

LETTERS TO THE EDITORS

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Negative group velocity electromagnetic waves and the energy–momentum tensor

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Abstract. **V G Veselago's results (*Usp. Fiz. Nauk* 179 689 (2009) [*Phys. Usp.* 52 649 (2009)]) on the electromagnetic (EM) energy–momentum tensor in a medium are analyzed. It is shown that Veselago's statements on the Abraham tensor are wrong (this is not actually a tensor, and the Abraham force was introduced into the theory as an artificial auxiliary device). In discussing the EM energy–momentum tensor in a dispersive medium, it seems to have escaped the author's attention that the problem was resolved a long time ago: the electromagnetic energy–momentum tensor for a dispersive isotropic medium at rest is a symmetric 4-tensor which includes the Brillouin energy density, the energy flux density (Umov–Poynting vector), the momentum density (the Umov–Poynting vector divided by c^2), and the Pitaevskii tension tensor. For a mechanically and thermally equilibrium medium, it is shown that the spatial components of the Polevoi–Rytov tensor which is discussed in the analyzed paper cannot be interpreted as the field-dependent part of the Pitaevskii total tension tensor, unless for quasimonochromatic plane wave propagation. It is also shown that for arbitrary (not necessarily zero) reflection, the force an EM wave in an isotropic medium exerts on a solid can be expressed in terms of an appropriate component of the Polevoi–Rytov tension tensor.**

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1. Introduction

As noted by V E Pafomov [4, 5], electromagnetic waves with a negative group velocity [1] exhibit interesting features in their behavior. The formula describing the Doppler effect — that is, the relation between the frequency ω of a wave emitted by a source moving with velocity $\mathbf{v}$ and the frequency ω_0 of a wave from the same source at rest — can be written in the form (see Ref. [6, § 48])

$$\omega_0 = \frac{\omega}{\sqrt{1 - v^2/c^2}} \left(1 \mp \frac{v \cos\theta}{v_{\mathrm{ph}}} \right), \tag{1.1}$$

where $\mathbf{v}_{\mathrm{ph}} = \omega \mathbf{k}/k^2$ is the phase velocity of the wave, $\mathbf{k}$ is the wave vector, θ is the angle between the source velocity $\mathbf{v}$ and the radiation propagation direction (i.e., the energy flux density vector or the wave group velocity $\mathbf{v}_{\mathrm{gr}}$), the upper sign refers to the positive ($\mathbf{v}_{\mathrm{gr}} \uparrow\uparrow \mathbf{v}_{\mathrm{ph}}$) group velocity, and the lower sign refers to negative ($\mathbf{v}_{\mathrm{gr}} \uparrow\downarrow \mathbf{v}_{\mathrm{ph}}$) group velocity. From Eqn (1.1) it is seen that for a wave with a negative group velocity the Doppler effect will be 'reversed' relative to that for a wave with a positive group velocity [4, 5].

The propagation direction of Cherenkov radiation is obtained from Eqn (1.1) by setting $\omega_0 = 0$, giving

$$\cos\theta = \pm \frac{v_{\mathrm{ph}}}{v} \,. \tag{1.2}$$

The angle between the radiation propagation direction and the particle velocity is acute or obtuse, depending on whether, respectively, the group velocity is positive ($\mathbf{v}_{\mathrm{gr}} \uparrow\uparrow \mathbf{v}_{\mathrm{ph}}$) or negative ($\mathbf{v}_{\mathrm{gr}} \uparrow\downarrow \mathbf{v}_{\mathrm{ph}}$) (i.e., the Vavilov–Cherenkov effect will be reversed [4, 5]).

The case of negative group velocity had been studied even earlier by L I Mandelstam [1; 2, p. 334; 3, p. 461] in connection with the reflection and refraction of an electromagnetic wave

[1] In an isotropic medium, the group and phase velocities of the wave are either in the same or in the opposite direction. We will assume, following Refs [1; 2, p. 334; 3, p. 461], that the group velocity is positive (negative) in the former (latter) case.

incident on a plane interface between two media. Let us take the interface to be the xy plane, and the z-axis to be directed from medium 1 to medium 2. Assume that a wave of frequency ω and wave vector $\mathbf{k}_0$ propagates undamped through medium 1. The y-axis is chosen such that the zx plane coincides with the plane of incidence ($k_{0y} = 0$), and the x-axis is directed so as to satisfy the condition $k_{0x} \geqslant 0$. (We specify that waves of frequency ω in medium 1 have a positive group velocity.) The uniform conditions in the plane of the interface imply that the following relations hold for the wave vectors of the incident ($\mathbf{k}_0$), reflected ($\mathbf{k}_1$), and refracted ($\mathbf{k}_2$) waves:

$$k_{1x} = k_{2x} = k_{0x} \geqslant 0\,, \qquad k_{1y} = k_{2y} = k_{0y} = 0\,. \tag{1.3}$$

According to the above-outlined conditions of the problem, $k_{0x} = k_1 \sin\theta_0 \geqslant 0$, where θ_0 is the angle of incidence, $k_{0z} = k_1\cos\theta_0 \geqslant 0$, and $k_0 = k_1$, so that $k_{1x} = k_1 \sin\theta_0$, $k_{1z} = -k_{0z}$, and the reflection angle $\theta_1 = \theta_0$. It will be assumed that the damping of the refracted wave is also negligible. Then the sign in the expression $k_{2z} = \pm(k_2^2 - k_1^2\sin^2\theta_0)^{1/2}$ is determined from the requirement that the energy of the refracted wave flow back from the boundary into the depths of medium 2 or that the projection of the group velocity of the refracted wave onto the z-axis be positive, $v_{\mathrm{gr}z} > 0$. Therefore, if the group velocity of the wave in medium 2 is positive ($\mathbf{v}_{\mathrm{gr}} \uparrow\uparrow \mathbf{v}_{\mathrm{ph}} \uparrow\uparrow \mathbf{k}_2$), then $k_{2x}, v_{\mathrm{gr}x} > 0$, $v_{\mathrm{gr}z}, k_{2z} > 0$, and the refracted and incident rays lie on different sides of the interface normal. If, on the contrary, the wave group velocity in medium 2 is negative ($\mathbf{v}_{\mathrm{gr}} \uparrow\downarrow \mathbf{v}_{\mathrm{ph}} \uparrow\uparrow \mathbf{k}_2$), then $k_{2x}, v_{\mathrm{ph}x} > 0$, $v_{\mathrm{gr}x} < 0$, $v_{\mathrm{gr}z} > 0$, $k_{2z}, v_{\mathrm{ph}z} < 0$, and the refracted and incident rays lie on the same side of the interface normal. Mandelstam termed the former case 'ordinary refraction'. As for the latter case, he called it 'unusual refraction'; most authors refer to the latter case as 'negative' refraction, and our suggestion [7] is 'reversed' refraction (in line with Refs [4, 5]).

Given that the electrodynamical properties of an isotropic nongyrotropic frequency-dispersive media are characterized by the dielectric $\varepsilon(\omega)$ and magnetic $\mu(\omega)$ permeabilities, it is the behavior of these functions $\varepsilon(\omega)$ and $\mu(\omega)$ which determines the sign of the group velocity of a wave of frequency ω. As noted in Ref. [8], a correct answer to this problem was first given in D V Sivukhin's work [9]: a wave in a medium propagates undamped only if (assuming the dielectric and magnetic permeabilities are real-valued) the product $\varepsilon(\omega)\,\mu(\omega) > 0$, with

$$\varepsilon(\omega), \mu(\omega) > 0 \Rightarrow \mathbf{v}_{\mathrm{gr}} \uparrow\uparrow \mathbf{v}_{\mathrm{ph}}\,, \qquad \varepsilon(\omega), \mu(\omega) < 0 \Rightarrow \mathbf{v}_{\mathrm{gr}} \uparrow\downarrow \mathbf{v}_{\mathrm{ph}}\,. \tag{1.4}$$

Notice, however, that Sivukhin later dismissed this result to state instead (see Ref. [10, § 64]): "For electromagnetic waves in isotropic media it can be shown that phase and energy propagate in the same direction." V E Pafomov [4, 5] later obtained conditions (1.4) by repeating the calculations performed — in a very simple and transparent manner, unlike Ref. [9] — in the first (1957) edition of *Electrodynamics of Continuous Media* (see Ref. [11], § 83), with the only modification being lifting the original restrictions on the signs of $\varepsilon(\omega)$ and $\mu(\omega)$ [see Section 4 below, which reports that $\varepsilon(\omega)$ and $\mu(\omega)$ can assume negative values only if dispersion is allowed for].

V G Veselago in Ref. [12] considers the well-known results of S M Rytov [13] concerning the energy–momentum tensor of a quasimonochromatic plane electromagnetic wave. According to Ref. [13], the momentum density of the wave is codirectional with the wave's phase (not group!) velocity. This result led Veselago to conclude that, when incident on a totally reflecting body, a wave of frequency ω in a medium with negative $\varepsilon(\omega)$ and $\mu(\omega)$ does not repel, but rather attracts the body.[2]

In his recent *Physics–Uspekhi* paper [15], Veselago proceeds to address the general aspects of the electrodynamics of continuous media in relation to the energy–momentum tensor of the electromagnetic field. Most of the statements in Ref. [15] disagree with the results of other studies (including ours [16]) and are, as will be shown below, erroneous.

At this point it is worth reminding ourselves of the exact essence of the problem of the energy–momentum tensor of an electromagnetic field in a continuous medium [17, § 35]. Maxwell's equations for the fields $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$ have the form [11, § 75]

$$\begin{aligned} &\operatorname{div}\mathbf{D} = 4\pi\rho^{\,\mathrm{ext}}\,, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\,, \\ &\operatorname{div}\mathbf{B} = 0\,, \qquad \operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j}^{\,\mathrm{ext}}\,, \end{aligned} \tag{1.5}$$

where $\rho^{\,\mathrm{ext}}$ and $\mathbf{j}^{\,\mathrm{ext}}$ are, respectively, the density and current density of the external (relative to the medium under consideration) charges. From Eqns (1.5) one rigorously obtains the equation for the work the field does on the external charges:

$$\mathbf{j}^{\mathrm{ext}}\mathbf{E} = -\frac{1}{4\pi}\left(\mathbf{E}\,\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\,\frac{\partial\mathbf{B}}{\partial t}\right) - \operatorname{div}\mathbf{S}^{\mathrm{P}}\,, \qquad \mathbf{S}^{\mathrm{P}} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}\,, \tag{1.6}$$

where $\mathbf{S}^{\mathrm{P}}$ is the Umov–Poynting vector, and the equation for the force $\mathbf{f}^{\mathrm{ext}} = \rho^{\,\mathrm{ext}}\mathbf{E} + (1/c)\,\mathbf{j}^{\mathrm{ext}}\times\mathbf{B}$ the field exerts on external charges in the unit volume is written out as[3]

$$\begin{aligned} &\rho^{\,\mathrm{ext}}E_i + \frac{1}{c}(\mathbf{j}^{\mathrm{ext}}\times\mathbf{B})_i = -\frac{1}{4\pi c}\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})_i + \sigma_i'^{\,\mathrm{M}} \\ &\quad - \frac{1}{8\pi}\left[\left(\mathbf{D}\,\frac{\partial\mathbf{E}}{\partial r_i} - \mathbf{E}\,\frac{\partial\mathbf{D}}{\partial r_i}\right) + \left(\mathbf{B}\,\frac{\partial\mathbf{H}}{\partial r_i} - \mathbf{H}\,\frac{\partial\mathbf{B}}{\partial r_i}\right)\right], \\ &\sigma_i'^{\,\mathrm{M}} = \frac{\partial\sigma_{ij}^{\mathrm{M}}}{\partial r_j}\,, \quad \sigma_{ij}^{\mathrm{M}} = \frac{1}{4\pi}\left[(E_iD_j + H_iB_j) - \frac{1}{2}\,\delta_{ij}(\mathbf{ED} + \mathbf{HB})\right], \end{aligned} \tag{1.7}$$

where σ_{ij}^{M} is the Minkowski stress tensor.

