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METHODOLOGICAL NOTES

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On description of a collisionless quantum plasma

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Contents

Abstract. **A plasma is regarded as quantum if its macroscopic properties are significantly affected by the quantum nature of its constituent particles. A proper description is necessary to comprehend when collective quantum plasma effects are important. In this paper, the most commonly used microscopic approaches to describe a collisionless quantum plasma are reviewed, together with their related assumptions and restrictions. In particular, the quantum plasma hydrodynamic approximation is analyzed in detail, and the analytical properties of the linear dielectric response function obtained from quantum plasma kinetic theory are investigated. Special attention is paid to what we consider to be the most important problems that have already appeared in the linear approximation and require further studies.**

1. Introduction

Most visible matter in the Universe is in the plasma state — the state of gas where significant numbers of atoms and molecules are ionized. A characteristic feature of the plasma state is the presence of essentially collective processes due to electromagnetic interactions of free charged particles of the matter.

Since a plasma contains particles that in general obey the laws of quantum mechanics, its most complete description should be based on the quantum mechanical approach to the system of interacting particles. However, such a general quantum mechanical approach is not always necessary since, in order to describe many plasma phenomena, it is often sufficient to consider plasma as a system of model classical particles, thus fully neglecting their quantum nature. A plasma can be regarded as quantum when the quantum nature of its particles significantly affects its macroscopic properties. This occurs, for example, for large number densities n or low temperatures T of the lightest plasma particles (usually, electrons and positrons/holes), when the characteristic de Broglie wavelengths for these particles, $\lambda_{\mathrm{B}} \sim \hbar/mv_{\mathrm{T}}$ (where $\hbar$ is the reduced Planck constant, m is the particle mass, $v_{\mathrm{T}} = \sqrt{T/m}$ is the particle thermal velocity; here and below we assume that the Boltzmann constant $k_{\mathrm{B}} = 1$ so that the temperature T is in energy units), become on the order of (or exceed) the mean interparticle distance $n^{-1/3}$.

Examples of quantum plasmas are: the gas of conductivity electrons in normal metals (whose charge is compensated by the ion lattice at equilibrium), electron and hole gas in semiconductors (especially in modern miniature semiconductor structures where the characteristic space scales of impurity variations are comparable to the characteristic electron and hole de Broglie wavelengths), and states of matter appearing at high compression as, for instance, in the fast ignition scenario of inertial confinement fusion processes where the deuterium-tritium mixture is compressed by powerful laser beams up to densities significantly exceeding the liquid hydrogen density [1–3]. The matter in the cores of dense astrophysical objects is in the quantum plasma state as well.

By the mid-20th century, a general understanding of the most important properties of quantum plasma was achieved. Recently, in the last decade, interest in quantum plasmas has significantly increased due to technological advances that allow us to directly study and use quantum plasma effects. For example, miniaturization of metallic and semiconductor structures to nanoscales (nanowires, thin metallic films, quantum dots) has reached a stage where the quantum tunneling effect of charge carriers has a significant influence

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on collective plasma processes in such structures. Impressive advances in obtaining ultra-short laser pulses (on the order of 1 attosecond) in principle open a possibility of studying the dynamics of charge carriers at time scales on the order of the plasma oscillation period (around 1 femtosecond in metallic structures and 1 picosecond in semiconductors). Moreover, applications of ultra-short petawatt X-ray laser pulses for compression of matter, in principle, should allow us to obtain and study high density quantum plasmas. Recent spectral measurements of X-ray Thomson scattering [4, 5] allow accurately determining the electron distribution function, temperature, etc. in the warm dense matter regime [6]. This method provides the chance to experimentally study strongly coupled (nonideal) and weakly coupled (ideal) degenerate quantum plasmas. All these developments call for proper theoretical descriptions of quantum plasmas in the corresponding regimes.

In order to describe quantum plasmas as ensembles of large numbers of interacting quantum particles, two approaches can be distinguished, depending on the strength of particle interactions. The first, phenomenological, approach was developed by Landau [7, 8] in his theory of Fermi liquid for a system of fermions whose interactions are not necessarily weak (this distinguishes a Fermi liquid from a Fermi gas). An important point in Landau's theory is that the properties of Fermi liquids can be described in terms of relatively weakly interacting quasiparticles — excited electron states above the Fermi level. The second approach, developed to describe a gas of weakly interacting fermions by the use of a quantum mechanical description of a system of interacting particles 'from first principles', is based on the methods of statistical theory.

In this methodological note, we consider only those ideal (almost ideal) collisionless quantum plasma models that are based on the second approach. We restrict ourselves by mostly considering the nonrelativistic case, when characteristic particle velocities are small compared with the speed of light in a vacuum.

The microscopic description from first principles of nonrelativistic quantum plasmas can be based on either Schrödinger's representation (in which the operators are time-independent, while the time dependence of physical quantities of the system is defined by the corresponding time dependence of the system's wavefunction or density matrix) or Heisenberg's representation (in which the time dependence is transferred from the wavefunctions to the operators). Most microscopic models of quantum plasmas [9] use the Schrödinger representation; the quantum plasma dynamics are then described by a set of Schrödinger equations for wavefunctions of separate groups of particles (so-called multistream model [10]), or by an equation for the density matrix of quantum plasma [11], or by an equation for the Wigner function [12–14] (also called the quantum distribution function and introduced due to its ideological similarity to the distribution function of classical plasma particles), or — and this approach is currently becoming increasingly popular — by a set of the so-called quantum hydrodynamics equations [15, 16]. Naturally, simplifying assumptions are made in all these models (thus leading to inevitable limitations of their applicability), which one should take into account when analyzing results obtained from them. However, the applicability limits of results obtained from a particular model are not always stated explicitly in the literature (this especially concerns the widely used model of quantum hydrodynamics [16–19]), which can lead to their incorrect interpretation. This has recently been pointed out, for example, by Melrose and Mushtaq [20] and by Kuzelev and Rukhadze [21].

Meanwhile, with the recent rapid growth in the number of publications on quantum plasmas, the lack of detailed analysis of the assumptions made in (and the limitations associated with) the most common quantum plasma models is becoming increasingly obvious. Therefore, it is useful to analyze the methods of the microscopic description of ideal quantum plasmas, as well as the related assumptions and limitations, in detail. Beyond that, an important issue of interest is that of macroscopically observable quantum phenomena in plasmas; this issue is also connected with the question of when (and which exactly) quantum plasma phenomena are important. In this paper, we provide a detailed analysis of the quantum hydrodynamics model, and study the kinetic features of the analytical properties of the linear dielectric response function in a quantum plasma. By doing so, we point out what, in our view, are the most important fundamental problems associated with the linear responses of quantum plasmas, which require further studies.

2. Basic methods of the microscopic description of quantum plasmas

Similar to the case of a classical plasma, the complete microscopic description of a quantum plasma as a system of many interacting particles is a practically hopeless task, not only because it would be impossible to solve the Schrödinger equation for the N-particle wavefunction of the system, but also due to the principal absence of such a wavefunction for a macroscopic system that interacts, however weakly, with its environment [21]. Even if this wavefunction existed, it would be necessary to obtain a full set of data for the system (i.e., the full set of initial conditions for the Schrödinger equation); that is completely unfeasible for systems with a large number of particles. However, there is actually no need in this detailed description; it is sufficient to statistically describe such a system in terms of its density matrix [11]; and once the density matrix is known, average values of the system's macroscopic parameters and their corresponding probabilities can be calculated — which is the ultimate purpose of the system's description. A convenient approach, based on the so-called mixed representation of the density matrix suggested by Wigner [14], in which the quantum distribution function (also called the Wigner function) is introduced [12, 13, 23], allows us to achieve the closest possible similarity with the description of classical plasmas in terms of the distribution function in the phase space of particles' coordinates and momenta. The Wigner function $f_N(\mathbf{q}, \mathbf{p}, t)$ is defined from the density matrix $\rho_N(\mathbf{q}, \mathbf{q}', t)$ of the system in the coordinate representation as follows [13, 23–25]:

$$f_N(\mathbf{q}, \mathbf{p}, t) = \frac{1}{(2\pi)^{3N}} \int \mathrm{d}\tau \exp(-\mathrm{i}\tau\mathbf{p})\, \rho_N\left(\mathbf{q} - \frac{1}{2}\hbar\tau,\ \mathbf{q} + \frac{1}{2}\hbar\tau,\ t\right), \tag{1}$$

where N is the number of particles in the system, and $\mathbf{q}$ and $\mathbf{p}$ are the $3N$-component vectors representing the set of coordinates and momenta of all particles in the system. In the limit $\hbar \to 0$, function $f_N(\mathbf{q}, \mathbf{p}, t)$ becomes the classical N-particle distribution function; hence, the plasma description in terms of the Wigner function covers both quantum and classical plasmas.

The equation that governs the evolution of the Wigner function can be obtained from the evolution equation for the density matrix in the coordinate representation; it is given by [13, 23, 25]

$$\frac{\partial f_N(\mathbf{q},\mathbf{p},t)}{\partial t} = \frac{1}{(2\pi)^{6N}}\frac{\mathrm{i}}{\hbar}\int\ldots\int \mathrm{d}\boldsymbol{\tau}\,\mathrm{d}\mathbf{k}\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}\mathbf{r}\exp\left\{\mathrm{i}\big[\mathbf{k}(\mathbf{r}-\mathbf{q})+\boldsymbol{\tau}(\boldsymbol{\eta}-\mathbf{p})\big]\right\} \times f_N(\mathbf{r},\boldsymbol{\eta},t)\left[H\left(\mathbf{r}-\frac{1}{2}\hbar\boldsymbol{\tau},\,\boldsymbol{\eta}+\frac{1}{2}\hbar\mathbf{k},\,t\right) - H\left(\mathbf{r}+\frac{1}{2}\hbar\boldsymbol{\tau},\,\boldsymbol{\eta}-\frac{1}{2}\hbar\mathbf{k},\,t\right)\right], \tag{2}$$

where $H(\mathbf{q},\mathbf{p},t)$ is the system's Hamiltonian containing exact (not averaged) fields through which the system's particles interact.

Function $f_N(\mathbf{q},\mathbf{p},t)$ depends on a huge, $6N+1$, number of variables; therefore, the plasma description in terms of the N-particle Wigner function f_N is still too complicated a task. For most practical problems, however, it is sufficient to know a one-particle distribution function that depends on coordinates and momenta of a single particle and does not depend on the coordinates and momenta of all other particles in the system:

$$f_1(\mathbf{q}_1,\mathbf{p}_1,t) = \int f_N(\mathbf{q},\mathbf{p},t)\,\mathrm{d}\mathbf{q}_2\ldots\mathrm{d}\mathbf{q}_N\,\mathrm{d}\mathbf{p}_2\ldots\mathrm{d}\mathbf{p}_N\,.$$

The equation for function $f_1(\mathbf{q}_1,\mathbf{p}_1,t)$ can be obtained from (2) by applying the Bogoliubov method [26] generalized for quantum systems [23]. This equation contains the two-particle distribution function $f_2(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,t)$, the equation for which contains, in turn, a three-particle distribution function, etc. Thus obtained, the Bogoliubov–Born–Kirkwood–Green–Yvon (BBGKY) set of coupled equations is equivalent to initial equation (2) for f_N, and its general solution is as complicated as the solution of (2).

Yet the problem of determining the one-particle distribution function $f_1(\mathbf{q}_1,\mathbf{p}_1,t)$ can be significantly simplified and reduced to obtaining the solution of just one (approximate) equation for f_1 if, first, we can neglect particle correlations that appear due to their identity [i.e., neglect particle exchange interactions so that $f_2(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,t) = f_1(\mathbf{q}_1,\mathbf{p}_1,t)\,f_1(\mathbf{q}_2,\mathbf{p}_2,t)$ for noninteracting particles] and, second, if we can neglect the two-particle correlation function $g_2(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,t)$, determined by the equation

$$f_2(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,t) = f_1(\mathbf{q}_1,\mathbf{p}_1,t)\,f_1(\mathbf{q}_2,\mathbf{p}_2,t) + g_2(\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2,t)\,, \tag{3}$$

which characterizes the statistical correlation of particles 1 and 2 due to their interaction. Function g_2 determines the collision integral in the equation for f_1, and neglecting it corresponds to the collisionless plasma approximation.

