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On the addition of velocities in the theory of relativity *

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Abstract. **Presented in translation from the German is Arnold Sommerfeld's paper "Über die Zusammensetzung der Geschwindigkeiten in der Relativtheorie", *Physikalische Zeitschrift* 10 826 (1909), which retains its relevance a hundred years after its publication, as is discussed by G B Malykin, *Usp. Fiz. Nauk* 180 965 (2010) [*Physics – Uspekhi* 53 (9) 923 (2010)].**

Minkowski has taught us to treat the Lorentz–Einstein transformation as 'rotation of space–time', i.e., as a transformation that takes the form of ordinary rotation, albeit not in the space xyz, but in the four-dimensional space $xyzl$, where $l = \mathrm{i}ct$ is also the length pertaining to the light path multiplied by imaginary unity. If a primed reference frame moves relative to an unprimed one with a constant velocity v in the direction of the x-axis, and if β denotes the ratio of velocities v/c, the transformation of coordinates is written as

$$\begin{aligned} x' &= x\cos\varphi + l\sin\varphi\,, \quad y' = y\,, \\ l' &= -x\sin\varphi + l\cos\varphi\,, \quad z' = z\,, \end{aligned} \tag{1}$$

whereas the imaginary rotation angle and the ratio of velocities are linked through the relationships

$$\tan\varphi = \mathrm{i}\beta\,, \quad \cos\varphi = \frac{1}{\sqrt{1-\beta^2}}\,, \quad \sin\varphi = \frac{\mathrm{i}\beta}{\sqrt{1-\beta^2}}\,. \tag{2}$$

Using a particular example, I would like to show how useful this analogy (or equality if one does the analytical treatment) between the rotations of space–time and ordinary rotations is for the kinematics of the theory of relativity.

If we perform two rotations with respect to one axis, or put another way, in the same rotation plane, the angles of rotation, and not their trigonometric functions, add up.

This also takes place for two translations in one and the same direction x (two rotations in space–time in the same plane xl); denoting (imaginary) angles of rotation as φ_1 and φ_2, the resulting angle of combined operation as φ, and the respective ratios of velocities v_1, v_2, and v to the speed of light as β_1, β_2, and β, we obtain

$$\varphi = \varphi_1 + \varphi_2\,,$$

and hence

$$\beta = \frac{1}{\mathrm{i}}\tan(\varphi_1+\varphi_2) = \frac{1}{\mathrm{i}}\,\frac{\tan\varphi_1+\tan\varphi_2}{1-\tan\varphi_1\tan\varphi_2} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\,,$$

or

$$v = \frac{v_1+v_2}{1+v_1 v_2/c^2}\,.$$

The last formula represents Einstein's famous velocity addition theorem; in Minkowski's interpretation it loses all its extraordinary character.

Two rotations in the same plane commute, i.e., their result is independent of the sequence of operations. The same holds true for two translations in the same direction because $\varphi_1 + \varphi_2 = \varphi_2 + \varphi_1$. Two rotations in different planes do not commute, nor do two translations in different directions. Apparently, the reason is that the first rotation will change the plane of the second rotation in the general case. Indeed, this occurs each time these planes do not coincide.

If we perform, for instance, the first rotation in the xl-plane through the angle φ_1 and then the second one in the plane yl with respect to the rotated system through the angle φ_2 defined in the rotated system, then, according to formula (1), it follows that

$$\begin{aligned} x_1 &= x\cos\varphi_1 + l\sin\varphi_1\,, \\ y_1 &= y\,, \\ l_1 &= -x\sin\varphi_1 + l\cos\varphi_1 \end{aligned}$$

and

$$\begin{aligned} x_2 &= x_1 = x\cos\varphi_1 + l\sin\varphi_1\,, \\ y_2 &= y_1\cos\varphi_2 + l_1\sin\varphi_2 = -x\sin\varphi_1\sin\varphi_2 + y\cos\varphi_2 \\ &\qquad + l\cos\varphi_1\sin\varphi_2\,, \\ l_2 &= -y_1\sin\varphi_2 + l_1\cos\varphi_2 = -x\sin\varphi_1\cos\varphi_2 - y\sin\varphi_2 \\ &\qquad + l\cos\varphi_1\cos\varphi_2\,. \end{aligned}$$

