [a, b]$. The field distributions inside and outside the layer are sought. The propagation of an electromagnetic monochromatic wave with $\mathbf{E}, \mathbf{H} \propto \exp(\mathrm{i}\omega t)$ (where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths) in such a medium is described by the Maxwell equations

$$\operatorname{rot} \mathbf{E} = -\mathrm{i}k_0\, \mu(z)\, \mathbf{H}\,, \qquad \operatorname{rot} \mathbf{H} = \mathrm{i}k_0 \varepsilon(z)\, \mathbf{E}\,,$$

where $k_0 = \omega/c$ is the vacuum value of the wave vector. For simplicity, we seek the solution of the Maxwell equations in the form of plane waves propagating along the z coordinate:

$$\mathbf{E} = E_x(z) \exp(\mathrm{i}\omega t)\, \mathbf{e}_x\,, \qquad \mathbf{H} = H_y(z) \exp(\mathrm{i}\omega t)\, \mathbf{e}_y\,.$$

The Maxwell equations then become

$$\partial_z E_x = -\mathrm{i}k_0 \mu H_y \,, \qquad \partial_z H_y = -\mathrm{i}k_0 \varepsilon E_x \,. \tag{1.1}$$

Here and hereafter, $\partial_z = \partial/\partial z$ denotes the derivative with respect to z. Outside the layer, the vacuum solutions for the fields are

$$E_x = E^+ \exp(-\mathrm{i}k_0 z) + E^- \exp(+\mathrm{i}k_0 z)\,,$$
$$H_y = E^+ \exp(-\mathrm{i}k_0 z) - E^- \exp(+\mathrm{i}k_0 z)\,,$$

where E^+ and E^- are the amplitudes of waves propagating along and opposite the z axis direction. Because the tangential field components are continuous, the vacuum solutions determine boundary conditions for the Maxwell equations inside the layer. Usually, the incident wave amplitude E^{inc} is given in front of the layer, and the condition of the absence of a backward wave is imposed behind the layer. This leads to the boundary conditions

$$E_x(a) + H_y(a) = E^{\mathrm{inc}}\,, \qquad E_x(b) - H_y(b) = 0\,. \tag{1.2}$$

Within the usual straightforward approach, a set of solutions of the Maxwell equations is typically found that satisfies one of the boundary conditions and depends on a single indefinite constant; this constant is then chosen such that the second boundary condition be satisfied [15, 16]. But a different approach is also possible based on the wave impedance equation. The plane wave impedance in our example is given by the ratio of orthogonal fields:

$$\xi(z) = \frac{E_x(z)}{H_y(z)}\,.$$

It is easy to see that $\xi = \pm 1$ corresponds to waves propagating along and opposite the z axis. Substituting $E_x = \xi H_y$ in Maxwell equations (1.1) yields a closed equation for the wave impedance [17]:

$$\partial_z \xi = -\mathrm{i}k_0(\mu - \varepsilon \xi^2)\,. \tag{1.3}$$

This nonlinear first-order differential equation no longer contains wave field components. Therefore, its solution is uniquely defined by a *single* initial condition, which can be conveniently adopted as the condition of the absence of a backward wave at the rear boundary of the layer, $\xi(b) = 1$; this condition is equivalent to the second boundary condition in (1.2). Substituting the known impedance $\xi(z)$ in the Maxwell equations, we arrive at a first-order equation for the wave field with the initial condition at the front boundary of the layer, e.g.,

$$\partial_z E_x = -\mathrm{i}k_0 \mu \xi^{-1} E_x\,, \qquad E_x(a) = \frac{E^{\mathrm{inc}}}{1 + \xi^{-1}(a)}\,.$$

Thus, the original boundary problem (1.1), (1.2) is divided into two evolutionary problems that can be solved consecutively. The above mathematical procedure is equivalent to the known method for lowering the order of second-order linear homogeneous differential equations by passing to a logarithmic derivative [18]. Indeed, elimination of one of the field components in Maxwell equation (1.1) leads to a one-dimensional Helmholtz equation for the remaining field component, e.g.,

$$\partial_z^2 E_x + k_0^2 \varepsilon \mu E_x = 0\,.$$

The change of the variable $V = (\partial_z E_z)/E_z$ reduces this equation to the complex Riccati equation

$$\partial_z V + V^2 + k_0^2 = 0\,.$$

Evidently, this equation is fully equivalent to the wave impedance equation $\xi = -\mathrm{i}k_0\mu/V$.

In most cases, instead of the impedance, it is convenient to introduce a 'local' reflection coefficient that couples forward and backward waves at a given point z inside the original layer:

$$R(z) \equiv \frac{E^- \exp(+\mathrm{i}k_0 z)}{E^+ \exp(-\mathrm{i}k_0 z)} = \frac{\xi(z) - 1}{\xi(z) + 1}\,.$$

Wave impedance equation (1.3) can be rewritten as an equation for the reflection coefficient,

$$\partial_z R = -\frac{\mathrm{i}k_0}{2}\left(\mu(1-R)^2 - \varepsilon(1+R)^2\right), \tag{1.4}$$

with the initial condition $R(b) = 0$. Due to the continuity of the tangential field components, the quantity $R(z)$ can also be interpreted as a coefficient defining the amplitude of the wave reflected from the 'reduced' layer $[z, b]$ to the vacuum. In what follows, the coordinate-dependent reflection coefficient thus introduced is sometimes referred to as the 'impedance' operator. Equation (1.4) more readily and naturally than (1.3) extends to the case of more complex media. Moreover, the impedance operator R, unlike the true wave impedance ξ or the logarithmic derivative V, is always finite in stable media, which significantly simplifies the numerical analysis in complicated cases. The above simplest example indicates that the wave impedance characterizes the local coupling of counterpropagating waves in an inhomogeneous medium. The possibility of finding this coupling by solving a *closed* evolution equation allows reducing the original boundary wave problem to an evolutionary-type problem.

In this paper, this method is extended to anisotropic and gyrotropic media with a tensorial dielectric or magnetic response. A specific feature of such media is the propagation of several normal waves of different polarizations. Such waves do not interact in a homogeneous or smoothly inhomogeneous medium. However, the geometric optics approximation can be violated in a spatially inhomogeneous medium in general, leading to a coupling between normal waves. Then the electromagnetic field distribution is described by a vector wave equation and, accordingly, a matrix operator arises instead of the scalar wave impedance, for which an evolution equation can be derived similar to Eqn (1.4) obtained for an isotropic plane-layered medium. This equation can also be obtained in the case of a three-dimensionally inhomogeneous (nonplanar-layered) medium, but in this case the 'impedance' is an integral linear operator, which is equivalent to a matrix of an infinite size. For a plane-layered anisotropic and gyrotropic medium without spatial dispersion, the proposed approach allows reducing the wave equation to a system of a finite number of first-order nonlinear differential equations for the components of the wave impedance matrix, with the boundary conditions admitting a simple formulation. Integration of these equations permits finding the coefficients of reflection, absorption, and transmission of a plane monochromatic wave incident on the plane-layered medium from the vacuum; thereafter, the wave field distribution can be reconstructed (from the wave impedance distribution inside the medium) for an arbitrary coordinate dependence of the components of the dielectric and magnetic permittivity tensors.

Even in the matrix formulation, this approach is a special case of a more general method known in the mathematical literature as the invariant embedding technique. The idea behind this method was suggested by Ambartsumyan [19, 20] and Chandrasekhar [21] to solve equations of the linear radiative transfer theory. At present, this method is extensively applied by both mathematicians and physicists to reduce boundary problems to evolutionary (initial value) problems that are easier to solve [19, 22–28]. The method is expounded in some detail in [27] and the references therein. It has been successfully used for a long time in radioprobing the ionosphere, which involves rather complicated calculations of electromagnetic fields and the reflection and transmission coefficients of waves propagating in an inhomogeneous magnetized plasma. The first attempts at direct integration of wave equations for the ionosphere [29–32] revealed the main computational problems arising from the large size of ionospheric layers compared with the probing wavelengths and the existence of evanescent wave regions responsible for numerical instabilities. These difficulties were obviated using different variants of the embedding technique, although the term itself is not widely used in the literature relevant to the ionosphere, where equations for wave admittance [33–35], the logarithmic field derivative [36, 37], and coefficients of reflection from the reduced layer [38–42] were used. Specifically, for gyrotropic plasma, a differential equation was obtained for the matrix of the reflection coefficients for a reduced layer [40, 41], generalizing Eqn (1.4). In quantum mechanics, the invariant embedding technique is known in the form of the variable phase method [43–45]. By analogy with the wave equation, the Schrödinger and Dirac equations can be reduced to a nonlinear Riccati-type equation. The physical content of such an approach is that at each point, a function satisfying the Riccati equation (phase function) has the meaning of the wave function phase shift (compared with the free motion case) acquired in scattering on the potential truncated at the given point. In this way, it is possible to directly, i.e., without finding the wave function, determine the scattering phase for a variety of applications. This method has been actively developed in recent years in application to interactions between quantum particles and a surface [46–49], determination of many-electron system spectra [50], etc. The use of a simplified impedance model for the solution of one-dimensional quantum mechanical problems was considered in [51].

The following advantages of the impedance technique over the straightforward solution of the boundary wave problem are worth noting. First, this method permits obviating the problem of the absence of 'dynamic causality' in the original problem when constructing the analytic theory (e.g., a wave reflected backward depends on the inhomogeneities still unpassed by the forward wave). Solutions of evolutionary equations in turn depend only on the preceding (in space) but not subsequent parameter values. This property permitted, inter alia, completing formulation of the statistical theory of wave propagation in randomly inhomogeneous media and substantiating phenomenological radiative transfer equations [28]. Second, the invariant embedding technique allows avoiding the well-known phenomenon of evanescent modes arising in numerical solutions of problems with a linear coupling of waves [29, 52]. Characteristic of such problems are evanescent regions with an exponential coordinate dependence of the field. Direct numerical integration inevitably results in exponentially growing solutions that suppress all other physical effects in a sufficiently extended evanescent region. Mathematically, they exemplify a 'stiff' system. Similar problems arise in simulating the wave propagation in the vicinity of medium resonances due to the presence of strongly absorbed modes. Reformulation of such problems into equations of the invariant impedance technique leads to the disappearance of exponentially growing solutions; in other words, the problem loses stiffness (in stable media). Therefore, computational methods based on the impedance technique may compete with the finite element method [53–55] or the FDTD method [56] typically used to simulate wave processes in stiff media. Finally, because the total energy is conserved in stable media, the impedance operator is always bounded, which allows efficiently calculating various 'resonances' of the medium in whose vicinity the wave fields are either unbounded or can take very large values.

In the aforecited studies, the invariant impedance technique was largely applied to the scalar wave equation. Its extension to a vector wave equation in an anisotropic and gyrotropic medium is ideologically transparent but involves technical complications caused by cumbersome calculations. For this reason, the ionosphere-related problems of coupled wave propagation in a plane-layered plasma were typically considered only along the inhomogeneity gradient [38–41]. Attempts to obtain more general impedance equations were fraught with mathematical errors [57, 58], corrected only in later work [59]. We distinguish two main causes of the above complications. On the one hand, the classical definition of the invariant embedding technique used, e.g., in [27], remains, for all its mathematical elegance, rather formal and unintelligible to a 'physically' thinking reader. On the other hand, physicists, in deriving equations of the impedance method, frequently proceed from geometro-optical modes of the homogeneous medium (containing a set of modes at each spatial point). In complex media, such as a gyrotropic plasma, this requires cumbersome calculations. In contrast, the impedance method does not necessarily imply separation of geometro-optical modes, and the problem is frequently simplified by passing to a set of vacuum modes [41, 59]. The primary objective of this paper is to present a physically comprehensible tool for the derivation of the impedance method unrelated to the choice of the modal representation of the electromagnetic field [59, 60]. The evolution equations of the invariant embedding technique are regarded as equations for the operator of reflection (from the reduced layer) that couples the waves counterpropagating in the medium along a certain distinguished direction. In the framework of this approach, the formal generalization to the vector wave problem, including that in three-dimensionally inhomogeneous media, is trivial, although it may also require cumbersome calculations in specific problems.

This paper is organized as follows. Section 2 describes a universal approach that allows passing from the linear boundary wave problem in the general form to a certain nonlinear evolutionary problem for the reflection operator. Based on this approach, a system of impedance equations is derived and interpreted in physical terms. In Section 3, this method is applied to the Maxwell equations in media without spatial dispersion. One-dimensionally and three-dimensionally inhomogeneous media are considered separately. Section 4 is focused on the analysis of certain general properties of the impedance equations. We discuss some important specific cases that admit analytic solutions, asymptotic series expansion of the general solution, conservation laws for the general solution, and so on. In Section 5, the developed methods are illustrated with the examples of electromagnetic wave propagation in a cold magnetized plasma with a sufficiently complicated density profile. The results of a numerical calculation of wave fields excited in a plasma layer by an incident wave with a given polarization are presented and analyzed qualitatively. An example is given of the analytic calculation of the electromagnetic radiation reflection matrix for a homogeneous half-space filled with a magnetized plasma. Our main results are summarized in the conclusion.

2. Transition from a boundary wave problem to an evolutionary problem

2.1 The boundary problem for counterpropagating wave equations

In this section, we expound the general idea of the method, regardless of the problem of electromagnetic wave propagation in complex media. We consider a spatially inhomogeneous linear and stationary medium in which a wave field $\mathcal{E}(t, \mathbf{r})$ propagates. We further assume the field to be monochromatic, $\mathcal{E} \sim \exp(\mathrm{i}\omega t)$, and suppose that the problem allows the z direction to be distinguished in the medium, and hence the initial field can be sought in the form of a superposition of two waves propagating forward ($\mathcal{E}^+$) and backward ($\mathcal{E}^-$) along the z axis:

$$\mathcal{E} = \mathcal{E}^+ + \mathcal{E}^- .$$

The specific methods for decomposing the field into counterpropagating waves are discussed in the next section using the Maxwell equations. For now, it suffices to postulate that such a decomposition is possible and that (in a linear medium) the original wave field propagation is described by equations of the form

$$\begin{cases} \partial_z \mathcal{E}^+ = \hat{t}^+ \mathcal{E}^+ + \hat{r}^- \mathcal{E}^- , \\ -\partial_z \mathcal{E}^- = \hat{t}^- \mathcal{E}^- + \hat{r}^+ \mathcal{E}^+ . \end{cases} \tag{2.1}$$

System (2.1) describes the propagation of coupled counterpropagating waves. We consider the first equation. Here, $\hat{t}^+$ is the 'forward' wave evolution operator in the absence of a 'backward' wave ($\partial_z \mathcal{E}^+ = \hat{t}^+ \mathcal{E}^+$) and $\hat{r}^-$ is the operator of the backward wave scattering into the forward wave. The second equation can be analyzed similarly. The minus sign in front of the derivative in the second equation arises because the wave propagation direction is opposite to the z axis direction.

We note that $\mathcal{E}^+$ and $\mathcal{E}^-$ may be vectors of arbitrary (but identical) dimensions. As shown below for the electromagnetic field in a medium without spatial dispersion, this dimension is two. Accordingly, the coefficients $\hat{t}^\pm$ and $\hat{r}^\pm$ in the counterpropagating wave equation in the general case are linear operators local with respect to the z coordinate. In the simplest case of a plane-layered medium (whose spatial inhomogeneity depends on z alone), these operators degenerate into mere multiplication by matrices of the relevant size, depending on z and on the wave number $\mathbf{k}_\perp$ transverse to the inhomogeneity direction and conserved in this geometry. In the case of additional transverse inhomogeneity, these operators can be presented in the form of an integral convolution over the transverse coordinates or transverse wave numbers, e.g.,

$$\begin{aligned} \hat{t}^+ \mathcal{E}^+(\mathbf{r}_\perp, z) &= \int \hat{t}^+(\mathbf{r}_\perp, \mathbf{r}'_\perp, z)\, \mathcal{E}^+(\mathbf{r}'_\perp, z)\, \mathrm{d}^2\mathbf{r}'_\perp , \\ \hat{t}^+ \mathcal{E}^+(\mathbf{k}_\perp, z) &= \int \hat{t}^+(\mathbf{k}_\perp, \mathbf{k}'_\perp, z)\, \mathcal{E}^+(\mathbf{k}'_\perp, z)\, \mathrm{d}^2\mathbf{k}'_\perp , \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathcal{E}^+(\mathbf{k}_\perp, z) &= \int \mathcal{E}^+(\mathbf{r}_\perp, z) \exp(\mathrm{i}\mathbf{k}_\perp \mathbf{r}_\perp)\, \mathrm{d}^2\mathbf{r}_\perp , \\ \hat{t}^+(\mathbf{k}_\perp, \mathbf{k}'_\perp, z) &= \int \hat{t}^+(\mathbf{r}_\perp, \mathbf{r}'_\perp, z) \exp(\mathrm{i}\mathbf{k}_\perp \mathbf{r}_\perp - \mathrm{i}\mathbf{k}'_\perp \mathbf{r}'_\perp)\, \mathrm{d}^2\mathbf{r}_\perp\, \mathrm{d}^2\mathbf{r}'_\perp . \end{aligned} \tag{2.3}$$

In what follows, it is more convenient to deal with the Fourier representation with respect to the transverse coordinates because it permits naturally separating waves propagating in opposite directions along the z axis.