In microscopic electrodynamics ($\mathbf{D} = \mathbf{E}, \mathbf{B} = \mathbf{H}$), Eqns (1.6) and (1.7) are reduced to the equations which are equivalent to the energy and momentum conservation laws (see Ref. [6, § 31–33]):

$$\mathbf{j}^{\mathrm{ext}}\mathbf{E} = -\frac{\partial w}{\partial t} - \operatorname{div}\mathbf{S}\,, \qquad \mathbf{f}^{\,\mathrm{ext}} = -\frac{\partial\mathbf{g}}{\partial t} + \boldsymbol{\sigma}'\,, \qquad \sigma_i' = \frac{\partial\sigma_{ij}}{\partial r_j}\,; \tag{1.8}$$

$$\begin{aligned} &w = \frac{1}{8\pi}(E^2 + H^2)\,, \qquad \mathbf{S} = \mathbf{S}^{\mathrm{P}}\,, \qquad \mathbf{g} = \frac{1}{c^2}\,\mathbf{S}^{\mathrm{P}}\,, \\ &\sigma_{ij} = \sigma_{ji} = \frac{1}{4\pi}\left[(E_iE_j + H_iH_j) - \frac{1}{2}\,\delta_{ij}(E^2 + H^2)\right]. \end{aligned} \tag{1.9}$$

[2] In Ref. [14], which reports the observation of the reversed Vavilov–Cherenkov effect, the prediction of this effect is credited to Ref. [12]. In actual fact, as noted above, the prediction was made by V E Pafomov [4, 5].

[3] Summation over all the twice repeating indices $i, j, k, \ldots = x, y, z$ is assumed to be taken throughout.

Here, w is the energy density, $\mathbf{S}$ is the energy density flux, $\mathbf{g}$ is the momentum density, σ_{ij} is the Maxwell stress tensor. All these quantities combine into the energy–momentum 4-tensor $T_{\alpha\beta}$ of the electromagnetic field ($r_\alpha = (\mathbf{r}, \mathrm{i}ct)$):

$$T_{ij} = \sigma_{ij}\,, \quad T_{j4} = -\mathrm{i}cg_j\,, \quad T_{4j} = -\frac{\mathrm{i}}{c}\,S_j\,, \quad T_{44} = w\,. \qquad (1.10)$$

Introducing the 4-force $f_\alpha^{\mathrm{ext}} = (\mathbf{f}^{\mathrm{ext}}, \mathrm{i}\,\mathbf{j}^{\mathrm{ext}}\mathbf{E}/c)$, equations (1.8) become[4]

$$f_\alpha^{\mathrm{ext}} = \frac{\partial T_{\alpha\beta}}{\partial r_\beta}\,. \qquad (1.11)$$

Following Minkowski and introducing the 4-tensor of the field, $F_{\alpha\beta} = -F_{\beta\alpha}$: $F_{ij} = e_{ijk}H_k$, $F_{4j} = -F_{j4} = \mathrm{i}E_j$, the energy–momentum tensor of the field in a vacuum, Eqns (1.9) and (1.10), takes the form

$$T_{\alpha\beta} = T_{\beta\alpha} = \frac{1}{4\pi}\left(F_{\alpha\gamma}F_{\gamma\beta} - \frac{1}{4}\,\delta_{\alpha\beta}F_{\gamma\nu}F_{\nu\gamma}\right). \qquad (1.12)$$

In macroscopic electrodynamics, equations like (1.8) *cannot* be obtained rigorously from Eqns (1.6), (1.7). Still, it is believed (see Ref. [17, § 35]) that equations equivalent to the energy–momentum conservation laws of the system 'matter plus field' *do* exist and have the following form [analogous to Eqns (1.8) and (1.11)]:

$$\mathbf{j}^{\mathrm{ext}}\mathbf{E} + \mathbf{f}\mathbf{v} = -\frac{\partial w}{\partial t} - \operatorname{div}\mathbf{S}\,, \qquad \mathbf{f}^{\mathrm{ext}} + \mathbf{f} = -\frac{\partial \mathbf{g}}{\partial t} + \boldsymbol{\sigma}'\,, \qquad (1.13)$$

$$\frac{\partial T_{\alpha\beta}}{\partial r_\beta} = f_\alpha^{\mathrm{ext}} + f_\alpha\,, \qquad (1.14)$$

where $(\mathbf{f}, \mathrm{i}\mathbf{f}\mathbf{v}/c)$ is the 4-force exerted by the field on the matter (per unit volume), and $\mathbf{v}$ is the velocity of matter.

The following are the requirements which necessarily have to be met when choosing the expression for $T_{\alpha\beta}$: (1) $T_{\alpha\beta}$ must really be a tensor in the sense that under all the transformations entering into the Lorentz group it must be transformed in the same manner as the product of two 4-vectors; (2) in the limiting case of no matter ($\mathbf{f} = 0, \mathbf{E} = \mathbf{D}, \mathbf{B} = \mathbf{H}$), $T_{\alpha\beta}$ must be identical to tensor (1.9), (1.10), or (1.12) of microscopic electrodynamics; (3) equations (1.13) must not be inconsistent with equations (1.6), (1.7) which rigorously follow from Maxwell's equations; (4) expressions for the force, which are determined from the two equations (1.13), must differ only in the term which is perpendicular to the velocity $\mathbf{v}$ of the medium (and does not contribute to the work $\mathbf{f}\mathbf{v}$), and (5) the force $\mathbf{f}$ obtained from equations (1.13) must not be inconsistent with the equations of motion of matter (i.e., the hydrodynamics or elasticity theory equations).

In his 1908 paper [18], H Minkowski introduced two 4-field tensors, $F_{\alpha\beta} = -F_{\beta\alpha}$ and $H_{\alpha\beta} = -H_{\beta\alpha}$ (see Ref. [11, § 76], [17, § 33]),

$$\begin{aligned} &F_{ij} = e_{ijk}B_k\,, \quad F_{4j} = -F_{j4} = \mathrm{i}E_j;\\ &H_{ij} = e_{ijk}H_k\,, \quad H_{4j} = -H_{j4} = \mathrm{i}D_j\,, \end{aligned} \qquad (1.15)$$

which allow Maxwell's equations (1.5) to be rewritten in explicitly relativistic-invariant form as

$$\frac{\partial F_{\alpha\beta}}{\partial r_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial r_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial r_\beta} = 0\,, \qquad \frac{\partial H_{\alpha\beta}}{\partial r_\beta} = \frac{4\pi}{c}\,j_\alpha^{\mathrm{ext}}\,. \qquad (1.16)$$

In his studies [18, 19], Minkowski chose the energy–momentum tensor of electromagnetic field in the form analogous to expression (1.12) in microscopic electrodynamics:

$$T_{\alpha\beta}^{\mathrm{M}} = \frac{1}{4\pi}\left(F_{\alpha\gamma}H_{\gamma\beta} - \frac{1}{4}\,F_{\gamma\nu}H_{\nu\gamma}\delta_{\alpha\beta}\right) \qquad (1.17)$$

or, according to formulas (1.10) and (1.15), σ_{ij}^{M} in the form (1.7) and

$$\begin{aligned} &\mathbf{g}^{\mathrm{M}} = \frac{1}{4\pi c}\,\mathbf{D}\times\mathbf{B}\,, \qquad \mathbf{S}^{\mathrm{M}} = \mathbf{S}^{\mathrm{P}} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}\,,\\ &w^{\mathrm{M}} = \frac{1}{8\pi}(\mathbf{E}\mathbf{D} + \mathbf{H}\mathbf{B})\,. \end{aligned} \qquad (1.18)$$

The tensor nature of $T_{\alpha\beta}^{\mathrm{M}}$ written in the form (1.17) is obvious, as is the fact that, in microscopic electrodynamics in the limit $\mathbf{E} = \mathbf{D}$, $\mathbf{B} = \mathbf{H}$, the tensor $T_{\alpha\beta}^{\mathrm{M}}$ reduces to the form (1.12). It should be noted that constitutive equations relating the fields $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$ are not used in Eqn (1.17), indicating that the Minkowski tensor is valid for any (not necessarily isotropic) medium and for arbitrarily strong fields.

Minkowski tensor (1.17), unlike the energy–momentum tensor in microscopic electrodynamics, formula (1.12), is not symmetric. M Abraham [20, 21] (see also Ref. [17, § 35]) suggested a symmetric form for the energy–momentum tensor of an electromagnetic field, $T_{\alpha\beta}^{\mathrm{A}}$, in a medium. In a comoving frame of reference (relative to which the medium is at rest) one has[5]

$$\begin{aligned} &\bar{\sigma}_{ij}^{\mathrm{A}} = \bar{\sigma}_{ji}^{\mathrm{A}} = \frac{1}{2}(\bar{\sigma}_{ij}^{\mathrm{M}} + \bar{\sigma}_{ji}^{\mathrm{M}})\,, \qquad \bar{\mathbf{g}}^{\mathrm{A}} = \frac{1}{c^2}\,\bar{\mathbf{S}}^{\mathrm{P}}\,,\\ &\bar{\mathbf{S}}^{\mathrm{A}} = \bar{\mathbf{S}}^{\mathrm{P}}\,, \qquad \bar{w}^{\mathrm{A}} = \bar{w}^{\mathrm{M}}\,. \end{aligned} \qquad (1.19)$$

The first two results of paper [15] refer to the Abraham tensor $T_{\alpha\beta}^{\mathrm{A}}$. These results are discussed in Sections 2 and 3 below.

2. The Abraham form of the energy–momentum tensor

The first major result of Ref. [15] as formulated in its Abstract is that "the Abraham energy–momentum tensor is actually not a tensor *because of its lack of relativistic invariance*" (the italics are ours), but in the body of the paper, the only argument given to support this statement reads (p. 693): "as direct calculation shows." Dropping the italics above makes the statement understandable, but as our 'direct calculation' will show, it is wrong. If the italicized words are also taken into consideration, the statement can only be considered to be due to some kind of misconception: the invariance property can only be spoken of with regard to a zero-rank tensor, which is a scalar.

An explicitly covariant form for the Abraham tensor — i.e., one valid in any inertial reference frame — was obtained in 1913 by R Grammel [22]. It can be found in, for example, Wolfgang Pauli's book [17, § 35] and is used in this form in current studies (see, for example, Ref. [23]). Importantly, however, this form involves the dielectric and magnetic

[4] Summation over all twice repeating indices $\alpha, \beta, \gamma, \ldots = 1, 2, 3, 4$ is assumed to be taken throughout.

[5] The quantities referring to the comoving coordinate system are marked with vincula.

permeabilities and, hence, implies that the fields involved are sufficiently weak (compared to the internal atomic field) and that the medium is isotropic. This may create the impression that the Minkowski tensor $T^{\mathrm{M}}_{\alpha\beta}$ is, in a sense, preferable to its Abraham counterpart. We will show below that, similarly to the Minkowski tensor (1.17), the Abraham tensor can also be written without invoking constitutive equations.