The point $x_2 = \text{const}$, $y_2 = \text{const}$ participating in the composite motion φ_1 and φ_2 also describes in the system xy a certain straight line, the direction of which is determined

* Arnold Sommerfeld's article "Über die Zusammensetzung der Geschwindigkeiten in der Relativtheorie" was sent by the author in 1909 simultaneously to two journals with the goal of reaching a wider scientific community, in full agreement with the practice accepted at that time, and was published almost simultaneously in *Physikalische Zeitschrift* [**10** 826 (1909)] and *Verhandlungen der Deutschen Physikalischen Gesselschaft* [**11** 557 (1909)].

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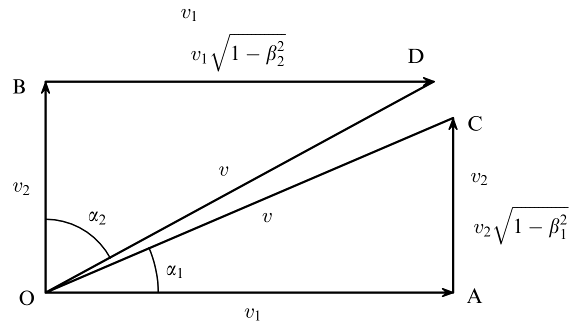

Figure 1.

from

$$0 = \mathrm{d}x\cos\varphi_1 + \mathrm{d}l\sin\varphi_1\,,$$
$$0 = -\mathrm{d}x\sin\varphi_1\sin\varphi_2 + \mathrm{d}y\cos\varphi_2 + \mathrm{d}l\cos\varphi_1\sin\varphi_2\,,$$

which leads to

$$\frac{\mathrm{d}x}{\mathrm{d}l} = -\tan\varphi_1\,,\qquad \frac{\mathrm{d}y}{\mathrm{d}l} = \tan\varphi_2\left(\frac{\mathrm{d}x}{\mathrm{d}l}\sin\varphi_1 - \cos\varphi_1\right) = -\frac{\tan\varphi_2}{\cos\varphi_1}\,, \tag{3}$$

or, according to formulas (2), to

$$\frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t} = \beta_1\,,\qquad \frac{1}{c}\frac{\mathrm{d}y}{\mathrm{d}t} = \beta_2\sqrt{1-\beta_1^2}\,. \tag{4a}$$

In contrast, if operations are performed in the reverse order, the result will be as follows

$$\frac{1}{c}\frac{\mathrm{d}y}{\mathrm{d}t} = \beta_2\,,\qquad \frac{1}{c}\frac{\mathrm{d}x}{\mathrm{d}t} = \beta_1\sqrt{1-\beta_2^2}\,. \tag{4b}$$

Let us place a right-angled ruler in the plane of Fig. 1 so that its sides coincide initially with OA and OB. We shall draw a line along the ruler side OB moving a pencil with the velocity of $v_2 = \beta_2 c$ and simultaneously displacing the ruler in the direction of its other side OA with the velocity $v_1 = \beta_1 c$. In this case, the pencil lead will describe a trajectory which differs from the one obtained if we move the pencil lead with the velocity v_1 along OA, while displacing the ruler with the velocity v_2 in the direction OB. The difference between both paths per unit time (OC in the first case, and OD in the second) is a small quantity of the 2nd-order if β_1 and β_2 are of the first order. The reason is that in the first case the velocity v_2 is estimated differently in the moving system (the ruler), because of the dependence of the notion of time on motion, than in the rest reference frame (the plane of Fig. 1); the same is true of the velocity v_1 in the second case. It is this situation that was alluded to above by arguing that in the space xyl the plane of the second rotation will be displaced after performing the space–time rotation. The velocities AC and BD are depicted in Fig. 1 so that they are seen with respect to the rest figure plane; the upper quantity given there pertains to the velocity seen from the ruler, and the lower one corresponds to the velocity determined with respect to the rest figure plane.