For a broad class of problems, equations for counterpropagating waves can be obtained from natural physical considerations. For infinite (nonwaveguide) media, it is enough if the solutions of Eqn (2.1) in a *homogeneous* medium correspond to the plane waves propagating in a definite direction. We find how the field interacts with an infinitely thin layer $[z, z + \mathrm{d}z]$ of the medium cut out transversely to the z axis and surrounded by a homogeneous medium (vacuum). As shown in Fig. 1, the wave $\mathcal{E}^+(z)$ incident on the layer from $-\infty$ uniquely determines the field $\mathcal{E}^+(z + \mathrm{d}z)$ in the wave transmitted behind the layer and the field $\mathcal{E}^-(z)$ in the reflected wave in front of the layer:

$$\begin{aligned} \mathcal{E}^+(z + \mathrm{d}z) &= \left[\hat{I} + \hat{t}^+(z)\, \mathrm{d}z\right] \mathcal{E}^+(z) , \\ \mathcal{E}^-(z) &= \hat{r}^+(z)\, \mathrm{d}z\, \mathcal{E}^+(z) . \end{aligned} \tag{2.4}$$

Here, $\hat{I} + \hat{t}^+(z)\,\mathrm{d}z$ and $\hat{r}^+(z)\,\mathrm{d}z$ are linear operators that generalize the field transmission and reflection coefficients for an infinitely thin layer, and $\hat{I}$ is the unit (identity) operator. The transmission and reflection operators for the field $\mathcal{E}^-(z)$ incident on the layer from $+\infty$ are defined similarly:

$$\begin{aligned} \mathcal{E}^-(z - \mathrm{d}z) &= \left[\hat{I} + \hat{t}^-(z)\, \mathrm{d}z\right] \mathcal{E}^-(z) , \\ \mathcal{E}^+(z) &= \hat{r}^-(z)\, \mathrm{d}z\, \mathcal{E}^-(z) . \end{aligned} \tag{2.5}$$

We note that the incidence of waves from 'right' and 'left' is not equivalent in general. For this reason, we had to introduce two pairs of transmission and reflection operators differing in the superscript $\pm$ depending on the wave propagation

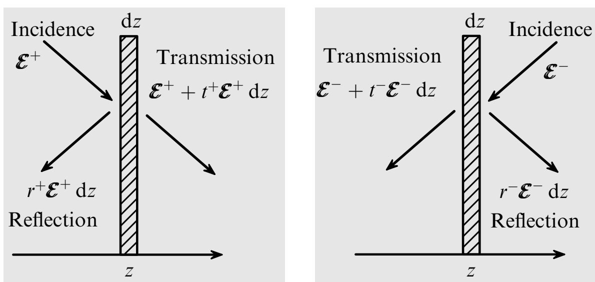

Figure 1. Illustration of the physical meaning of the differential reflection and transmission operators.

direction. By combining relations (2.4) and (2.5), it is easy to obtain equations for the amplitudes of counterpropagating waves in form (2.1). In other words, these equations have a clear physical meaning: the amplitude of each of the waves $\boldsymbol{\mathcal{E}}^{\mp}$ changes as a result of nonideal passage through the medium, $\pm\hat{t}^{\pm}\boldsymbol{\mathcal{E}}^{\pm}$, and reflection of the oncoming wave, $\mp\hat{r}^{\mp}\boldsymbol{\mathcal{E}}^{\mp}$. These processes are responsible for variations in the wave field polarization during propagation in an inhomogeneous medium (including those related to the linear conversion of normal waves) and for field absorption in a dissipative medium.

To find the wave field, the counterpropagating wave equations must be supplemented with boundary conditions. We consider the problem of wave incidence from $z \to -\infty$ on a medium layer lying in the range $z \in [a, b]$. In this case, the incident wave amplitude $\boldsymbol{\mathcal{E}}^{+}$ must be specified in the plane $z = a$ in front of the layer and the condition of the absence of the reflected wave $\boldsymbol{\mathcal{E}}^{-}$ must be specified in the plane $z = b$ behind it; we thus obtain the boundary conditions

$$\boldsymbol{\mathcal{E}}^{+}(a) = \boldsymbol{\mathcal{E}}^{\mathrm{inc}}\,, \qquad \boldsymbol{\mathcal{E}}^{-}(b) = 0\,, \tag{2.6}$$

where $\boldsymbol{\mathcal{E}}^{\mathrm{inc}}$ denotes the incident wave field distribution over the layer surface. We have therefore arrived at the problem in which the initial amplitudes of coupled counterpropagating waves are specified at different ends of the relevant region. As mentioned in the Introduction, a direct solution of such a problem may encounter difficulties in both theoretical analysis and numerical simulation.

2.2 The impedance technique for counterpropagating wave equations

There is a simple method to reduce the boundary problem under consideration to an evolutionary problem with initial conditions. We formally define the impedance operator relating the counterpropagating wave amplitudes in a given section z:

$$\boldsymbol{\mathcal{E}}^{-}(z) = \hat{R}(z)\,\boldsymbol{\mathcal{E}}^{+}(z)\,. \tag{2.7}$$

This operator can be regarded as the operator of reflection from the reduced layer $[z, b]$, relating the wave incident from the side of small z to the reflected wave under the condition that the space outside the layer is filled with a homogeneous medium. For example, $\hat{R}(a)$ defines the operator of reflection from the initial layer $[a, b]$. The operator $\hat{R}(z)$ is linear because the original system of equations for counterpropagating waves is linear. Therefore, this operator has the structure of type (2.2). Substitution of (2.7) in Eqns (2.1) and (2.6) gives

$$\begin{aligned} &\partial_z \boldsymbol{\mathcal{E}}^{+} = (\hat{t}^{+} + \hat{r}^{-}\hat{R})\,\boldsymbol{\mathcal{E}}^{+}\,, && \boldsymbol{\mathcal{E}}^{+}(a) = \boldsymbol{\mathcal{E}}^{\mathrm{inc}}\,, \\ &\partial_z(\hat{R}\boldsymbol{\mathcal{E}}^{+}) = -(\hat{t}^{-}\hat{R} + \hat{r}^{+})\,\boldsymbol{\mathcal{E}}^{+}\,, && \hat{R}(b)\,\boldsymbol{\mathcal{E}}^{+}(b) = 0\,. \end{aligned} \tag{2.8}$$

The second equation can be transformed into

$$\begin{aligned} \partial_z(\hat{R}\boldsymbol{\mathcal{E}}^{+}) &= (\partial_z \hat{R})\,\boldsymbol{\mathcal{E}}^{+} + \hat{R}\,\partial_z \boldsymbol{\mathcal{E}}^{+} \\ &= (\partial_z \hat{R} + \hat{R}\hat{t}^{+} + \hat{R}\hat{r}^{-}\hat{R})\,\boldsymbol{\mathcal{E}}^{+} = -(\hat{t}^{-}\hat{R} + \hat{r}^{+})\,\boldsymbol{\mathcal{E}}^{+} \end{aligned}$$

by eliminating the derivative $\partial_z \boldsymbol{\mathcal{E}}^{+}$. This relation must hold at each point for any $\boldsymbol{\mathcal{E}}^{+}$; therefore, it can be rewritten as an *operator* equation for $\hat{R}$:

$$-\partial_z \hat{R} = \hat{R}\hat{r}^{-}\hat{R} + \hat{R}\hat{t}^{+} + \hat{t}^{-}\hat{R} + \hat{r}^{+}\,, \qquad \hat{R}(b) = 0\,. \tag{2.9}$$

This equation is the generalization of scalar equation (1.4) to the vector and non-one-dimensional problems and contains no counterpropagating wave amplitudes. It leads to a new evolutionary problem that may prove more convenient than the original boundary problem.

We formulate the main algorithm of the impedance method. Variations of the operator $\hat{R}(z)$ along the z coordinate are described by a first-order equation; therefore, the initial condition in (2.9) uniquely defines the solution. Next, the solution found for the reflection operator can be introduced into Eqn (2.8) to obtain the second evolutionary problem for the distribution of the forward wave $\boldsymbol{\mathcal{E}}^{+}(z)$ for a given structure of incident radiation. We note that these two evolutionary problems are characterized by different integration directions: 'from right to left,' i.e., from b to a, for $\hat{R}(z)$ and 'from left to right' for $\boldsymbol{\mathcal{E}}^{+}(z)$; the second evolutionary problem can be solved only after the first one. Finally, with $\hat{R}(z)$ and $\boldsymbol{\mathcal{E}}^{+}(z)$ known, it is possible to reconstruct the $\boldsymbol{\mathcal{E}}^{-}(z)$ wave field distribution from (2.7). In this way, the wave field is determined over the entire region of interest.

Transition from a boundary problem to an evolutionary problem comes at a price. First, a nonlinear equation is obtained for $\hat{R}(z)$ instead of the original system of linear equations. Second, this is an operator equation. This circumstance is not too burdensome for a plane-layered medium in which all the coefficients in Eqn (2.9) are matrices, and the operator $\hat{R}(z)$ is also represented in the form of a numerical matrix with z-dependent elements, and Eqns (2.8) and (2.9) are reduced to systems of ordinary differential equations. But in the general case, we are dealing with operators of type (2.2), which lead to integro-differential equations. For clarity, we write them explicitly (in the Fourier representation) as

$$\begin{aligned} &\boldsymbol{\mathcal{E}}^{-}(\mathbf{k}_\perp, z) = \int \hat{R}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z)\,\boldsymbol{\mathcal{E}}^{+}(\mathbf{k}'_\perp, z)\,\mathrm{d}^2\mathbf{k}'_\perp\,, \\ &-\partial_z \hat{R}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z) \\ &\quad = \iint \hat{R}(\mathbf{k}_\perp, \mathbf{k}''_\perp, z)\,\hat{r}^{-}(\mathbf{k}''_\perp, \mathbf{k}'''_\perp, z)\,\hat{R}(\mathbf{k}'''_\perp, \mathbf{k}'_\perp, z)\,\mathrm{d}^2\mathbf{k}''_\perp\,\mathrm{d}^2\mathbf{k}'''_\perp \\ &\quad + \int \Big[\hat{R}(\mathbf{k}_\perp, \mathbf{k}''_\perp, z)\,\hat{t}^{+}(\mathbf{k}''_\perp, \mathbf{k}'_\perp, z) \\ &\quad + \hat{t}^{-}(\mathbf{k}_\perp, \mathbf{k}''_\perp, z)\,\hat{R}(\mathbf{k}''_\perp, \mathbf{k}'_\perp, z)\Big]\,\mathrm{d}^2\mathbf{k}''_\perp + \hat{r}^{+}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z)\,, \\ &\partial_z \boldsymbol{\mathcal{E}}^{+}(\mathbf{k}_\perp, z) = \int\Big[\int \hat{r}^{-}(\mathbf{k}_\perp, \mathbf{k}''_\perp, z)\,\hat{R}(\mathbf{k}''_\perp, \mathbf{k}'_\perp, z)\,\mathrm{d}^2\mathbf{k}''_\perp \\ &\quad + \hat{t}^{+}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z)\Big]\,\boldsymbol{\mathcal{E}}^{+}(\mathbf{k}'_\perp, z)\,\mathrm{d}^2\mathbf{k}'_\perp\,. \end{aligned} \tag{2.10}$$

2.3 An alternative method for the derivation of impedance equations

To clarify the physical meaning of the above transformations, we first consider an auxiliary problem. We suppose that the incident radiation propagates along z from $-\infty$ in a medium layer $z \in [a, b]$, which can be split into two adjacent sublayers, $[a, c]$ and $[c, b]$. We seek the reflection $\hat{R}^{+}$ and transmission $\hat{T}^{+}$ operators for the initial layer $[a, b]$ under the assumption that all reflection and transmission characteristics are known separately for $[a, c]$ and $[c, b]$.

We introduce some notation. The isolated $[a, c]$ sublayer is characterized by operators of reflection $\hat{R}_1^{+}$, $\hat{R}_1^{-}$ and transmission $\hat{T}_1^{+}$, $\hat{T}_1^{-}$ for the waves incident along the z axis in the 'positive' and 'negative' directions. The second isolated sublayer is characterized by operators $\hat{R}_2^+$, $\hat{R}_2^-$ and $\hat{T}_2^+$, $\hat{T}_2^-$. Collectively, the two sublayers make up a kind of a Fabry–Perot interferometer in which multibeam interference occurs. Summing the contributions of all interfering beams gives the characteristics of the transmitted $\mathcal{E}^{\rm pass}$ and reflected $\mathcal{E}^{\rm refl}$ radiation for the combined $[a, b]$ layer:

$$\mathcal{E}^{\rm pass} \equiv \hat{T}^+\mathcal{E}^{\rm inc} = \hat{T}_2^+\hat{T}_1^+\mathcal{E}^{\rm inc} + \hat{T}_2^+\hat{R}_1^-\hat{R}_2^+\hat{T}_1^+\mathcal{E}^{\rm inc} + \hat{T}_2^+\hat{R}_1^-\hat{R}_2^+\hat{R}_1^-\hat{R}_2^+\hat{T}_1^+\mathcal{E}^{\rm inc} + \ldots,$$

$$\mathcal{E}^{\rm refl} \equiv \hat{R}^+\mathcal{E}^{\rm inc} = \hat{R}_1^+\mathcal{E}^{\rm inc} + \hat{T}_1^-\hat{R}_2^+\hat{T}_1^+\mathcal{E}^{\rm inc} + \hat{T}_1^-\hat{R}_2^+\hat{R}_1^-\hat{R}_2^+\hat{T}_1^+\mathcal{E}^{\rm inc} + \hat{T}_1^-\hat{R}_2^+\hat{R}_1^-\hat{R}_2^+\hat{R}_1^-\hat{R}_2^+\hat{T}_1^+\mathcal{E}^{\rm inc} + \ldots .$$

The right-hand sides contain operator geometric series of the form

$$\sum_{k=0}^{\infty}(\hat{R}_1^-\hat{R}_2^+)^k = (\hat{I} - \hat{R}_1^-\hat{R}_2^+)^{-1},$$

where $(\ldots)^{-1}$ denotes operator inversion. Using this property, it is easy to obtain expressions for the reflection and transmission operators in the initial layer:

$$\begin{aligned}\hat{R}^+ &= \hat{R}_1^+ + \hat{T}_1^-\hat{R}_2^+(\hat{I} - \hat{R}_1^-\hat{R}_2^+)^{-1}\hat{T}_1^+,\\ \hat{T}^+ &= \hat{T}_2^+(\hat{I} - \hat{R}_1^-\hat{R}_2^+)^{-1}\hat{T}_1^+.\end{aligned} \tag{2.11}$$

We show that these operator relations lead to Eqns (2.8) and (2.9) of the impedance method. For example, if one of the sublayers is infinitely thin,

$$[a, c] = [z - {\rm d}z, z], \quad [c, b] = [z, b],$$

then, in accordance with the above definitions,

$$\begin{aligned}&\hat{R}_1^\pm = \hat{r}^\pm\,{\rm d}z, && \hat{T}_1^\pm = \hat{I} + \hat{t}^\pm\,{\rm d}z,\\ &\hat{R}_2^+ = \hat{R}(z), && \hat{T}_2^+ = \hat{T}(z),\\ &\hat{R}^+ = \hat{R}(z - {\rm d}z), && \hat{T}^+ = \hat{T}(z - {\rm d}z).\end{aligned}$$

In the second line, by analogy with the reflection operator $\hat{R}(z)$ introduced above, we define the operator of wave transmission $\hat{T}(z)$ through the reduced layer $[z, b]$. Substitution of the operators thus obtained in relations (2.11) yields

$$\hat{R}(z - {\rm d}z) = \hat{r}^+\,{\rm d}z + (\hat{I} + \hat{t}^-\,{\rm d}z)\,\hat{R}(z)\big(\hat{I} - \hat{r}^-\hat{R}(z)\,{\rm d}z\big)^{-1}(\hat{I} + \hat{t}^+\,{\rm d}z),$$

$$\hat{T}(z - {\rm d}z) = \hat{T}(z)\big(\hat{I} - \hat{r}^-\hat{R}(z)\,{\rm d}z\big)^{-1}(\hat{I} + \hat{t}^+\,{\rm d}z).$$

The Taylor series expansion of these relations in dz leads to expressions for the first-order variation in the reflection and transmission operators:

$$\begin{aligned}-{\rm d}\hat{R} &= (\hat{R}\hat{r}^-\hat{R} + \hat{R}\hat{t}^+ + \hat{t}^-\hat{R} + \hat{r}^+)\,{\rm d}z,\\ -{\rm d}\hat{T} &= \hat{T}(\hat{t}^+ + \hat{r}^-\hat{R})\,{\rm d}z.\end{aligned} \tag{2.12}$$

The first equation exactly corresponds to expression (2.9). As expected, the basic evolution equation describes a change in the transmission operator upon the addition of an infinitely thin layer from the side of incident radiation. Taking the identity ${\rm d}(\hat{T}\hat{T}^{-1}) = {\rm d}\hat{T}\hat{T}^{-1} + \hat{T}\,{\rm d}\hat{T}^{-1} = 0$ into account, the second equation can be written in the form

$${\rm d}\hat{T}^{-1} = (\hat{t}^+ + \hat{r}^-\hat{R})\,\hat{T}^{-1}\,{\rm d}z,$$

equivalent to traveling wave equation (2.8). This coincidence is due to the conservation of $\hat{T}(z)\mathcal{E}^+(z)$, which, by the definition of $\hat{T}(z)$, characterizes the transmitted field at the rear boundary of the layer. Thus, considering the problem of adding an infinitely thin layer to an inhomogeneous layer brings us back to Eqns (2.8) and (2.9). This alternative approach is mathematically less elegant but appears more natural from the physical standpoint.