As a result, the equation for one-particle distribution function (one-particle Wigner function) $f_1(\mathbf{q},\mathbf{p},t)$ is reduced to Eqn (2) with $N=1$, where Hamiltonian H now does not contain exact fields but rather contains self-consistent fields averaged over the one-particle distribution function [23]. This approximation, in which a quantum plasma is considered as an ensemble of quantum particles interacting via average collective fields and which is analogous to the mean-field approximation for classical plasmas suggested by Vlasov [27], is called the Hartree mean-field approximation [28]. Taking into account particle correlations due to their identity (i.e., taking into account exchange interactions) leads to an additional term in equation for f_1 [13]; the corresponding approximation is called the Hartree–Fock (mean-field) approximation [29].

One important parameter determining plasma properties is the so-called coupling parameter — the ratio of the characteristic potential energy of plasma particle interactions to their characteristic kinetic energy. For quantum plasmas with degenerate electrons [9], the coupling parameter is given by

$$\Gamma_{\mathrm{q}} = \frac{U_{\mathrm{int}}}{\varepsilon_{\mathrm{F}}} = \frac{e^2 n^{1/3}}{\varepsilon_{\mathrm{F}}} \sim \left(\frac{\hbar\omega_{\mathrm{p}}}{\varepsilon_{\mathrm{F}}}\right)^2 \ll 1\,,$$

where $\varepsilon_{\mathrm{F}} = (\hbar^2/2m)(3\pi^2 n)^{2/3}$ is the Fermi energy, $\omega_{\mathrm{p}} = (4\pi e^2 n/m)^{1/2}$ is the plasma frequency, and e, m, and n are the electron or hole (positron) charge, mass, and number density. For small values of the coupling parameter, $\Gamma_{\mathrm{q}} \ll 1$ (i.e., when a plasma can be considered as an almost ideal gas), the role of plasma particle collisions is small compared with the role of collective processes determined by self-consistent plasma fields, and therefore only in this case can the plasma be approximately considered as a collisionless one, at least for those collective processes whose characteristic times are small compared with the characteristic time between collisions [16].

Thus the collisionless plasma approximation (in which the collision integral is neglected in equation for f_1) is justified only for weakly coupled (ideal) plasmas, $\Gamma_{\mathrm{q}} \ll 1$. We note here that in plasmas with $\Gamma_{\mathrm{q}} \gtrsim 1$, particle correlations are significant and cannot be neglected — and such plasmas are called strongly coupled plasmas. It is remarkable that in degenerate quantum plasmas, the coupling parameter Γ_{q} decreases with increasing density as $\Gamma_{\mathrm{q}} \propto n^{-1/3}$, i.e., such quantum plasmas become increasingly ideal and collisionless with increasing densities, in contrast to classical plasmas, which become increasingly strongly coupled with increasing densities.

As mentioned above, the most commonly used recently [9] are the following models of quantum plasmas (which are, as a rule, collisionless, i.e., formally correct only in the approximation of ideal plasmas, $\Gamma_{\mathrm{q}} \ll 1$): the quantum analog of the multistream model [10], the kinetic model based on the Wigner equation for the one-particle quantum distribution function [12, 24], and, finally, the quantum hydrodynamics model. All these models, in one way or another, are based on the Schrödinger equation for the wavefunctions of plasma particles, and therefore they are essentially nonrelativistic; hence, they can only be used for describing nonrelativistic ideal plasmas and, strictly speaking, for describing waves with small (nonrelativistic) phase velocities $\omega/k \ll c$ [20] (here, ω and k are, respectively, the wave frequency and wavenumber, and c is the speed of light in a vacuum). We should also mention more general relativistic models of 'quantum plasmadynamics' [30]; we expect that these models will be more widely used in the future, notably, due to their logically consistent description of both quantum particles and quantized fields. However, here we will only consider the class of nonrelativistic models, as they are the most widely used in the recent literature on quantum plasmas, mainly due to their relative simplicity.

It is necessary to make the following comment in regard to the applicability of the collisionless plasma models considered here. It may seem at first glance that collisionless plasma models are not capable of describing irreversible processes such as the relaxation of nonequilibrium plasmas. Nevertheless, plasma relaxation processes can occur at time scales that are short in comparison to the characteristic times between particle collisions, i.e., within the applicability range of collisionless models [31]. The irreversibility of such relaxation processes occurs not due to collisions, but rather due to collective plasma processes.

A special role here is played by resonances between collective plasma oscillations (waves) and plasma particles. These wave-particle resonances lead to Landau damping of plasma waves in the case of equilibrium distributions, and to instabilities of the corresponding waves in the case of nonequilibrium plasma particle distributions. As a result of the resonant interaction between waves and plasma particles, the latter can become trapped by the corresponding wave fields; hence, the particle phase (coordinate and momentum) space is partitioned into regions containing the trajectories of trapped and free particles divided by separatrices. Due to the development of dynamical chaos in the vicinity of every separatrix, the phase trajectories of trapped and free particles become entangled and the stochastic layer is formed around the corresponding separatrix [32]. In the case of several wave harmonics (a wave packet) propagating in a plasma, the separatrices corresponding to resonances with these wave harmonics and, hence, their stochastic layers, can overlap, thus leading to a wider region of stochastic motions being formed in the particle phase space; the wave phases are random and plasma particle motions acquire a diffusive character (diffusion in the momentum space) in this stochastic region. A good example of such a process is the quasilinear relaxation of the plasma beam instability that is correctly described within the collisionless kinetic model [33, 34]. In this process, the role of collisions is played by resonant interactions of plasma particles, with collective plasma excitations leading to particle diffusion in the momentum space and, as a consequence, to the irreversible process of instability relaxation [31, 33].

Thus, collisionless models can be applied for a description of processes, including irreversible relaxation processes, whose characteristic time scales are short compared with the characteristic times between collisions of plasma particles.

2.1 Collisionless kinetic models of quantum plasmas with electrostatic and electromagnetic interactions of particles

Collisionless kinetic models of quantum plasmas with electrostatic and electromagnetic interactions of particles are based on the time evolution equation for one-particle Wigner function $f_1(\mathbf{q}, \mathbf{p}, t)$ in the mean-field approximation, in which the self-consistent electrostatic or electromagnetic fields are described by either Poisson's equation or Maxwell's equations, respectively. The one-particle Wigner function represents the quasi-density of quantum particle probability distribution in the coordinate-momentum phase space (we call it quasi-density because the Wigner function can acquire negative values, due to the noncommutativity of particle position and momentum operators in quantum mechanics, i.e., due to the uncertainty principle) [12]. The Wigner function f_1 is normalized as

$$n(\mathbf{q}, t) = \int f_1(\mathbf{q}, \mathbf{p}, t)\, \mathrm{d}\mathbf{p}\,, \tag{4}$$

where $n(\mathbf{q}, t)$ is the number density of plasma particles. For a system of charged particles interacting via self-consistent electrostatic fields with potential $\phi(\mathbf{q})$, the Hamiltonian is $H(\mathbf{q}, \mathbf{p}, t) = \mathbf{p}^2/2m + e\phi(\mathbf{q}, t)$, where $\mathbf{p}$ is the kinetic momentum of a particle (which in this case coincides with its canonical momentum $\mathbf{P}$), and the equation for $f_1(\mathbf{q}, \mathbf{p}, t)$ (for simplicity, below we omit the subscript 1 for the Wigner function, so f connotes the one-particle distribution function) becomes

$$\frac{\partial f(\mathbf{q}, \mathbf{p}, t)}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial f}{\partial \mathbf{q}} = \frac{1}{(2\pi)^3}\frac{\mathrm{i}e}{\hbar}\int \mathrm{d}\boldsymbol{\tau}\, \mathrm{d}\boldsymbol{\eta} \exp\left[\mathrm{i}\boldsymbol{\tau}(\boldsymbol{\eta} - \mathbf{p})\right]$$
$$\times f(\mathbf{q}, \boldsymbol{\eta}, t)\left[\phi\left(\mathbf{q} - \frac{1}{2}\hbar\boldsymbol{\tau}, t\right) - \phi\left(\mathbf{q} + \frac{1}{2}\hbar\boldsymbol{\tau}, t\right)\right]. \tag{5}$$

The quantum kinetic Wigner equation (5), together with the Poisson equation for electrostatic potential $\phi(\mathbf{q}, t)$ [in which the particle density is defined by Eqn (4)], describes quantum plasmas with electrostatic interactions of particles. The set of coupled Wigner and Poisson equations is also called the Wigner–Poisson (model) set of equations.

For a system of spinless charged particles interacting via self-consistent electromagnetic fields, the Hamiltonian is

$$H(\mathbf{q}, \mathbf{P}, t) = \frac{\left[\mathbf{P} - (e/c)\mathbf{A}(\mathbf{q}, t)\right]^2}{2m} + e\phi(\mathbf{q}, t)\,,$$

where $\mathbf{P}$ is the particle canonical momentum, and $\phi(\mathbf{q}, t)$ and $\mathbf{A}(\mathbf{q}, t)$ are the scalar and vector electromagnetic field potentials, respectively. By changing variables according to $\mathbf{P} \to \mathbf{p} + e\mathbf{A}(\mathbf{q}, t)/c$, where $\mathbf{p}$ is the particle kinetic momentum, the equation for the one-particle Wigner function [Eqn (2) for $N = 1$] is reduced to

$$\frac{\partial f(\mathbf{q}, \mathbf{p}, t)}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial f}{\partial \mathbf{q}} + e\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{mc}\right)\frac{\partial f}{\partial \mathbf{p}}$$
$$= \frac{1}{(2\pi)^3}\frac{1}{m}\int \mathrm{d}\boldsymbol{\tau}\, \mathrm{d}\boldsymbol{\xi} \exp\left[\mathrm{i}\boldsymbol{\tau}(\boldsymbol{\xi} - \mathbf{p})\right]$$
$$\times\left\{-\mathrm{i}\frac{e}{c} f(\mathbf{q}, \boldsymbol{\xi})\left[\left(\boldsymbol{\tau}\frac{\partial}{\partial \mathbf{q}}\right)(\boldsymbol{\xi}\mathbf{A}(\mathbf{q}))\right.\right.$$
$$\left. - \frac{1}{\hbar}\boldsymbol{\xi}\left(\mathbf{A}\left(\mathbf{q} + \frac{\hbar\boldsymbol{\tau}}{2}\right) - \mathbf{A}\left(\mathbf{q} - \frac{\hbar\boldsymbol{\tau}}{2}\right)\right)\right]$$
$$+ \frac{\mathrm{i}em}{\hbar}\left[\phi\left(\mathbf{q} - \frac{\hbar\boldsymbol{\tau}}{2}\right) - \phi\left(\mathbf{q} + \frac{\hbar\boldsymbol{\tau}}{2}\right)\right] f(\mathbf{q}, \boldsymbol{\xi})$$
$$+ \mathrm{i}em\left(\boldsymbol{\tau}\frac{\partial\phi(\mathbf{q})}{\partial \mathbf{q}}\right) f(\mathbf{q}, \boldsymbol{\xi}) - \frac{e}{2c}\left(\frac{\partial f(\mathbf{q}, \boldsymbol{\xi})}{\partial \mathbf{q}}\right.$$
$$\left. + f(\mathbf{q}, \boldsymbol{\xi})\frac{\partial}{\partial \mathbf{q}}\right)\left[2\mathbf{A}(\mathbf{q}) - \mathbf{A}\left(\mathbf{q} - \frac{\hbar\boldsymbol{\tau}}{2}\right) - \mathbf{A}\left(\mathbf{q} + \frac{\hbar\boldsymbol{\tau}}{2}\right)\right]$$
$$- \mathrm{i}\frac{e^2}{2c^2} f(\mathbf{q}, \boldsymbol{\xi})\left[2\mathbf{A}(\mathbf{q}) - \mathbf{A}\left(\mathbf{q} - \frac{\hbar\boldsymbol{\tau}}{2}\right) - \mathbf{A}\left(\mathbf{q} + \frac{\hbar\boldsymbol{\tau}}{2}\right)\right]$$
$$\left.\times\left(\frac{\partial\left(\boldsymbol{\tau}\mathbf{A}(\mathbf{q})\right)}{\partial \mathbf{q}} + \frac{1}{\hbar}\left[\mathbf{A}\left(\mathbf{q} - \frac{\hbar\boldsymbol{\tau}}{2}\right) - \mathbf{A}\left(\mathbf{q} + \frac{\hbar\boldsymbol{\tau}}{2}\right)\right]\right)\right\} \tag{6}$$

[note that the corresponding equation (30) of Ref. [24] has typos in some signs], where $\mathbf{E} = -(1/c)\,\partial\mathbf{A}/\partial t - \nabla\phi$ and $\mathbf{B} = \nabla \times \mathbf{A}$ are the self-consistent electric and magnetic fields, respectively. It is important to note that the right hand side of Eqn (6) can be cast in a more compact form that is especially convenient for numerical calculations (see Ref. [35]). Kinetic Wigner equation (6), coupled with the Maxwell equations for the self-consistent electromagnetic field potential ϕ and $\mathbf{A}$ in the corresponding gauge (e.g., Coulomb gauge $\nabla\mathbf{A} = 0$ for vector potential $\mathbf{A}$), describes quantum plasmas with electromagnetic interactions of spinless particles, and is called the Wigner–Maxwell (model) set of equations.