Tensor (1.19) in the comoving reference frame can be presented in the form

$$\bar{T}^{\mathrm{A}}_{\alpha\beta} = \frac{1}{2}\left(\bar{T}^{\mathrm{M}}_{\alpha\beta} + \bar{T}^{\mathrm{M}}_{\beta\alpha}\right) + \bar{A}_{\alpha\beta}\,, \tag{2.1}$$

where, using Eqns (1.15) and (1.18), one has

$$\bar{A}_{ij} = \bar{A}_{44} = 0\,, \qquad \bar{A}_{j4} = \bar{A}_{4j} = \frac{\mathrm{i}}{8\pi}\left(\bar{\mathbf{D}}\times\bar{\mathbf{B}} - \bar{\mathbf{E}}\times\bar{\mathbf{H}}\right)_j$$
$$= \frac{1}{8\pi}\left(\bar{H}_{jk}\bar{F}_{k4} - \bar{F}_{jk}\bar{H}_{k4}\right). \tag{2.2}$$

Let us compose the quantity

$$A_{\alpha\beta} = L_{\alpha\gamma}L_{\beta\nu}\bar{A}_{\gamma\nu}\,, \tag{2.3}$$

where $L_{\alpha\beta}$ are the Lorentz transformation coefficients in passing from the comoving frame to the laboratory frame:

$$r_\alpha = L_{\alpha\beta}\,\bar{r}_\beta\,, \quad \bar{r}_\alpha = L_{\beta\alpha}\,r_\beta\,, \quad L_{\alpha\gamma}L_{\beta\gamma} = L_{\gamma\alpha}L_{\gamma\beta} = \delta_{\alpha\beta}\,. \tag{2.4}$$

By substituting $\bar{A}_{\alpha\beta}$ from Eqn (2.2) into Eqn (2.3) we obtain, after a little algebra, the following expression

$$A_{\alpha\beta} = \frac{1}{8\pi}\left(L_{\alpha4}L_{\beta j} + L_{\alpha j}L_{\beta4}\right)\left(\bar{H}_{j\gamma}\bar{F}_{\gamma4} - \bar{F}_{j\gamma}\bar{H}_{\gamma4}\right). \tag{2.5}$$

Using Eqn (2.4) and noting the antisymmetry of the 4-tensors $F_{\alpha\beta}$ and $H_{\alpha\beta}$ reduces Eqn (2.5) to the form

$$A_{\alpha\beta} = \frac{1}{8\pi}\,L_{\gamma4}\big[L_{\alpha4}(H_{\beta\nu}F_{\nu\gamma} - F_{\beta\nu}H_{\nu\gamma})$$
$$+ L_{\beta4}(H_{\alpha\nu}F_{\nu\gamma} - F_{\alpha\nu}H_{\nu\gamma})\big]. \tag{2.6}$$

Lorentz transformations for the case in which the comoving frame has velocity $\mathbf{v}$ directed arbitrarily with respect to the coordinate axes take the form (see Ref. [17, § 4])

$$\mathbf{r} = \bar{\mathbf{r}} + \gamma\mathbf{v}\bar{t} + \frac{\gamma-1}{v^2}\,\mathbf{v}(\mathbf{v}\bar{\mathbf{r}})\,,$$
$$t = \gamma\left(\bar{t} + \frac{\mathbf{v}\bar{\mathbf{r}}}{c^2}\right), \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}\,. \tag{2.7}$$

From this and Eqn (2.4), it follows that $L_{\alpha4}$ can be expressed in terms of the 4-velocity u_α:

$$L_{\alpha4} = -\mathrm{i}u_\alpha\,, \qquad u_\alpha = \gamma\left(\frac{\mathbf{v}}{c}\,,\,\mathrm{i}\right). \tag{2.8}$$

From Eqns (2.6) and (2.8) we find that

$$A_{\alpha\beta} = A_{\beta\alpha} = \frac{1}{8\pi}\,u_\gamma\big[u_\alpha(F_{\beta\nu}H_{\nu\gamma} - H_{\beta\nu}F_{\nu\gamma})$$
$$+ u_\beta(F_{\alpha\nu}H_{\nu\gamma} - H_{\alpha\nu}F_{\nu\gamma})\big]\,. \tag{2.9}$$

Because u_α is a 4-vector, and $F_{\alpha\beta}$ and $H_{\alpha\beta}$ are 4-tensors, it follows that $A_{\alpha\beta}$ defined by formula (2.9) is a symmetric second-rank tensor, and hence so is

$$T^{\mathrm{A}}_{\alpha\beta} = \frac{1}{2}\left(T^{\mathrm{M}}_{\alpha\beta} + T^{\mathrm{M}}_{\beta\alpha}\right) + A_{\alpha\beta}\,. \tag{2.10}$$

In the comoving reference frame, $\bar{T}^{\mathrm{A}}_{\alpha\beta}$ is written out in the form of Eqn (1.19) or (2.1).

Thus, the Abraham energy–momentum tensor $T^{\mathrm{A}}_{\alpha\beta}$ is indeed a tensor — exactly as is the case with its Minkowski counterpart $T^{\mathrm{M}}_{\alpha\beta}$ — that is, both transform as the product of two 4-vectors under any Lorentz group transformation.

In an attempt to explain the result of Ref. [15] concerning the Abraham tensor, the authors of a letter to *Physics–Uspekhi* [24] (written as a response to our work [16]) present some calculations which, they believe, show that "the Minkowski tensor in any inertial frame of reference depends in like manner on field components in the same frame, whereas the Abraham tensor does not" (the fields they speak of are **E**, **H**, **D**, and **B**). In actual fact, the Abraham tensor $T^{\mathrm{A}}_{\alpha\beta}$ (2.10) has the same form in all inertial reference frames [as has the Minkowski tensor (1.17)]. However, unlike the Abraham tensor $T^{\mathrm{A}}_{\alpha\beta}$ (2.10), the Minkowski tensor $T^{\mathrm{M}}_{\alpha\beta}$ (1.17) does not contain the velocity of the medium — as do not, by the way, Maxwell's equations (1.5) or (1.16) for the four fields **E**, **H**, **D**, and **B**. It is apparently the independence of the Minkowski tensor (1.17) of the velocity of the medium which, in Veselago's view, is the advantage of $T^{\mathrm{M}}_{\alpha\beta}$ over the Abraham tensor $T^{\mathrm{A}}_{\alpha\beta}$.

However, it can be argued, in line with Pauli [17, § 33], that Maxwell's equations and energy–momentum tensors (of any form) containing all four fields **E**, **H**, **D**, and **B** "are only a hollow notion unless relations between **E**, **H** and **D**, **B** are established," i.e., unless the constitutive equations — which do depend on the medium velocity — are introduced (see Ref. [11, § 76], [17, § 33]). With this done, Maxwell's equations (only for two fields, **E** and **H**), the Minkowski energy–momentum tensor, and the Abraham tensor will be dependent on the velocity of the medium.

The real difference between the Minkowski and Abraham tensors is that $T^{\mathrm{A}}_{\alpha\beta}$ is symmetric, whereas $T^{\mathrm{M}}_{\alpha\beta}$ is not. The symmetry of the energy–momentum tensor is related to the conservation of the 4-tensor of the moment of momentum (see Ref. [6, § 32]), so that the lack of symmetry is a serious drawback for the Minkowski tensor. Pauli's view [17, § 35] was that the macroscopic energy–momentum tensor cannot be asymmetric because it is obtained by averaging the microscopic energy–momentum tensor, which is symmetric, and "averaging does not detract from the symmetry of a tensor." Notice also that in the relevant chapter of the *Electrodynamics of Continuous Media* [11, § 75] the Minkowski tensor (as an alternative to the Abraham tensor) is not discussed at all — exactly because, we believe, it is asymmetric.

As regards the calculations performed in Ref. [24], we note that they can be made correct by properly supplementing the setting up of the problem. The authors write: "Let us consider two reference frames, K and K′, where K′ moves at the velocity v with respect to K along the Ox-axis." In reality, though, these two frames are not arbitrary: K is a comoving reference frame (with respect to which the matter is at rest), whereas K′ is a laboratory frame, so that the velocity v has a different sign than the velocity of matter.

3. Force acting on matter in an electromagnetic field in the absence of dispersion

A second conclusion of paper [15] concerns equation (1.14) [labelled (22) in Ref. [15][6]] for an isotropic nongyrotropic medium at rest with nondispersive ε and μ in the absence of external charges ($\rho^{\text{ext}} = 0$, $\mathbf{j}^{\text{ext}} = 0$). It reads as follows: "Whereas the Minkowski tensor causes no problems when using Eqn (22), determining the forces by means of the Abraham tensor turns out to require ... that Eqn (22) be modified by introducing the so-called Abraham force.... *Another reason why the Abraham tensor cannot be directly used in Eqn* (22) *is that it is not relativistically invariant*" (the italics are ours). The lack of meaning in the italicized sentence has been noted in Section 2. The result itself, the author of Ref. [15] argues, is proved in Ref. [25] — but it is not. So our task here will be to show by direct calculation what Eqn (1.14) will actually yield if the Minkowski tensor $T^{\text{M}}_{\alpha\beta}$ or the Abraham tensor $T^{\text{A}}_{\alpha\beta}$ is substituted into it.

We start by writing out Eqns (1.6), (1.7) for the case we are considering here ($\rho^{\text{ext}} = 0$, $\mathbf{j}^{\text{ext}} = 0$), remembering in addition that for a medium at rest[7] $\partial\varepsilon/\partial t = \partial\mu/\partial t = 0$ and $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$:

$$\frac{1}{8\pi}\frac{\partial}{\partial t}(\varepsilon E^2 + \mu H^2) + \operatorname{div}\mathbf{S}^{\text{P}} = 0\,, \tag{3.1}$$

$$-\frac{\varepsilon\mu}{4\pi c}\frac{\partial}{\partial t}\,\mathbf{E}\times\mathbf{H} + \boldsymbol{\sigma}'^{\text{M}} + \frac{1}{8\pi}(\mathbf{E}^2\nabla\varepsilon + \mathbf{H}^2\nabla\mu) = 0\,, \tag{3.2}$$

where now

$$\sigma^{\text{M}}_{ij} = \sigma^{\text{M}}_{ji} = \frac{1}{4\pi}\left[(\varepsilon E_i E_j + \mu H_i H_j) - \frac{1}{2}(\varepsilon E^2 + \mu H^2)\delta_{ij}\right]. \tag{3.3}$$

If the energy–momentum tensor is taken in the Minkowski form, Eqn (1.17) or Eqn (1.18), then equations (1.13) or equation (1.14) with $\mathbf{j}^{\text{ext}} = 0$, $\mathbf{f}^{\text{ext}} = 0$, $\mathbf{v} = 0$, $\mathbf{D} = \varepsilon\mathbf{E}$, and $\mathbf{B} = \mu\mathbf{H}$ reduce to equation (3.1) and to

$$\mathbf{f}^{\text{M}} = -\frac{\varepsilon\mu}{4\pi c}\frac{\partial}{\partial t}\,\mathbf{E}\times\mathbf{H} + \boldsymbol{\sigma}'^{\text{M}}\,, \tag{3.4}$$

which, together with equation (3.2), defines the force in the Minkowski approach (see Refs [25, 26]):

$$\mathbf{f}^{\text{M}} = -\frac{1}{8\pi}(\mathbf{E}^2\nabla\varepsilon + \mathbf{H}^2\nabla\mu)\,. \tag{3.5}$$

Using the Abraham tensor (1.19) for $T_{\alpha\beta}$ in equation (1.13) or equation (1.14), we obtain the same equation (3.1) and, instead of equation (3.4), the equation

$$\mathbf{f}^{\text{A}} = -\frac{1}{4\pi c}\frac{\partial}{\partial t}\,\mathbf{E}\times\mathbf{H} + \boldsymbol{\sigma}'^{\text{M}}\,, \tag{3.6}$$

which, with equation (3.2) taken into account, gives the force in the Abraham approach (see Refs [25, 26]):

$$\mathbf{f}^{\text{A}} = -\frac{1}{8\pi}(\mathbf{E}^2\nabla\varepsilon + \mathbf{H}^2\nabla\mu) + \frac{\varepsilon\mu - 1}{c^2}\frac{\partial\mathbf{S}^{\text{P}}}{\partial t} = \mathbf{f}^{\text{M}} + \mathbf{f}_{\text{A}}\,, \tag{3.7}$$

differing from $\mathbf{f}^{\text{M}}$ by an additional term, the so-called Abraham force

$$\mathbf{f}_{\text{A}} = \frac{\varepsilon\mu - 1}{4\pi c}\frac{\partial}{\partial t}\,\mathbf{E}\times\mathbf{H}\,. \tag{3.8}$$

Thus, one and the same equation (1.14) produces both equations for the force $\mathbf{f}^{\text{M}}$, formula (3.5), in the Minkowski approach [when the field momentum density is taken to be $\mathbf{g}^{\text{M}}$, formula (1.18)] and for the force $\mathbf{f}^{\text{A}}$, formula (3.7), in the Abraham approach [when the field momentum density $\mathbf{g}^{\text{A}}$ is given by Eqn (1.19)] — and this happens without any additions suggested in Ref. [15]. The difference between the forces in these approaches is determined by the Abraham force (3.8).