We emphasize once again that the described method for the solution of boundary wave problems is universal. Although a certain main direction of wave propagation along the z axis was distinguished in deriving the impedance equations, the method is not confined to plane-layered media (even if the equations are simplified in them). To apply the impedance technique to a specific problem, it suffices to find the reflection and transmission operators for a differentially thin layer of the medium. In the next section, we show how to reach this goal in the case of electromagnetic waves propagating in a medium characterized by anisotropic and gyrotropic dielectric and magnetic responses.

3. The impedance technique for Maxwell equations in media without spatial dispersion

In this section, we consider the procedures to reduce the Maxwell equations to equivalent equations for counterpropagating waves (2.1), i.e., to the form adapted for practical application of the impedance technique. We note at the very beginning that the solution of this problem is not unique because of a freedom in choosing the field $\mathcal{E}$ describing electromagnetic waves. Physically, the derivation of counterpropagating wave equations amounts to finding the reflection $\hat{r}^\pm$ and transmission $\hat{t}^\pm$ operators of monochromatic electromagnetic waves interacting with an infinitely thin layer of a medium. Certainly, these operators can be directly calculated for each specific problem based on matching the solutions of the Maxwell equations. Examples of such calculations are given in most textbooks on electrodynamics (see, e.g., Refs [15–17]. But the calculations become extremely cumbersome for complex anisotropic and gyrotropic media. We consider a more elegant and rather compact general method for the derivation of counterpropagating wave equations based on formal transformations of the Maxwell equations.

3.1 Derivation of equations for tangential fields

We consider a medium characterized by anisotropic and gyrotropic dielectric and magnetic responses given by matter equations of the form $\mathbf{D} = \hat{\varepsilon}\mathbf{E}$ and $\mathbf{B} = \hat{\mu}\mathbf{H}$. Here, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$ are complex electromagnetic field vectors proportional to $\exp({\rm i}\omega t)$, and $\hat{\varepsilon}$ and $\hat{\mu}$ are tensors of the dielectric and magnetic permittivity of the medium, depending on the frequency ω and three space coordinates. Because spatial dispersion is disregarded,[1] the right-hand sides of the matter

[1] The extension of the proposed method to the case of spatial dispersion encounters no serious difficulty but strongly depends on the specific dispersion model. Examples of the application of the impedance technique to simulation of wave processes in a hot magnetized plasma with spatial dispersion can be found in [59, 60].

equations contain local operators (multiplication of the field vector by a matrix). In expanded form, these equations are to be understood as

$$D_\alpha(\mathbf{r}) = \sum_\beta \varepsilon_{\alpha\beta}(\mathbf{r})\, E_\beta(\mathbf{r})\,, \qquad B_\alpha(\mathbf{r}) = \sum_\beta \mu_{\alpha\beta}(\mathbf{r})\, H_\beta(\mathbf{r})\,,$$

where the indices α and β, ranging over x, y, and z, denote projections on the respective Cartesian coordinates.

Propagation of monochromatic electromagnetic waves in the above medium is described by the Maxwell equations

$$\operatorname{rot}\mathbf{E} = -\mathrm{i}k_0\mathbf{B}\,, \qquad \operatorname{rot}\mathbf{H} = \mathrm{i}k_0\mathbf{D}$$

or, in the expanded form,

$$\partial_z E_x = -\mathrm{i}k_0 B_y + \partial_x E_z\,, \quad \partial_z E_y = \mathrm{i}k_0 B_x + \partial_y E_z\,, \tag{3.1}$$

$$\partial_z H_x = \mathrm{i}k_0 D_y + \partial_x H_z\,, \quad \partial_z H_y = -\mathrm{i}k_0 D_x + \partial_y H_z\,, \tag{3.2}$$

$$\mathrm{i}k_0 B_z = \partial_y E_x - \partial_x E_y\,, \quad \mathrm{i}k_0 D_z = \partial_x H_y - \partial_y H_x\,. \tag{3.3}$$

We recall that $k_0 = \omega/c$ corresponds to the wave vector modulus of an electromagnetic wave propagating in the vacuum. In accordance with the general idea of the impedance technique, we distinguish a direction along the z axis relative to which we introduce incoming and outgoing waves necessary to define the boundary conditions. The Maxwell equations contain no derivative with respect to longitudinal field components along this direction, which allows expressing the longitudinal fields in terms of transverse fields and their derivatives. It follows from Eqns (3.3) that

$$E_z = -\varepsilon_{zz}^{-1}(\varepsilon_{zx}E_x + \varepsilon_{zy}E_y + \hat{n}_x H_y - \hat{n}_y H_x)\,,$$

$$H_z = -\mu_{zz}^{-1}(\mu_{zx}H_x + \mu_{zy}H_y - \hat{n}_x E_y + \hat{n}_y E_x)\,,$$

where $\hat{n}_x = -\mathrm{i}k_0^{-1}\,\partial/\partial x$ and $\hat{n}_y = -\mathrm{i}k_0^{-1}\,\partial/\partial y$. Substituting the longitudinal fields in the remaining equations (3.1) and (3.2), we obtain a closed differential equation for the evolution of transverse fields. In matrix form, this equation becomes

$$\partial_z\boldsymbol{\Psi} = \mathrm{i}k_0\hat{M}\boldsymbol{\Psi}\,, \qquad \boldsymbol{\Psi} = [E_x, E_y, H_x, H_y]\,, \tag{3.4}$$

where $\boldsymbol{\Psi}$ is the vector of transverse field components and the matrix operator $\hat{M}$ is

$$\hat{M} = \begin{bmatrix} \hat{n}_x\varepsilon'_{zx} + \mu'_{yz}\hat{n}_y & \hat{n}_x\varepsilon'_{zy} - \mu'_{yz}\hat{n}_x & -\hat{n}_x\varepsilon_{zz}^{-1}\hat{n}_y - \mu''_{yx} & \hat{n}_x\varepsilon_{zz}^{-1}\hat{n}_x - \mu''_{yy} \\ \hat{n}_y\varepsilon'_{zx} - \mu'_{xz}\hat{n}_y & \hat{n}_y\varepsilon'_{zy} + \mu'_{xz}\hat{n}_x & -\hat{n}_y\varepsilon_{zz}^{-1}\hat{n}_y + \mu''_{xx} & \hat{n}_y\varepsilon_{zz}^{-1}\hat{n}_x + \mu''_{xy} \\ \varepsilon''_{yx} + \hat{n}_x\mu_{zz}^{-1}\hat{n}_y & \varepsilon''_{yy} - \hat{n}_x\mu_{zz}^{-1}\hat{n}_x & \varepsilon'_{yz}\hat{n}_y + \hat{n}_x\mu'_{zx} & -\varepsilon'_{yz}\hat{n}_x + \hat{n}_x\mu'_{zy} \\ -\varepsilon''_{xx} + \hat{n}_y\mu_{zz}^{-1}\hat{n}_y & -\varepsilon''_{xy} - \hat{n}_y\mu_{zz}^{-1}\hat{n}_x & -\varepsilon'_{xz}\hat{n}_y + \hat{n}_y\mu'_{zx} & \varepsilon'_{xz}\hat{n}_x + \hat{n}_y\mu'_{zy} \end{bmatrix}, \tag{3.5}$$

where $\varepsilon'_{\alpha\beta} = \varepsilon_{zz}^{-1}\varepsilon_{\alpha\beta}$, $\varepsilon''_{\alpha\beta} = \varepsilon_{\alpha\beta} - \varepsilon_{\alpha z}\varepsilon_{zz}^{-1}\varepsilon_{z\beta}$, $\mu'_{\alpha\beta} = \mu_{zz}^{-1}\mu_{\alpha\beta}$, and $\mu''_{\alpha\beta} = \mu_{\alpha\beta} - \mu_{\alpha z}\mu_{zz}^{-1}\mu_{z\beta}$ are new functions depending on the three spatial coordinates. We note that $\hat{M}$ is a second-order differential operator in the transverse coordinates.

These relations were obtained in the **r**-space. The Fourier transformation over the transverse coordinates similar to (2.3) permits readily passing to the **k**-space. For example, we find the first diagonal element of this operator:

$$M_{11}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z) = -\mathrm{i}k_0^{-1}\int\Big[\delta'(x - x')\,\varepsilon'_{zx}(\mathbf{r}'_\perp)$$
$$+ \delta'(y - y')\,\mu'_{yz}(\mathbf{r}_\perp)\Big]\exp\big[\mathrm{i}(\mathbf{k}_\perp\mathbf{r}_\perp - \mathbf{k}'_\perp\mathbf{r}'_\perp)\big]\,\mathrm{d}^2\mathbf{r}_\perp\,\mathrm{d}^2\mathbf{r}'_\perp$$
$$= \frac{k_x}{k_0}\int\varepsilon_{zz}^{-1}(\mathbf{r}_\perp)\,\varepsilon_{zx}(\mathbf{r}_\perp)\exp\big[\mathrm{i}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\mathbf{r}_\perp\big]\,\mathrm{d}^2\mathbf{r}_\perp$$
$$+ \frac{k'_y}{k_0}\int\mu_{zz}^{-1}(\mathbf{r}_\perp)\,\mu_{yz}(\mathbf{r}_\perp)\exp\big[\mathrm{i}(\mathbf{k}_\perp - \mathbf{k}'_\perp)\mathbf{r}_\perp\big]\,\mathrm{d}^2\mathbf{r}_\perp\,.$$

In the general case, we arrive at the integro-differential evolution equation for tangential field components

$$\partial_z\boldsymbol{\Psi}(\mathbf{k}_\perp, z) = \mathrm{i}k_0\int\hat{M}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z)\,\boldsymbol{\Psi}(\mathbf{k}'_\perp, z)\,\mathrm{d}^2\mathbf{k}'_\perp\,.$$

The convolution over $\mathbf{k}'_\perp$ reflects the dependence of the Fourier harmonics in a transversely inhomogeneous medium. In the simplest (but important for applications) case of a plane-layered medium inhomogeneous only along the z axis, each Fourier harmonic propagates independently with a constant transverse wave vector $\mathbf{k}_\perp$. Formally, this manifests itself as the degeneration of $\hat{M}$ into the operator of multiplication by a numerical matrix dependent on $\mathbf{k}_\perp$ and z. This matrix is given by formula (3.5) in which $\hat{n}_x$ and $\hat{n}_y$ are the corresponding projections of the conserved vector $\mathbf{k}_\perp/k_0$.

Equation (3.4) can be reduced to an equivalent form describing interactions between the counterpropagating waves. We assume that we know the representation of the initial wave field $\boldsymbol{\Psi}$ through counterpropagating waves $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ in the form

$$\boldsymbol{\Psi} = \hat{U}\begin{bmatrix}\boldsymbol{\mathcal{E}}^+\\ \boldsymbol{\mathcal{E}}^-\end{bmatrix}, \tag{3.6}$$

where $\hat{U}$ is a linear operator governing the transformation of the 'new' fields into the 'old' ones, $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ are the two vector fields that uniquely characterize the distribution of the original wave field and correspond to the waves propagating along and opposite the z axis, and the term in the square brackets is a vector composed of the $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ field components. The operator $\hat{U}$ may depend on the longitudinal coordinate only locally (i.e., it does not contain z derivatives). After the substitution of (3.6) in (3.4), it follows that the new fields must satisfy the equation

$$\partial_z\left(\hat{U}\begin{bmatrix}\boldsymbol{\mathcal{E}}^+\\ \boldsymbol{\mathcal{E}}^-\end{bmatrix}\right) = \mathrm{i}k_0\hat{M}\hat{U}\begin{bmatrix}\boldsymbol{\mathcal{E}}^+\\ \boldsymbol{\mathcal{E}}^-\end{bmatrix}.$$

This equation can be represented as

$$\partial_z\begin{bmatrix}\boldsymbol{\mathcal{E}}^+\\ \boldsymbol{\mathcal{E}}^-\end{bmatrix} = \hat{D}\begin{bmatrix}\boldsymbol{\mathcal{E}}^+\\ \boldsymbol{\mathcal{E}}^-\end{bmatrix}, \tag{3.7}$$

where

$$\hat{D} = \mathrm{i}k_0\hat{U}^{-1}\hat{M}\hat{U} - \hat{U}^{-1}\,\partial_z\hat{U}\,. \tag{3.8}$$

Evidently, Eqn (3.7) is exactly equivalent to the equations for counterpropagating waves (2.1) with the coefficients $\hat{t}^\pm$ and $\hat{r}^\pm$ determined by the relation

$$\hat{D} = \begin{bmatrix} +\hat{t}^+ & +\hat{r}^- \\ -\hat{r}^+ & -\hat{t}^- \end{bmatrix}. \tag{3.9}$$

We see that relation (3.8) provides a relatively simple general method to calculate the reflection and transmission operators for a differentially thin layer; these operators fully determine counterpropagating wave equations for a medium with the given $\hat{\varepsilon}(\mathbf{r})$ and $\hat{\mu}(\mathbf{r})$. For this, it is necessary to find the operator matrix $\hat{D}$ and then to split it into four equal-size quadrants, each to be matched with one of the four reflection and transmission operators in (3.9). For example, the $\hat{D}$ operator in a plane-layered medium is a 4×4 matrix; in this case,

$$\hat{t}^+ = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \qquad \hat{t}^- = -\begin{bmatrix} D_{33} & D_{34} \\ D_{43} & D_{44} \end{bmatrix},$$

$$\hat{r}^+ = -\begin{bmatrix} D_{31} & D_{32} \\ D_{41} & D_{42} \end{bmatrix}, \qquad \hat{r}^- = \begin{bmatrix} D_{13} & D_{14} \\ D_{23} & D_{24} \end{bmatrix}.$$

The problem stated at the beginning of this section is thus formally solved.

To apply the impedance method to the Maxwell equations with boundary conditions (2.6), it only remains to choose a particular way to represent the initial wave field $\mathbf{\Psi}$ in terms of counterpropagating waves. Two approaches to such representation are discussed in Sections 3.2–3.4.

3.2 Field decomposition in terms of local waveguide modes

The wave propagation direction can be found by considering Eqn (3.4) in the geometric optics approximation, i.e., neglecting the dependence of the wave operator on the longitudinal coordinate. We suppose that

$$\mathbf{\Psi} = \tilde{\mathbf{\Psi}}(z) \exp\left(-\mathrm{i}\int k_z(z)\,\mathrm{d}z\right),$$

where $\tilde{\mathbf{\Psi}}$ is a slowly varying function of the z coordinate in the **r**- or **k**-space. Then the initial equation becomes

$$-\mathrm{i}k_z(z)\,\tilde{\mathbf{\Psi}}(z) = \mathrm{i}k_0\hat{M}(z)\,\tilde{\mathbf{\Psi}}(z)\,. \tag{3.10}$$

The longitudinal coordinate z enters these equations as a parameter. In each section z, these equations are to be regarded as equations for waveguide eigenmodes in a *locally homogeneous medium along the z coordinate*. The theory of such equations has been developed in Refs [17, 61–63]. It may be assumed for most physical problems that the equations describe a discrete spectrum of eigenmodes $\tilde{\mathbf{\Psi}}_i$ forming the complete basis in the space of $\mathbf{\Psi}$ vectors in each section along the longitudinal coordinate (see the Appendix for more details).

We divide the set of all eigenmodes into two groups in accordance with the direction of propagation along the z axis. Group 1 includes propagating or exponentially decaying evanescent modes in the positive direction of the z axis, and group 2 contains modes propagating or exponentially decaying in the negative direction. We introduce the notation $\tilde{\mathbf{\Psi}}_i^+$ and $k_{z,i}^+$ for the modes of group 1 and the respective eigenvalues, and $\tilde{\mathbf{\Psi}}_i^-$ and $k_{z,i}^-$ for the modes and eigenvalues of group 2. Generally, classification of modes in terms of propagation direction must reflect the direction of the group velocity vector, which is not necessarily coincident with the wave vector direction in anisotropic and gyrotropic dispersive media. In many cases, this discrepancy can be neglected and classification can be based on the analysis of the wave vector direction. In this case, the $\tilde{\mathbf{\Psi}}_i^+$ modes correspond to wave vectors with $\mathrm{Re}\,k_z^+ > 0$ (forward propagation along the z axis) and with $\mathrm{Re}\,k_z^+ = 0$, $\mathrm{Im}\,k_z^+ < 0$ (exponential decay in the positive direction along the same axis) and the $\tilde{\mathbf{\Psi}}_i^-$ modes correspond to wave vectors with $\mathrm{Re}\,k_z^- < 0$ (backward propagation along the z axis) and with $\mathrm{Re}\,k_z^- = 0$, $\mathrm{Im}\,k_z^- > 0$ (exponential decay in the negative direction along the same axis). Evidently, all the modes fall into one of these categories. Except in special cases, for each mode propagating in a given direction, there is a counterpropagating wave.