If the kinetic distribution function f_1 is known, then such macroscopic plasma characteristics as the density, momentum flux velocity, and corresponding electric current density and pressure tensor, can be obtained by calculating the corresponding moments of the distribution function, and linear and nonlinear plasma responses to electromagnetic fields can be determined, which is often the ultimate goal of plasma description. Singularities, occurring in the responses due to the contribution of particles that are resonant with plasma waves, can be removed by applying the causality principle and the corresponding Landau rule of integration (see more details in Section 4).

The main assumptions for the collisionless kinetic models based on the Wigner–Poisson or Wigner–Maxwell set of equations are the following.

(1) The plasma is ideal,

$$\Gamma_{\mathrm{q}} = \frac{U_{\mathrm{int}}}{\varepsilon_{\mathrm{F}}} = \frac{e^2 n^{1/3}}{\varepsilon_{\mathrm{F}}} \sim \left(\frac{\hbar\omega_{\mathrm{p}}}{\varepsilon_{\mathrm{F}}}\right)^2 \ll 1 \,.$$

We note that this condition is not satisfied for electron gas in metals, where $\Gamma_{\mathrm{q}} \sim 1$; therefore, the collisionless kinetic models, strictly speaking, are not applicable to metals.

(2) Plasma particles interact only via average classical collective fields that satisfy Poisson's or Maxwell's equations (the self-consistent (classical) mean-field approximation).

(3) Particle collisions are not taken into account (the models are collisionless); the same holds for particle correlations due to their identity (exchange interactions are ignored).

(4) Usually, particle spins are not taken into account. In general, corrections due to particle spins can still be accounted for in the nonrelativistic approximation, for example, by introducing the spin distribution function and writing the corresponding kinetic equation for this function, similar to the Wigner equation for f_1; this has been done, for example, by Silin and Rukhadze [36].

(5) The nonrelativistic approximation is used; see our note after Eqn (24) in Section 4.

2.2 Multistream model

The multistream model is based on the Hartree mean-field approximation; here, plasma is considered as a collection of 'cold beams' formed by groups of particles with the same momenta (see also Section 4, where the dielectric response tensor of a quantum plasma is obtained on the basis of this model). Using linearized equations of 'cold hydrodynamics' with self-consistent fields for each of these groups of particles (beams), their current density is calculated and thus the corresponding dielectric permittivity tensor is determined. Then, adding up the contributions of all groups of plasma particles with the corresponding 'weight functions', i.e., averaging these contributions over plasma equilibrium distribution function $f_0(\mathbf{p})$, one obtains the dielectric permittivity tensor of the whole plasma. This procedure is equivalent to calculating the dielectric permittivity tensor directly from the quantum kinetic equation for the one-particle Wigner function in the mean-field approximation. The contribution of groups of particles resonant with collective plasma excitations leads (similarly to kinetic models) to a singularity when calculating the sum (which becomes an integral in the limit of a infinite number of particle beams with an infinitely narrow velocity spread) of the total contribution of all particle groups in the plasma response. This singularity can also be removed by accounting for the causality of plasma excitations and applying the corresponding Landau integration rule (see Section 4 for more details).

The *main assumptions for the multistream model* are the same as those for kinetic models based on the collisionless quantum kinetic equation (see Section 2.1).

2.3 Quantum hydrodynamics model

This model, described already in Ref. [15], is constructed similarly to the multistream model (note that it can also be derived from the kinetic equation for the one-particle Wigner function [16]): the wavefunctions of plasma particles are represented as $\psi_\alpha(\mathbf{r}, t) = a_\alpha(\mathbf{r}, t) \exp\left(\mathrm{i}S_\alpha(\mathbf{r}, t)/\hbar\right)$ [37], where $a_\alpha(\mathbf{r}, t)$ and $S_\alpha(\mathbf{r}, t)$ are the real functions of space and time, and the number density n_α and velocity $\mathbf{v}_\alpha$ of particle group α are defined as $n_\alpha = |\psi_\alpha(\mathbf{r}, t)|^2 = a_\alpha^2(\mathbf{r}, t)$ and $\mathbf{v}_\alpha = \nabla S_\alpha(\mathbf{r}, t)/m$. Then the macroscopic plasma density $n(\mathbf{r}, t) = \langle n_\alpha \rangle$ and velocity $\mathbf{u}(\mathbf{r}, t) = \langle \mathbf{v}_\alpha \rangle$ are introduced, where $\langle \ldots \rangle$ stands for the average over an ensemble of plasma particles, and the first two hydrodynamic equations are written for $n(\mathbf{r}, t)$ and $\mathbf{u}(\mathbf{r}, t)$: the continuity equation and the equation of motion, with the latter containing two pressure-like terms, namely [16] the classical pressure defined by $P^{\mathrm{cl}} = mn(\langle v_\alpha^2 \rangle - \langle v_\alpha \rangle^2)$, and the quantum pressure given by

$$P^{\mathrm{q}} = \frac{\hbar^2}{2m} \left\langle (\nabla a_\alpha)^2 - a_\alpha(\nabla^2 a_\alpha) \right\rangle .$$

In order to close the set of these two equations, the following *two assumptions* are made for P^{q} and P^{cl}.

(1) Wavefunctions of all plasma electrons have equal amplitudes, $a_\alpha(\mathbf{r}, t) = a(\mathbf{r}, t)$, that, nevertheless, can vary in space and time while having different phases $S_\alpha(\mathbf{r}, t)$. This assumption agrees with the assumption of uncorrelated plasma particles: indeed, the spatial distribution of each quantum particle, defined by amplitude $a_\alpha(\mathbf{r}, t)$, does not depend on spatial distributions of other particles in the system. Equality $a_\alpha(\mathbf{r}, t) = a(\mathbf{r}, t)$ implies that spatial distribution density $n_\alpha = |a_\alpha|^2$ of each quantum particle is proportional to density n of the whole particle system, i.e., all particles are 'smeared' over the whole system in the same way; in other words, the size of the wave packet representing each particle in the system is equal to the size of the whole system. (We note that this assumption does not impose any restriction on the spatial and temporal scales of wave phenomena that can be correctly described by this model in the system.) This assumption leads to the following relation between P^{q} and n [16]:

$$P^{\mathrm{q}} = \frac{\hbar^2}{2m} \left[\left(\nabla\sqrt{n}\right)^2 - \sqrt{n}\left(\nabla^2\sqrt{n}\right) \right]; \qquad (7)$$

(2) There is an equation of state that relates 'classical' pressure $P^{\rm cl} \equiv mn(\langle v_\alpha^2 \rangle - \langle v_\alpha \rangle^2)$ of a quantum plasma to its macroscopic density $n(\mathbf{r}, t) \equiv \langle n_\alpha \rangle$.

By assuming a particular equation of state for the classical pressure $P^{\rm cl}$, we impose corresponding limitations on the applicability of the thus obtained closed hydrodynamics model (this is discussed in more detail below in Section 3). Of course, besides the assumptions 1 and 2 stated above, the construction of the quantum plasma hydrodynamics model also includes all the main assumptions characteristic for the Wigner kinetic model.

To summarize, we list the full set of *main assumptions for the quantum plasma hydrodynamics model*:

(1) The plasma is ideal,

$$\Gamma_{\rm q} = \frac{U_{\rm int}}{\varepsilon_{\rm F}} = \frac{e^2 n^{1/3}}{\varepsilon_{\rm F}} \sim \left(\frac{\hbar\omega_{\rm p}}{\varepsilon_{\rm F}}\right)^2 \ll 1\,.$$

As already noted, this condition is not satisfied for electron gas in metals ($\Gamma_{\rm q} \sim 1$).

(2) Plasma particles interact only via average classical collective fields described by Maxwell's equations.

(3) Particle collisions are not taken into account (the models are collisionless).

(4) Exchange interactions are ignored.

(5) The nonrelativistic approximation is used [see discussion in Section 4 after Eqn (23)].

(6) Wavefunctions $\psi_\alpha(\mathbf{r}, t) = a_\alpha(\mathbf{r}, t)\exp(\mathrm{i}S_\alpha(\mathbf{r}, t)/\hbar)$ have equal amplitudes $a_\alpha(\mathbf{r}, t) = a(\mathbf{r}, t)$ (which nevertheless can vary in space and time) for all plasma particles, while differing only in their phases $S_\alpha(\mathbf{r}, t)$. This imposes a relation between $P^{\rm q}$ and n, i.e., an 'equation of state' (7) for the quantum pressure. We note that this assumption limits the size of wave packets of plasma particles (requiring them to be equal to the whole system size), but does not limit the frequencies or wavenumbers of plasma waves that can be adequately described by this model.

(7) Some equation of state is assumed, which relates the 'classical' gas pressure $P^{\rm cl} \equiv mn(\langle v_\alpha^2 \rangle - \langle v_\alpha \rangle^2)$ to the macroscopic gas density $n(\mathbf{r}, t) \equiv \langle n_\alpha \rangle$. Usually, the adiabatic equation of state is postulated, $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$, with $P_0^{\rm cl} = n_0\varepsilon_{\rm F}$ for degenerate electrons (i.e. for $T_{\rm e} \ll \varepsilon_{\rm F}$, where $T_{\rm e}$ is the electron temperature in energy units), or with $P_0^{\rm cl} = n_0 T_{\rm e}$ for nondegenerate electrons (i.e., for $T_{\rm e} \gg \varepsilon_{\rm F}$). From this equation of state follows a restriction on the lengths of waves that can be correctly described within such hydrodynamics model: $k\lambda_{\rm F} \ll 1$ for degenerate electrons, where $\lambda_{\rm F} = v_{\rm F}/\sqrt{3}\,\omega_{\rm p}$ is the Thomas–Fermi length and $v_{\rm F} = \sqrt{2\varepsilon_{\rm F}/m}$ is the Fermi velocity of electrons, or $k\lambda_{\rm D} \ll 1$ for nondegenerate electrons, where $\lambda_{\rm D} = \sqrt{T_{\rm e}/2\pi e^2 n}$ is the electron Debye length (see Section 3 below).

3. Applicability range of quantum hydrodynamics equations

Due to its relative simplicity, the quantum hydrodynamics model has a significant advantage over the more exhaustive kinetic model — a lesser number of variables on which the plasma characteristics depend (i.e., four variables, $\mathbf{r}$, and t in the hydrodynamics model, instead of seven variables, $\mathbf{r}$, $\mathbf{p}$, and t in the kinetic model). This advantage allows considering nonlinear plasma phenomena relatively easy; it is eventually the reason why the hydrodynamic approach is preferred for describing such phenomena in quantum plasmas [18, 38]. However, it is necessary to remember the applicability ranges of this approach when analyzing results obtained through its use.

