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Noncommutative nature of the addition of noncollinear velocities in special relativity and the geometric phase method

(commemorating the publication centennial of A Sommerfeld's work)*

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Contents

Abstract. In 1909, Arnold Sommerfeld used geometric calculations to show that the relativistic addition of two noncollinear velocities on an imaginary-radius sphere is a noncommutative operation. Sommerfeld was the first to use the geometric phase method to calculate the angle between the resulting velocities depending on the order in which they are added. For this, he related the value of this angle to the excess of the spherical triangle formed by the two original velocities and their sum. In 1931, Sommerfeld applied his method to analyze the Thomas precession.

1. Introduction

In 1909, Arnold Sommerfeld (1968–1951) disclosed that the operation of the addition of relativistic velocities is noncommutative [1, 2]. More than twenty years later, he returned to this question [3] and showed that the method developed by him in papers [1, 2] allows computing the magnitude of the Thomas precession (TP) [4–11]. The results produced in Refs [1–3] were considered earlier in reviews [12, 13]; however, their main focus was on the mathematical side of the question.

* This article was written to commemorate the publication centennial of A Sommerfeld's paper "Über die Zusammensetzung der Geschwindigkeiten in der Relativtheorie" (On the addition of velocities in relativity theory) in *Physikalische Zeitschrift* [1].

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The goal of this paper is to consider studies [1–3] from the physical viewpoint and in particular demonstrate that Sommerfeld was the first to apply the method of geometrical phase (GP) (the topological phase sometimes referred to as the Berry phase) [14–23] to compute the magnitude of relativistic rotation of the velocity of a physical body or the spin of a material particle, and was the second, after William Rowan Hamilton (1805–1865) [24], to apply this computing technique. As far as we know, Sommerfeld's role in creating the GP method has never been discussed before.

2. The relativistic law of adding velocities (1900–1908)

In the initial development phase of the special theory of relativity (STR), the main efforts of researchers were devoted to formulating the transformations that would connect three spatial coordinates and time in a rest inertial reference frame (IRF) K, wherein the observer is stationed, with those in an IRF K' moving with velocity v; the relativistic law of addition of velocities received much less attention. The exception was the monograph by Joseph Larmor (1857–1942) [25] published in 1900, but its author was limited to considering the addition of collinear velocities in the limit $v \ll c$.

The fundamental work [26] by Albert Einstein (1879–1955) issued in 1905 proposed for the first time an expression for the addition of the body velocity $\mathbf{u}$ with the velocity $\mathbf{v}$ of the IRF for an arbitrary angle between $\mathbf{u}$ and $\mathbf{v}$. Einstein devoted a special section (§ 5) in Ref. [26] to this question, where he notes, "Thus the law of velocity parallelogram in our theory holds only in the first approximation." In 1906, this question was also addressed by Henri Poincaré (1854–1912) [27].

Applications of Einstein's relativistic law of adding velocities [26] followed practically immediately: already in 1908, Jakob Laub (1872–1962) derived an expression for the drag coefficient when an optical medium executes a translatory motion, and in doing so its boundaries are not resting but move, in contrast to those in the experiment by Fizeau [28]. (For details on the Laub drag coefficient, see Ref. [29].)

3. Sommerfeld's first study (1909)

The most interesting corollaries from the results of Ref. [26] were obtained in 1909 by Sommerfeld [1, 2], who considered the general case of adding noncollinear velocities. The result proved to be so important that Sommerfeld submitted his work to two German scientific journals at once: it reached the editorial office of *Physicalische Zeitschrift* on 30 September 1909, and that of *Verhandlungen der Deutschen Physicalischen Gesellschaft* on 21 October. Both journals published it before the end of 1909. In order to compute the absolute value and the direction of the sum of two orthogonal velocities, Sommerfeld made use of an original mathematical approach — the geometrical interpretation of Lorentz transformations (LTs) in terms of rotations through imaginary angles. The mathematical side of Refs [1, 2] was analyzed in sufficient detail in Refs [12, 13]; we give in this paper only a brief explanation of the Sommerfeld method. Instead of two velocities v_1 and v_2 and their relativistic sum v_3, Refs [1, 2] proposed considering three imaginary angles φ_1, φ_2, and φ_3, where $\tan\varphi_{1,2,3} = \mathrm{i}(v_{1,2,3}/c)$, c is the speed of light in vacuum, and $\mathrm{i} = \sqrt{-1}$. In this way, any LT with velocity v is associated with the rotation through the angle φ, while the length of the corresponding segment of a great-circle arc on a sphere with a unit imaginary radius is numerically equal to φ. Then, the orientation of the arc depends on the direction of $\mathbf{v}$ in the real space (or in the Minkowski space).[1] Sommerfeld showed that $\varphi_3 = \varphi_1 + \varphi_2$ only if v_1 and v_2 are collinear. In the case when v_1 and v_2 are orthogonal, one finds $\cos\varphi_3 = \cos\varphi_1 + \cos\varphi_2$, as follows from spherical trigonometry [31], with the direction of total velocity v_3 being dependent on the sequence of adding v_1 and v_2 [1, 2]. As a consequence, two rotations in the same plane of Minkowski space (corresponding to the addition of two collinear velocities) commute with each other, while two rotations in different planes of Minkowski space (corresponding to the addition of noncollinear velocities) does not do so [1, 2]. In particular, if one performs two LTs in sequence, first in the direction of the x-axis and then in the direction of the y-axis, one arrives at a certain result, but if the same transformations were performed in the reversed order, first in the direction of the y-axis and then in the direction of the x-axis, the result would differ from the other one. The absolute values of the resulting velocities will coincide in both cases, but their directions will be distinct.