Because the set of eigenmodes forms a complete basis, the solution of Eqn (3.4) in each section along the longitudinal coordinate can be presented as

$$\mathbf{\Psi}(\mathbf{r}_\perp, z) = \sum_i \mathscr{E}_i^+(z)\,\tilde{\mathbf{\Psi}}_i^+(\mathbf{r}_\perp, z) + \sum_i \mathscr{E}_i^-(z)\,\tilde{\mathbf{\Psi}}_i^-(\mathbf{r}_\perp, z)\,, \tag{3.11a}$$

or

$$\mathbf{\Psi}(\mathbf{k}_\perp, z) = \sum_i \mathscr{E}_i^+(z)\,\tilde{\mathbf{\Psi}}_i^+(\mathbf{k}_\perp, z) + \sum_i \mathscr{E}_i^-(z)\,\tilde{\mathbf{\Psi}}_i^-(\mathbf{k}_\perp, z)\,, \tag{3.11b}$$

where $\mathscr{E}_i^+$ and $\mathscr{E}_i^-$ are new variables uniquely characterizing the field distribution and corresponding to the waves propagating in a definite direction. This procedure is the most straightforward way to decompose the initial wave field into waves propagating forward and backward along the z axis. We now update the notation in agreement with formula (3.6); the variables $\mathscr{E}_i^+$ and $\mathscr{E}_i^-$ are to be viewed as elements of the respective vectors $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$, and the $\tilde{\mathbf{\Psi}}_i^+$ and $\tilde{\mathbf{\Psi}}_i^-$ vectors as columns of a matrix $\hat{U}$:

$$\boldsymbol{\mathcal{E}}^+ = [\mathscr{E}_1^+, \mathscr{E}_2^+, \ldots]\,, \qquad \boldsymbol{\mathcal{E}}^- = [\mathscr{E}_1^-, \mathscr{E}_2^-, \ldots]\,,$$

$$\hat{U} = [\tilde{\mathbf{\Psi}}_1^+, \tilde{\mathbf{\Psi}}_2^+, \ldots, \tilde{\mathbf{\Psi}}_1^-, \tilde{\mathbf{\Psi}}_2^-, \ldots]\,.$$

Using relations (3.8) and (3.9), it is now possible to find the $\hat{D}$ operator and obtain the impedance method equations.

Because the counterpropagating waves $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ are composed of the amplitudes of normal modes, which are eigenvectors of the wave operator $\hat{M}$, the equation for counterpropagating waves and the impedance equations are generally a system of an infinite number of coupled *ordinary* differential equations involving no nonlocal operators (integral convolutions or differentiation with respect to transverse coordinates). Indeed, in the normal wave representation, the first term in (3.8) is a diagonal matrix composed of the wave operator eigenvalues:

$$\hat{D} = \left[\begin{array}{c:c} -\mathrm{i}\hat{k}_z^+ & 0 \\ \hdashline 0 & -\mathrm{i}\hat{k}_z^- \end{array}\right] - \hat{U}^{-1}\,\partial_z \hat{U}\,, \qquad \hat{k}_z^\pm = \mathrm{diag}\,(k_{z,1}^\pm, k_{z,2}^\pm, \ldots)\,. \tag{3.12}$$

It can be seen that the operator $\hat{D}$ is a *numerical* matrix. The same refers to the reflection and transmission operators (matrices) of a differentially thin layer, $\hat{t}^\pm$ and $\hat{r}^\pm$. The physical meaning of the elements of the scattering matrix is as follows:

r_{ij}^+ is the coefficient of reflection of $\tilde{\mathbf{\Psi}}_i^+$ into $\tilde{\mathbf{\Psi}}_j^-$;
r_{ij}^- is the coefficient of reflection of $\tilde{\mathbf{\Psi}}_i^-$ into $\tilde{\mathbf{\Psi}}_j^+$;
t_{ij}^+ is the coefficient of transmission of $\tilde{\mathbf{\Psi}}_i^+$ into $\tilde{\mathbf{\Psi}}_j^+$;
t_{ij}^- is the coefficient of transmission of $\tilde{\mathbf{\Psi}}_i^-$ into $\tilde{\mathbf{\Psi}}_j^-$.

The off-diagonal terms in the transition matrices define the coupling of modes with different indices i and j (mode cross-polarization). We note that both reflection and cross polarization of eigenmodes are possible only in a z-inhomogeneous medium. This fact, which is obvious from the physical standpoint, becomes readily apparent from formula (3.12). Indeed, its first term is responsible for the propagation described by the matrices $\hat{t}^{\pm} = \mp\mathrm{i}\hat{k}_z^{\pm}$ and $\hat{r}^{\pm} = 0$ in agreement with geometric optics laws. All corrections due to deviations from geometric optics are taken into account by the second term originating from spatial variations in the polarization of local eigenmodes of the medium. These corrections precisely reduce to the mode reflection and linear interaction.

The matrices $\hat{t}^{\pm}$ and $\hat{r}^{\pm}$ can be associated with Eqns (2.8) and (2.9), while the impedance method described in the preceding section can then be used to reconstruct electromagnetic fields. In this case, the operator $\hat{R}$ is a numerical 4×4 matrix composed of the coefficients of reflection of the ith mode into the jth mode. Although the wave field decomposition implies the use of the modes obtained in the geometric optics *approximation*, the impedance method yields a *rigorous* solution of the original wave problem as long as boundary conditions (2.6) are strictly satisfied for the $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ fields. This goal is achieved if the region of interest is embedded in a homogeneous medium; then the geometric optics approximation becomes rigorous at its boundaries.

We somewhat detail the case of a plane-layered medium changing only in the longitudinal direction. The solution of the wave equation can then be sought in the form $\tilde{\boldsymbol{\Psi}} \sim \exp(-\mathrm{i}\mathbf{k}_\perp \mathbf{r}_\perp)$, where $\mathbf{k}_\perp$ plays the role of a conserved parameter. Equation (3.10) then degenerates into a linear *algebraic* equation for eigenmodes of a locally homogeneous medium, and the numerical matrix $\hat{M}$ has size 4×4. Hence, there are four eigenvectors corresponding to two counterpropagating modes (except in special cases of degeneration). In other words, the counterpropagating waves $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ are given by two-component vectors composed of complex amplitudes of the normal modes corresponding to a definite propagation direction, the operators $\hat{U}$ and $\hat{D}$ are represented by numerical 4×4 matrices, and the reflection and transmission operators $\hat{t}^{\pm}$, $\hat{r}^{\pm}$, and $\hat{R}$ are represented by numerical 2×2 matrices. The first equation of the impedance method (2.9) degenerates into a system of four nonlinear ordinary differential equations and Eqn (2.8) into a system of two linear ordinary differential equations.

If the medium is inhomogeneous in a certain transverse direction, then instead of a wave vector conserved along this direction, we have an infinite set of discrete modes corresponding to 'quantization' along this direction. Conversely, if the medium is homogeneous in a certain transverse direction, then the corresponding discrete index is replaced by a conserved wave number. This corresponds to a two-dimensionally inhomogeneous problem. We therefore have $\tilde{\boldsymbol{\Psi}}_i^{\pm}(k_x, k_y)$ in a plane-layered medium, $\tilde{\boldsymbol{\Psi}}_{ij}^{\pm}(k_x)$ in a two-dimensionally inhomogeneous medium, and $\tilde{\boldsymbol{\Psi}}_{ijk}^{\pm}$ in a three-dimensionally inhomogeneous medium. Here, k_x and k_y are the conserved transverse wave vectors, the index i takes only two values, and j and k range over an infinite number of discrete values. This means that the representation of the initial field through local wave modes in the transverse inhomogeneity problem automatically implies the use of the 'spectral' method for solving the original (integral or differential) problem, leading to impedance equations in the form of a system of infinitely many coupled ordinary differential equations (although a finite number of modes may be sufficient for practical calculations).

3.3 Field decomposition in terms of boundary modes

The representation of counterpropagating waves in terms of the amplitudes of local normal modes considered in the previous section is not unique. It may be convenient for the purpose of practical calculations to use a simpler representation of counterpropagating waves through the amplitudes of 'boundary' modes whose structure does not change along the z coordinate. We note before considering this representation that the use of the impedance method requires a correspondence of the fields $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ with a definite propagation direction *only at the boundaries of the layer*, where this is necessary for setting the boundary conditions. At all other points, it is sufficient that these fields uniquely characterize the original wave field distribution, even though they do not correspond to waves with a definite propagation direction. From this standpoint, the fields $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^-$ can be regarded as a formal change of variables in the Maxwell equations chosen so as to enable the boundary conditions to be written in the simplest possible form.

We assume that the transverse inhomogeneity disappears at the boundaries of the layer. This is a natural and virtually nonrestrictive physical assumption because it is always possible to refine the definition of the inhomogeneous layer at the boundaries so as to satisfy it. Then the wave field can be expanded in terms of eigenmodes of the homogeneous problem at the layer boundaries:

$$\boldsymbol{\Psi}(\mathbf{k}_\perp, z) = \sum_{i=1,2} \mathscr{E}_{0i}^{+}(\mathbf{k}_\perp, z)\, \tilde{\boldsymbol{\Psi}}_{0i}^{+}(\mathbf{k}_\perp) + \sum_{i=1,2} \mathscr{E}_{0i}^{-}(\mathbf{k}_\perp, z)\, \tilde{\boldsymbol{\Psi}}_{0i}^{-}(\mathbf{k}_\perp)\,.$$

As a basis in this expansion, we can choose the eigenmodes $\tilde{\boldsymbol{\Psi}}_{0i}^{+}$ corresponding to $\mathrm{Re}\, k_z > 0$ or $\mathrm{Re}\, k_z = 0$, $\mathrm{Im}\, k_z < 0$ at one layer boundary, $z = a$,

$$-\mathrm{i}k_z(\mathbf{k}_\perp)\, \tilde{\boldsymbol{\Psi}}_0^{+}(\mathbf{k}_\perp) = \mathrm{i}k_0 \hat{M}(\mathbf{k}_\perp, a)\, \tilde{\boldsymbol{\Psi}}_0^{+}(\mathbf{k}_\perp)\,, \qquad (3.13\mathrm{a})$$

and the eigenmodes $\tilde{\boldsymbol{\Psi}}_{0i}^{-}$ corresponding to $\mathrm{Re}\, k_z < 0$ or $\mathrm{Re}\, k_z = 0$, $\mathrm{Im}\, k_z > 0$ at the other boundary, $z = b$,

$$-\mathrm{i}k_z(\mathbf{k}_\perp)\, \tilde{\boldsymbol{\Psi}}_0^{-}(\mathbf{k}_\perp) = \mathrm{i}k_0 \hat{M}(\mathbf{k}_\perp, b)\, \tilde{\boldsymbol{\Psi}}_0^{-}(\mathbf{k}_\perp)\,. \qquad (3.13\mathrm{b})$$

These modes are independent of the longitudinal coordinate and have a simpler structure than the waveguide modes; they result from the solution of an algebraic problem and occur in a finite number (four, except for degenerate cases). These modes define the transition matrix

$$\hat{U}_0 = \left[\tilde{\boldsymbol{\Psi}}_{01}^{+}, \tilde{\boldsymbol{\Psi}}_{02}^{+}, \tilde{\boldsymbol{\Psi}}_{01}^{-}, \tilde{\boldsymbol{\Psi}}_{02}^{-}\right]$$

from the fields $\boldsymbol{\mathcal{E}}_0^+$ and $\boldsymbol{\mathcal{E}}_0^-$ to the original field by a formula analogous to (3.6). The $\boldsymbol{\mathcal{E}}_0^+$ and $\boldsymbol{\mathcal{E}}_0^-$ fields are introduced such that they satisfy boundary conditions (2.6), which allows the impedance technique to be applied to them. The fact that neither $\boldsymbol{\mathcal{E}}_0^+$ nor $\boldsymbol{\mathcal{E}}_0^-$ corresponds to the counterpropagating waves *inside the layer* is not a serious limitation, and the local modes can be reconstructed whenever necessary (see Section 4.1). Because the matrix $\hat{U}_0$ is independent of the longitudinal coordinate, expression

(3.8) is simplified to

$$\hat{D} = \mathrm{i}k_0 \hat{U}_0^{-1} \hat{M} \hat{U}_0 \,. \qquad (3.14)$$

Both the $\hat{D}$ matrix itself and the impedance equations are then much simpler than in the local waveguide mode representation, because of the much simpler mode structure at the boundary.

One more advantage of this method is that it permits determining counterpropagating waves $\boldsymbol{\mathcal{E}}_0^+(\mathbf{k}_\perp, z)$ and $\boldsymbol{\mathcal{E}}_0^-(\mathbf{k}_\perp, z)$ and the transition matrix $\hat{U}_0(\mathbf{k}_\perp)$ in a transverse inhomogeneity problem in exactly the same way as in a plane-layered medium; in fact, the counterpropagating waves are given by two-dimensional vectors and the transition matrix has the finite size 4×4. The difference from the one-dimensional case is apparent only in the structure of the operator $\hat{D}$ defined in (3.4). In a non-one-dimensional case, this operator is nonlocal because the original operator $\hat{M}$ is also nonlocal. The action of the operator on an arbitrary function $\phi(\mathbf{k}_\perp)$ can be explicitly represented as

$$\hat{D}\phi(\mathbf{k}_\perp) = \mathrm{i}k_0 \int \hat{U}_0^{-1}(\mathbf{k}_\perp)\, \hat{M}(\mathbf{k}_\perp, \mathbf{k}'_\perp, z)\, \hat{U}_0(\mathbf{k}'_\perp)\, \phi(\mathbf{k}'_\perp)\, \mathrm{d}^2\mathbf{k}'_\perp \,.$$

The nonlocality reflects that the transverse wave vector is not conserved in an inhomogeneous medium. By splitting this matrix operator into four 2×2 matrices in accordance with formula (2.9), we find the counterpropagating wave equation and impedance equations in the Fourier representation. These equations are obtained in the form of integro-differential relations for vectors and matrices of finite size (2 and 2×2). In a plane-layered medium, these equations degenerate into a system of ordinary differential equations.

3.4 Field decomposition in terms of vacuum modes

We consider the most natural scenario for the implementation of the proposed method, when both ends of the layer adjoin with the vacuum. For clarity, we first consider the case of plane-layered geometry under the assumption that a plane electromagnetic wave $\propto \exp(\mathrm{i}\omega t - \mathrm{i}\mathbf{kr})$ is incident on the layer from the vacuum at an angle ϑ to the normal. To simplify calculations, the transverse coordinates x and y are oriented such that the incident wave vector lies in the xz plane, i.e.,

$$k_x = k_0 \sin\vartheta\,, \quad k_y = 0\,, \quad k_z = k_0 \cos\vartheta\,.$$

We recall that k_x and k_y are conserved during wave propagation because the problem is one-dimensional in the xy plane. As a result, the wave operator $\hat{M}$ in the vacuum becomes

$$\hat{M}_0 = \begin{bmatrix} 0 & 0 & 0 & -\cos^2\vartheta \\ 0 & 0 & 1 & 0 \\ 0 & \cos^2\vartheta & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

which corresponds to the simple eigenmode matrix

$$\hat{U}_0 = \begin{bmatrix} 0 & \cos\vartheta & 0 & -\cos\vartheta \\ 1 & 0 & 1 & 0 \\ -\cos\vartheta & 0 & \cos\vartheta & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \qquad (3.15)$$
$$\quad\;\; \tilde{\boldsymbol{\Psi}}_{01}^+ \quad \tilde{\boldsymbol{\Psi}}_{02}^+ \quad \tilde{\boldsymbol{\Psi}}_{01}^- \quad \tilde{\boldsymbol{\Psi}}_{02}^-$$

Due to the continuity of tangential electromagnetic fields forming the $\boldsymbol{\Psi}$ vector, the same eigenmodes can be used to decompose the fields into counterpropagating waves at the layer boundary. Using transformations (3.14), we find the matrix and split it into four parts, in accordance with (3.9). This gives the four desired reflection and transmission operators of a differentially thin layer:

$$\hat{r}^+ = \frac{\mathrm{i}k_0}{2} \times \begin{bmatrix} M_{23}\cos\vartheta - M_{32}\sec\vartheta - M_{22} + M_{33}, & -M_{21}\cos\vartheta - M_{34}\sec\vartheta - M_{24} - M_{31} \\ M_{43}\cos\vartheta + M_{12}\sec\vartheta - M_{13} - M_{42}, & -M_{41}\cos\vartheta + M_{14}\sec\vartheta + M_{11} - M_{44} \end{bmatrix},$$

$$\hat{r}^- = \frac{\mathrm{i}k_0}{2} \times \begin{bmatrix} M_{23}\cos\vartheta - M_{32}\sec\vartheta + M_{22} - M_{33}, & -M_{21}\cos\vartheta - M_{34}\sec\vartheta + M_{24} + M_{31} \\ M_{43}\cos\vartheta + M_{12}\sec\vartheta + M_{13} + M_{42}, & -M_{41}\cos\vartheta + M_{14}\sec\vartheta - M_{11} + M_{44} \end{bmatrix},$$

$$\hat{t}^+ = \frac{\mathrm{i}k_0}{2} \times \begin{bmatrix} -M_{23}\cos\vartheta - M_{32}\sec\vartheta + M_{22} + M_{33}, & M_{21}\cos\vartheta - M_{34}\sec\vartheta + M_{24} - M_{31} \\ -M_{43}\cos\vartheta + M_{12}\sec\vartheta - M_{13} + M_{42}, & M_{41}\cos\vartheta + M_{14}\sec\vartheta + M_{11} + M_{44} \end{bmatrix},$$

$$\hat{t}^- = \frac{\mathrm{i}k_0}{2} \times \begin{bmatrix} -M_{23}\cos\vartheta - M_{32}\sec\vartheta - M_{22} - M_{33}, & M_{21}\cos\vartheta - M_{34}\sec\vartheta - M_{24} + M_{31} \\ -M_{43}\cos\vartheta + M_{12}\sec\vartheta + M_{13} - M_{42}, & M_{41}\cos\vartheta + M_{14}\sec\vartheta - M_{11} - M_{44} \end{bmatrix}.$$