We note that in some studies (see reviews by Manfredi et al. [39, p. 26], and by Manfredi [9, p. 14]) the following statement is made: "it can be shown that, for distances larger than the Thomas–Fermi screening length ($\lambda_{\rm F}$ in our notations — S.V.V. and Y.O.T.), *one can replace* n_α (the space density of one electron — S.V.V. and Y.O.T.) with $n = \langle n_\alpha \rangle = \sum_\alpha p_\alpha |\psi_\alpha|^2$ (the macroscopic number density of an electron gas — S.V.V. and Y.O.T.)" in the quantum pressure

$$P^{\rm q} = \frac{\hbar^2}{2m}\sum_\alpha p_\alpha \left[\left(\frac{\partial\sqrt{n_\alpha}}{\partial x}\right)^2 - \sqrt{n_\alpha}\,\frac{\partial^2\sqrt{n_\alpha}}{\partial x^2}\right]. \tag{8}$$

As a result of this replacement, the set of hydrodynamic equations becomes (in the one-dimensional case)

$$\frac{\partial n}{\partial t} + \frac{\partial(nu)}{\partial x} = 0\,, \tag{9}$$

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} = \frac{e}{m}\frac{\partial\phi}{\partial x} - \frac{1}{mn}\frac{\partial P^{\rm cl}}{\partial x} - \frac{1}{mn}\frac{\partial P^{\rm q}}{\partial x}\,, \tag{10}$$

where $P^{\rm q} = P^{\rm q}(n)$ is defined by Eqn (7). This statement, in fact, implies that replacing $n_\alpha \to n = \langle n_\alpha \rangle$ in (8), which is equivalent to postulating the 'equation of state' for quantum pressure (7), is only valid at lengths larger than $\lambda_{\rm F}$. This essentially means that it is the equation of state (7) for quantum pressure $P^{\rm q}$ that imposes the condition that the wavelengths of wave phenomena described by Eqns (9) and (10) be large compared to the Thomas–Fermi length, $k\lambda_{\rm F} \ll 1$ (for degenerate electrons, i.e., when $T_{\rm e} \ll \varepsilon_{\rm F}$). The proof of this statement is based merely on the fact that only for $k\lambda_{\rm F} \ll 1$ does the hydrodynamics model (9), (10) correctly describe the dispersion of longitudinal oscillations in a degenerate electron gas (this follows from the comparison of dispersion relations derived hydrodynamically and kinetically that was done in Refs [9, 39]). Hence, the limitation $k\lambda_{\rm F} \ll 1$ must be imposed somewhere during the derivation of the closed set of hydrodynamic equations (9) and (10) (which is correct), namely (and this is incorrect) — when replacing $n_\alpha \to n$ in Eqn (8), i.e., when postulating equation of state (7) for the quantum pressure.

It seems appropriate to elucidate this issue here. Let us show that, for plasmas with degenerate electrons (i.e. for $T_{\rm e} \ll \varepsilon_{\rm F}$), limitation $k\lambda_{\rm F} \ll 1$ on the wavelengths described by quantum hydrodynamics equations (9) and (10) appears not as a consequence of postulating equation of state (7) for quantum pressure $P^{\rm q}$, but rather as a result of postulating a particular equation of state for classical pressure $P^{\rm cl} = P^{\rm cl}(n)$, namely the adiabatic equation of state $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$, with $P_0^{\rm cl} = n_0\varepsilon_{\rm F}$ (where n_0 is the equilibrium electron density). Whereas the possibility of replacing $n_\alpha \to n = \langle n_\alpha \rangle$ in Eqn (8) (which leads to equation of state (7) for the quantum pressure) is, in fact, implied by supposing the plasma to be ideal and the wave packet sizes of all particles to coincide with the system's size (assumptions 1 and 6 in the list of quantum hydrodynamics assumptions in Section 2). Neither of these two assumptions, in turn, leads to any limitations on the lengths of waves described by the hydrodynamics model.

Without any additional assumptions, except the assumption of an ideal nonrelativistic plasma, equations for the plasma macroscopic density $n(x,t)$ and hydrodynamic velocity $u(x,t)$ in the one-dimensional case are given by Eqns (9) and (10), with 'quantum pressure' $P^{\rm q} = P^{\rm q}(n_\alpha)$ defined by Eqn (8), and 'classical pressure' $P^{\rm cl}$ defined as $P^{\rm cl}(\mathbf{r},t) \equiv mn(\langle v_\alpha^2\rangle - \langle v_\alpha\rangle^2)$ (it is called 'classical' because it corresponds to the gas pressure in the classical limit $\hbar \to 0$, while 'quantum' pressure $P^{\rm q}$, from which the Bohm diffusion term appears in (10), does not have any analog in classical plasmas).

As already noted, in order to close this set of equations, we need to introduce two simplifying assumptions: (1) to postulate an equation of state for the classical pressure — a relation between $P^{\rm cl}$ and n, and (2) to postulate equation of state (7) for quantum pressure $P^{\rm q}$, which is equivalent to replacing n_α by n in Eqn (8). Let us consider these two assumptions separately and show which limitations they impose on the applicability of the resulting closed set of hydrodynamic equations.

First, we consider assumptions implied by postulating an equation of state for the classical pressure; to do this, we consider the classical limit of Eqns (9) and (10) (to exclude, for now, the Bohm diffusion term with $P^{\rm q}$, which vanishes in the classical limit). We note that even in the classical case, nothing prevents us from considering the Fermi–Dirac distribution of electrons as an equilibrium distribution — in the classical case this is just a constructed electron distribution, and we are free to construct any equilibrium distribution at will. As indicated in the textbook by Alexandrov et al. [17], in collisionless plasmas (those for which Eqns (9) and (10) are written), there are two cases when the (classical) pressure can be calculated explicitly, and therefore the set of hydrodynamic equations (9) and (10) can be closed. The first case corresponds to processes with characteristic length scales L and time scales τ, whose characteristic velocity significantly exceeds either the electron thermal velocity (for a Maxwellian distribution of electrons) or the electron Fermi velocity (for a Fermi–Dirac distribution of electrons),[1]

$$\frac{L}{\tau} \sim \frac{\omega}{k} \gg \max\{v_{\rm T}, v_{\rm F}\}\,. \tag{11}$$

In this case, following Ref. [34], we can completely neglect the thermal (or Fermi, in the case of 'emulating' a degenerate electron gas in a classical plasma) spread of particle velocities, and obtain, from Eqn (10), the Euler equation with zero pressure, $P^{\rm cl} = 0$. However, this approximation does not yield the correction to the dispersion of plasma waves due to the thermal (in the case of nondegenerate electrons) or Fermi (in the case of degenerate electrons) velocity spread of electrons; this does not suit us, and in order to obtain this correction, we need to take into account the thermal or Fermi electron velocity spread, which is done further below.

The second case corresponds to processes for which

$$v_{\rm T_i} \ll \frac{L}{\tau} \sim \frac{\omega}{k} \ll \max\{v_{\rm T}, v_{\rm F}\}\,, \tag{12}$$

where $v_{\rm T_i}$ is the characteristic (thermal) velocity of heavy plasma particles (ions).[2] In this case, the effect of electron inertia is negligibly small, and after excluding the electric field from the momentum equations for electrons and ions, one obtains the set of one-fluid hydrodynamics equations. These equations are suitable for the description of processes such as ion sound [with limitation (12)], but are not suitable for describing electron oscillations, for which the electron inertia is essential. Therefore, since we are mainly interested in the dispersion of electron oscillations, we are not considering this case here.

We recall now the first of the two cases mentioned above — the case of fast processes satisfying the condition (11) — and take into account the effect of electron velocity spread (thermal in the nondegenerate case, and Fermi in the degenerate case; the ions are still regarded as cold). Thus, we consider small perturbations of equilibrium that in the degenerate case is characterized by the Fermi–Dirac distribution of electrons, so that the classical pressure at the equilibrium is $P_0^{\rm cl} = n_0\varepsilon_{\rm F}$ (dropping the factor on the order of unity). We write the equation for the electron energy density (the second moment of the distribution function f) in the collisionless one-dimensional case as

$$\frac{\partial P^{\rm cl}}{\partial t} + u\,\frac{\partial P^{\rm cl}}{\partial x} + 3P^{\rm cl}\,\frac{\partial u}{\partial x} + 2\,\frac{\partial Q}{\partial x} = 0\,, \tag{13}$$

where Q is the heat flux defined as $Q = (m/2)\int (v-u)^3 f\,{\rm d}v$. With the assumption of fast processes [under condition (11)], the term with heat flux $\partial Q/\partial x$ is small compared to $\partial P^{\rm cl}/\partial t$, and can be neglected (hence, condition (11) implies that the corresponding process is adiabatic). As a result, taking into account the continuity equation (9), Eqn (13) becomes

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)\frac{P^{\rm cl}}{n^3} = 0\,, \tag{14}$$

from which follows $P^{\rm cl}/n^3 = \text{const}$, or $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$ — the equation of state for electron gas for adiabatically fast processes with $P_0^{\rm cl} \sim n_0\varepsilon_{\rm F}$. Substituting $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$ into (10) (in the classical limit, i.e., without the Bohm term), we obtain the hydrodynamic momentum equation for electron gas that accounts for the velocity spread of electrons in equilibrium (unlike the approximation of cold electrons in [17]). It is this equation that yields the correction $\sim k^2 v_{\rm F}^2$ (or $\sim k^2 v_{\rm T}^2$ in the nondegenerate case) to the dispersion of longitudinal electron oscillations.

Therefore, quantum hydrodynamics equation (10) with $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$ and $P_0^{\rm cl} \sim n_0\varepsilon_{\rm F}$, which yields the dispersion of longitudinal electron oscillations, follows from the kinetic theory in the approximation of adiabatically fast processes with $\omega \gg kv_{\rm F}$. Applying this to electron oscillations, for which $\omega \sim \omega_{\rm p}$, this approximation is valid for long wavelengths, $k\lambda_{\rm F} \ll 1$ for degenerate electrons or $k\lambda_{\rm D} \ll 1$ for fully nondegenerate electrons.

Let us go back to the quantum case. Here, first, the degeneracy of electrons is no longer a mere result of an arbitrary construction of the distribution function, but is an effect of quantum statistics (due to Pauli's exclusion princi-

[1] By fulfilling condition (11), we imply that Landau damping appearing in the kinetic description of collisionless plasmas [8, 40] due to resonant interactions between plasma particles — in our case, electrons — and collective plasma excitations is exponentially small (since the number of resonant particles is small), and can be neglected. Therefore, the hydrodynamics model of a collisionless plasma in which Landau damping is not taken into account in principle, is nevertheless correct under condition (11).

[2] We note that Landau damping (now on plasma ions) is also negligibly small under condition (12), i.e., the collisionless plasma hydrodynamics model that does not, in principle, take into account Landau damping is nevertheless correct in this case as well.

ple), and, second, the following (Bohm) term appears in Eqn (10):

$$\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\sum_\alpha p_\alpha \frac{\partial^2\sqrt{n_\alpha}/\partial x^2}{\sqrt{n_\alpha}}$$

(the remaining terms of Eqn (10) are the same as in the classical case and, for them, the discussion of the previous paragraphs can be repeated). It is assumed for the Bohm term that

$$\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\sum_\alpha p_\alpha \frac{\partial^2\sqrt{n_\alpha}/\partial x^2}{\sqrt{n_\alpha}} = \frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\frac{\partial^2\sqrt{n}/\partial x^2}{\sqrt{n}}\right), \qquad (15)$$

i.e., n_α is simply replaced by n in the Bohm term (thus, for the quantum pressure, equation of state (7), which relates $P^{\rm q}$ and n, is postulated).

It can be easily seen, upon simple substitution $n_\alpha \to |\psi_\alpha|^2$ with $\psi_\alpha(x,t) = a_\alpha(x,t)\exp(\mathrm{i}S_\alpha(x,t)/\hbar)$ on the left-hand side of Eqn (15), that (15) is equivalent to assumption 6 in the quantum hydrodynamics list of assumptions of Section 2.3, which by itself does not impose any constraints on the lengths of waves described within the quantum hydrodynamics framework, as already noted in Section 2.

Thus, the quantum hydrodynamics equations that were used in a number of studies to obtain the dispersion of longitudinal electron oscillations in ideal quantum plasmas, i.e., the set of Eqns (9) and (10) with $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$ and $P_0^{\rm cl} \sim n_0\varepsilon_{\rm F}$, can be obtained from the kinetic theory under assumptions 1–7 listed at the end of Section 2.3, and in the approximation of adiabatically fast processes, $\omega \gg kv_{\rm F}$, which for longitudinal oscillations in degenerate electron plasmas is equivalent to condition $k\lambda_{\rm F} \ll 1$. It is important to stress that this condition does not occur as a limitation of validity of Eqn (15) (i.e., that of equation of state (7) for the quantum pressure), but as a consequence of the approximation of adiabatically fast processes, for which the equation of state for the classical electron gas pressure is $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$ with $P_0^{\rm cl} \sim n_0\varepsilon_{\rm F}$ (in the one-dimension case).