For the combined velocity $v = \beta c$ determined with respect to the rest reference frame, Eqns (4a) and (4b) give the same result:

$$\beta^2 = \frac{1}{c^2}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right] = \beta_1^2 + \beta_2^2 - \beta_1^2\beta_2^2\,,$$

or

$$1 - \beta^2 = (1-\beta_1^2)(1-\beta_2^2)\,,$$

or, if we introduce, in addition to φ_1 and φ_2, the resultant angle φ, one will find with account for formulas (2):

$$\cos\varphi = \cos\varphi_1\cos\varphi_2\,. \tag{5}$$

If α_1 and α_2 represent the slopes of trajectories presented in the figure, it follows from formulas (3) that

$$\tan\alpha_1 = \frac{\tan\varphi_2}{\sin\varphi_1}$$

and, respectively,

$$\tan\alpha_2 = \frac{\tan\varphi_1}{\sin\varphi_2}\,, \tag{6}$$

while the following inequality is always satisfied:

$$\alpha_1 + \alpha_2 < \frac{\pi}{2}\,. \tag{7}$$

Although they seem somewhat strange at first glance, these results also become transparent if one departs from the Minkowski viewpoint. Indeed, if we identify the rotation angles φ_1 and φ_2 with arcs on a unit sphere (Fig. 2) so that the vertex angle A of a triangle OAC, and B of a triangle OBD are right angles, the resultant angle $\mathrm{OC} = \mathrm{OD} = \varphi$ follows directly from the law of cosines as the hypotenuse of congruent spherical triangles, in agreement with equality (5). The fact that the resultant rotation plane depends on the sequence of two rotations is apparent from Fig. 2. Indeed, the great circle passing through B perpendicular to OB does not, obviously, pass through C, but intersects AC. The so-called Napier's rule for right-angled spherical triangles gives formula (6) for the angles α_1 and α_2 with catheti. The sum of the angles in a spherical triangle exceeds two right angles, and the sum of the angles with catheti in a right-angled spherical triangle exceeds the right angle, as seen from Fig. 2. Moreover, the difference (spherical excess) is equal to the area of a triangle on a sphere of radius one. The sides of our spherical triangles in the case of space–time rotations are purely imaginary, so that their area is negative.

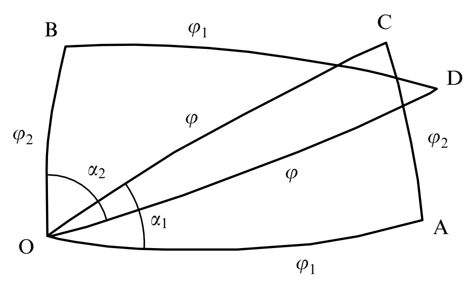

Figure 2.

The spherical excess transforms to the spherical defect; this underlies inequality (7) and the departure of trajectories OC and OD in Fig. 1.[1]

Summing up, we can argue that in order to add velocities in the theory of relativity one should apply formulas of spherical, not planar, trigonometry (with imaginary sides). Bearing this statement in mind, the cumbersome algebra of transformations, an example of which is given above, becomes redundant and can be replaced by a transparent construction on a sphere. We shall provide one more example.