These expressions can be represented in a more compact form as

$$\begin{aligned} \hat{r}^\pm &= \hat{\mu}_1 \cos\vartheta + \hat{\mu}_2 \sec\vartheta \mp \hat{\mu}_3 \pm \hat{\mu}_4\,, \\ \hat{t}^\pm &= -\hat{\mu}_1 \cos\vartheta + \hat{\mu}_2 \sec\vartheta \pm \hat{\mu}_3 \pm \hat{\mu}_4\,, \end{aligned} \qquad (3.16)$$

where

$$\hat{\mu}_1 = \frac{\mathrm{i}k_0}{2}\begin{bmatrix} M_{23} & -M_{21} \\ M_{43} & -M_{41} \end{bmatrix}, \qquad \hat{\mu}_2 = \frac{\mathrm{i}k_0}{2}\begin{bmatrix} -M_{32} & -M_{34} \\ M_{12} & M_{14} \end{bmatrix},$$
$$\hat{\mu}_3 = \frac{\mathrm{i}k_0}{2}\begin{bmatrix} M_{22} & M_{24} \\ M_{42} & M_{44} \end{bmatrix}, \qquad \hat{\mu}_4 = \frac{\mathrm{i}k_0}{2}\begin{bmatrix} M_{33} & -M_{31} \\ -M_{13} & M_{11} \end{bmatrix}, \qquad (3.17)$$

and M_{ij} denotes elements of matrix (3.5) with $n_x = \sin\vartheta$ and $n_y = 0$. In the case under consideration, $M_{21} = M_{43} = 0$, but these terms are preserved for symmetry and generalization to the case of a three-dimensionally inhomogeneous medium. We note that the reflection matrices at $\hat{\mu}_3 = \hat{\mu}_4$ and the transmission matrices at $\hat{\mu}_3 = -\hat{\mu}_4$ are independent of the wave propagation direction (the $\pm$ sign). Interestingly, the fulfillment of these conditions automatically ensures the equality of the reflection and transmission coefficients in cross polarization because the matrix $\hat{\mu}_1$ is diagonal in the plane-layered geometry.

The proposed representation of the field in the form of a vacuum mode superposition has a clear physical meaning. For plane waves in the vacuum, $\mathbf{E} \perp \mathbf{H} \perp \mathbf{k}$, which readily implies that the eigenmodes $\tilde{\boldsymbol{\Psi}}_{01}^\pm$ given by the first and third columns of matrix (3.15) correspond to vacuum plane waves with a unit amplitude in which the electric field vector has no component along the normal to the layer boundary. Such waves are usually called transverse electric (TE)-polarized with respect to the planar boundary of the layer. The eigenmodes $\tilde{\boldsymbol{\Psi}}_{02}^\pm$ given by the second and fourth columns of matrix (3.15) correspond to transverse magnetic (TM)-polarized vacuum plane waves with a unit amplitude.[2] Corresponding to such a set of eigenmodes are reflection matrices composed of reflection coefficients in the amplitude for TE and TM waves. For example,

r_{11}^{+} is the reflection of a TE wave incident from $-\infty$ into a TE wave,

r_{11}^{-} is the reflection of a TE wave incident from $+\infty$ into a TE wave,

r_{22}^{+} is the reflection of a TM wave incident from $-\infty$ into a TM wave,

r_{12}^{+} is the reflection of a TM wave incident from $-\infty$ into a TE wave,

r_{21}^{+} is the reflection of a TE wave incident from $-\infty$ into a TM wave, etc.

Elements of the transmission matrices $\hat{t}^{\pm}$ are interpreted similarly. Clearly, this representation leads to the classical statement of the problem of wave reflection from and transmission through a layer, with the radiation outside the layer decomposed into TE- and TM-polarized waves. We note that nondiagonal elements of the reflection and transmission operators determine cross polarization of the TE and TM waves, which can be regarded as a linear coupling of these waves incident on an anisotropic and/or gyrotropic medium.

To conclude, we consider a transversely non-one-dimensional medium. In a plane-layered medium, we are free to choose the direction of x and y axes; therefore, the vacuum eigenmodes were defined in the case where the vector $\mathbf{k}_{\perp}$ was directed along the x axis. A non-one-dimensional problem gives no such freedom, and the expressions for TE and TM modes have to be generalized to the case of an arbitrary orientation of the wave vector:

$$k_x = k_0 \cos\varphi \sin\vartheta\,, \quad k_y = k_0 \sin\varphi \sin\vartheta\,, \quad k_z = k_0 \cos\vartheta\,.$$

It is easy to see that such a generalization is achieved by simple rotation of the transverse coordinates through an angle φ. The matrix of vacuum modes can be obtained as

$$\hat{U}_0(\varphi,\vartheta) = \begin{bmatrix} \hat{T}_{\varphi} & 0 \\ 0 & \hat{T}_{\varphi} \end{bmatrix} \hat{U}_0(0,\vartheta)\,, \qquad \hat{T}_{\varphi} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix},$$

where $\hat{U}_0(0,\vartheta)$ is defined by formula (3.15) and $\hat{T}_{\varphi}$ is the matrix of rotation through the angle φ in the xy plane. Applying some simple algebra, we can represent the kernel of the integral operator $\hat{D}$ as

$$\hat{D}(\mathbf{k}_{\perp},\mathbf{k}'_{\perp},z) = \hat{U}_0^{-1}(0,\vartheta)\begin{bmatrix} \hat{T}_{-\varphi} & 0 \\ 0 & \hat{T}_{-\varphi} \end{bmatrix} \times \mathrm{i}k_0 \hat{M}(\mathbf{k}_{\perp},\mathbf{k}'_{\perp},z)\begin{bmatrix} \hat{T}_{\varphi'} & 0 \\ 0 & \hat{T}_{\varphi'} \end{bmatrix} \hat{U}_0(0,\vartheta')\,.$$

Here, φ' and ϑ' correspond to the $\mathbf{k}'_{\perp}$ vector. Hence, the reflection and transmission operators determining the coefficients in nonlocal equations (2.10) of the impedance method are given by

$$\hat{r}^{\pm}(\mathbf{k}_{\perp},\mathbf{k}'_{\perp},z) = \hat{T}_{-\varphi}(\hat{\mu}_1\cos\vartheta' + \hat{\mu}_2\sec\vartheta \mp \hat{\mu}_3 \pm \hat{\mu}_4)\,\hat{T}_{\varphi'}\,,$$
$$\hat{t}^{\pm}(\mathbf{k}_{\perp},\mathbf{k}'_{\perp},z) = \hat{T}_{-\varphi}(-\hat{\mu}_1\cos\vartheta' + \hat{\mu}_2\sec\vartheta \pm \hat{\mu}_3 \pm \hat{\mu}_4)\,\hat{T}_{\varphi'}\,,$$

[2] For TE waves, $E_x = E_z = H_y = 0$; for TM waves, $H_x = H_z = E_y = 0$. Therefore, the amplitudes of TE and TM waves in the chosen system of coordinates are respectively characterized by E_y and H_y. It is easy to see that these field components form the vector $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^{+} + \boldsymbol{\mathcal{E}}^{-}$ for which the impedance method described in Section 2 was developed.

where $\hat{\mu}_j = \hat{\mu}_j(\mathbf{k}_{\perp},\mathbf{k}'_{\perp},z)$ are the kernels of the integral matrix operators in (3.17).

3.5 Summary

In the foregoing, we have considered two ways of field decomposition in terms of local waveguide and boundary vacuum modes. The field representation in a one-dimensional plane-layered problem through local and vacuum modes leads to similar impedance equations differing only in coefficients. The representation in terms of local modes is physically more sound because it explicitly discriminates between the effects of geometric optics, wave scattering, and cross polarization arising from medium inhomogeneity. At the same time, such a representation often involves bulky expressions due to the complex polarization of normal modes and the necessity to differentiate polarization vectors. The second approach is more convenient for analytic purposes because it permits avoiding calculation of normal modes. However, the decomposition into vacuum modes encounters additional difficulties in the interpretation of these solutions, which typically need to be 'translated' into medium modes. From the standpoint of the numerical solution of impedance equations, both problems are technically equivalent, i.e., the complication of the impedance equations upon transition to medium modes has no serious consequences. Therefore, the choice of a specific representation is not crucial.

The situation changes in the non-one-dimensional case because the two representations result in a different structure of impedance equations. Field decomposition in terms of local waveguide modes automatically implies the use of the spectral method based on decomposing the solutions into eigenfunctions of a nonlocal wave operator and reducing the problem to a set of ordinary differential equations for the decomposition coefficients. In the non-one-dimensional problem, the second approach with the use of boundary vacuum modes seems to be more flexible because it leads to integro-differential equations. The method of the solution of the resultant nonlocal problem can be chosen at will, e.g., as the variational method [64], the method of sequential approximations, or optimized schemes for numerical solution of integro-differential problems [65]. Also, the spectral method based on the decomposition of solutions in terms of eigenfunctions of operator (3.14) can be used; it yields an infinite system of equations for the coefficients analogous to the equations in the first method.

4. General properties of the impedance equations

In this section, we consider some general properties of solutions of the wave equation that follow from the impedance method for an arbitrary anisotropic and gyrotropic medium. Apart from being of general physical interest, these properties may be useful for verifying numerical algorithms for solving the impedance equations.

4.1 The relation between representations in terms of local waveguide and boundary vacuum modes

In Section 3, the Maxwell equations were reduced to two equivalent formulations, the problem of coupling of the fields $\boldsymbol{\mathcal{E}}^{+}$ and $\boldsymbol{\mathcal{E}}^{-}$ determining the decomposition of the original field in polarizations of local geometro-optical modes and the problem of coupling of the fields $\boldsymbol{\mathcal{E}}_0^{+}$ and $\boldsymbol{\mathcal{E}}_0^{-}$ determining the decomposition of the original field in terms of polarizations of vacuum TE and TM modes. These two problems are related as

$$\mathbf{\Psi} = \hat{U}\begin{bmatrix}\boldsymbol{\mathcal{E}}^{+}\\ \boldsymbol{\mathcal{E}}^{-}\end{bmatrix} = \hat{U}_0\begin{bmatrix}\boldsymbol{\mathcal{E}}_0^{+}\\ \boldsymbol{\mathcal{E}}_0^{-}\end{bmatrix}.$$

This relation allows introducing a transition matrix $\hat{G}$ that describes the transition from one representation to the other and plays an important role in the theory:

$$\begin{bmatrix}\boldsymbol{\mathcal{E}}_0^{+}\\ \boldsymbol{\mathcal{E}}_0^{-}\end{bmatrix} = \hat{G}\begin{bmatrix}\boldsymbol{\mathcal{E}}^{+}\\ \boldsymbol{\mathcal{E}}^{-}\end{bmatrix}, \quad \hat{G} = \hat{U}_0^{-1}\hat{U} = \begin{bmatrix}\hat{g}_{11} & \hat{g}_{12}\\ \hat{g}_{21} & \hat{g}_{22}\end{bmatrix}. \tag{4.1}$$

In particular, if the reflection matrix $\hat{R}$ for the local modes of the medium is known, then the reflection matrix $\hat{R}_0$ for the vacuum TE and TM modes can be found from the formula

$$\hat{R}_0 = (\hat{g}_{21} + \hat{g}_{22}\hat{R})(\hat{g}_{11} + \hat{g}_{12}\hat{R})^{-1}, \tag{4.2}$$

which follows from the relation

$$\begin{bmatrix}\boldsymbol{\mathcal{E}}_0^{+}\\ \hat{R}_0\boldsymbol{\mathcal{E}}_0^{+}\end{bmatrix} = \hat{G}\begin{bmatrix}\boldsymbol{\mathcal{E}}^{+}\\ \hat{R}\boldsymbol{\mathcal{E}}^{+}\end{bmatrix} = \begin{bmatrix}(\hat{g}_{11} + \hat{g}_{12}\hat{R})\,\boldsymbol{\mathcal{E}}^{+}\\ (\hat{g}_{21} + \hat{g}_{22}\hat{R})\,\boldsymbol{\mathcal{E}}^{+}\end{bmatrix},$$

where $\hat{g}_{ij}$ denotes one of the four quadrants of the matrix $\hat{G}$. Calculation of the matrices $\hat{g}_{ij}$ is in a sense equivalent to the calculation of the reflection and transmission matrices of a differentially thin layer in the representation of the fields $\boldsymbol{\mathcal{E}}_0^{+}$ and $\boldsymbol{\mathcal{E}}_0^{-}$ because their simple coupling follows from relations (3.12) and (3.14):

$$\begin{bmatrix}+\hat{t}^{+} & +\hat{r}^{-}\\ -\hat{r}^{+} & -\hat{t}^{-}\end{bmatrix}\begin{bmatrix}\hat{g}_{11} & \hat{g}_{12}\\ \hat{g}_{21} & \hat{g}_{22}\end{bmatrix} = \begin{bmatrix}\hat{g}_{11} & \hat{g}_{12}\\ \hat{g}_{21} & \hat{g}_{22}\end{bmatrix}\begin{bmatrix}-\mathrm{i}\hat{k}_z^{+} & 0\\ 0 & -\mathrm{i}\hat{k}_z^{-}\end{bmatrix}. \tag{4.3}$$

Relation (3.4) reflects the fact that $\hat{G}^{-1}\hat{D}\hat{G} = \mathrm{i}k_0\hat{U}^{-1}\hat{M}\hat{U}$ is a diagonal matrix.

4.2 Reflection from a half-space filled with a homogeneous medium

A problem that is encountered rather frequently is to find the reflection matrix for a semi-infinite space filled with a medium homogeneous along the z coordinate. We assume for definiteness that the reflection matrix is formulated for the vacuum TE and TM modes. It is easy to see that this matrix must satisfy the quadratic matrix equation

$$\hat{R}_0\hat{r}^{-}\hat{R}_0 + \hat{R}_0\hat{t}^{+} + \hat{t}^{-}\hat{R}_0 + \hat{r}^{+} = 0 \tag{4.4}$$

derived from impedance equation (2.9) with $\partial_z\hat{R} = 0$. Indeed, the introduction of a layer of the same medium in front of the half-space makes no change, and just this property of the matrix is expressed by the above equation. The coefficients in this equation are operator 2×2 matrices calculated from formulas (3.9) and (3.14). If the medium is homogeneous in both the longitudinal and transverse directions, these coefficients are also numerical matrices.

The solution of this matrix equation is given by (4.2), with $\hat{R} = 0$ in the right-hand side:

$$\hat{R}_0 = \hat{g}_{21}\hat{g}_{11}^{-1}. \tag{4.5}$$

This solution can be deduced from general physical considerations. The condition $\hat{R} = 0$ means the absence of reflection in a homogeneous medium in the *local normal mode representation* because no wave arrives to the half-space boundary from the medium. The continuous tangential fields at this boundary must be related by expression (4.1), which implies the applicability of formula (4.2) for re-expressing the reflection matrix in the vacuum mode representation. We note that reflection matrix (4.5) can be obtained directly from (4.1) by substituting the vacuum fields $\boldsymbol{\mathcal{E}}_0^{+}$ and $\boldsymbol{\mathcal{E}}_0^{-} = \hat{R}_0\boldsymbol{\mathcal{E}}_0^{+}$ in the left-hand side and the medium fields expressed through the normal wave amplitudes $\boldsymbol{\mathcal{E}}^{+}$ and $\boldsymbol{\mathcal{E}}^{-} = 0$ in the right-hand side, and eliminating $\boldsymbol{\mathcal{E}}^{+}$.

Evidently, expression (4.5) satisfies the original equation (4.4) and the coefficients in (4.4) satisfy matrix relation (4.3). Eliminating $\hat{k}_z^{+}$ from this relation, we obtain

$$\hat{t}^{+}\hat{g}_{11} + \hat{r}^{-}\hat{g}_{21} = -\hat{g}_{11}\hat{g}_{21}^{-1}(\hat{t}^{-}\hat{g}_{21} + \hat{r}^{+}\hat{g}_{11}).$$

Multiplication of this equation by $\hat{g}_{21}\hat{g}_{11}^{-1}$ from the left and $\hat{g}_{11}^{-1}$ from the right leads exactly to Eqn (4.4) for $\hat{R}_0 = \hat{g}_{21}\hat{g}_{11}^{-1}$.

Interestingly, solution (4.5) permits obtaining conditions under which the reflection is absent at a certain polarization of incident radiation, $\hat{R}_0\boldsymbol{\mathcal{E}}_0^{+} = 0$. This phenomenon in isotropic media is known as the Brewster effect [16]. The necessary and sufficient condition for reflectionless polarization of the incident radiation is that the reflection matrix be degenerate, $\det\hat{R}_0 = 0$, if simultaneously $\det\hat{g}_{21} = 0$, but $\det\hat{g}_{11} \neq 0$. These two conditions determine the Brewster angle in an arbitrary anisotropic and gyrotropic medium.