In the case of longitudinal oscillations (see Section 5 below) in degenerate electron plasmas, condition $k\lambda_{\rm F} \ll 1$ corresponds to the Langmuir part of the spectrum [9, 18]:

$$\omega^2 = \omega_{\rm p}^2 + \frac{3}{5}k^2 v_{\rm F}^2 + \frac{\hbar^2 k^4}{4m^2}, \qquad (16)$$

which is correctly described by the quantum hydrodynamics model (9) and (10) (with $P^{\rm cl} = P_0^{\rm cl}(n/n_0)^3$ and $P_0^{\rm cl} \sim n_0\varepsilon_{\rm F}$, and Eqn (7) for $P^{\rm q}$), unlike the essentially kinetic part of the spectrum for $\omega_{\rm p}/v_{\rm F} \ll k \ll mv_{\rm F}/\hbar$ that is associated with the (kinetic) resonance $\omega \approx kv_{\rm F} + \hbar k^2/2m$ and is analogous to the zero sound in an almost ideal Fermi gas (see Section 5).

4. Dielectric permittivity of a collisionless quantum plasma

The linear response of a quantum plasma to electromagnetic disturbances is described by dielectric permittivity tensor $\epsilon_{ij}(\omega, \mathbf{k})$. To obtain the dielectric permittivity tensor, one has to solve the equations that govern the medium (plasma) dynamics in the presence of self-consistent electromagnetic fields; these equations need to be linearized by assuming the fields and perturbations (of the medium by these fields) to be small. For the case of collisionless quantum plasma with electromagnetic interaction of particles, such an equation is quantum kinetic equation (6). The procedure for calculating $\epsilon_{ij}(\omega,\mathbf{k})$ from this equation is rather cumbersome; however, we can obtain the same result in a somewhat simpler way by using the quantum multistream model mentioned above in Section 2.2.

It is well known (see, e.g., Ref. [41]) that the linear dielectric permittivity tensor can be relatively easily calculated for classical plasmas by using the (classical) multistream model. In this model, the plasma is considered as a collection of uncorrelated groups of classical particles with the same momenta (i.e., cold particle beams), and for each of these groups, the current density and the corresponding dielectric permittivity tensor of the group is calculated. The contributions of all groups of particles are then multiplied by the momentum distribution function of all plasma particles, $f_0(\mathbf{p})$, then summed, thus yielding the dielectric permittivity tensor of the whole plasma. This procedure is equivalent to calculating the dielectric permittivity tensor directly from the linearized Vlasov kinetic equation with self-consistent electromagnetic fields, since this approach also assumes uncorrelated plasma particles, weakly coupled with each other only by collective self-consistent mean fields.

Obviously, this procedure can be generalized to the case of quantum plasma as well, as long as the quantum plasma can be adequately described as a collection of weakly correlated quantum particles, i.e., if the coupling parameter $\Gamma_{\rm q} \sim (\hbar\omega_{\rm p}/\varepsilon_{\rm F})^2$ is small. This generalization was done, for example, by Kuzelev and Rukhadze [42]. Starting from the set of 'cold' quantum hydrodynamics equations,

$$\begin{aligned}
&\frac{\partial n_\alpha}{\partial t} + \nabla(n_\alpha \mathbf{v}_\alpha) = 0\,,\\
&\frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha\nabla)\mathbf{v}_\alpha = \frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\\
&\qquad + \frac{\hbar^2}{4m^2}\nabla\left(\frac{1}{n_\alpha}\left[\nabla^2 n_\alpha - \frac{1}{2n_\alpha}(\nabla n_\alpha)^2\right]\right), \qquad (17)\\
&\mathbf{j}_\alpha = en_\alpha\mathbf{v}_\alpha\,, \qquad \mathbf{v}_\alpha = \nabla S_\alpha - \frac{e}{c}\mathbf{A}\,,
\end{aligned}$$

for a group α of quantum particles with wavefunctions $\psi_\alpha(\mathbf{r},t) = a_\alpha(\mathbf{r},t)\exp(\mathrm{i}S_\alpha(\mathbf{r},t)/\hbar)$ (not accounting for spin, i.e., for spinless particles), it is not difficult to calculate the conductivity tensor $\sigma^\alpha_{ij}(\omega,\mathbf{k})$ for this group of particles by expressing their current density via the electric field as $j_{\alpha i}(\omega,\mathbf{k}) = \sigma^\alpha_{ij}(\omega,\mathbf{k})\,E_j(\omega,\mathbf{k})$ (for the corresponding Fourier components). The dielectric permittivity tensor for the αth group of particles is then obtained as

$$\begin{aligned}
\epsilon^\alpha_{ij}(\omega,\mathbf{k}) &= \delta_{ij} + \frac{4\pi\mathrm{i}}{\omega}\sigma^\alpha_{ij}(\omega,\mathbf{k})\\
&= \delta_{ij} - \frac{4\pi e^2 n_{0\alpha}}{m\omega^2}\left\{\delta_{ij} + \frac{\omega - \mathbf{k}\mathbf{v}_\alpha}{(\omega-\mathbf{k}\mathbf{v}_\alpha)^2 - \omega_k^2}(k_i v_{\alpha j} + k_j v_{\alpha i})\right.\\
&\quad\left. + \frac{k^2 v_{\alpha i}v_{\alpha j} + \omega_k^2\kappa_i\kappa_j}{(\omega-\mathbf{k}\mathbf{v}_\alpha)^2 - \omega_k^2}\right\},
\end{aligned}$$

where $n_{0\alpha}$ is the unperturbed number density of the αth group of particles, $\omega_k = \hbar k^2/2m$, and $\boldsymbol{\kappa} = \mathbf{k}/|\mathbf{k}|$ is the unit vector along $\mathbf{k}$. Summing up the contributions of all plasma particle groups, with corresponding unperturbed densities $n_{0\alpha} = \int \mathrm{d}\mathbf{p}_\alpha\, f_{0\alpha}(\mathbf{p}_\alpha)$, we obtain the linear dielectric permittivity tensor of a quantum plasma: [3]

$$\epsilon_{ij}(\omega,\mathbf{k}) = \delta_{ij} - \frac{4\pi e^2}{m\omega^2}\int \mathrm{d}\mathbf{p}\, f_0(\mathbf{p})\, \frac{(\omega-\mathbf{kv})^2}{(\omega-\mathbf{kv})^2-\omega_k^2} \times \left\{\delta_{ij} + \frac{k_i v_j + k_j v_i}{\omega-\mathbf{kv}} + \frac{k^2 v_i v_j}{(\omega-\mathbf{kv})^2} + \frac{\omega_k^2(\kappa_i\kappa_j-\delta_{ij})}{(\omega-\mathbf{kv})^2}\right\}, \quad (18)$$

where $\mathbf{v} = \mathbf{p}/m$. By writing $\epsilon_{ij}(\omega,\mathbf{k})$ for an isotropic plasma as [34]

$$\epsilon_{ij}(\omega,\mathbf{k}) = \epsilon^{\mathrm{l}}(\omega,\mathbf{k})\kappa_i\kappa_j + \epsilon^{\mathrm{tr}}(\omega,\mathbf{k})(\delta_{ij}-\kappa_i\kappa_j)\,,$$

we obtain from Eqn (18) the following expressions for the longitudinal (l) and transverse (tr) dielectric permittivities of isotropic quantum plasma [42]:

$$\epsilon^{\mathrm{l}}(\omega,\mathbf{k}) = 1 + \frac{4\pi e^2}{\hbar k^2}\int \mathrm{d}\mathbf{p}\, \frac{\hat{D}[f_0(\mathbf{p})]}{\omega-\mathbf{kv}}\,, \quad (19)$$

$$\epsilon^{\mathrm{tr}}(\omega,\mathbf{k}) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} + \frac{2\pi e^2}{\hbar\omega^2}\int \mathrm{d}\mathbf{p}\, \frac{v_\perp^2}{\omega-\mathbf{kv}}\, \hat{D}[f_0(\mathbf{p})]\,, \quad (20)$$

where $\hat{D}[f_0(\mathbf{p})]$ is the difference operator defined as $\hat{D}[f_0(\mathbf{p})] = f_0(\mathbf{p}+\hbar\mathbf{k}/2) - f_0(p-\hbar\mathbf{k}/2)$, and $v_\perp$ is the particle velocity component perpendicular to vector $\mathbf{k}$. Equations (19) and (20) are equivalent to the expressions obtained directly by linearizing the kinetic equation (6) [36],

$$\epsilon^{\mathrm{l}}(\omega,\mathbf{k}) = 1 + \frac{4\pi e^2}{\omega\hbar k^2}\int \mathrm{d}\mathbf{p}\, \frac{\mathbf{kv}}{\omega-\mathbf{kv}}\, \hat{D}[f_0(\mathbf{p})]\,, \quad (21)$$

$$\epsilon^{\mathrm{tr}}(\omega,\mathbf{k}) = 1 + \frac{2\pi e^2}{\omega^2 k^2}\int \mathrm{d}\mathbf{p}\, [\mathbf{k}\times\mathbf{v}]^2 \left\{ f_0'(\mathbf{p}) + \frac{1}{\hbar}\, \frac{\hat{D}[f_0(\mathbf{p})]}{\omega-\mathbf{kv}} \right\}, \quad (22)$$

where $f_0'(\mathbf{p})$ is the derivative of the distribution function with respect to particle kinetic energy $\varepsilon = p^2/2m$. Indeed, it is easy to demonstrate that, in isotropic plasmas, Eqns (21) and (22) coincide with Eqns (19) and (20), respectively. Also, the same expressions (19) and (20) have been obtained for an isotropic quantum plasma by Klimontovich and Silin [24], as well as by Kuz'menkov and Maksimov [43] (from their Eqns (27) and (28) for $\epsilon^{\mathrm{l}}(\omega,\mathbf{k})$ and $\epsilon^{\mathrm{tr}}(\omega,\mathbf{k})$ in the limit of no exchange interactions; see Ref. [43]).

[3] For real ω and $\mathbf{k}$, the problem of integrating over the momentum component $p_\parallel$ along $\mathbf{k}$ appears in Eqn (18) due to pole singularity $(\omega - kp_\parallel/m)^2 - \omega_k^2 = 0$ on the integration path over real $p_\parallel$. This problem can be eliminated by considering only those plasma perturbations that obey the causality principle, i.e., that appear at an initial time instant t_0 while being absent at earlier $t < t_0$. For such perturbations, the temporal Fourier components of functions that describe these perturbations [e.g., the temporal Fourier component of electrostatic potential $\phi(t)$] are determined by the one-sided Fourier integral (for the potential — by $\phi_\omega = \int_0^\infty \phi(t')\exp(\mathrm{i}\omega t')\,\mathrm{d}t'$, where $t' = t - t_0$) defined only for complex ω with positive imaginary part $\mathrm{Im}(\omega) > 0$. To define these integrals for real ω, it is necessary to analytically continue the corresponding functions of ω from the upper half-plane of complex ω on the real axis. To do that, it is necessary to set $\omega = \omega + \mathrm{i}\delta$ in Eqn (18), and tend δ to zero from above. This procedure defines the rule (known as the Landau rule) of bypassing the pole on the integration path over real $p_\parallel$, and eliminates the problem of integrating over $p_\parallel$ in Eqn (18). The contribution of integration over $p_\parallel$ in the vicinity of this pole gives us the imaginary part of $\epsilon_{ij}(\omega,\mathbf{k})$ even for real ω, which, in turn, leads to Landau damping [8, 40]. Further in this paper, the Landau (bypass) rule $\omega = \omega + \mathrm{i}0$ is assumed in the corresponding integrals over momenta, unless the opposite is specifically stated.