If velocities v_1 and v_2 are at arbitrary angle α with respect to each other, the outer angle A in spherical triangle OAC in Fig. 2 is also α, and from the law of cosines for spherical geometry it follows for the resultant angle φ that

$$\cos\varphi = \cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 \cos\alpha\,. \qquad (8)$$

Should we rewrite this expression in terms of velocities v_1, v_2, and v, then according to Eqn (2) the result is

$$\frac{1}{\sqrt{1-\beta^2}} = \frac{1+\beta_1\beta_2\cos\alpha}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}\,,$$
$$v_2 = \frac{v_1^2 + v_2^2 + 2v_1v_2\cos\alpha - (1/c^2)v_1^2v_2^2\sin^2\alpha}{\left(1+(1/c^2)v_1v_2\cos\alpha\right)^2}\,, \qquad (9)$$

the formula already derived by Einstein [1] from the transformation relationships. As can be seen, formula (8) is more transparent than formula (9), and in the same fashion Fig. 2 turns out to be more helpful than Fig. 1. This is explained by the fact that consideration in terms of rotation angles and (in real circumstances) respective constructions better reflect the sense of the theory of relativity than operations in terms of projections on the tangent plane comprising velocities v.

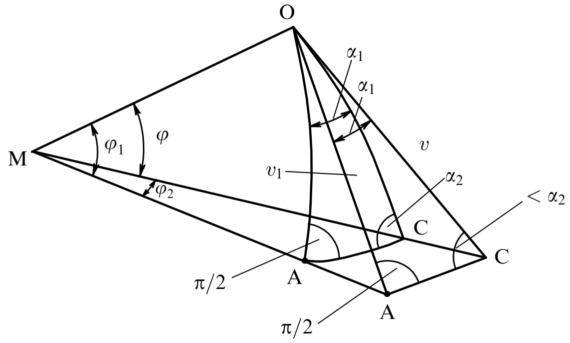

Figure 3.

The only goal of this short note was to demonstrate that Minkowski's profound viewpoint on space–time not only facilitates the basic construction of the theory of relativity in the methodological sense, but also serves as a convenient principle in considering special questions.

Reference

1. Einstein A *Jahrbuch Radioaktivität Elektronik* **411** (1908) p. 423

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[1] The following may serve as an explanation of the relationship between the figures. Figure 1 is obtained when we perform a central projection of Fig. 2 from the center of sphere M on a tangent plane drawn through the point of contact O. In this case, vertex angles at O and right angles at A and B are preserved, and sides φ_1, φ_2, and φ issuing from O are replaced by their projections on the tangent plane [1*] and, respectively, by velocities v_1, v_2, and v proportional to them. In contrast, vertex angles at C and D will be modified by the projection (they are equal to α_2 and α_1, respectively, in Fig. 1 but the respective angles are larger than α_2 and α_1), and sides CA and BD cannot be simply replaced by their tangents. Figure 3 shows how this projection is obtained in reality for triangle OAC.[2*]

[1*] The word 'Tangenten', the German for tangents, is translated here as 'projections on the tangent plane' as namely they are implied. (*Translator's comment*).

[2*] Formulas (5) and (6) are valid for real-valued angles for right triangles on the surface of the sphere drawn in Figs 2 and 3. In that case, one has

$$\tan\alpha_1 \tan\alpha_2 = \frac{1}{\cos\varphi_1\cos\varphi_2}\,.$$

In order to change to right triangles on the surface of the pseudosphere presented in Fig. 1, the angles φ_1, φ_2, and φ should be made purely imaginary. In that case, the right hand side of the equality just written will be less than one and, consequently,

$$\sin\alpha_1 \sin\alpha_2 < \cos\alpha_1\cos\alpha_2\,,$$

i.e., $\cos(\alpha_1+\alpha_2) > 0$, which leads to formula (7): $\alpha_1+\alpha_2 < \pi/2$.

The vertex angles C and D of these triangles in Fig. 1 (we denote them as α_2' and α_1', respectively) satisfy, obviously, equalities

$$\alpha_1 + \alpha_2' = \frac{\pi}{2}\,, \qquad \alpha_1' + \alpha_2 = \frac{\pi}{2}\,,$$

which lead to $\alpha_2' > \alpha_2$ and $\alpha_1' > \alpha_1$ when compared to formula (7), which is what Sommerfeld argues in footnote 1. (*Comment by V I Ritus*)