This method for determining the matrix of reflection from a longitudinally homogeneous half-space is based on the calculation of medium eigenmodes necessary to find the $\hat{U}$ matrix, which is sometimes inconvenient. In the case of a homogeneous medium, it is possible to avoid expansion in the eigenmodes by using the general method for the solution of a quadratic matrix equation proposed in [66]. However, this leads to the problem of choosing the 'physical' solution among spurious solutions [Eqn (4.4) may have from 1 to 6 solutions in the absence of degeneracy].

4.3 Reflection from a layer filled with a homogeneous medium

A similar approach is applicable to the derivation of an analytic solution for the reflection matrix of a finite-thickness layer filled with a homogeneous medium along the z coordinate. In this case, it is more convenient to work in the local medium polarization representation, i.e., in the absence of reflection inside the layer, where the transmission coefficients are defined by diagonal matrices $\hat{k}_z^{\pm}$ [see Eqn (3.12)]. As a result, impedance equation (2.9) in a homogeneous medium degenerates into the linear equation

$$\partial_z\hat{R} = \mathrm{i}\hat{k}_z^{+}\hat{R} - \mathrm{i}\hat{k}_z^{-}\hat{R},$$

admitting the explicit solution

$$\hat{R}(z) = \hat{Q}^{-}(z)\,\hat{R}(b)\hat{Q}^{+}(z), \quad \hat{Q}^{\pm}(z) = \exp\left[\pm\mathrm{i}\hat{k}_z^{\pm}(z-b)\right]. \tag{4.6}$$

Here, the exponentials of diagonal matrices are themselves diagonal matrices. The initial value of $\hat{R}(b)$ is found from the physical condition of the absence of the reflected wave behind the layer. This means that the reflection matrix in (4.2) vanishes at the layer boundary, which is possible if $\hat{R}(b) = -\hat{g}_{22}^{-1}\hat{g}_{21}$. Indeed, the tangential fields at the boundary must be related, due to their continuity, by expression (4.1) with the vacuum fields $\boldsymbol{\mathcal{E}}_0^+$ and $\boldsymbol{\mathcal{E}}_0^- = 0$ in the left-hand side and the medium fields $\boldsymbol{\mathcal{E}}^+$ and $\boldsymbol{\mathcal{E}}^- = \hat{R}(b)\boldsymbol{\mathcal{E}}^+$ in the right-hand side; this implies the above initial value of the reflection matrix. Applying Eqn (4.2) to the reflection matrix $\hat{R}(z)$ found in the local mode representation, we can find the reflection matrix in the vacuum wave representation:

$$\hat{R}_0(z) = \left[\hat{g}_{21} - \hat{g}_{22}\hat{Q}^-(z)\,\hat{g}_{22}^{-1}\hat{g}_{21}\hat{Q}^+(z)\right] \times \left[\hat{g}_{11} - \hat{g}_{12}\hat{Q}^-(z)\,\hat{g}_{22}^{-1}\hat{g}_{21}\hat{Q}^+(z)\right]^{-1}. \tag{4.7}$$

This is a generalization of the known relation for the Fabry–Perot resonator to an arbitrary anisotropic and gyrotropic medium. As $b \to +\infty$ in the case of infinitesimal dissipation, this formula turns into relation (4.5) for the matrix of reflection from a half-space.

4.4 A formal general solution for the reflection operator

Nonlinear equation (2.9) for the $\hat{R}(z)$ operator is a Riccati-type equation [67]. The general solution of such equations can be found if any particular solution $\hat{R}_1(z)$ is known. This is also valid in our case, despite the presence of noncommuting operators. We seek the solution in the form

$$\hat{R} = \hat{R}_\nabla + \hat{R}_1 .$$

If $\hat{R}_1$ is a solution of (2.9), then $\hat{R}_\nabla$ satisfies the equation

$$-\partial_z \hat{R}_\nabla = \hat{R}_\nabla \hat{r}^- \hat{R}_\nabla + \hat{R}_\nabla \hat{t}_\nabla^+ + \hat{t}_\nabla^- \hat{R}_\nabla , \tag{4.8}$$

where $\hat{t}_\nabla^+ = \hat{t}^+ + \hat{r}^-\hat{R}_1$ and $\hat{t}_\nabla^- = \hat{t}^- + \hat{R}_1\hat{r}^-$. This equation is of the same type as the original one, but does not contain the free term $\hat{r}^+$. Direct substitution shows that Eqn (4.8) has the general solution

$$\hat{R}_\nabla(z) = \hat{Q}^-\left[\hat{R}_\nabla^{-1}(b) + \int_b^z \hat{Q}^-\hat{r}^-\hat{Q}^+\,\mathrm{d}z'\right]^{-1}\hat{Q}^+ , \qquad \hat{Q}^\pm = \exp\left(-\int_b^z \hat{t}_\nabla^\pm\,\mathrm{d}z'\right). \tag{4.9}$$

Calculating operator exponentials is equivalent to solving linear differential problems $-\partial_z\hat{Q}^\pm = \hat{t}_\nabla^\pm\hat{Q}^\pm$ with the 'unit' initial conditions $\hat{Q}^\pm(b) = \hat{I}$.

For a piecewise-homogeneous medium, this solution turns into the solutions deduced previously from physical considerations, as is easy to see using the local mode representation in which $\hat{r}^\pm = 0$ and $\hat{t}_\nabla^\pm = \hat{t}^\pm = \mathrm{const}$, while the formula for $\hat{R}_\nabla$ exactly transforms into (4.6). The same is slightly more difficult to demonstrate in the vacuum mode representation. For this, it is possible to use homogeneous distribution (4.5) satisfying the *stationary* impedance equation and choose $\hat{R}_\nabla^{-1}(b) = -\hat{g}_{11}\hat{g}_{21}^{-1}$. After some algebraic transformations with the use of relations (4.3), we then return to Eqn (4.7).

In a non-one-dimensional case, the general solution may be helpful if a particular solution of the impedance equation is known. This solution can then be 'improved' using formulas (4.8) and (4.9), and a physical solution can be obtained satisfying any boundary condition specified beforehand. For example, if we choose $\hat{R}_\nabla(b) = -\hat{R}_1(b)$, then $\hat{R}$ satisfies the boundary condition required for the applicability of the impedance method, $\hat{R}(b) = 0$. Another illustration is provided by the following problem. We assume that the solution of the impedance equation for a certain layer $[a, b]$ surrounded by the vacuum is known. We let $\hat{R}_1(z)$ denote this solution and seek a solution for this layer assuming that the vacuum in the region $z > b$ is replaced by a substrate, e.g., a half-space with the known reflection operator $\hat{R}_0$ that can be found, for instance, by the method described in Section 4.2. The solution for the layer with the substrate is then given by formulas (4.8) and (4.9) if we set $\hat{R}_\nabla(b) = \hat{R}_0$.

4.5 Solution in the form of a series expansion in powers of reflection

In applications, the reflected wave is often small compared with the forward one. In calculating the forward wave propagation in the zeroth approximation, it is then possible to disregard the reflected wave and develop a theory of perturbations in powers of the differential reflection operators $\hat{r}^\pm$. Counterpropagating wave equation (2.1) with boundary conditions (2.6) can be represented in the integral form

$$\boldsymbol{\mathcal{E}}^+ = \int_a^z \exp\left(\int_{z'}^z \hat{t}^+(z'')\,\mathrm{d}z''\right)\hat{r}^-(z')\,\boldsymbol{\mathcal{E}}^-(z')\,\mathrm{d}z' + \exp\left(\int_a^z \hat{t}^+\,\mathrm{d}z'\right)\boldsymbol{\mathcal{E}}^{\mathrm{inc}} ,$$

$$\boldsymbol{\mathcal{E}}^- = \int_z^b \exp\left(\int_z^{z'} \hat{t}^-(z'')\,\mathrm{d}z''\right)\hat{r}^+(z')\,\boldsymbol{\mathcal{E}}^+(z')\,\mathrm{d}z' .$$

By analogy with the scattering theory, the solution of these equations can be constructed by iterations as a sum $\boldsymbol{\mathcal{E}}^\pm = \sum \boldsymbol{\mathcal{E}}_n^\pm$, where

$$\boldsymbol{\mathcal{E}}_0^+ = \exp\left(\int_a^z \hat{t}^+\,\mathrm{d}z'\right)\boldsymbol{\mathcal{E}}^{\mathrm{inc}} , \qquad \boldsymbol{\mathcal{E}}_0^- = 0 ,$$

and each subsequent summand follows from the preceding one by the transformation

$$\boldsymbol{\mathcal{E}}_{n+1}^+ = \int_a^z \exp\left(\int_{z'}^z \hat{t}^+\,\mathrm{d}z''\right)\hat{r}^-\boldsymbol{\mathcal{E}}_n^-\,\mathrm{d}z' ,$$

$$\boldsymbol{\mathcal{E}}_{n+1}^- = \int_z^b \exp\left(\int_z^{z'} \hat{t}^-\,\mathrm{d}z''\right)\hat{r}^+\boldsymbol{\mathcal{E}}_n^+\,\mathrm{d}z' .$$

From the physical standpoint, this procedure is the expansion in terms of the multiplicity of mutual scattering of counterpropagating waves, with each item $\boldsymbol{\mathcal{E}}_n^\pm \sim O[(\hat{r}^\pm)^n]$ being the contribution of n-fold scattering to Eqns (2.1).

If we are interested in the absorption and linear interaction of waves propagating only in the positive direction along the z axis, we may confine ourselves to the zeroth approximation $\boldsymbol{\mathcal{E}}_0^+$ for the forward waves. The corresponding backward wave $\boldsymbol{\mathcal{E}}_1^-$ appears only in the first order and the correction for the forward wave due to the presence of reflection appears in the second order. The applicability condition for this approximation is $\|\boldsymbol{\mathcal{E}}_0^+\| \gg \|\hat{r}^-\boldsymbol{\mathcal{E}}_1^-\| \sim O[\hat{r}^-\hat{r}^+]$.

It is easy to verify that the forward and backward waves are related as

$$\boldsymbol{\mathcal{E}}_1^-(z) = \left[\int_z^b \exp\left(\int_z^{z'} \hat{t}^-\,\mathrm{d}z''\right) \times \hat{r}^+\exp\left(\int_z^{z'} \hat{t}^+\,\mathrm{d}z''\right)\mathrm{d}z'\right]\boldsymbol{\mathcal{E}}_0^+(z) .$$

The operator in the square brackets is exactly the reflection operator $\hat{R}_1$ in the first approximation resulting from the solution of impedance equation (2.9) with $\hat{r}^- = 0$, i.e., with the nonlinear term neglected. The next terms of the expansion can be found from the recurrent formula

$$\hat{R}_{n+2} = \int_z^b \exp\left(\int_z^{z'} \hat{t}^- \, \mathrm{d}z''\right)\left(\sum_{k=1}^{n} \hat{R}_k \hat{r}^+ \hat{R}_{n-k+1}\right) \times \exp\left(\int_z^{z'} \hat{t}^+ \, \mathrm{d}z''\right) \mathrm{d}z'$$

for odd n, where the terms with even indices are equal to zero. All three factors in the integrand have a clear physical meaning (forward wave propagation up to the reflection point z', and reflection and propagation of the backward wave to the observation point z).

5. An example. Propagation of electromagnetic waves in a cold magnetized plasma

In this methodological paper, we do not attempt to review all possible applications of the impedance technique in electrodynamic problems. The reader interested in this topic is referred to special publications cited in the Introduction. In what follows, we confine ourselves to a familiar class of problems pertaining to the propagation of electron cyclotron (EC) electromagnetic waves in a cold magnetized plasma. To simplify the discussion, we consider a one-dimensional configuration in which the parameters of the plasma vary only along the z coordinate. Other applications of the impedance method for this class of problems are considered at greater length in Refs [38–42, 58–60].

5.1 Problem setup for an inhomogeneous plasma layer

To illustrate the potential of the method, we consider a plasma layer with the relatively complicated profile shown in Fig. 2. We assume that the external magnetic field is constant and directed along the x axis. We are interested in the reflection, transmission, and absorption of high-frequency electromagnetic waves with a given polarization incident on the plasma layer. This configuration roughly simulates the geometry of toroidal magnetic traps in the EC resonance region and the coherent modulation of plasma density reflects the influence of magnetohydrodynamic perturbations of a plasma column.

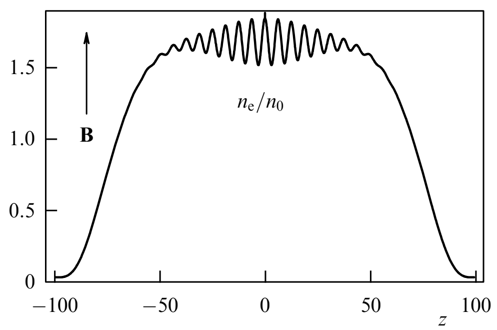

Figure 2. The plasma concentration profile $n_e(z) = 1.7 n_0[(1 - z^6)^4 + 0.1\exp(-z^2/100)\cos z]$ used in the numerical example in Section 5. Here and in the remaining figures the coordinate z is normalized to a certain $k_{00} = \omega_0/c$ that also defines the characteristic density $n_0 = m_e\omega_0^2/4\pi e^2$ or the so-called critical density for a cut-off of electromagnetic radiation in an isotropic plasma.

For the waves with frequencies much higher than all ion frequencies, the medium being considered is described by the dielectric permittivity tensor [1–4]

$$\hat{\varepsilon}(z,\omega) = \begin{bmatrix} \varepsilon_\parallel & 0 & 0 \\ 0 & \varepsilon_\perp & -\mathrm{i}g \\ 0 & \mathrm{i}g & \varepsilon_\perp \end{bmatrix}, \tag{5.1}$$

where

$$\varepsilon_\parallel = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \mathrm{i}\nu)}, \qquad \varepsilon_\perp = 1 - \frac{\omega_{pe}^2(1 - \mathrm{i}\nu/\omega)}{(\omega - \mathrm{i}\nu)^2 - \omega_{ce}^2},$$

$$g = \frac{\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{(\omega - \mathrm{i}\nu)^2 - \omega_{ce}^2}, \qquad \varepsilon_\perp' = \varepsilon_\perp - \sin^2\vartheta,$$

$\omega_{ce} = eB/m_e c$ and $\omega_{pe} = [4\pi e^2 n_e(z)/m_e]^{1/2}$ are the electron cyclotron and plasma frequencies, and $\nu \ll \omega$ is the dissipation introduced to eliminate a singularity occurring in numerical integration in the region of the upper hybrid resonance. We recall that the incident wave vector lies in the xz plane; in other words, $k_x = k_0 \sin\vartheta$ and $k_y = 0$ are conserved. In accordance with (3.5), the wave operator $\hat{M}$ is uniquely defined by the dielectric tensor as

$$\hat{M} = \begin{bmatrix} 0 & \mathrm{i}g\sin\vartheta/\varepsilon_\perp & 0 & -\varepsilon_\perp'/\varepsilon_\perp \\ 0 & 0 & 1 & 0 \\ 0 & \varepsilon_\perp' - g^2/\varepsilon_\perp & 0 & \mathrm{i}g\sin\vartheta/\varepsilon_\perp \\ -\varepsilon_\parallel & 0 & 0 & 0 \end{bmatrix}.$$

This matrix can also be obtained as a special case of the Clemmow–Budden–Heading operator [33].

In calculations, we use the field decomposition in terms of the vacuum TE and TM waves. In this representation, reflection and transmission matrices (3.16) become

$$\hat{r}^\pm = \mathrm{i}k_0 \begin{bmatrix} (g^2/\varepsilon_\perp - \varepsilon_\perp + 1)\sec\vartheta & -\mathrm{i}g\tan\vartheta/\varepsilon_\perp \\ \mathrm{i}g\tan\vartheta/\varepsilon_\perp & \varepsilon_\parallel\cos\vartheta - \varepsilon_\perp'\sec\vartheta/\varepsilon_\perp \end{bmatrix},$$
$$\hat{t}^\pm = \hat{r}^\pm - \mathrm{i}k_0\cos\vartheta \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon_\parallel \end{bmatrix}. \tag{5.2}$$

In the present case, the 'forward' and 'backward' wave propagation is characterized by the same matrices.