Equations (19) and (20) are obtained by using the nonrelativistic model of quantum plasma (formally correct for $c \to \infty$), which, strictly speaking, is not applicable for a description of plasma waves with relativistic phase velocities, $\omega/k \gtrsim c$, regardless of whether the plasma particle velocities are relativistic or not [20]. Indeed, it is easy to show that conservation of the total energy and momentum in the process of emission or absorption of a radiation quantum (i.e., of a longitudinal or transverse wave) with wave frequency ω and wave vector $\mathbf{k}$ by a plasma particle with momentum $\mathbf{p}$ and energy ε,

$$\varepsilon' = \varepsilon \pm \hbar\omega\,, \qquad \mathbf{p}' = \mathbf{p} \pm \hbar\mathbf{k}\,, \quad (23)$$

in the relativistic approach, i.e., for

$$\varepsilon = \sqrt{m^2c^4 + p^2c^2}$$

(where m is the rest mass of the particle, e.g., an electron), leads to the resonance condition

$$\omega - \mathbf{kv} \pm \frac{\hbar}{2m\gamma}\left(k^2 - \frac{\omega^2}{c^2}\right) = 0\,, \quad (24)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. For nonrelativistic particle velocities $\gamma \approx 1$, we obtain

$$\omega - \mathbf{kv} \pm \frac{\hbar}{2m}\left(k^2 - \frac{\omega^2}{c^2}\right) = 0\,. \quad (25)$$

However, if, in the nonrelativistic approximation, we formally take $\varepsilon = p^2/2m$ in Eqn (23), then the term ω^2/c^2 does not appear at all in resonance condition (25), which, strictly speaking, is incorrect in the case of the relativistic phase velocity of a wave that is in resonance with the particle, i.e., for $\omega/k \gtrsim c$. And though the effect of the term ω^2/c^2 turns out to be insignificant for most processes in unmagnetized plasma, the corresponding contribution can be crucially important for some processes in the presence of an external magnetic field, such as cyclotron maser radiation [44]. One should remember that, strictly speaking, the nonrelativistic approximation is justified *a priori* only when the wave phase velocities (as well as the plasma particle velocities) are nonrelativistic, i.e., for $\omega/k \ll c$ [41]. Indeed, energy conservation (23) in wave-particle interactions does not preclude the possibility that, e.g., an initially nonrelativistic particle may become relativistic after interacting with a wave quantum with sufficiently large energy/momentum.

In the relativistic treatment, the dielectric permittivity tensor of an isotropic electron–positron plasma (where electrons and positrons are assumed to be unpolarized) is given by [20, 30]

$$\epsilon_{ij}^{\mathrm{rel}}(\omega,\mathbf{k}) = \delta_{ij} - \frac{4\pi e^2}{m\omega^2}\int \frac{\mathrm{d}\mathbf{p}}{\gamma}\, f_0(\mathbf{p})\, \frac{(\omega-\mathbf{kv})^2}{(\omega-\mathbf{kv})^2 - \Delta_k^2} \times \left\{\delta_{ij} + \frac{k_i v_j + k_j v_i}{\omega-\mathbf{kv}} + \frac{(k^2-\omega^2/c^2)v_i v_j}{(\omega-\mathbf{kv})^2}\right\}, \quad (26)$$

where $\mathbf{p} = m\gamma\mathbf{v}$, $f_0(\mathbf{p}) = 2\bar{n}(\mathbf{p})/(2\pi\hbar)^3$, $\bar{n}(\mathbf{p})$ is the sum of occupation numbers for electrons and positrons, and

$$\Delta_k = \frac{\hbar}{2m\gamma}\left(k^2 - \frac{\omega^2}{c^2}\right) = \frac{\omega_k}{\gamma}\left(1 - \frac{\omega^2}{c^2k^2}\right). \quad (27)$$

We note that the integrands' denominators for the electron and positron contributions in Eqn (26) correspond to the products of relativistic resonance conditions (24) for the emission and absorption of wave quanta with frequency ω and wavenumber k by relativistic particles (electrons or positrons, respectively) with energy $\varepsilon = \sqrt{m^2c^4 + p^2c^2}$ and momentum $\mathbf{p}$. Indeed, symmetrizing (26) on $\mathbf{p}$ due to $f_0(-\mathbf{p}) = f_0(\mathbf{p})$ for an isotropic plasma and equating the denominator of the obtained integrand to zero,

$$\left[(\omega - \mathbf{kv})^2 - \Delta_k^2\right]\left[(\omega + \mathbf{kv})^2 - \Delta_k^2\right] = 0\,,$$

we obtain four conditions

$$\omega \pm \mathbf{kv} = \pm\Delta_k\,, \qquad (28)$$

which correspond to the resonance conditions in the processes of photon (or plasmon) emission and absorption by electrons and positrons, as well as in the processes of one-photon or one-plasmon electron–positron pair creation and annihilation [30]. It is easy to see that the longitudinal $\epsilon^{\mathrm{l}}_{\mathrm{rel}}(\omega, \mathbf{k})$ and transverse $\epsilon^{\mathrm{tr}}_{\mathrm{rel}}(\omega, \mathbf{k})$ dielectric permittivities of an isotropic plasma, obtained from Eqn (26), match the corresponding expressions obtained by Tsytovich [45] (adjusting for sign typos in Ref. [45] that were corrected in Eqns (9.1.12) and (9.1.13) of Ref. [30], as well as neglecting the polarization effects).

For electron plasmas (without positrons), in the nonrelativistic limit, $c \to \infty$ (i.e., for nonrelativistic plasmas with $\gamma \to 1$ and waves with $\omega/k \ll c$), Eqn (26) changes into

$$\epsilon^{\mathrm{NR}}_{ij}(\omega, \mathbf{k}) = \delta_{ij} - \frac{4\pi e^2}{m\omega^2}\int \mathrm{d}\mathbf{p}\, f_0(\mathbf{p})\, \frac{(\omega - \mathbf{kv})^2}{(\omega - \mathbf{kv})^2 - \omega_k^2} \times \left\{\delta_{ij} + \frac{k_i v_j + k_j v_i}{\omega - \mathbf{kv}} + \frac{k^2 v_i v_j}{(\omega - \mathbf{kv})^2}\right\}, \qquad (29)$$

which differs from $\epsilon_{ij}(\omega, \mathbf{k})$ in Eqn (18) by the absence of the term

$$-\frac{4\pi e^2}{m\omega^2}\int \mathrm{d}\mathbf{p}\, f_0(\mathbf{p})\, \frac{\omega_k^2}{(\omega - \mathbf{kv})^2 - \omega_k^2}\,(\kappa_i\kappa_j - \delta_{ij})\,. \qquad (30)$$

The disagreement between the nonrelativistic limit, Eqn (29), of Eqn (26) and expression (18) obtained in several different ways from nonrelativistic quantum plasma models occurs due to the fact that Eqn (26) is obtained for a gas of unpolarized particles (electrons and/or positrons) with spins $\pm 1/2$, while Eqn (18) is obtained from those models that do not account for spin at all. We should note that term (30), by whose absence Eqn (29) differs from Eqn (18), contributes only to the plasma transverse dielectric permittivity, $\epsilon^{\mathrm{tr}}(\omega, \mathbf{k})$, and has no effect on the longitudinal dielectric permittivity, $\epsilon^{\mathrm{l}}(\omega, \mathbf{k})$. Therefore, the nonrelativistic theories not accounting for particle spins are not quite correct for plasmas composed of particles with half-integer spins (e.g., for electron plasmas): while correctly describing longitudinal oscillations in such plasmas, they incorrectly describe their transverse modes. Note that accounting for paramagnetic effects (related to particle spins) in the nonrelativistic model [36] indeed leads to the disappearance of term (30) in tensor (18); as a result, the latter becomes exactly equal to the nonrelativistic limit, Eqn (29), of tensor (26). The noted discrepancy between the nonrelativistic responses, Eqns (18) and (29), related to plasma particle spins, is another important example demonstrating the necessity of careful consideration of all relevant effects, however small they may seem *a priori* in the nonrelativistic approximation.

To obtain the relativistic generalization of the collective linear response, Eqn (18), of a gas of spinless charged particles (e.g., a gas of Cooper electron pairs with zero total spins, or a gas of bosons with spin 0), one needs to construct the corresponding relativistic theory based on the Klein–Gordon equation (which is a special case of the Dirac equation for spinless particles). However, we can instead apply a phenomenological approach to obtain the relativistic generalization of the dielectric tensor (18). We note that the only difference between the relativistic response sought for a gas of spinless charged particles (which should turn into (18) when $c \to \infty$) and the relativistic response of a gas of unpolarized electrons is in the term that turns into (30) in the limit $c \to \infty$. The denominator of expression (30), in turn, corresponds to the product of two resonance conditions (24) for emission and absorption of a wave quantum with energy $\hbar\omega$ and momentum $\hbar\mathbf{k}$ in the nonrelativistic limit $c \to \infty$. Therefore, in the relativistic case, this denominator should turn into the product of the corresponding relativistic resonance conditions, Eqns (23), i.e., ω_k^2 should turn into Δ_k^2, and $(\omega - \mathbf{kv})^2 - \omega_k^2$ should turn into $(\omega - \mathbf{kv})^2 - \Delta_k^2$. Hence, Eqn (30) in the relativistic theory should turn into the following expression (we also have to replace the nonrelativistic mass (rest mass), m, by the relativistic mass, γm):

$$-\frac{4\pi e^2}{m\omega^2}\int \frac{\mathrm{d}\mathbf{p}}{\gamma}\, f_0(\mathbf{p})\, \frac{\Delta_k^2}{(\omega - \mathbf{kv})^2 - \Delta_k^2}\,(\kappa_i\kappa_j - \delta_{ij})\,. \qquad (31)$$

By adding expression (30) to Eqn (26), we obtain the following phenomenological expression for the dielectric permittivity tensor of a relativistic quantum plasma composed of spinless particles:

$$\epsilon^{\mathrm{rel}}_{ij}(\omega, \mathbf{k}) = \delta_{ij} - \frac{4\pi e^2}{m\omega^2}\int \frac{\mathrm{d}\mathbf{p}}{\gamma}\, f_0(\mathbf{p})\, \frac{(\omega - \mathbf{kv})^2}{(\omega - \mathbf{kv})^2 - \Delta_k^2} \times \left\{\delta_{ij} + \frac{k_i v_j + k_j v_i}{\omega - \mathbf{kv}} + \frac{(k^2 - \omega^2/c^2) v_i v_j}{(\omega - \mathbf{kv})^2} + \frac{\Delta_k^2(\kappa_i\kappa_j - \delta_{ij})}{(\omega - \mathbf{kv})^2}\right\}. \qquad (32)$$

It is easy to verify that Eqn (32) turns into Eqn (18) in the limit $c \to \infty$, as required.

We stress again that plasma responses obtained from the relativistic approach (the response of an unpolarized electron or electron-positron gas, as well as the response of a gas of spinless charged particles) differ from the corresponding plasma responses obtained from the nonrelativistic models, among other things, by the presence of a term proportional to ω^2/c^2. The effect of this term on the dispersion of longitudinal and transverse waves in nonrelativistic quantum plasmas is usually small in most cases (e.g., for electron gas in metals), becoming significant for high density plasmas, when the process of pair creation by photons and/or plasmons is allowed in terms of energy. In the latter case, an additional mechanism of plasma longitudinal and transverse wave damping, associated with the electron–positron pair creation process, takes place (in addition to Landau damping). It follows from the energy balance consideration that this damping occurs for waves (both longitudinal and transverse) with superluminous phase velocities, and, in addition, has the energy threshold [45],

$$(\hbar\omega)^2 > 4(mc^2)^2 + (\hbar k)^2 c^2\,. \qquad (33)$$

It follows from (33) that, in the long wavelength limit (when $\omega \sim \omega_p$), the longitudinal and transverse wave damping due to the creation of real electron-positron pairs becomes significant for plasma densities $n \gtrsim 10^{32}$ cm^{-3} [45]. This is, however, a rather exotic case, as such large densities can exist perhaps only in the cores of dense astrophysical objects, e.g., white dwarfs. On the other hand, the effects associated with the creation of virtual electron–positron pairs do not have the strict energy threshold (33), and thus can affect the waves in plasmas with lower densities. These effects can also have an influence on the analytic properties of plasma linear response functions.

5. Quantum kinetic effects and analytic properties of the linear longitudinal response of a quantum plasma

Consider now essentially quantum effects that occur in the kinetic description of collective modes in quantum plasmas. For simplicity, we dwell on longitudinal oscillations resulting from an initial perturbation of a quantum plasma.