In the statical limit, the Abraham force (3.8) vanishes, making the expressions for the force in both approaches identical. These expressions, however, do not include the so-called striction forces (see Ref. [11, §§ 15, 35]) obtained by Helmholtz [27] (Helmholtz's paper [27] is cited in monograph [28, p. 135]):

$$\begin{aligned}\mathbf{f}^{\text{H}} &= -\frac{1}{8\pi}(E^2\nabla\varepsilon + H^2\nabla\mu) + \mathbf{f}^{\text{str}}\,,\\ \mathbf{f}^{\text{str}} &= \frac{1}{8\pi}\,\nabla\rho\left(\frac{\partial\varepsilon}{\partial\rho}\,E^2 + \frac{\partial\mu}{\partial\rho}\,H^2\right),\end{aligned} \tag{3.9}$$

where ρ is the density of matter. The force $\mathbf{f}^{\text{str}}$ is taken into account (in the case of a medium at rest) by adding the tensor

$$\sigma^{\text{str}}_{ij} = \frac{1}{8\pi}\,\delta_{ij}\,\rho\left(\frac{\partial\varepsilon}{\partial\rho}\,E^2 + \frac{\partial\mu}{\partial\rho}\,H^2\right) \tag{3.10}$$

to $T^{\text{M}}_{\alpha\beta}$ and $T^{\text{A}}_{\alpha\beta}$. So modified, the tensors are, as before, called the Minkowski and Abraham energy–momentum tensors [25]. In our view, if we neglect dispersion, the energy–momentum tensor of an electromagnetic field in a medium at rest should be more properly called the Helmholtz–Abraham tensor, $T^{\text{HA}}_{\alpha\beta}$. Its components can be defined in accordance with Eqn (1.10) as follows (see Ref. [11, §§ 15, 35, 75]):

$$\begin{aligned}&T^{\text{HA}}_{ij} = \sigma^{\text{H}}_{ij}\,, \qquad T^{\text{HA}}_{j4} = T^{\text{HA}}_{4j} = -\frac{\text{i}}{c}\,S^{\text{P}}_j\,,\\ &T^{\text{HA}}_{44} = w = \frac{1}{8\pi}(\varepsilon E^2 + \mu H^2)\,,\end{aligned} \tag{3.11}$$

$$\sigma^{\text{H}}_{ij} = \sigma^{\text{H}}_{ji} = \frac{\varepsilon}{4\pi}\left(E_i E_j - \frac{1}{2}\,E^2\delta_{ij}\right) + \frac{\mu}{4\pi}\left(H_i H_j - \frac{1}{2}\,H^2\delta_{ij}\right) + \sigma^{\text{str}}_{ij}\,, \tag{3.12}$$

where σ^{str}_{ij} is the tensor (3.10). The expression for the force, accordingly, has the form

$$\mathbf{f}^{\text{HA}} = \mathbf{f}^{\text{H}} + \mathbf{f}_{\text{A}}\,, \tag{3.13}$$

with $\mathbf{f}^{\text{H}}$ and $\mathbf{f}_{\text{A}}$ given by formulas (3.9) and (3.8), respectively.

The force acting on a unit volume of the medium in the absence of a field is $-\nabla P_0$, where P_0 is the pressure in the medium, and its corresponding stress tensor is defined as $-P_0\delta_{ij}$ (see Ref. [11, § 15]). Therefore, the total stress tensor in a medium in the presence of an electromagnetic field is

$$\sigma^{\text{tot}}_{ij} = -P_0\delta_{ij} + \sigma^{\text{H}}_{ij}\,, \tag{3.14}$$

[6] Similar to Ref. [25], we use an imaginary temporal variable; Ref. [15] uses a real temporal variable ct, in which case one should take into account the difference between the covariant, contravariant, and mixed tensor components.

[7] Because the discussion below is limited to media at rest, we will simplify our work by removing the vincula from variables. For example, instead of $\bar{\mathbf{E}}$ we will write simply $\mathbf{E}$.

and the total force acting on a unit volume of the medium is given by

$$\mathbf{f}^{\,\mathrm{tot}} = -\nabla P_0 + \mathbf{f}^{\,\mathrm{HA}}\,, \tag{3.15}$$

where $P_0(\rho, T)$ is the pressure which would be found in the medium in the absence of an electromagnetic field for the given values of density and temperature (see Ref. [11, § 15]).

It should be noted that, for any problem, the tensor σ_{ij} cannot enter the final answer unless together with $(-P_0\delta_{ij})$, and the force $\mathbf{f}$, unless together with $(-\nabla P_0)$, implying the obvious fact that only the total stress tensor $\sigma_{ij}^{\mathrm{tot}}$ and the total force $\mathbf{f}^{\,\mathrm{tot}}$ are physically meaningful.

4. Abraham–Brillouin–Pitaevskii and Polevoi–Rytov tensors

Reference [15] then proceeds to discuss the energy–momentum tensor of an electromagnetic field allowing for dispersive dielectric and magnetic permeabilities. As is well known (see Ref. [11, § 80]), in this case one is considering a quasimonochromatic field which comprises a set of monochromatic components with frequencies in a narrow interval $\delta\omega$ about a certain average frequency $\omega > 0$:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \operatorname{Re} \mathbf{E}_0(\mathbf{r}, t) \exp(-\mathrm{i}\omega t)\,, \\ \mathbf{H}(\mathbf{r}, t) &= \operatorname{Re} \mathbf{H}_0(\mathbf{r}, t) \exp(-\mathrm{i}\omega t)\,, \end{aligned} \tag{4.1}$$

where $\mathbf{E}_0(\mathbf{r}, t)$ and $\mathbf{H}_0(\mathbf{r}, t)$ vary slowly in time compared to $\exp(-\mathrm{i}\omega t)$: if $\tau_0 \sim 1/(\delta\omega)$ is the time scale of $\mathbf{E}_0(\mathbf{r}, t)$ and $\mathbf{H}_0(\mathbf{r}, t)$, then

$$\frac{1}{\omega\tau_0} \ll 1\,. \tag{4.2}$$

The field inductions $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ can be written out in a similar form:

$$\begin{aligned} \mathbf{D}(\mathbf{r}, t) &= \operatorname{Re} \mathbf{D}_0(\mathbf{r}, t) \exp(-\mathrm{i}\omega t)\,, \\ \mathbf{B}(\mathbf{r}, t) &= \operatorname{Re} \mathbf{B}_0(\mathbf{r}, t) \exp(-\mathrm{i}\omega t)\,. \end{aligned} \tag{4.3}$$

The functions $\mathbf{D}_0$ and $\mathbf{B}_0$ are related to the functions $\mathbf{E}_0$ and $\mathbf{H}_0$ by constitutive equations which can be written, correct to first-order terms in the parameter (4.2), as (see Ref. [11], § 102)

$$\begin{aligned} \mathbf{D}_0(\mathbf{r}, t) &= \varepsilon(\omega)\,\mathbf{E}_0(\mathbf{r}, t) + \mathrm{i}\,\frac{\partial\varepsilon}{\partial\omega}\,\frac{\partial\mathbf{E}_0}{\partial t}\,, \\ \mathbf{B}_0(\mathbf{r}, t) &= \mu(\omega)\,\mathbf{H}_0(\mathbf{r}, t) + \mathrm{i}\,\frac{\partial\mu}{\partial\omega}\,\frac{\partial\mathbf{H}_0}{\partial t}\,. \end{aligned} \tag{4.4}$$

The energy density of an electromagnetic field in matter, i.e., the component T_{44} of the energy–momentum tensor [to a zero-order approximation in parameter (4.2)], is found to be given by the Brillouin relation (see Ref. [11, § 80])[8]

$$\begin{aligned} w^{\mathrm{B}} &= \frac{1}{8\pi}\left(\frac{\partial\omega\varepsilon}{\partial\omega}\,\langle E^2\rangle + \frac{\partial\omega\mu}{\partial\omega}\,\langle H^2\rangle\right) \\ &= \frac{1}{16\pi}\left[\frac{\partial\omega\varepsilon}{\partial\omega}\,|\mathbf{E}_0^2| + \frac{\partial\omega\mu}{\partial\omega}\,|\mathbf{H}_0^2|\right]. \end{aligned} \tag{4.5}$$

Reference [15, p. 693] argues that "...no question usually arises as to whether all the other components of the energy–momentum tensor should also be modified in some way in the presence of dispersion." In reality, the question of the remaining components of the energy–momentum tensor not only has arisen but has long been answered (see Ref. [11, §§ 75, 80, 81]). We have the following relations

$$w = w^{\mathrm{B}}\,, \quad \mathbf{S} = \langle\mathbf{S}^{\mathrm{P}}\rangle\,, \quad \mathbf{g} = \mathbf{g}^{\mathrm{A}} = \frac{\langle\mathbf{S}^{\mathrm{P}}\rangle}{c^2}\,, \quad \sigma_{ij}^{\mathrm{tot}} = -P_0\delta_{ij} + \sigma_{ij}^{\mathrm{P}}\,,$$

$$\begin{aligned} \sigma_{ij}^{\mathrm{P}} = \sigma_{ji}^{\mathrm{P}} &= \frac{1}{4\pi}\left[\varepsilon(\omega)\langle E_iE_j\rangle - \frac{1}{2}\left(\varepsilon - \rho\,\frac{\partial\varepsilon}{\partial\rho}\right)\langle E^2\rangle\delta_{ij}\right] \\ &+ \frac{1}{4\pi}\left[\mu(\omega)\langle H_iH_j\rangle - \frac{1}{2}\left(\mu - \rho\,\frac{\partial\mu}{\partial\rho}\right)\langle H^2\rangle\delta_{ij}\right]; \end{aligned} \tag{4.6}$$

the stress tensor σ_{ij}^{P} was obtained as far back as 1960 by L P Pitaevskii [29]. What is remarkable about the Pitaevskii tensor is that, unlike the energy w^{B} in formula (4.5) — and also to zero-order approximation in the parameter (4.2) — it does not contain derivatives $\partial\varepsilon/\partial\omega$ and $\partial\mu/\partial\omega$ and is obtained from the stress tensor in the absence of dispersion (3.12) by the simple formal replacement

$$\begin{aligned} &\varepsilon \to \varepsilon(\omega)\,, \qquad E_iE_j \to \langle E_iE_j\rangle\,, \\ &\mu \to \mu(\omega)\,, \qquad H_iH_j \to \langle H_iH_j\rangle\,. \end{aligned} \tag{4.7}$$

Therefore, the force $\mathbf{f}$ which the field exerts on the medium has as one of its terms the force $\mathbf{f}^{\mathrm{HA}}$, (3.13), with the same replacement (4.7) and, accordingly, with $\mathbf{S}^{\mathrm{P}} \to \langle\mathbf{S}^{\mathrm{P}}\rangle$ (see Ref. [11, § 81]). Accounting for dispersion leads to appearing a certain additional force which was studied for a nonmagnetic medium ($\mu = 1$) by Washimi and Karpman [30] (see also Refs [11, § 81] and by Barash and Karpman [31]). The force related to the dispersion $\varepsilon(\omega)$ was found in Ref. [31] to be given by

$$\mathbf{f}_{\mathrm{d}}^{\mathrm{BK}} = \mathbf{f}_{\mathrm{d}}^{\mathrm{WK}} + \frac{1}{8\pi}\,\frac{\partial\varepsilon}{\partial\omega}\,\operatorname{Im}\left(\frac{\partial\mathbf{E}_0^*}{\partial t}\,\nabla\right)\mathbf{E}_0\,, \tag{4.8}$$

where the first term is identical to the expression

$$\mathbf{f}_{\mathrm{d}}^{\mathrm{WK}} = \frac{\omega}{8\pi c}\,\frac{\partial\varepsilon}{\partial\omega}\,\operatorname{Re}\left(\frac{\partial\mathbf{E}_0^*}{\partial t} \times \mathbf{H}_0\right) \tag{4.9}$$

obtained earlier in Ref. [30].