To interpret the results, we find a relation between vacuum waves and electromagnetic normal modes of the cold plasma. The plasma modes are defined as eigenvectors of the wave operator. The corresponding dispersion equation

$$\det(\mathrm{i}k_0\hat{M} + \mathrm{i}k_z\hat{I}) = 0$$

can be presented in the form of a biquadratic equation for the 'transverse refractive index' $N = k_z/k_0$,

$$N^4 - pN^2 + q = 0, \qquad p = \left(\frac{\varepsilon_\parallel}{\varepsilon_\perp} + 1\right)\varepsilon_\perp' - \frac{g^2}{\varepsilon_\perp},$$
$$q = (\varepsilon_\perp'^2 - g^2)\frac{\varepsilon_\parallel}{\varepsilon_\perp}, \tag{5.3}$$

with the solutions $N^2 = p/2 \pm \operatorname{sign}\varepsilon_\perp\sqrt{p^2/4 - q}$. The plus and minus signs correspond to ordinary (O) and extraordin-

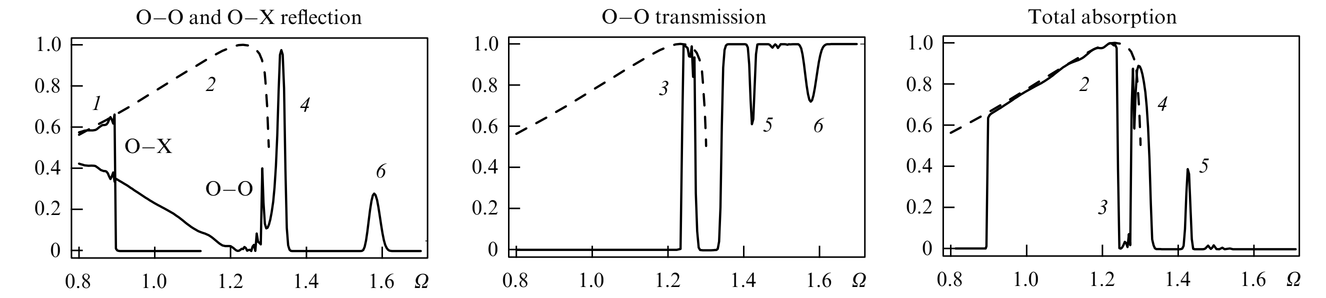

Figure 3. Coefficients of reflection into ordinary $|R^2_{\rm O-O}|$ and extraordinary $|R^2_{\rm O-X}|$ waves (in terms of intensity), the coefficient of transmission into an ordinary wave $|T^2_{\rm O-O}|$, and the total absorption $Q = 1 - |R^2_{\rm O-O}| - |R^2_{\rm O-X}| - |T^2_{\rm O-O}| - |T^2_{\rm O-X}|$ depending on the frequency for an ordinary wave incident at the fixed angle $\vartheta = 40°$ on the plasma layer shown in Fig. 2. The dashed line shows the coupling coefficient between the ordinary and extraordinary waves ensuing from formula (5.6). The coefficient of transmission of the ordinary wave into the extraordinary one is very low, $|T^2_{\rm O-X}| \sim 10^{-4}|T^2_{\rm O-O}|$, and is not shown. The magnetic field corresponds to $\omega_{\rm ce}/\omega_0 = 0.9$, the normalized frequency is $\Omega = \omega/\omega_0$.

ary (X) waves of the magnetized plasma.[3] We note that in the framework of geometric optics, only extraordinary waves are subject to absorption in the vicinity of the upper hybrid resonance, while ordinary ones do not 'recognize' it. Polarization vectors of normal waves follow from the expression

$$\tilde{\mathbf{\Psi}} = [{\rm i}gN\sin\vartheta,\ \varepsilon_{\parallel}\varepsilon'_{\perp} - N^2\varepsilon_{\perp},\ -n_z(\varepsilon_{\parallel}\varepsilon'_{\perp} - N^2\varepsilon_{\perp}),\ {\rm i}g\varepsilon_{\parallel}\sin\vartheta]\,, \tag{5.4}$$

into which four roots of the dispersion relation must be substituted and the resulting vectors then normalized. Once the polarization vectors are known, it is possible to construct the matrix $\hat{G} = \hat{U}_0^{-1}\hat{U}$ relating normal waves to the amplitudes of vacuum TE and TM waves (4.1). The general expression for this matrix is rather cumbersome, but we need to know only the so-called limiting polarizations of normal waves incoming from and outgoing to the vacuum. By calculating $\lim \hat{G}$ as $\omega_{\rm pe} \to 0$, we find a simple relation between the amplitudes $\mathscr{E}_{\rm O}$, $\mathscr{E}_{\rm X}$ of the ordinary and extraordinary waves and the amplitudes $\mathscr{E}_{\rm TE}$, $\mathscr{E}_{\rm TM}$ of the vacuum waves at the boundaries of the plasma layer smoothly passing into the vacuum:

$$\begin{bmatrix} \mathscr{E}_{\rm O}^{\pm} \\ \mathscr{E}_{\rm X}^{\pm} \end{bmatrix} = \frac{1}{\sqrt{1+\gamma^2}} \begin{bmatrix} 1 & {\rm i}\gamma \\ {\rm i}\gamma & 1 \end{bmatrix} \begin{bmatrix} \mathscr{E}_{\rm TE}^{\pm} \\ \mathscr{E}_{\rm TM}^{\pm} \end{bmatrix}, \qquad \gamma = \frac{2\omega\sin\vartheta}{\omega_{\rm ce}\cos^2\vartheta - \sqrt{\omega_{\rm ce}^2\cos^4\vartheta + 4\omega^2\sin^2\vartheta}}\,. \tag{5.5}$$

This formula coincides with the expression obtained in [1–4].

To summarize, the following procedure for the solution of the problem of incidence of an electromagnetic wave on a plasma layer is proposed. For a given plasma density distribution, impedance equation (2.9) with matrix coefficients (5.2) and the zero initial condition and Eqn (2.8) for a forward wave with two different initial conditions corresponding to the incident vacuum TE and TM modes are integrated numerically. This gives the reflection and transmission matrices for these modes, as well as the field structure in the layer corresponding to the incidence of TE and TM modes. As a rule, solutions in terms of plasma modes are needed for practical applications. Because the original problem is linear, the solutions for the incident ordinary or extraordinary waves can be constructed as linear combinations in accordance with the relation for limiting polarizations in (5.5). This also applies to the reflection and transmission matrices.

[3] Generally speaking, these definitions may not coincide with the standard nomenclature of normal waves based on the Appleton–Hartree solution for a fixed wave propagation angle [1–3] in the narrow vicinity of the plasma resonance $\omega_{\rm pe} = \omega$ (see [68] for the details).

5.2 Linear coupling of ordinary and extraordinary waves and the Bragg resonance scattering of the ordinary wave

The most interesting case from the physical standpoint is the incidence of an ordinary wave onto a layer of an overdense plasma. The numerical calculation of this case is exemplified in Figs 3 and 4. Figure 3 presents the reflection, transmission, and absorption coefficients in terms of intensity as functions of the frequency of the ordinary wave with a fixed incidence angle. Characteristic regions in the spectra are labeled by numerals from *1* to *6*; the dispersion curves and field distributions corresponding to these regions are shown in Fig. 4.

Regions *1–3* correspond to the effective ordinary-to-extraordinary wave linear transformation in the vicinity of the plasma resonance $\omega_{\rm pe} = \omega$. In this case, the linear transformation reduces to the 'tunneling' of electromagnetic radiation through the nontransparency zone with $k_z^2 < 0$. The coupling coefficient between normal waves (in terms of intensity) is well approximated by the known quantum mechanical formula for the probability of underbarrier tunneling [69] (in electromagnetic problems [2, 70, 71]):

$$|C^2_{\rm O-X}| = \exp\left\{\int_{{\rm Im}\,k_z<0} 2\,{\rm Im}\,k_z\,{\rm d}z\right\}.$$

In our case, this formula reduces to the expression

$$|C^2_{\rm O-X}| = \exp\left\{-\pi\sqrt{2}\,k_0\left(\frac{{\rm d}\ln n_{\rm e}}{{\rm d}z}\right)^{-1}\left(\frac{\omega_{\rm ce}}{\omega}\right)^{3/2} \times\left(\sqrt{\frac{\omega}{\omega_{\rm ce}}+1}\,\sin\vartheta - 1\right)^2\right\}, \tag{5.6}$$

obtained in Refs [72–74]. The coupling coefficient between ordinary and extraordinary waves (5.6) is shown in Fig. 3 by the dashed line. The results of numerical calculations are in excellent agreement with the analytic formula for both the reflection and absorption spectra (cases *1* and *2*, respectively). Case *1* implies partial conversion of an incident

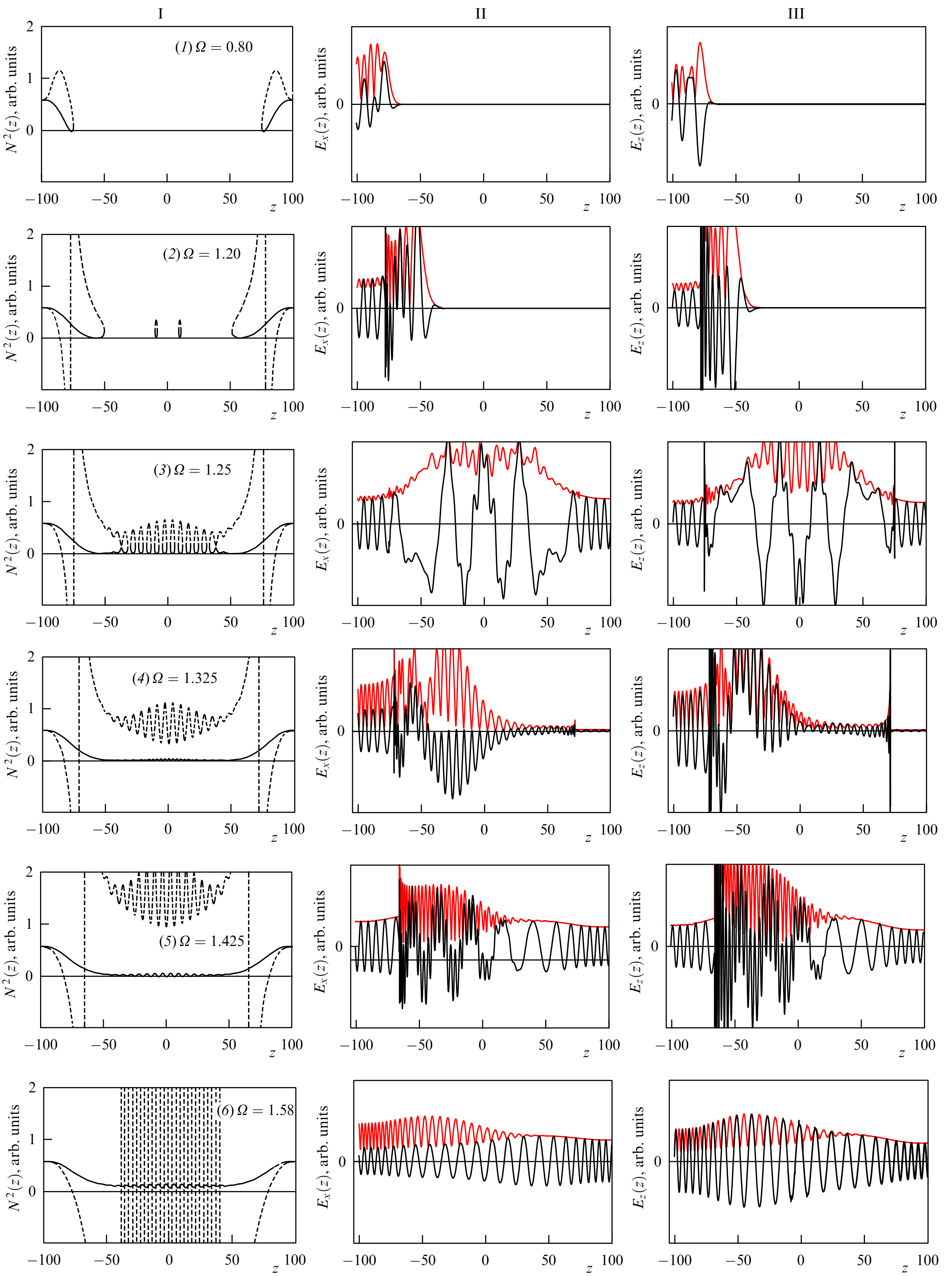

Figure 4. I — dispersion curves $N^2(z) = \omega^2 k_z^2/c^2$ for ordinary (solid lines) and extraordinary (dashed lines) waves corresponding to regions *1–6* in Fig. 3. Field distribution Re $E_x(z)$ (II) and Re $E_z(z)$ (III). Light lines show $|E_x|$ and $|E_z|$; the absence of their modulation corresponds to propagating waves and pronounced harmonic modulation to standing waves.

ordinary wave into the fast extraordinary wave propagating in the reverse direction and freely escaping into the vacuum [see the plots of $N^2(z)$ in Fig. 4 (*1*)]. This case corresponds to $\omega < \omega_{ce}$. Transition to case *2* occurs in the range $\omega > \omega_{ce}$. The region of the upper hybrid resonance formed in the plasma appears as a singularity in the $N^2(z)$ plots in Fig. 4. As the ordinary wave reflects from the overdense plasma layer, it partially transforms into the slow extraordinary wave that turns back toward the incident wave and propagates to the upper hybrid resonance, where it is efficiently absorbed (Fig. 4 (*2*)). The absorption in a dense hot plasma is accompanied by the generation of quasi-electrostatic plasma oscillations that propagate in an overdense plasma. This effect is used to heat and diagnose the dense plasma in spherical tokamaks and optimized stellarators [75]. The turning point of the extraordinary wave disappears as the frequency increases, which enables the wave to cross the plasma layer and reach the symmetric plasma resonance surface $\omega_{pe} = \omega$, where it transforms back into an ordinary wave. In this way, case *3* is realized where the main part of the radiation leaves the layer as an ordinary wave even though it propagates as an extraordinary wave inside the layer [as can be seen from the field structure in Fig. 4 (*3*)]. This case is characterized by a low intensity of the radiation absorption and reflection into the cross mode (O−X). The fine structure in the reflection and transmission spectra of the incident mode arises from the mirror symmetry of the linear transformation regions that form a kind of a Fabry–Perot interferometer whose resonances are apparent as a fine structure of the lines.

We note that a linear wave transformation in cases *1* and *2* occurs at the density profile slope where the periodic modulation is insignificant. Starting from case *3*, radiation penetrates into the central part of the plasma layer and scattering becomes possible in the presence of a Bragg resonance with a spatial density grating [76]. An almost complete resonance scattering of the ordinary wave into the oncoming extraordinary wave occurs in case *4* at $k_z^{\rm X} - k_z^{\rm O} = k_z^{\rm pl}$, where the indices X, O, and pl denote the wave vectors of the extraordinary wave, ordinary wave, and plasma density grating. As can be seen from the electromagnetic field distributions in Fig. 4 (*4*), the backward scattering dominates. The scattered extraordinary wave locked inside the dense plasma is completely absorbed in the vicinity of the upper hybrid resonance. The unscattered radiation is reflected from the surface $\omega_{pe} = \omega$ as an ordinary wave, giving rise to a peak in the reflection spectrum.

As the frequency increases further, the plasma layer becomes transparent to the ordinary wave, i.e., the inequality $\omega < \omega_{pe}$ is fulfilled everywhere. Region *5* corresponds to a Bragg resonance at the second spatial harmonic $k_z^{\rm X} - k_z^{\rm O} = 2k_z^{\rm pl}$. In this case, only part of the incident radiation scatters into an ordinary wave even under optimal conditions, which accounts for a dip in the transmission spectrum. The scattered radiation reaches the upper hybrid resonance and becomes completely absorbed, giving rise to a peak in the absorption spectrum.

In region *6*, the 'classical' Bragg scattering occurs when the ordinary wave scatters into an oncoming ordinary wave under the conditions $2k_z^{\rm O} = k_z^{\rm pl}$. Hence, there is a peak in the reflection spectrum and a dip in the transmission spectrum in the absence of perturbation in the absorption spectrum.

These resonance scattering effects in a magnetized plasma may be important for thermonuclear plasma diagnostics by microwave resonance Doppler reflexometry [77, 78] and the collective scattering of microwave radiation [79–81].

5.3 Resonance scattering of the extraordinary wave

The case of the incidence of an extraordinary wave on a plasma layer with density modulation is not as abundant in physical effects as the previous one. Nevertheless, it is equally important in the context of practical applications, e.g., for microwave plasma probing. This case becomes especially simple in terms of the parameters used in Section 5.2: the extraordinary wave reflects from the plasma layer either with partial conversion into the ordinary wave at $\omega < \omega_{ce}$ in accordance with formula (5.6) or without conversion at $\omega > \omega_{ce}$ and $\omega_{pe}^2 > (\omega - \omega_{ce})\omega$. The radiation does not penetrate into the central part of the layer where resonant Bragg scattering is possible.