In the case of an initial perturbation of one-particle Wigner function $f(\mathbf{r},\mathbf{p},0)$ in a uniform isotropic plasma, the time evolution for spatial Fourier components of the electrostatic plasma potential,

$$\phi_{\mathbf{k}}(t,\mathbf{k}) = \int \phi(t,\mathbf{r}) \exp(\mathrm{i}\mathbf{kr})\,\mathrm{d}\mathbf{r}\,,$$

is given by the following expression:

$$\phi_{\mathbf{k}}(t,\mathbf{k}) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\sigma}^{+\infty+\mathrm{i}\sigma} \phi_\omega(\omega,\mathbf{k}) \exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega\,, \qquad (34)$$

where the integration is done in the plane of complex ω along the horizontal contour located in the upper half-plane, $\operatorname{Im}\omega = \sigma > 0$, and

$$\phi_\omega(\omega,\mathbf{k}) = \frac{4\pi\mathrm{i}e}{mk^2\epsilon^1(\omega,\mathbf{k})}\int_{-\infty}^{+\infty} \frac{g(\mathbf{k},p_x)}{\omega - kp_x/m}\,\mathrm{d}p_x\,, \qquad (35)$$

where p_x is the component of the particle momentum along $\mathbf{k}$, $g(\mathbf{k},p_x) = \int g(\mathbf{k},\mathbf{p})\,\mathrm{d}p_y\,\mathrm{d}p_z$, $g(\mathbf{k},\mathbf{p})$ is the Fourier transform of the initial perturbation $f(\mathbf{r},\mathbf{p},0)$, and $\epsilon^1(\omega,\mathbf{k})$ is the longitudinal dielectric permittivity of plasma, defined by Eqn (19).

To find the behavior of the potential at large times t, we need to integrate over ω in (34) along the path formed from the initial contour by taking the limit $\sigma = \operatorname{Im}\omega$ to $-\infty$, while preserving the analyticity of function $\phi_\omega(\omega,\mathbf{k})$ in the integral. (In order to do so, this function $\phi_\omega(\omega,\mathbf{k})$, in turn, should be analytically continued from its definition region $\operatorname{Im}\omega > 0$ to the region $\operatorname{Im}\omega \leqslant 0$; for that, the contours of integration over p_x in the numerator and denominator of (35) must be displaced from the real axis $\operatorname{Im} p_x = 0$ to the lower half-plane $\operatorname{Im} p_x < 0$ in such a way that the pole $p_x = m\omega/k$ is passed from below [40].) The integration [over ω in (34)] path must pass above all singularities of function $\phi_\omega(\omega,\mathbf{k})$ (analytically continued to the region $\operatorname{Im}\omega \leqslant 0$) that are on or under the real axis $\operatorname{Re}\omega$ [40].

In the case of a classical plasma, the equilibrium distribution function f_0 is an entire function of p_x (i.e., f_0 has no singularities at finite p_x), and the analytic continuation of $\epsilon^1(\omega,\mathbf{k})$ to the region $\operatorname{Im}\omega \leqslant 0$ is also an entire function of ω; the same argument applies to the analytic continuation to the region $\operatorname{Im}\omega \leqslant 0$ of the function

$$\int_{-\infty}^{+\infty} \frac{g(\mathbf{k},p_x)}{\omega - kp_x/m}\,\mathrm{d}p_x\,, \qquad (36)$$

in the numerator of Eqn (35) for $\phi_\omega(\omega,\mathbf{k})$, if the initial perturbation $g(\mathbf{k},p_x)$ is also an entire function of p_x. Thus, provided $f_0(p_x)$ and $g(\mathbf{k},p_x)$ are entire functions of p_x, the function $\phi_\omega(\omega,\mathbf{k})$ (35), which is analytically continued on the whole plane of complex ω, is the ratio of two entire functions of ω. In this case, the only singularities of the function $\phi_\omega(\omega,\mathbf{k})$ are the poles $\epsilon^1(\omega,\mathbf{k}) = 0$ defined by zeros in the denominator in Eqn (35) for $\phi_\omega(\omega,\mathbf{k})$. The contribution of these poles in integral (33) completely determines the evolution of $\phi(t,\mathbf{r})$, which, in this case, is a superposition of oscillations, exponentially damped (or growing, in the case of nonequilibrium plasma) with time.

However, the solution to the initial value problem in quantum plasmas turns out to be more complicated [46], since the equilibrium distribution function f_0 is no longer an entire function of p_x. Indeed, an electron gas obeys the Fermi statistics, and its equilibrium distribution function (Wigner function) corresponds to the Fermi–Dirac distribution,

$$f_0(p) = \frac{2}{(2\pi\hbar)^3}\left\{\exp\left[\frac{p^2/2m - \mu(T)}{T}\right] + 1\right\}^{-1}, \qquad (37)$$

where T is the electron temperature and $\mu(T)$ is the chemical potential. In the one-dimensional case ($p = p_x$), this function (shown in Fig. 1 by dashed lines) has singular points (first-

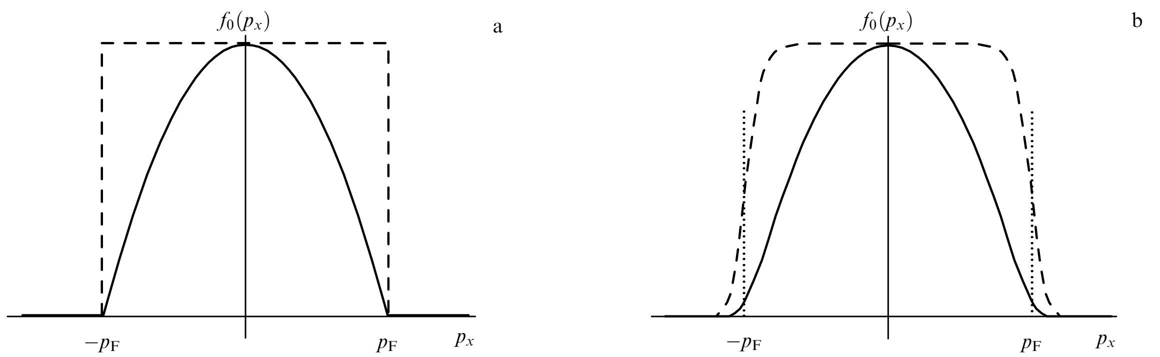

Figure 1. Distribution functions $f_0^{1D}(p_x)$ for the one-dimensional, $p = p_x$ (shown with dashed lines), and three-dimensional, $p = (p_x^2 + p_\perp^2)^{1/2}$ (shown with solid lines), Fermi–Dirac distribution (37) in the case of: (a) complete degeneracy ($\mu(T)/T \to \infty$), and (b) partial degeneracy ($\mu(T)/T = 10$). The functions are normalized on their corresponding values $f_0^{1D}(0)$.

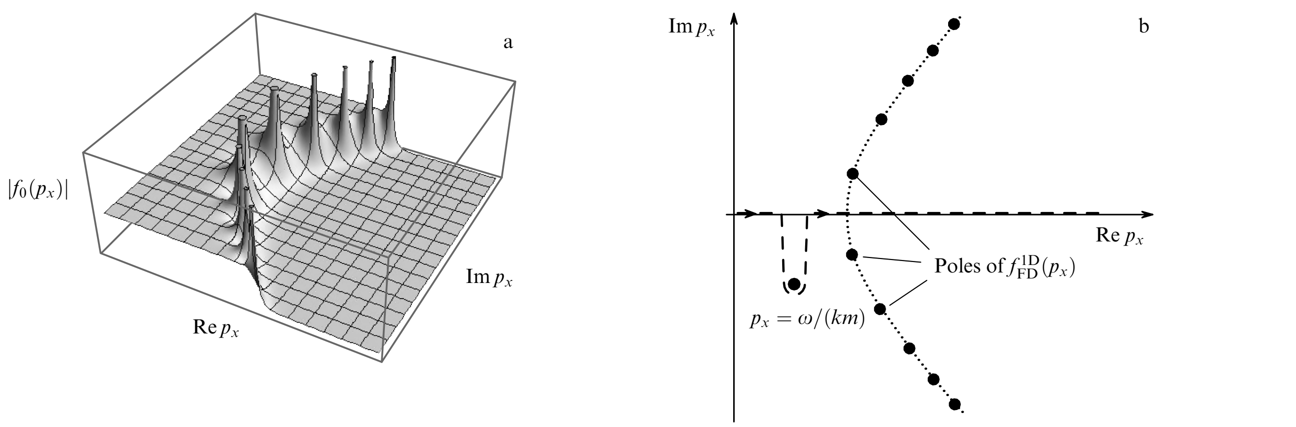

Figure 2. (a) Absolute value of the one-dimensional Fermi–Dirac distribution $f_0(p_x) \propto \{\exp[(p_x^2/2m - \mu(T))/T] + 1\}^{-1}$ as a function of complex p_x. (b) Poles of the one-dimensional Fermi–Dirac distribution $f_0(p_x) \propto \{\exp[(p_x^2/2m - \mu(T))/T] + 1\}^{-1}$ and the integration path over p_x (dashed line) for the analytic continuation of the function $\epsilon^l(\omega, \mathbf{k})$ in the region Im $\omega \leqslant 0$; the hyperbole — the locus of the poles of the function $f_0(p_x)$ — is shown by the dotted line.

order poles) that are on a hyperbola in the complex p_x plane (see Fig. 2), intersecting the real axis Im $p_x = 0$ at points $\pm[2\mu(T)/m]^{1/2}$ and with the distance between adjacent singular points proportional to the temperature T [46], so that in the limit $T \to 0$ these points completely fill the hyperbola. As discussed above, when analytically continuing function $\epsilon^l(\omega, \mathbf{k})$ into the region Im $\omega \leqslant 0$, the path of integration over p_x in $\epsilon^l(\omega, \mathbf{k})$ must pass the pole $p_x = m\omega/k$ from below; however, this integration path should not intersect any pole of the function $f_0(p_x)$. Therefore, the contour of integration over p_x is 'pinched' between the pole $p_x = m\omega/k$ lying in the lower half-plane of complex p_x and the nearest pole p_{x0} of the function $f_0(p_x)$. For $m\omega/k \to p_{x0}$, i.e., when these two poles coincide, the contour unavoidably passes through (intersects) these two poles, and hence the function $\epsilon^l(\omega, \mathbf{k})$ thus has singularities at $\omega_j = kp_{x0j}/m$, where p_{x0j} are the poles of function $f_0(p_x)$.

Thus, the function $\epsilon^l(\omega, \mathbf{k})$ is no longer an entire function of ω in quantum plasmas (in contrast to that in classical Maxwellian plasmas), and singularities of $\epsilon^l(\omega, \mathbf{k})$, along with zeros of $\epsilon^l(\omega, \mathbf{k})$, contribute to $\phi(t, \mathbf{r})$. The contribution of singularities of $\epsilon^l(\omega, \mathbf{k})$, unlike the contribution of its zeros, may not be an exponentially damped oscillating function, but rather a relatively slowly (by power law) decaying function of time [46, 47], not necessarily an oscillating one.