Results presented in our recently published paper [32] support the validity of using the Abraham–Brillouin–Pitaevskii form (4.5), (4.6) for the energy–momentum tensor. Reference [32] addresses two problems: the quantum-mechanical problem of a force acting on an atom in the quasi-monochromatic field (4.1), and the classical (i.e., Newtonian) problem of a force acting on an isotropic harmonic oscillator in the same field. In both problems, the force is taken to be

$$\begin{aligned} \mathbf{F}(\mathbf{R}, t) &= \frac{\alpha}{4}\,\nabla|\mathbf{E}_0|^2 + \frac{\alpha}{2c}\,\frac{\partial}{\partial t}\,\operatorname{Re}(\mathbf{E}_0^* \times \mathbf{H}_0) \\ &+ \frac{1}{2}\,\frac{\mathrm{d}\alpha}{\mathrm{d}\omega}\left[\frac{\omega}{c}\,\operatorname{Re}\left(\frac{\partial\mathbf{E}_0^*}{\partial t} \times \mathbf{H}_0\right) + \operatorname{Im}\left(\frac{\partial\mathbf{E}_0^*}{\partial t}\,\nabla\right)\mathbf{E}_0\right], \end{aligned} \tag{4.10}$$

with $\alpha(\omega)$ being the electric polarizability of the atom (or oscillator). It can be argued that expression (4.10) defines the field period-averaged force acting on any nonrelativistic particle with negligible magnetic polarizability $\beta(\omega)$.

[8] The angle brackets denote averaging over the period of the field.

The dielectric and magnetic permeabilities of a rarefied gas of particles are presented in the form (see Ref. [11, § 15])

$$\varepsilon(\omega) = 1 + 4\pi n\,\alpha(\omega)\,, \qquad \mu(\omega) = 1 + 4\pi n\,\beta(\omega)\,, \qquad (4.11)$$

where n is the number of particles per unit volume. Hence, the force acting on a unit volume of gas of uniform ($\nabla n = 0$) number density can be represented as the sum of the Helmholtz–Abraham force (3.13) [with the replacement (4.7)] and the Barash–Karpman force (4.8).

The author of Ref. [15] does not use the tensor (4.6) and suggests instead "the Polevoi–Rytov modification of the energy–momentum tensor" [33]. In Ref. [33], a wave packet in a uniform medium at rest ($\partial\varepsilon/\partial t = \partial\mu/\partial t = 0$, $\nabla\varepsilon = \nabla\mu = 0$) is approximated by a quasimonochromatic plane wave with functions $\mathbf{E}_0(\mathbf{r}, t)$ and $\mathbf{H}_0(\mathbf{r}, t)$ in formulas (4.1) having the form

$$\begin{aligned} \mathbf{E}_0(\mathbf{r}, t) &= \mathbf{E}_{00}(\mathbf{r}, t)\exp(\mathrm{i}\mathbf{kr})\,, \\ \mathbf{H}_0(\mathbf{r}, t) &= \mathbf{H}_{00}(\mathbf{r}, t)\exp(\mathrm{i}\mathbf{kr})\,, \end{aligned} \qquad (4.12)$$

where the coordinate functions $\mathbf{E}_{00}(\mathbf{r}, t)$ and $\mathbf{H}_{00}(\mathbf{r}, t)$ slowly vary compared to $\exp(\mathrm{i}\mathbf{kr})$: if l_0 is the length scale of $\mathbf{E}_{00}(\mathbf{r}, t)$ and $\mathbf{H}_{00}(\mathbf{r}, t)$, then the parameter

$$\frac{1}{kl_0} \sim \frac{c}{\omega l_0} \sim \frac{\lambda}{l_0} \ll 1\,, \qquad (4.13)$$

where $\lambda \sim 1/k$ is the wavelength.

To the zero-order approximation in the small parameters (4.2) and (4.13), Maxwell's equations (1.5), together with the constitutive equations (4.4), determine the relation between the field amplitudes and give the dispersion relation (see Ref. [11, § 83]):

$$\mathbf{H}_{00} = \frac{c}{\omega\mu(\omega)}\,\mathbf{k}\times\mathbf{E}_{00}\,, \qquad \mathbf{E}_{00} = -\frac{c}{\omega\varepsilon(\omega)}\,\mathbf{k}\times\mathbf{H}_{00}\,, \qquad (4.14)$$

$$\mathbf{k}^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,\mu(\omega)\,. \qquad (4.15)$$

From relation (4.15) it follows that for real ε and μ, the vector $\mathbf{k}$ can be real [and hence the wave $\sim\exp(\mathrm{i}\mathbf{kr})$ can be undamped] only if, for a given frequency, either $\varepsilon(\omega) > 0$ and $\mu(\omega) > 0$ or $\varepsilon(\omega) < 0$ and $\mu(\omega) < 0$ [4, 5, 9]. If the wave is undamped, relations (4.14) become

$$\begin{aligned} &\mathbf{H}_{00} = \pm\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{l}\times\mathbf{E}_{00}\,, \qquad \mathbf{E}_{00} = \mp\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{l}\times\mathbf{H}_{00}\,, \\ &\mathbf{l} = \frac{\mathbf{k}}{k}\,, \qquad k = \frac{n\omega}{c}\,, \qquad n = \sqrt{\varepsilon\mu}\,, \end{aligned} \qquad (4.16)$$

where, as before (see Section 1), the upper (lower) sign refers to the case $\varepsilon, \mu > 0$ ($\varepsilon, \mu < 0$).

For the energy (4.5), we obtain (see Ref. [11, § 83]) the relations

$$w^{\mathrm{B}} = \frac{1}{16\pi\mu\omega}\,\frac{\partial\omega^2\varepsilon\mu}{\partial\omega}\,|\mathbf{E}_{00}|^2 = \frac{1}{16\pi\varepsilon\omega}\,\frac{\partial\omega^2\varepsilon\mu}{\partial\omega}\,|\mathbf{H}_{00}|^2\,, \qquad (4.17)$$

and for the Umov–Poynting vector we find

$$\begin{aligned} \langle\mathbf{S}^{\mathrm{P}}\rangle &= \frac{c}{8\pi}\,\mathrm{Re}\,\mathbf{E}^*_{00}\times\mathbf{H}_{00} = \pm\frac{c}{8\pi}\sqrt{\frac{\varepsilon}{\mu}}\,|\mathbf{E}_{00}|^2\,\mathbf{l} \\ &= \pm\frac{c}{8\pi}\sqrt{\frac{\mu}{\varepsilon}}\,|\mathbf{H}_{00}|^2\,\mathbf{l}\,. \end{aligned} \qquad (4.18)$$

The expression for the wave group velocity follows from Eqn (4.15) as

$$\mathbf{v}_{\mathrm{gr}} = \frac{\partial\omega}{\partial\mathbf{k}} = \frac{c}{\partial(n\omega)/\partial\omega}\,\mathbf{l} = \frac{2\omega\varepsilon\mu}{\partial(\omega^2\varepsilon\mu)/\partial\omega}\,\mathbf{v}_{\mathrm{ph}} = \frac{1}{w^{\mathrm{B}}}\,\langle\mathbf{S}^{\mathrm{P}}\rangle\,, \qquad (4.19)$$

where $\mathbf{v}_{\mathrm{ph}}$ is the phase velocity of the wave.

Because $w^{\mathrm{B}} \geqslant 0$ (see Ref. [11, § 80]), it follows from formula (4.17) that for all frequencies ($\omega > 0$) one has

$$\frac{1}{\varepsilon}\,\frac{\partial\omega^2\varepsilon\mu}{\partial\omega} > 0\,. \qquad (4.20)$$

Formula (4.19), together with inequality (4.20), yields conditions (1.4). Notice that the dielectric and magnetic permeabilities can assume negative values only in the presence of dispersion. Indeed, neglecting dispersion, inequality (4.20) reduces to $\mu > 0$ (and $\varepsilon > 0$ because $\varepsilon\mu > 0$).

The Polevoi–Rytov tensor [33] is written out as

$$T^{\mathrm{PR}}_{\alpha\beta} = -w\,\frac{c}{\omega}\sqrt{1 - \frac{v^2_{\mathrm{gr}}}{c^2}}\,k_\alpha u_{\mathrm{gr}\beta}\,, \qquad (4.21)$$

where $k_\alpha = (\mathbf{k}, \mathrm{i}\omega/c)$ is the 4-wave vector, $u_{\mathrm{gr}\alpha}$ is the 4-group velocity of the wave, and $w = w^{\mathrm{B}}$ is the energy density (4.17). Using Eqn (1.10) and noting that $\langle\mathbf{S}^{\mathrm{P}}\rangle = w^{\mathrm{B}}\mathbf{v}_{\mathrm{gr}}$ [see expression (4.19)], the tensor (4.21) can be written in the form

$$\begin{aligned} &\sigma^{\mathrm{PR}}_{ij} = -\frac{1}{\omega}\,k_i\langle S^{\mathrm{P}}_j\rangle = -g^{\mathrm{PR}}_i v_{\mathrm{gr}j}\,, \\ &\mathbf{g}^{\mathrm{PR}} = \frac{1}{\omega}\,w^{\mathrm{B}}\mathbf{k}\,, \quad \mathbf{S}^{\mathrm{PR}} = \langle\mathbf{S}^{\mathrm{P}}\rangle = w^{\mathrm{B}}\mathbf{v}_{\mathrm{gr}}\,, \quad w^{\mathrm{PR}} = w^{\mathrm{B}} \end{aligned} \qquad (4.22)$$

(as was also done in Ref. [33]). These results were earlier given in this form by S M Rytov [13] who did not introduce the 4-group velocity of the wave, though.

The authors of review [33] believed that their results (4.22) are a direct consequence of Maxwell's equations. The only equations that rigorously follow from Maxwell's equations for quantities quadratic in the field intensity are equations (1.6) and (1.7). In these we have to set (as in Ref. [33])

$$\begin{aligned} &\mathbf{j}^{\mathrm{ext}} = 0\,, \quad \mathbf{f}^{\mathrm{ext}} = 0\,, \quad \mathbf{v} = 0\,, \quad \rho^{\mathrm{ext}} = 0\,, \\ &\frac{\partial\varepsilon}{\partial t} = \frac{\partial\mu}{\partial t} = 0\,, \quad \nabla\varepsilon = \nabla\mu = 0\,, \end{aligned}$$

and to represent fields in the form (4.1), (4.3), (4.4), and (4.12). On averaging over the field period, Eqn (1.6) becomes (see Ref. [11, § 80])

$$\frac{\partial w^{\mathrm{B}}}{\partial t} + \mathrm{div}\,\langle\mathbf{S}^{\mathrm{P}}\rangle = 0\,. \qquad (4.23)$$

Because the field does not perform work (the volume under consideration is at rest and contains no external charges), it follows uniquely [see Eqn (1.13)] that w^{B} has the meaning of the energy density, and $\langle S^{\mathrm{P}}\rangle$ of the energy flux density. Hence, the energy density and the energy flux density in the Polevoi–Rytov tensor (4.22) are determined correctly in the sense of being identical to the corresponding expressions in the tensor (4.6).