Figure 5 presents a different example characterized by a stronger magnetic field satisfying the inequality $\omega_{ce} > \omega_{pe}$. In this case, the plasma has a transparency window in which resonance scattering is effectively manifested. The figure shows reflection and transmission spectra of the extraordinary wave. The absorption coefficient is not shown because it does not exceed 1%. We distinguished three characteristic spectral regions. Region *1* corresponds to the scattering of the extraordinary wave into itself, $2k_z^{\rm X} = k_z^{\rm pl}$; region *2* to the scattering of the extraordinary wave into itself at the second

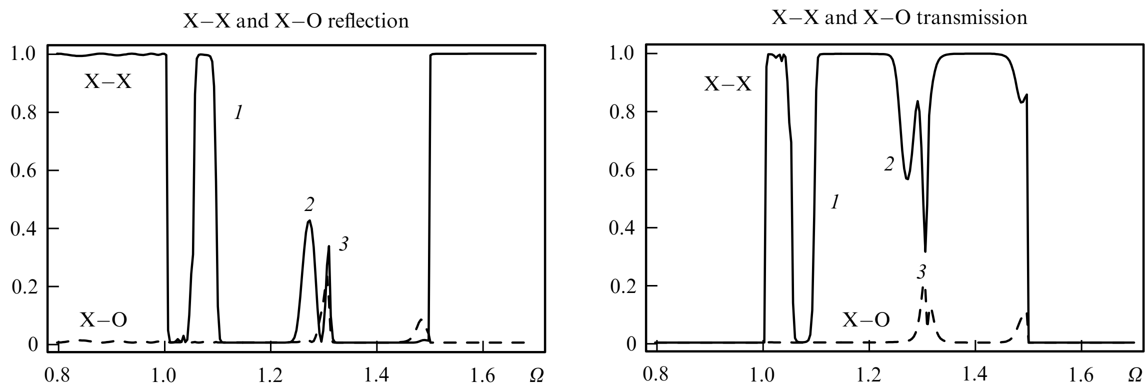

Figure 5. Frequency-dependent coefficients of reflection and transmission (in terms of intensity) of an extraordinary wave into extraordinary (solid line) and ordinary (dashed line) ones for the plasma layer shown in Fig. 2. The incidence angle is $\vartheta = 35°$, the magnetic field corresponds to $\omega_{ce}/\omega_0 = 1.5$, and the normalized frequency is $\Omega = \omega/\omega_0$.

harmonic, $2k_z^{\rm X} = 2k_z^{\rm pl}$; region 3 to the scattering of the extraordinary wave into a concurrent ordinary wave, $k_z^{\rm O} - k_z^{\rm X} = k_z^{\rm pl}$, as is apparent from the appearance of the cross-polarization peak in the transmission spectrum. Resonances 2 and 3 overlap; therefore, the scattered extraordinary wave undergoes secondary scattering into an ordinary one. This accounts for the cross-polarization peak in the reflection spectrum (with simultaneous suppression of the reflection into the main mode).

5.4 Reflection from a plasma-filled homogeneous half-space

To conclude, we consider an example of an analytic calculation based on the method described in Section 4. We consider reflection into the vacuum from a semi-infinite layer filled with a homogeneous plasma with dielectric tensor (5.1). The reflection matrix is given by Eqn (4.6). In this case, the computation can be brought to completion: first, the matrix $\hat{G}$ is calculated from the known polarization vectors (5.4), then the upper ($\hat{g}_{11}$) and bottom ($\hat{g}_{21}$) left quadrants of this matrix are identified, and the reflection matrix $\hat{R}_0 = \hat{g}_{21}\hat{g}_{11}^{-1}$ is finally derived. After simple algebraic transformations, components of the reflection matrix can be represented in the compact form

$$R_{0,11} = \frac{\varepsilon_\parallel p(\cos\vartheta - p) - pq\sec\vartheta + (\varepsilon_\parallel + 1)(q + \varepsilon_\parallel\varepsilon'_\perp/\varepsilon_\perp)}{\varepsilon_\parallel p(\cos\vartheta + p) + pq\sec\vartheta - (\varepsilon_\parallel - 1)(q + \varepsilon_\parallel\varepsilon'_\perp/\varepsilon_\perp)} ,$$

$$R_{0,22} = \frac{\varepsilon_\parallel p(\cos\vartheta + p) - pq\sec\vartheta - (\varepsilon_\parallel + 1)(q + \varepsilon_\parallel\varepsilon'_\perp/\varepsilon_\perp)}{\varepsilon_\parallel p(\cos\vartheta + p) + pq\sec\vartheta - (\varepsilon_\parallel - 1)(q + \varepsilon_\parallel\varepsilon'_\perp/\varepsilon_\perp)} ,$$

$$R_{0,12} = -R_{0,21} = \frac{\mathrm{i}\varepsilon_\perp}{2g\sin\vartheta} \times \frac{\left[p^2 - 2q - \varepsilon_\parallel\varepsilon'_\perp/\varepsilon_\perp\right]^2 - p^2(p^2 - 4q)}{\varepsilon_\parallel p(\cos\vartheta + p) + pq\sec\vartheta - (\varepsilon_\parallel - 1)(q + \varepsilon_\parallel\varepsilon'_\perp/\varepsilon_\perp)} ,$$

where p and q are the coefficients of biquadratic dispersion equation (5.3). We recall that this reflection matrix is formulated for the vacuum TE and TM modes. In a similar way, it is possible to analytically calculate the reflection matrix for the Fabry–Perot interferometer filled with a homogeneous magnetized plasma.

6. Conclusion

We have considered the invariant embedding technique applied to the problems of electromagnetic wave propagation in complex media with tensorial electromagnetic response. A specific feature of such media is the possibility of propagation of several coupled normal waves with different polarizations. In this method, the set of normal waves is divided into two groups of counterpropagating waves. If the problem admits such a division, it is possible to find a matrix impedance operator that characterizes the coupling of different counterpropagating waves in an inhomogeneous medium. This goal is achieved by solving a nonlinear evolution equation with the universal (zero) boundary conditions unrelated to the characteristics of incident radiation. Accordingly, the impedance operator is also independent of them. Once the impedance operator is known, the field for any given structure of incident radiation can be reconstructed over the entire space by a trivial linear operation.

In one-dimensionally inhomogeneous problems, the proposed method allows reducing the wave equation to a system of first-order nonlinear differential equations for the components of the reflection matrix. Such a system is characterized by a low dimension (equal to the squared number of normal modes propagating unidirectionally in the medium). In almost every conceivable situation, the system is easy to integrate numerically on a personal computer. By virtue of technical simplicity and low demand for computational resources, the method provides a convenient tool for studying the propagation and linear transformation of electromagnetic waves in situations where the geometric optics approximation is violated by a strong spatial dependence of medium characteristics. Moreover, the impedance technique can be used in the one-dimensional setting to rapidly check the results of more sophisticated electrodynamic calculations when standard approximate methods based on ray tracing or a quasioptical approach are inapplicable.

In two- or three-dimensionally inhomogeneous media, the impedance method leads to either a system of ordinary differential equations of infinite size (corresponding to an infinite set of transverse modes) or a system of low-dimension integro-differential equations. Although both problems require a nontrivial numerical solution, the method in question is more convenient for dealing with a large class of similar problems than the straightforward solution of the boundary problem for the Maxwell equations, first and foremost due to the use of evolutionary equations. Moreover, the proposed approach automatically and exactly accounts for the radiation condition for the passing wave behind the layer, regardless of the structure of incident radiation.

The proposed method can be extended to media with spatial dispersion described by finite-order derivatives in the dielectric and magnetic permittivity tensors. The idea behind such a generalization is rather simple: the increase of the order of the Maxwell equations due to spatial dispersion is taken into account by increasing the vector $\mathbf{\Psi}$ dimension in Eqn (3.4). As a result, the Maxwell equations reduce to formula (3.4), but the wave operator $\hat{M}$ becomes more complicated, depending on a particular form of the dispersion. The impedance method can be further developed as described above. Interestingly, this approach automatically solves the problem of additional boundary conditions related to the increased dimension of wave equations [82], at least for media smoothly passing to homogeneous ones at the boundaries. Such an analysis for a magnetized plasma is described in [59].

This work was done under the state contract No. 14.740.0607 with the Russian Ministry of Education and Science and supported by the RFBR grants 09-02-00972 and 10-02-00411.

7. Appendix. The energy conservation law in the impedance technique

The electromagnetic energy flux density in a medium without spatial dispersion is determined by the Umov–Poynting vector $\mathbf{S} \equiv (c/4\pi)[\mathbf{EH}]$ [15]. Therefore, the energy flux along the z coordinate averaged over field oscillations can be estimated as

$$S_z = \frac{c}{8\pi}\int \mathrm{Re}\,(E_x H_y^* - E_y H_x^*)\,\mathrm{d}^2\mathbf{r}_\perp .$$

A change in the energy flux determines the electromagnetic energy dissipation density per unit length $\partial_z S_z = q_z$, where

$$q_z = \frac{1}{16\pi} \int \mathrm{Im}\,(\mathbf{ED} + \mathbf{BH})\,\mathrm{d}^2\mathbf{r}_\perp\,. \tag{A.1}$$

The energy flux is conserved in a nondissipative medium. Formally, this follows from the hermiticity of the $\hat{\varepsilon}$ and $\hat{\mu}$ tensors responsible for the vanishing of dissipation (A.1). We consider the relation $\partial_z S_z = q_z$ in the variables introduced in Section 3.

We introduce a scalar product of vector fields depending on transverse coordinates or wave vectors in the Fourier representation:

$$(\mathbf{a}, \mathbf{b}) \equiv \int \sum_i a_i b_i^*\,\mathrm{d}^2\mathbf{r}_\perp \quad \text{or} \quad \int \sum_i a_i b_i^*\,\mathrm{d}^2\mathbf{k}_\perp\,.$$

This scalar product defines the notion of Hermitian conjugation of an operator, $(\mathbf{a}, \hat{A}\mathbf{b}) = (\hat{A}^*\mathbf{a}, \mathbf{b})$. It is easy to see that the energy flux can be expressed in terms of the scalar product as

$$S_z = (\mathbf{\Psi}, \hat{\sigma}\mathbf{\Psi})\,, \qquad \hat{\sigma} = \frac{c}{16\pi}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The easiest way to derive the dissipation of the electromagnetic energy per unit length is to use Eqn (3.4), from which the law of energy flux variation follows in the form

$$q_z = \partial_z(\mathbf{\Psi}, \hat{\sigma}\mathbf{\Psi}) = (\mathrm{i}k_0\hat{M}\mathbf{\Psi}, \hat{\sigma}\mathbf{\Psi}) + (\mathbf{\Psi}, \mathrm{i}k_0\hat{\sigma}\hat{M}\mathbf{\Psi})$$
$$= -2k_0\,\mathrm{Im}\,(\mathbf{\Psi}, \hat{\sigma}\hat{M}\mathbf{\Psi})\,.$$

Here, we use the hermiticity of the matrix $\hat{\sigma}$. In terms of transition matrix (3.6) in Section 3, both the energy flux and the dissipation can be expressed through the counterpropagating wave amplitudes:

$$S_z = \left(\begin{bmatrix} \boldsymbol{\mathcal{E}}^+ \\ \boldsymbol{\mathcal{E}}^- \end{bmatrix}, \hat{U}^*\hat{\sigma}\hat{U}\begin{bmatrix} \boldsymbol{\mathcal{E}}^+ \\ \boldsymbol{\mathcal{E}}^- \end{bmatrix}\right),$$
$$q_z = -2k_0\,\mathrm{Im}\left(\begin{bmatrix} \boldsymbol{\mathcal{E}}^+ \\ \boldsymbol{\mathcal{E}}^- \end{bmatrix}, \hat{U}^*\hat{\sigma}\hat{M}\hat{U}\begin{bmatrix} \boldsymbol{\mathcal{E}}^+ \\ \boldsymbol{\mathcal{E}}^- \end{bmatrix}\right). \tag{A.2}$$

These relations show that a medium has no dissipation if the operator $\hat{\sigma}\hat{M}$ is Hermitian. The same conclusion can be drawn in a more complicated way, directly from wave operator (3.5) or from the hermiticity of the tensors $\hat{\varepsilon}$ and $\hat{\mu}$ in a nondissipative media. If we require that the wave field be bounded in the transverse directions, the hermiticity of $\hat{\sigma}\hat{M}$ guarantees the existence of the full discrete spectrum of eigenvectors in Eqn (3.10), used to determine the normal modes. A dissipative medium may have no discrete basis; in that case, the normal modes can be determined disregarding the dissipation. For this, the operator $\hat{M}$ must be replaced by $\hat{M}'$ corresponding to the Hermitian part of $\hat{\sigma}\hat{M}$:

$$\hat{M}' = \frac{\hat{\sigma}^{-1}\left[\hat{\sigma}\hat{M} + (\hat{\sigma}\hat{M})^*\right]}{2} = \hat{M} + \frac{\hat{\sigma}\hat{M}^*\hat{\sigma} - \hat{M}}{2}\,.$$

The decomposition in terms of vacuum modes is complete because it automatically satisfies the hermiticity condition.

The conservation laws can be instructively derived directly from equations for counterpropagating waves (2.1). For simplicity, we assume that the modes used for decomposing the field into counterpropagating waves are normalized such that the unit amplitude $\mathscr{E}_i^\pm$ corresponds to a mode with the 'unit' energy flux. Specifically, this condition is satisfied for the vacuum TE and TM modes. Then the total energy flux is (up to a dimensional normalization factor) the difference between the energy flows of all modes propagating forward and backward along the z axis. In our notation,

$$S_z = \mathrm{const}\left\{(\boldsymbol{\mathcal{E}}^+, \boldsymbol{\mathcal{E}}^+) - (\boldsymbol{\mathcal{E}}^-, \boldsymbol{\mathcal{E}}^-)\right\}.$$

For the TE and TM modes defined by matrix (3.15), the constant is $(c/8\pi)\cos\vartheta$. Using the rule for differentiation of the scalar product

$$\partial_z(\boldsymbol{\mathcal{E}}^\pm, \boldsymbol{\mathcal{E}}^\pm) = (\partial_z\boldsymbol{\mathcal{E}}^\pm, \boldsymbol{\mathcal{E}}^\pm) + (\boldsymbol{\mathcal{E}}^\pm, \partial_z\boldsymbol{\mathcal{E}}^\pm) = 2\,\mathrm{Re}\,(\partial_z\boldsymbol{\mathcal{E}}^\pm, \boldsymbol{\mathcal{E}}^\pm)\,,$$

it is easy to derive the law of energy flux variation in the form

$$\partial_z\{(\boldsymbol{\mathcal{E}}^+, \boldsymbol{\mathcal{E}}^+) - (\boldsymbol{\mathcal{E}}^-, \boldsymbol{\mathcal{E}}^-)\}$$
$$= 2\,\mathrm{Re}\left\{(\boldsymbol{\mathcal{E}}^+, \hat{t}^+\boldsymbol{\mathcal{E}}^+) + (\boldsymbol{\mathcal{E}}^-, \hat{t}^-\boldsymbol{\mathcal{E}}^-) + \left(\boldsymbol{\mathcal{E}}^-, \left[\hat{r}^+ + (\hat{r}^-)^*\right]\boldsymbol{\mathcal{E}}^+\right)\right\}. \tag{A.3}$$

This is a special case following from the general relations (A.2). The right-hand side contains dissipation. Direct substitution in (3.16) and (3.17) shows that if the operator $\hat{\sigma}\hat{M}$ is Hermitian, then the differential transmission operators are anti-Hermitian and the reflection operators are Hermitian conjugate:

$$\hat{t}^+ + (\hat{t}^+)^* = 0\,, \quad \hat{t}^- + (\hat{t}^-)^* = 0\,, \quad \hat{r}^+ + (\hat{r}^-)^* = 0\,. \tag{A.4}$$

This means that the right-hand side of Eqn (A.3) vanishes in a nondissipative medium, which ensures the energy flux conservation in each section across the transverse coordinate:

$$(\boldsymbol{\mathcal{E}}^+, \boldsymbol{\mathcal{E}}^+) - (\boldsymbol{\mathcal{E}}^-, \boldsymbol{\mathcal{E}}^-) = \mathrm{const}\,.$$

An even stronger assertion holds for the reflection $\hat{R}(z)$ and transmission $\hat{T}(z)$ operators in a finite layer $[z, b]$. It follows from (2.12) that

$$-\partial_z(\hat{R}^*\hat{R}) = \left[\hat{R}^*\hat{R}(\hat{t}^+ + \hat{r}^-\hat{R}) + \hat{R}^*\hat{t}^-\hat{R} + \hat{R}^*\hat{r}^+\right] + \text{h.c.},$$
$$-\partial_z(\hat{T}^*\hat{T}) = \hat{T}^*\hat{T}(\hat{t}^+ + \hat{r}^-\hat{R}) + \text{h.c.},$$

where h.c. stands for the Hermitian conjugation of the preceding term. We emphasize that these are operator relations; in particular, their left-hand sides involve not the scalar products but new operators obtained by consecutively applying reflection or transmission operators and their Hermitian conjugates. By virtue of the properties of (A.4), the term $R^*\hat{t}^-\hat{R}$ in a nondissipative medium cancels with its Hermitian conjugate and can therefore be omitted, while the term $\hat{R}^*\hat{r}^+$ can be replaced with $(\hat{R}^*\hat{r}^+)^* - \hat{t}^+ = -(\hat{t}^+ + \hat{r}^-\hat{R})$; here, $\hat{t}^+$ and its Hermitian conjugate also cancel. Adding the two equations gives

$$-\partial_z(\hat{R}^*\hat{R} + \hat{T}^*\hat{T}) = (\hat{R}^*\hat{R} + \hat{T}^*\hat{T} - \hat{I})(\hat{t}^+ + \hat{r}^-\hat{R}) + \text{h.c.}$$

The solution of this equation must satisfy the physical initial conditions under which a zero-thickness layer reflects nothing, $\hat{R}(b) = 0$, and transmits everything, $\hat{T}(b) = \hat{I}$. The obvious solution satisfying these conditions is

$$\hat{R}^*\hat{R} + \hat{T}^*\hat{T} = \hat{I}\,.$$

This universal property of the reflection and transmission operators in an inhomogeneous nondissipative medium has a clear physical meaning: the sum of energy fluxes of the reflected and transmitted waves must be equal to the energy flux in the incident wave. For example, if the operators are given by $n \times n$ matrices, this relation implies n^2 algebraic constraints.

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