We note, however, that in most cases an electron gas is described by the three-dimensional, rather than the one-dimensional, Fermi–Dirac distribution function (37). In this case, the one-dimensional distribution function under the integral over the momentum component p_x along the wave vector in Eqn (19) is defined as the three-dimensional distribution function $f_0(p)$ integrated over momenta $\mathbf{p}_\perp$ perpendicular to $\mathbf{k}$:

$$f_0^{\rm 1D}(p_x) = \int f_0(p)\,\mathrm{d}\mathbf{p}_\perp$$

(it is shown in Fig. 1 by solid lines), while the difference operator $\hat{D}[f_0(\mathbf{p})] = f_0(\mathbf{p} + \hbar\mathbf{k}/2) - f_0(\mathbf{p} - \hbar\mathbf{k}/2)$ acts only on $f_0^{\rm 1D}(p_x)$, i.e.,

$$\hat{D}[f_0(\mathbf{p})] = \left[f_0(p_x + \hbar k/2) - f_0(p_x - \hbar k/2)\right] f_0(\mathbf{p}_\perp)\,.$$

As a result, in the case of the three-dimensional Fermi-Dirac distribution, Eqn (19) for $\epsilon^l(\omega, \mathbf{k})$ is given by

$$\epsilon^l(\omega, \mathbf{k}) = 1 + \frac{4\pi e^2}{\hbar k^2} \frac{4\pi m T}{(2\pi\hbar)^3} \int_{-\infty}^{+\infty} \mathrm{d}p_x \, \frac{1}{(\omega - kp_x/m)} \times \left\{ \ln\left[1 + \exp\left(-\frac{(p_x + \hbar k/2)^2 - 2m\mu(T)}{2mT}\right)\right] - \ln\left[1 + \exp\left(-\frac{(p_x - \hbar k/2)^2 - 2m\mu(T)}{2mT}\right)\right]\right\}, \qquad (38)$$

where the path of integration over p_x is again chosen to pass the pole $p_x = m\omega/k$ from below. The logarithmic functions in (38) are not entire functions — each of them has singularities like branching points and branch cuts in the complex p_x plane, shown in Fig. 3. These singularities lead to the function $\epsilon^l(\omega, \mathbf{k})$ having similar singularities in the complex ω plane, shown in Fig. 4, which can contribute to the integral over ω in Eqn (34) for $\phi(t, \mathbf{r})$, along with the zeros of $\epsilon^l(\omega, \mathbf{k})$, i.e., with the poles of $1/\epsilon^l(\omega, \mathbf{k})$ in Eqn (35). It is this contribution that can also lead to a power law, i.e., the nonexponential character of temporal attenuation of an initial perturbation, similar to the case of the one-dimensional Fermi distribution [46, 47]. Note that this is a purely kinetic effect absent in the quantum hydrodynamics approximation.

If, under some conditions, the contribution to $\phi(t, \mathbf{r})$ due to singularities of $\epsilon^l(\omega, \mathbf{k})$ in a Fermi gas dominates over the contribution of zeros of $\epsilon^l(\omega, \mathbf{k})$ in $\phi(t, \mathbf{r})$ at large times t, then the physical picture of longitudinal collective modes in quantum plasmas is changed: in this case, the evolution of an initial perturbation of quantum plasmas at large times t can be qualitatively different from that of classical plasmas. In this regard, of most interest are those regions where the contribution along the corresponding integration path has the smallest (in absolute value) imaginary part. The question of when this can happen is of fundamental interest, also related to the macroscopic observability of quantum plasma phenomena, and requires further studies.

We note, however, that even in those cases where the contribution from integrating over ω along the cuts in Fig. 4 is small or zero, the quantum kinetic effects nevertheless significantly affect the dispersion and damping of longitudinal oscillations. In the case of completely degenerate

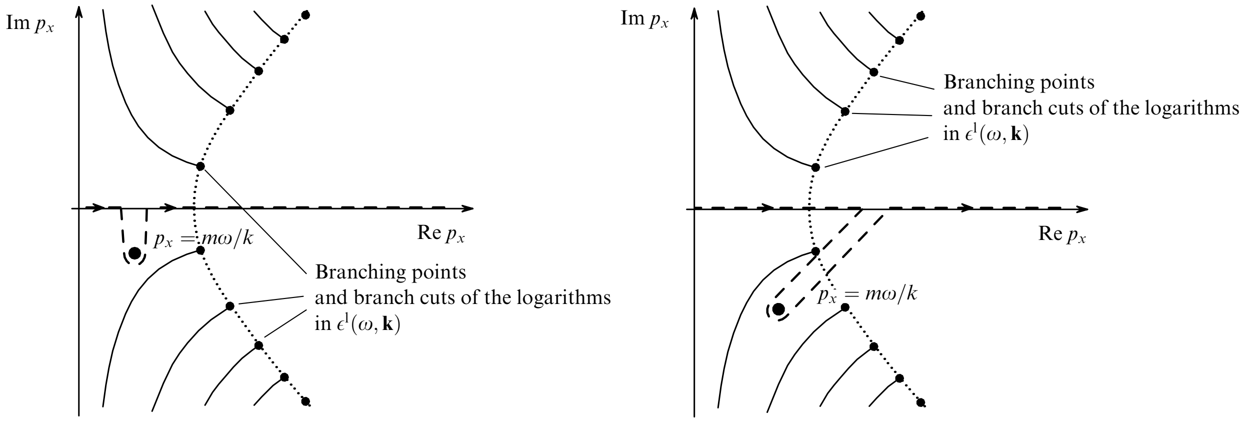

Figure 3. Singularities (branching points and branch cuts, shown by solid lines) of the logarithmic functions under the integral in (38), and the paths of integration over p_x (shown by dashed lines) for the analytic continuation of the function $\epsilon^{\mathrm{l}}(\omega, \mathbf{k})$ in the region Im $\omega \leqslant 0$, in two cases illustrating possible location of the pole $p_x = m\omega/k$ (shown by a bold dot) relative to the branch cuts. The dotted line shows the locus of the branching points (the hyperbole).

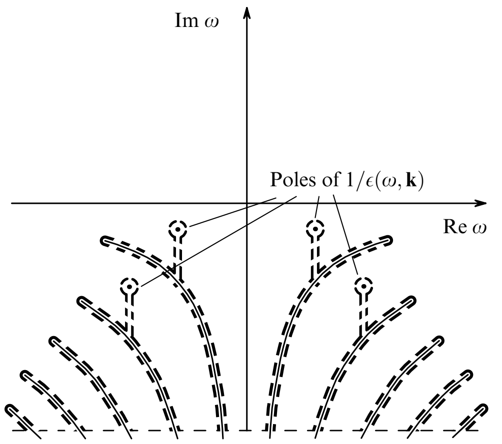

Figure 4. Singularities (branching points and branch cuts, shown by solid lines) of the function $\epsilon^{\mathrm{l}}(\omega, \mathbf{k})$ in the complex ω plane, and the path of integration (shown by the dashed line) over ω in Eqn (34), displaced in the lower half-plane to infinity, which bypasses the singularities of $\epsilon^{\mathrm{l}}(\omega, \mathbf{k})$ that are in the lower half-plane. Both the poles of the function $1/\epsilon^{\mathrm{l}}(\omega, \mathbf{k})$, as well as its branch cuts in the lower half-plane, contribute to the integral in (34).

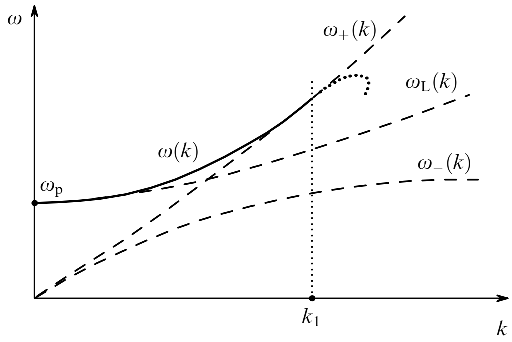

Figure 5. Dispersion of longitudinal oscillations of a degenerate ($T_{\mathrm{e}} = 0$) electron gas. Approximate solutions (39) and (40) are shown by dashed lines. Landau damping occurs for the part ($k > k_1$) of the dispersion curve shown by the dotted line.

($T_{\mathrm{e}} = 0$) Fermi electron distribution, equation $\epsilon^{\mathrm{l}}(\omega, \mathbf{k}) = 0$ yields the following approximate solutions at long wavelengths (note the typos in the signs in Ref. [48] in the corresponding dispersion relation of the 'kinetic mode'; see Eqn (40) below):

$$\omega_{\mathrm{L}}(k) = \left(\omega_{\mathrm{p}}^2 + \frac{3}{5}\,k^2 v_{\mathrm{F}}^2\right)^{1/2} \quad \text{for} \quad k \ll \frac{\omega_{\mathrm{p}}}{v_{\mathrm{F}}}\,, \tag{39}$$

$$\omega_{\pm}(k) = (k v_{\mathrm{F}} \pm \omega_k)\left[1 + \exp\left(-2 - 2k^2\lambda_{\mathrm{F}}^2\right)\right] \quad \text{for} \quad \frac{\omega_{\mathrm{p}}}{v_{\mathrm{F}}} \ll k \ll \frac{m v_{\mathrm{F}}}{\hbar}\,. \tag{40}$$

These limiting solutions are shown in Fig. 5.

For $k \ll \omega_{\mathrm{p}}/v_{\mathrm{F}}$, dispersion (39) of the longitudinal mode of collective oscillations corresponds to that of the usual Langmuir mode, which can also be obtained from the hydrodynamics model; see Eqn (16). Thus, we can also call this the 'hydrodynamic' mode. Yet, for $\omega_{\mathrm{p}}/v_{\mathrm{F}} \ll k \ll m v_{\mathrm{F}}/\hbar$, dispersion (40) of the longitudinal oscillation mode is mainly determined by the kinetic resonances $\omega - k v_{\mathrm{F}} \pm \omega_k = 0$ [see Eqn (25)] between plasmons and electrons whose velocities along $\mathbf{k}$ are equal to v_{F}. For those values of k for which the frequency of longitudinal oscillations defined by the standard Langmuir dispersion (39) becomes close to the kinetic resonance frequency $\omega = k v_{\mathrm{F}} \pm \omega_k$ for particles on the Fermi sphere (which occurs in the vicinity of the intersection of dispersion curve (39) with the resonance line $\omega = k v_{\mathrm{F}} + \omega_k$), the kinetic effects become dominant and hydrodynamic mode (39) 'switches' to kinetic mode (40). We also note that the quantum recoil leads to an imaginary part of the longitudinal dielectric permittivity in the degenerate case [46] for

$$k > k_1 \approx \frac{\omega_{\mathrm{p}}}{v_{\mathrm{F}}}\sqrt{\frac{3}{2}\left(|\ln\eta| - 1\right)}\,,$$

where $\eta = \hbar\omega_{\mathrm{p}}/4\varepsilon_{\mathrm{F}} \sim \sqrt{\Gamma_{\mathrm{q}}}$, which, in turn, leads to Landau damping of longitudinal oscillation modes (zero sound) with $k > k_1 \approx (\omega_{\mathrm{p}}/v_{\mathrm{F}})\sqrt{(3/2)(|\ln\eta| - 1)}$, first demonstrated in Ref. [46] (while Landau damping is zero for $k < k_1$). Thus, quantum kinetic effects play an important (and for certain wavenumbers, the dominant) role in the dispersion and damping of longitudinal collective oscillation modes in quantum plasmas.

6. Conclusion

To conclude, first of all we have analyzed in this paper the applicability limitations that follow from the basic assumptions of microscopic models of ideal quantum plasmas, i.e., the most widely used models in the recent literature. Lack of understanding of (or ignoring) these limitations can lead to an incorrect interpretation of results obtained with a particular model. For example, the description of longitudinal collective oscillations of a degenerate electron gas within the quantum hydrodynamics model yields dispersion relation (16) that is only valid in the long wavelength limit, $k\lambda_{\mathrm{F}} \ll 1$, which is not always stated explicitly. Moreover, the very reason for this limitation on the lengths of waves described by quantum hydrodynamics is not always clearly understood. Therefore, in Section 3, we have considered this problem and shown that this limitation appears in the quantum hydrodynamics approximation as a result of postulating a particular (adiabatic) equation of state for the 'classical' plasma pressure, and not as a result of postulating equation of state (7) for the 'quantum' plasma pressure.

We have also discussed the linear collective responses of quantum plasmas, and pointed out that it is conceptually incorrect to use the responses obtained from the nonrelativistic plasma models to describe waves with relativistic phase velocities, $\omega/k \gtrsim c$, regardless of whether the plasma itself is relativistic or not. Furthermore, we have pointed out that even for unmagnetized plasmas, ignoring the effects related to particle spins leads to an incorrect dielectric permittivity tensor, which, among other things, results in an incorrect dispersion relation for transverse waves in quantum plasmas — a not often mentioned but nevertheless important fact.

Finally, we have discussed the quantum kinetic effects associated with the nontrivial analytic properties of the complex linear plasma response functions, occurring both due to the quantum degeneracy of electron distribution and due to the quantum recoil. These effects are hardly discussed in the literature, with rare exceptions, while being of fundamental interest. In particular, the correct account of the analytic properties of quantum plasma linear response functions can significantly change the physical picture of the evolution of collective perturbations in both unbounded and bounded quantum plasmas (the Landau initial value and boundary value problems [40]).

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