We now proceed by averaging equation (1.7) over the field period to obtain

$$\langle\boldsymbol{\sigma}'^{\mathrm{M}}\rangle = \frac{1}{4\pi c}\,\frac{\partial}{\partial t}\,\langle\mathbf{D}\times\mathbf{B}\rangle + \mathbf{f}_{\mathrm{d}}\,, \qquad (4.24)$$

where

$$f_{\mathrm{d}i} = \frac{1}{8\pi}\left\langle\left(\mathbf{D}\,\frac{\partial\mathbf{E}}{\partial r_i} - \mathbf{E}\,\frac{\partial\mathbf{D}}{\partial r_i}\right) + \left(\mathbf{B}\,\frac{\partial\mathbf{H}}{\partial r_i} - \mathbf{H}\,\frac{\partial\mathbf{B}}{\partial r_i}\right)\right\rangle\bigg|_{\nabla\varepsilon=\nabla\mu=0}\,, \tag{4.25}$$

with the subscript 'd' indicating that $\mathbf{f}_{\mathrm{d}} \neq 0$ only if dispersion is taken into account.

Following averaging, the tensor σ_{ij}^{M} [see equation (1.7)] is expressed in terms of the field amplitudes $\mathbf{E}_{00}$ and $\mathbf{H}_{00}$ and their derivatives $\partial\mathbf{E}_{00}/\partial t$ and $\partial\mathbf{H}_{00}/\partial t$. To the zero-order approximation in the parameter (4.2), one finds

$$\langle\sigma_{ij}^{\mathrm{M}}\rangle = \frac{1}{8\pi}\,\mathrm{Re}\left[\varepsilon(\omega)\left(E_{00i}^{*}E_{00j} - \frac{1}{2}\,|\mathbf{E}_{00}|^2\delta_{ij}\right) + \mu(\omega)\left(H_{00i}^{*}H_{00j} - \frac{1}{2}\,|\mathbf{H}_{00}|^2\delta_{ij}\right)\right]. \tag{4.26}$$

We proceed by expressing $\mathbf{H}_{00}$ in formula (4.26) in terms of $\mathbf{E}_{00}$ [according to Eqn (4.16)] and subsequently using the well-known relation (see Ref. [6, § 6]) between the product of two antisymmetric third-rank unit tensors e_{ijk} and the second-rank unit tensor δ_{ij}. A straightforward calculation yields

$$\langle\sigma_{ij}^{\mathrm{M}}\rangle = -\frac{1}{8\pi}\,\varepsilon(\omega)|\mathbf{E}_{00}|^2 l_i l_j\,. \tag{4.27}$$

Making use of formulas (4.18), (4.16), and (4.22), the equality (4.27) can also be written out in the form

$$\langle\sigma_{ij}^{\mathrm{M}}\rangle = -\frac{1}{\omega}\,k_i\langle S_j^{\mathrm{P}}\rangle = \sigma_{ij}^{\mathrm{PR}}\,, \tag{4.28}$$

i.e., the Polevoi–Rytov stress tensor is identical to the wave period-averaged Minkowski tensor (1.7).

To the same approximation, one derives the relations

$$\frac{1}{4\pi c}\,\langle\mathbf{D}\times\mathbf{B}\rangle = \frac{1}{8\pi c}\,\varepsilon(\omega)\,\mu(\omega)\,\mathrm{Re}\,\mathbf{E}_{00}^{*}\times\mathbf{H}_{00} = \frac{1}{8\pi\omega}\,\mathbf{k}|\mathbf{E}_{00}|^2\,, \tag{4.29}$$

$$\mathbf{f}_{\mathrm{d}} = \frac{1}{16\pi}\,\mathbf{k}\left(\frac{\partial\varepsilon}{\partial\omega}\,\frac{\partial|\mathbf{E}_{00}|^2}{\partial t} + \frac{\partial\mu}{\partial\omega}\,\frac{\partial|\mathbf{H}_{00}|^2}{\partial t}\right) = \frac{1}{16\pi\mu}\,\frac{\partial\varepsilon\mu}{\partial\omega}\,\mathbf{k}\,\frac{\partial|\mathbf{E}_{00}|^2}{\partial t}\,. \tag{4.30}$$

Using formulas (4.29), (4.30), (4.17), and (4.22), the right-hand side of the equality (4.24) can be written as

$$\frac{1}{4\pi c}\,\frac{\partial}{\partial t}\langle\mathbf{D}\times\mathbf{B}\rangle + \mathbf{f}_{\mathrm{d}} = \frac{1}{\omega}\,\frac{\partial w^{\mathrm{B}}}{\partial t}\,\mathbf{k} = \frac{\partial\mathbf{g}^{\mathrm{PR}}}{\partial t}\,. \tag{4.31}$$

As follows from Eqns (4.28) and (4.31), equality (4.24) takes the form

$$\boldsymbol{\sigma}'^{\,\mathrm{PR}} = \frac{\partial\mathbf{g}^{\mathrm{PR}}}{\partial t}\,. \tag{4.32}$$

Notice that neglecting dispersion yields $\mathbf{v}_{\mathrm{gr}} = \mathbf{v}_{\mathrm{ph}}$, $\mathbf{g}^{\mathrm{PR}} = \langle\mathbf{D}\times\mathbf{B}\rangle/(4\pi c)$, according to Eqn (4.22), i.e., $\mathbf{g}^{\mathrm{PR}}$ is identical to the field momentum density $\mathbf{g}^{\mathrm{M}}$ in the Minkowski form [see Eqn (1.18)]. It should also be pointed out that Eqn (4.32) is not independent but follows from equation (4.23) and the definitions of $\sigma_{ij}^{\mathrm{PR}}$ and $\mathbf{g}^{\mathrm{PR}}$ in Eqn (4.22). Indeed, using initially the first equality in Eqn (4.22), then equation (4.23), and finally the definition of $\mathbf{g}^{\mathrm{PR}}$ in Eqn (4.22), we arrive at Eqn (4.32).

Thus, equations (4.23) and (4.32) are correct because they indeed directly follow from Maxwell's equations and the constitutive equations for a medium at rest. What remains to be figured out is the meaning of the tensor $\sigma_{ij}^{\mathrm{PR}}$ and vector $\mathbf{g}^{\mathrm{PR}}$. As noted in review [33], equations (4.23) and (4.32) or the equations $\partial T_{\alpha\beta}^{\mathrm{PR}}/\partial r_\beta = 0$ [labelled (38) in Ref. (33)] "take the form of the continuity equations in which it is natural to regard the tensor $T_{\alpha\beta}^{\mathrm{PR}}$, which is bilinear in the field, as the energy–momentum tensor of the system (field + medium)." Choosing $\sigma_{ij}^{\mathrm{PR}}$ and $\mathbf{g}^{\mathrm{PR}}$ as a stress tensor and a wave momentum density tensor implies, on account of Eqns (1.13) and (4.32), that the force the wave exerts on the medium is zero, $\mathbf{f} = 0$. But in a fluid at rest, the total force [see Eqn (3.15)] is $\mathbf{f}^{\,\mathrm{tot}} = -\nabla P_0 + \mathbf{f} = 0$, i.e., the force $\mathbf{f}$ has to compensate for the force $-\nabla P_0$. Equation (4.32) can also be written in the form [see Eqn (4.31)]

$$\boldsymbol{\sigma}'^{\,\mathrm{PR}} = \frac{\partial\mathbf{g}^{\mathrm{M}}}{\partial t} + \mathbf{f}_{\mathrm{d}}\,. \tag{4.33}$$

Notice that, as before, $\sigma_{ij}^{\mathrm{PR}}$ 'can naturally be considered' as a stress tensor, but what can equally naturally be considered as a momentum density is the Minkowski form of the field momentum density [see Eqn (1.18)], in which case $\mathbf{f} = \mathbf{f}_{\mathrm{d}}$. Taking into consideration that $\partial(\mathbf{g}^{\mathrm{M}} - \mathbf{g}^{\mathrm{A}})/\partial t = \mathbf{f}_{\mathrm{A}}$, where $\mathbf{g}^{\mathrm{A}}$ is the Abraham form (4.6) of the field momentum density and $\mathbf{f}_{\mathrm{A}}$ is the Abraham force [see Eqn (3.8)], equation (4.32) or (4.33) can also be written out as

$$\boldsymbol{\sigma}'^{\,\mathrm{PR}} = \frac{\partial\mathbf{g}^{\mathrm{A}}}{\partial t} + (\mathbf{f}_{\mathrm{A}} + \mathbf{f}_{\mathrm{d}})\,. \tag{4.34}$$

We could then suppose that $\sigma_{ij}^{\mathrm{PR}}$, $\mathbf{g}^{\mathrm{A}}$, and $\mathbf{f}_{\mathrm{A}} + \mathbf{f}_{\mathrm{d}}$ are, respectively, the stress tensor, the wave momentum density, and the force the wave exerts on the medium. Finally, adding to both sides of equation (4.32) the striction force $\mathbf{f}^{\mathrm{str}}$ (3.9) averaged over the wave period, and bearing in mind that according to Eqns (4.6), (4.26), and (4.28) the stress tensor is given by

$$\sigma_{ij}^{\mathrm{P}} = \sigma_{ij}^{\mathrm{PR}} + \langle\sigma_{ij}^{\mathrm{str}}\rangle\,, \tag{4.35}$$

we arrive at the equation

$$\boldsymbol{\sigma}'^{\,\mathrm{P}} = \frac{\partial\mathbf{g}^{\mathrm{A}}}{\partial t} + (\mathbf{f}^{\mathrm{HA}} + \mathbf{f}_{\mathrm{d}})\,, \tag{4.36}$$

where $\mathbf{f}^{\mathrm{HA}}$ [see formula (3.13)] is the sum of the Abraham force (3.8) and the Helmholtz force (3.9), both wave period-averaged, the latter of which in this case ($\nabla\varepsilon = \nabla\mu = 0$) reduces to just the striction force $\mathbf{f}^{\mathrm{str}}$. Polevoi–Rytov's equations (4.32) are correct, but so are equations (4.33), (4.34), and (4.36), implying that Maxwell's equations and constitutive equations for a medium at rest are insufficient to determine either the stress tensor or the force acting on the medium (see, for example, review [25]) and that it is also necessary to consider the displacement of the medium and its associated change in the energy (if the process is adiabatic) or in the free energy (if the process is isothermal). It is in this way that the Helmholtz tensor and Pitaevskii tensor are obtained (see Refs [11, §§ 15, 75, 81] and [29]).

Review article [33] does not compare the tensor $T_{\alpha\beta}^{\mathrm{PR}}$ it introduces — Eqns (4.21) and (4.22) here — with the energy–

momentum tensor (4.6), nor indeed does it mention Ref. [29] (or Ref. [25], either). It was already noted that the energy density w^{PR} and the energy flux density $\mathbf{S}^{\mathrm{PR}}$ in the Polevoi–Rytov tensor (4.22) are identical to the counterpart expressions in tensor (4.6), and that the tensor $\sigma_{ij}^{\mathrm{PR}}$, Eqn (4.22), unlike the stress tensor σ_{ij}^{P}, does not contain the striction force tensor [see formula (4.35)]. The vector $\mathbf{g}^{\mathrm{PR}}$ in Eqn (4.22) also differs from the field momentum density in Eqn (4.6):

$$\mathbf{g}^{\mathrm{PR}} = \pm \mathbf{g}^{\mathrm{A}} \varepsilon(\omega)\, \mu(\omega)\, \frac{v_{\mathrm{ph}}}{v_{\mathrm{gr}}} . \tag{4.37}$$

We can say here nothing else about the vector $\mathbf{g}^{\mathrm{PR}}$. As regards the tensor $\sigma_{ij}^{\mathrm{PR}}$, its physical meaning will be clarified in Section 5, where we will show that $\sigma_{ij}^{\mathrm{PR}}$ is identical to the field-dependent part of the total stress tensor $\sigma_{ij}^{\mathrm{tot}} = \sigma_{ij}^{\mathrm{P}} - P_0 \delta_{ij}$ (4.6) in a mechanically and thermally equilibrium medium, provided a single quasimonochromatic plane wave propagates in the medium. We recall that P_0 is the pressure which would exist in the medium in the absence of a field at given values of its density and temperature (i.e., at those found in the presence of a field) (see Ref. [11, § 15]). Clearly, Maxwell's equations and the constitutive equations alone do not imply this conclusion, as they do not contain the pressure P_0.

5. Light pressure on solids

Consider a solid completely immersed in liquid medium in the presence of a quasimonochromatic electromagnetic field (4.1); we take into consideration the force of gravity and assume that the body is kept in the fluid by forces that are at rest and external with respect to the fluid (they can, for example, be related to threads by which the solid hangs). The force $\mathbf{P}$ acting on a unit area of the surface of a solid at rest is given by the momentum flux density or the stress tensor (see Ref. [11, § 16]) as

$$P_i = -\sigma_{ij}^{\mathrm{tot}} N_j , \tag{5.1}$$

where $\mathbf{N}$ is the inward unit normal to the surface of the body, and $\sigma_{ij}^{\mathrm{tot}}$ is the Pitaevskii stress tensor (4.6).

Reference [11, §§ 16, 35] presents formulas for the total force and total moment of the force which static fields in a uniform-density uniform-temperature fluid at rest exert on a solid. These formulas comprise

$$\begin{gathered} \tilde{P}_i = -\tilde{\sigma}_{ij} N_j , \\ \tilde{\sigma}_{ij} = \frac{1}{4\pi} \left\{ \varepsilon \left(E_i E_j - \frac{1}{2} E^2 \delta_{ij} \right) + \mu \left(H_i H_j - \frac{1}{2} H^2 \delta_{ij} \right) \right\} \end{gathered} \tag{5.2}$$

as a force acting on a unit surface. In Section 4 we noted that stress tensor (4.6) is obtained formally from the corresponding tensor for a static field, Eqns (3.12), (3.10), by the substitution (4.7). It can therefore be conjectured that expressions for the force and for the moment of force in an alternating field (and with allowance for dispersion) are also identical to their static field counterparts, provided the same substitution (4.7) is made in $\tilde{\sigma}_{ij}$, Eqn (5.2). This conjecture[9] turns out to be true under very general assumptions.

[9] As expressed by L P Pitaevskii.

Assuming, as in Ref. [11, § 16], that the fluid resides in thermal equilibrium, we have

$$\begin{gathered} \nabla \varepsilon = \left(\frac{\partial \varepsilon}{\partial \rho} \right)_T \nabla \rho + \left(\frac{\partial \varepsilon}{\partial T} \right)_\rho \nabla T = \left(\frac{\partial \varepsilon}{\partial \rho} \right)_T \nabla \rho , \\ \nabla \mu = \left(\frac{\partial \mu}{\partial \rho} \right)_T \nabla \rho . \end{gathered} \tag{5.3}$$

Expression (3.9) for the force then becomes

$$\mathbf{f}^{\mathrm{H}} = \frac{\rho}{8\pi} \nabla \left[\frac{\partial \varepsilon}{\partial \rho} \langle E^2 \rangle + \frac{\partial \mu}{\partial \rho} \langle H^2 \rangle \right] . \tag{5.4}$$

We noted in Section 4 that the expression for the force $\mathbf{f}_{\mathrm{d}}$ due to the inclusion of dispersion has so far been obtained only for media with $\mu = 1$ (see Refs [11, § 81], [30–32]) or, if for an arbitrary medium, then for a single quasimonochromatic plane wave [see Eqn (4.30)]. In can be argued that in the general case the force $\mathbf{f}_{\mathrm{d}}$ [as well as the Abraham force $\mathbf{f}_{\mathrm{A}}$ (3.8)] to an order of magnitude equals $|E_0|^2/(c\tau_0)$. If the field (4.1) is such that $c\tau_0 \gg l_0$, where l_0 is the distance characterizing functions $|\mathbf{E}_0|$ and $|\mathbf{H}_0|$, then the Abraham force $\mathbf{f}_{\mathrm{A}}$ and the force $\mathbf{f}_{\mathrm{d}}$ can be neglected compared with $\mathbf{f}^{\mathrm{H}}$, Eqn (5.4) [see expressions (3.13) and (3.15)], giving

$$\mathbf{f}^{\mathrm{tot}} = -\nabla P_0 + \mathbf{f}^{\mathrm{H}} + \rho \mathbf{g} , \tag{5.5}$$

where the force of gravity was also included.

We further assume as in Ref. [11, §§ 15, 16] that the fluid in an electromagnetic field is also in mechanical equilibrium, i.e., $\mathbf{f}^{\mathrm{tot}} = 0$. From Eqns (5.5) and (5.4) we then obtain the equation for P_0:

$$\nabla P_0 = \rho \nabla \left\{ \mathbf{gr} + \frac{1}{8\pi} \left[\frac{\partial \varepsilon}{\partial \rho} \langle E^2 \rangle + \frac{\partial \mu}{\partial \rho} \langle H^2 \rangle \right] \right\} . \tag{5.6}$$

For the case of a noncompressible fluid ($\nabla \rho = 0$), the last equation is easily solved (see Ref. [11, § 15]) to give

$$P_0(\mathbf{r}, t) = p(\mathbf{r}) + \frac{\rho}{8\pi} \left[\left(\frac{\partial \varepsilon}{\partial \rho} \right)_T \langle E^2 \rangle + \left(\frac{\partial \mu}{\partial \rho} \right)_T \langle H^2 \rangle \right] , \tag{5.7}$$

$$p(\mathbf{r}) = p_0 + \rho \mathbf{gr} , \tag{5.8}$$

where $p(\mathbf{r})$ is the pressure in the fluid in the absence of an electromagnetic field, and $p_0 = p(0)$.

For another type of medium, a sufficiently rarefied gas, the dielectric and magnetic permeabilities are given by formulas (4.11). Using the equation of state $P_0 = nT$ (Clapeyron's equation with the Boltzmann constant set to 1), equation (5.6) with $\varepsilon(\omega)$ and $\mu(\omega)$ from Eqn (4.11) reduces to

$$T \nabla n = n \nabla \left\{ m \mathbf{gr} + \frac{1}{2} \left[\alpha(\omega) \langle E^2 \rangle + \beta(\omega) \langle H^2 \rangle \right] \right\} , \tag{5.9}$$

where m is the molecular mass. It is seen that the gas density varies quadratically with the field intensity. To stay within linear electrodynamics ($\mathbf{D} \sim \mathbf{E}$, $\mathbf{B} \sim \mathbf{H}$), Eqn (4.11) and, hence, terms with $\langle E^2 \rangle$ and $\langle H^2 \rangle$ in Eqn (5.9), should be modified by replacing the density n by its zero-field counterpart n_0 (see Ref. [11, § 15]), which relates to the zero-field pressure p through $p = n_0 T$. This gives

$$n(\mathbf{r}, t) = n_0(\mathbf{r})\left[1 + \frac{\alpha(\omega)\langle E^2\rangle + \beta(\omega)\langle H^2\rangle}{2T}\right],$$
$$n_0(\mathbf{r}) = n_0(0)\exp\frac{m\mathbf{gr}}{T}. \tag{5.10}$$

The pressure $P_0 = nT$ is found to be given by the same expression (5.7) obtained for an incompressible liquid medium, but with formula (5.8) for the zero-field pressure replaced by the expression

$$p(\mathbf{r}) = n_0 T = p_0 \exp\frac{m\mathbf{gr}}{T}, \qquad p_0 = n_0(0)T. \tag{5.11}$$

By substituting P_0 from formula (5.7) into $\sigma_{ij}^{\mathrm{tot}}$, Eqn (4.6), we obtain the stress tensor in the form

$$\sigma_{ij}^{\mathrm{tot}} = -p(\mathbf{r})\delta_{ij} + \tilde{\sigma}_{ij},$$
$$\tilde{\sigma}_{ij} = \frac{1}{4\pi}\left\{\varepsilon(\omega)\left[\langle E_i E_j\rangle - \frac{1}{2}\langle E^2\rangle\delta_{ij}\right] + \mu(\omega)\left[\langle H_i H_j\rangle - \frac{1}{2}\langle H^2\rangle\delta_{ij}\right]\right\}, \tag{5.12}$$

with tensor $\tilde{\sigma}_{ij}$ really obtained from tensor (5.2) by the replacement (4.7). The first term in $\sigma_{ij}^{\mathrm{tot}}$, Eqn (5.12), contributes an amount $p(\mathbf{r})N_i$ to the force P_i (5.1), which, after integrating over the entire surface of the solid, leads, as it should, to the Archimedes force $(-\mathbf{g}M)$, where $M = \int_V \rho_0(\mathbf{r})\,\mathrm{d}V$ is the mass of fluid (gas) in the volume of the solid. Notice that the tensor $\tilde{\sigma}_{ij}$ which is identical to the tensor $\langle\sigma_{ij}^{\mathrm{M}}\rangle$ — the Minkowski tensor (1.7) averaged over the field period (4.1) — can be obtained from the Pitaevskii tensor σ_{ij}^{P}, Eqn (4.6), by dropping the striction terms $\langle\sigma_{ij}^{\mathrm{str}}\rangle$ (3.10): $\tilde{\sigma}_{ij} = \langle\sigma_{ij}^{\mathrm{M}}\rangle$. From this and formula (4.28) it follows that, for a single quasimonochromatic plane wave, $\tilde{\sigma}_{ij}$ is identical to the Polevoi–Rytov tensor $\sigma_{ij}^{\mathrm{PR}}$ [see Eqn (4.22)]. This result was presented at the end of Section 4.

As the simplest example, consider the case where a linearly polarized electromagnetic wave impinges normally on the planar surface of a solid. We choose the coordinate axes in the same way as in Section 1 (the z-axis is along the inward normal to the surface of the body), then the pressure [see Eqn (5.2)] is defined as

$$\tilde{P}_i = -\tilde{\sigma}_{iz}. \tag{5.13}$$

The fields $\mathbf{E}$ and $\mathbf{H}$ in Eqn (5.12) should, of course, be understood as the strengths of the total field in the fluid near the wall (for the incident and the reflected waves). For the unit vectors $\mathbf{l}_0$ and $\mathbf{l}_1$ of the incident and reflected waves, we have the relations [see Eqns (4.16) and (4.18)] $l_{0i} = \pm\delta_{iz}$, $l_{1i} = \mp\delta_{iz}$, and for the amplitudes of incident and reflected waves one finds

$$E_{00i}^{(0)} = E_{00}^{(0)}\delta_{ix}, \qquad H_{00i}^{(0)} = \sqrt{\frac{\varepsilon}{\mu}}\,E_{00}^{(0)}\delta_{iy},$$
$$E_{00i}^{(1)} = rE_{00}^{(0)}\delta_{ix}, \qquad H_{00i}^{(1)} = -\sqrt{\frac{\varepsilon}{\mu}}\,rE_{00}^{(0)}\delta_{iy}, \tag{5.14}$$

where r is the amplitude reflective index. Substituting these expressions into Eqns (5.12) and (5.13), we obtain after simple algebra

$$\tilde{P}_i = \tilde{P}\delta_{iz}, \qquad \tilde{P} = \frac{\varepsilon(\omega)}{8\pi}(1+R)\left|E_{00}^{(0)}\right|^2, \tag{5.15}$$

where $R = |r|^2$ is the power reflective index.

Pressure (5.15) can be given another form by introducing the energy flux density $\langle S_z^{\mathrm{P}}\rangle$ [see Eqn (4.18)] and the incident wave phase velocity v_{ph} and then the corresponding component of the Polevoi–Rytov tensor (4.22), giving

$$\tilde{P} = \pm(1+R)\,\frac{\langle S_z^{\mathrm{P}}\rangle}{v_{\mathrm{ph}}} = -(1+R)\sigma_{zz}^{\mathrm{PR}}. \tag{5.16}$$

From formulas (5.15) and (5.16) it is seen that the pressure $\tilde{P}$ has its sign determined by the sign of $\varepsilon(\omega)$: if $\varepsilon(\omega) > 0$ [and $\mu(\omega) > 0$], i.e., the wave group velocity is positive, then $\tilde{P} > 0$ (the pressure on the body). If $\varepsilon(\omega) < 0$ [and $\mu(\omega) < 0$], i.e., the wave group velocity is negative, then $\tilde{P} < 0$ (attraction). As noted in Section 1, this feature of a negative group velocity wave was first pointed out by Veselago [12].

Reference [12] considers a wave incident normally on an ideally reflecting body and makes the assumptions that (1) the force the incident and reflected wave exert on the body is codirectional with the incident wave momentum, and (2) the momentum (or more precisely, the momentum density) of the wave is equal to the vector $\mathbf{g}^{\mathrm{PR}}$ in tensor (4.22). In tensor (4.22) we always have $\mathbf{g}^{\mathrm{PR}}\uparrow\uparrow\mathbf{k}$, so that if the group velocity is positive, then $\mathbf{g}^{\mathrm{PR}}\uparrow\uparrow\mathbf{S}^{\mathrm{PR}}$ and we deal with 'light pressure', and if the group velocity is negative, then $\mathbf{g}^{\mathrm{PR}}\uparrow\downarrow\mathbf{S}^{\mathrm{PR}}$, and the 'light pressure' is replaced by 'light attraction'.[10] Both starting assumptions are wrong: the force is determined not by the momentum but by its change rate, i.e., the momentum flux (or the opposite-in-sign stress tensor), and the field momentum density is not $\mathbf{g}^{\mathrm{PR}}$ but rather $\mathbf{g}^{\mathrm{A}}$ [see Eqns (4.6) and (4.37)]. Still, Ref. [12] predicts a correct sign for the force [see Eqn (5.16) with $R = 1$].

In his analysis in Ref. [15] of a wave incident normally on a body that completely absorbs radiation, Veselago lifts both his earlier [12] assumptions and admits that the force is determined by the momentum flux density rather than by the momentum itself. The momentum flux density of a wave (now only one, incident) is written by Veselago as a ratio of the energy density flux to the phase velocity of the wave — without any justification whatsoever and seemingly ignoring the fact that this is nothing other than the component $(-\sigma_{zz}^{\mathrm{PR}})$ of the Polevoi–Rytov tensor [see our formula (5.16)], which Veselago discusses at the end of Ref. [15].

6. Conclusions

The key points of the above discussion can be summarized generally as follows: most of the results of Ref. [15] concerning the energy–momentum tensor of an electromagnetic field in a medium are wrong.

(1) Contrary to what is stated in Ref. [15], the Abraham form of the energy–momentum tensor is indeed a tensor in the sense of being Lorentz transformed as the product of two

[10] The authors of Ref. [33] exclude this case because, as they write with reference to the first edition (1957) of the *Electrodynamics of Continuous Media* (see also Ref. [11, § 84]), the group velocity of a wave "in an isotropic medium is directed along the wave vector" [33, p. 552]. The overlooked point here is that Ref. [11] proves this result only for nonmagnetic ($\mu = 1$) media.

4-vectors. The Minkowski tensor, unlike the Abraham tensor, does not depend on the velocity of the medium — but only until constitutive equations are introduced; after that, the Minkowski tensor also becomes velocity-dependent. On the other hand, the asymmetric character of the Minkowski tensor should be considered as its major disadvantage. The author of the recent *Physics–Uspekhi* paper [34], who also discusses the energy–momentum tensor of an electromagnetic field in a medium, claims that he "proved the necessity of using the Minkowski form for the momentum density in a medium." He writes [34, p. 637]: "A recent publication [Ref. [15] — *Translator's note*] demonstrated the relativistic covariance of the Minkowski energy–momentum tensor, thus providing further evidence in its favor." As regards Ref. [34], two comments suffice here: first, the Minkowski tensor is written by Minkowski himself in a form [see Eqn (1.17)] which makes the covariance of the tensor *obvious* (in the literal sense of this word), and, second, the Abraham tensor is also written in the explicitly covariant form [see Eqn (2.10)].

(2) Expressions for the force are obtained in both Minkowski's and Abraham's approaches from one and the same equation equivalent to the momentum conservation law. Because the tensors differ from each other, the corresponding expressions for the force also differ, specifically by a term commonly referred to as the Abraham force [11, § 75] [see Eqn (3.8)]. The Abraham tensor, like the Minkowski tensor, does not include striction forces; the correct energy–momentum tensor of an electromagnetic field in an isotropic medium at rest without allowance for dispersion is the Helmholtz–Abraham tensor, Eqns (3.10)–(3.15) [11, §§ 15, 35, 75].

(3) The energy–momentum tensor $T_{\alpha\beta}$ of a (quasimonochromatic) electromagnetic field in a medium at rest with allowance for dispersion — which is left undiscussed in Ref. [15] — is something already known for half a century: its component T_{44} is the energy w^{B} determined by Brillouin's formula [11, § 80], the components $T_{4j} = T_{j4}$ are proportional to the wave period-averaged Umov–Poynting vector $\langle S_j^{\mathrm{P}} \rangle$, and the components T_{ij} form the Pitaevskii tensor σ_{ij}^{P} [11, § 81], [29] [see Eqn (4.6)]. The force due to dispersion, with which the quasimonochromatic plane wave acts on a volume unit of matter (with $\mu = 1$), is determined by Eqn (4.9) obtained by Washimi and Karpman in Ref. [30] (see also Ref. [11, § 81]). Neglecting dispersion, tensor (4.6) reduces to tensor (3.11), (3.12).

(4) Tensor $\sigma_{ij}^{\mathrm{PR}}$, Eqn (4.22), is the field-dependent part of the Pitaevskii form of the total stress tensor $\sigma_{ij}^{\mathrm{tot}}$, Eqn (4.6), in a medium which is in thermal and mechanical equilibrium ($T = \mathrm{const}$, $\mathbf{f}^{\mathrm{tot}} = 0$), and in which a quasimonochromatic plane wave propagates.

(5) A quasimonochromatic plane wave incident normally on a solid body at rest acts on the body with a force (per unit area) which is expressed in terms of the stress tensor $\sigma_{ij}^{\mathrm{PR}}$ [see Eqn (5.16)] for any reflective index (not only in the absence of reflection, as in Ref. [15]).

Finally, the question we cannot avoid is whether negative group velocity waves can exist at all (as before, isotropic nongyrotropic media are assumed). As before (see Ref. [7]), we believe that fulfilling Sivukhin–Pafomov's conditions (1.4) for the negative group velocity of a wave "would be an almost improbable random event" for the following simple reason. The frequency ranges for which $\varepsilon(\omega) < 0$ are near those eigenfrequencies ω_0 of the substance whose corresponding electric dipole transitions, with $\omega > \omega_0$ (see Ref. [11, § 84]), do not contribute to $\mu(\omega)$, so that for frequencies for which $\varepsilon(\omega) < 0$, the magnetic permeability $\mu(\omega) \approx 1$. An interesting, though not entirely uncontroversial, study [8] makes an even stronger statement: "In the more than 50 years that have passed [since Sivukhin's paper [9] — *Translator's note*] the situation has not changed and, I am sure, will never change. No continuous homogeneous media with $\varepsilon < 0$ and $\mu < 0$ can exist in the optical spectrum range." It should be kept in mind, however, that Ref. [8] (following Refs [12, 35]) considers that "in a homogeneous isotropic medium without field absorption and amplification, the only reason for opposite directions of $\mathbf{k}$ and $\langle \mathbf{S}^{\mathrm{P}} \rangle$ for uniform waves can be negative values of ε and μ." Actually, though, the Sivukhin–Pafomov conditions (1.4) are only sufficient but not necessary for a wave to have a negative group velocity [7, 36, 37]. The dielectric and magnetic permeabilities of an isotropic nongyrotropic medium are, strictly speaking, functions not only of the frequency but also of the absolute magnitude of the wave vector (see Ref. [11, Ì 103], [38, Ì 2]): $\varepsilon = \varepsilon(\omega, k)$, $\mu = \mu(\omega, k)$. Even if the dependence of ε on k can be neglected in a large frequency range, at frequencies ω sufficiently close to the eigenfrequencies ω_0 of the medium (which are exactly the ones we are interesting in, in this particular case), this dependence becomes significant and has to be taken into account. The conditions determining the sign of the group velocity of the wave were given in Ref. [36]; in terms of the dielectric and magnetic permeabilities they are written out as

$$
\begin{aligned}
&\varepsilon\left(\frac{\omega^2}{c^2}\frac{\partial\varepsilon\mu}{\partial k^2} - 1\right) < 0 \Rightarrow \mathbf{v}_{\mathrm{gr}} \uparrow\uparrow \mathbf{v}_{\mathrm{ph}}\,, \\
&\varepsilon\left(\frac{\omega^2}{c^2}\frac{\partial\varepsilon\mu}{\partial k^2} - 1\right) > 0 \Rightarrow \mathbf{v}_{\mathrm{gr}} \uparrow\downarrow \mathbf{v}_{\mathrm{ph}}\,,
\end{aligned}
\tag{6.1}
$$

where, needless to say, the condition $\varepsilon\mu > 0$ should also be satisfied. Neglecting the dependences of ε and μ on k, conditions (6.1) reduce, as they should, to conditions (1.4). But, according to Eqn (6.1), a wave can also have a negative group velocity at positive ε and μ. In particular, setting $\mu = 1$ (and hence, of course, $\varepsilon > 0$), as is usual for the optical range, the condition for the group velocity to be negative becomes

$$
\frac{\omega^2}{c^2}\frac{\partial\varepsilon}{\partial k^2} > 1\,. \tag{6.2}
$$

Notice that condition (6.2) follows directly from the well-known expression

$$
\langle \mathbf{S}^{\mathrm{P}} \rangle = \frac{c^2}{8\pi\omega}\,|\mathbf{E}_{00}|^2\left(1 - \frac{\omega^2}{c^2}\frac{\partial\varepsilon}{\partial k^2}\right)\mathbf{k} \tag{6.3}
$$

for the energy flux density of a quasimonochromatic plane transverse wave (see Ref. [11, § 103]). It is shown in Ref. [36] that condition (6.2) can hold for a monatomic gas in sufficiently narrow ranges of frequency ω adjacent on the long-wavelength side to the frequencies ω_0 of electric dipole transitions.

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