In our opinion, the best interpretation of this phenomenon was suggested in a paper by V I Ritus (b. 1927) [11]: "Asymmetry of the relativistic law of adding two noncollinear velocities with respect to their permutation leads to two modified triangles which depict on an Euclidean plane the addition of nonpermuted and permuted velocities and the appearance of nonzero angle between the two resulting velocities. The particle spin turns over the same angle as well if the particle velocity is changed by a Lorentz boost with a velocity not aligned with that of particle." Hence, it becomes apparent that the phenomenon of spin rotation accompanying the curvilinear motion of a material particle — the Thomas precession (TP) [4–11] — follows from the results of Sommerfeld's work [1, 2].

Sommerfeld also revealed [1, 2] that the sum of angles φ_1, φ_2, and φ_3 of a spherical triangle formed by arcs of a great circle is less than π since the spherical excess (the positive difference between the sum of the angles of a spherical triangle and π) [31] on a sphere with an imaginary radius is negative. Moreover, Sommerfeld demonstrated that in Euclidean space the angle between the velocities $\mathbf{v}_3^{\mathrm{a}}$ and $\mathbf{v}_3^{\mathrm{b}}$ ($\mathbf{v}_3^{\mathrm{a}} = \mathbf{v}_1 \oplus \mathbf{v}_2$, $\mathbf{v}_3^{\mathrm{b}} = \mathbf{v}_2 \oplus \mathbf{v}_1$, where the operator $\oplus$ denotes the relativistic addition of velocity vectors) for two LTs with orthogonal velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, carried out in a different sequence, is numerically equal to the spherical excess taken with the opposite sign [1, 2]. Since the spherical excess of a triangle is numerically equal to the area of this triangle on a sphere of unit radius [31], it is obvious that this angle represents a characteristic manifestation of the GP in STR. At the present time, manifestations of the GP are being discovered in various branches of physics, such as classical and quantum mechanics, polarization optics, and some others [14–23]. A characteristic feature of the GP is that the most convenient way of evaluating it consists in computing the area confined by a closed curve on a sphere, describing the evolution of the state of a certain system parameter, in particular, the area of an appropriate spherical triangle, for instance, in the velocity space or on the Poincaré sphere [14–23]. It is noteworthy that the angle between the directions of total velocities can be computed by parallel translation of the velocity vector on a sphere of unit imaginary radius along arcs φ_1 and φ_2, and in the reverse order along φ_2 and φ_1.

Thus, as early as 1909 Sommerfeld applied the GP method [1, 2] to compute the relativistic rotation of the velocity of a body, resulting from two orthogonal LTs, and developed the mathematical apparatus to compute the magnitude of TP.

4. Further application of the methods of non-Euclidean geometry in special relativity. Studies by É Borel, and L Föppl and P Daniell (1910–1914)

A period of rapid development of the methods of non-Euclidean geometry in STR [13] followed the publication of Sommerfeld's studies [1, 2]. By way of example, we can mention the work by Sommerfeld himself [32, 33], V Varičak (1865–1942) [34–37], M von Laue (1879–1960) [38], E B Wilson (1879–1964) and G N Lewis (1875–1945) [39], A A Robb (1873–1936) [40], É Borel (1871–1956) [41–43], K Ogura (1885–1962) [44], L Föppl (1887–1976) and P Daniell (1889–1946) [45], and L Silberstein (1872–1948) [46]. Studies [34–43, 45, 46] were addressed in sufficient detail in Ref. [13].[2] The works by É Borel [41–43], and L Föppl and P Daniell [45] are the most interesting.

Borel, who was aware of the results of Sommerfeld's work [1], used the Lobachevskian geometry instead of a sphere with

[1] Sommerfeld mentioned later that the spherical geometry with imaginary arcs is equivalent to the planar Lobachevskian geometry [3]. Sommerfeld was the second, after Minkowski [30], to apply non-Euclidean geometry in STR [13].

[2] In particular, Sommerfeld's studies [32, 33] deal with vector analysis in the Minkowski space; V Varičak [34–37] was the first to pass from the Sommerfeld trigonometry on a sphere with a unit imaginary radius [1, 2] to the Lobachevskian geometry, well known at that time (see also Refs [12, 13]), while E B Wilson and G N Lewis considered the rotation of vectors and planes in non-Euclidean space [39].

a unit imaginary radius. In Ref. [41], Borel considered relativistic kinematic rotation which was later termed the TP, proposed an intuitive physical explanation for it, and obtained an approximate expression for the TP angular velocity in the limit $v \ll c$. Had Borel not worked in this limit, he could have obtained the expression for the TP in the general case. Later on, Ya A Smorodinskii (1917–1992) derived the correct expression for the TP [47, 48] (see also Ref. [9]) using the Lobachevskian geometry.

The work by Föppl and Daniell [45], which considers relativistic rotation of a rigid body exerting circular motion, as well as more recent work [49, 50], used the parallel translation method for conical motion in the Minkowski space to compute the relativistic rotation of an electron (i.e., the authors in fact made use of the well-known solid angle theorem [24, 51]). In essence, if a body exerts circular motion in the (x, y) plane of physical space, its world line in the Minkowski space looks like a helicoidal spiral around the ict-axis. Using this approach, the authors of Ref. [45] obtained values larger by the factor γ for the angular frequency of the TP [6–11]. As shown in Ref. [9], the error committed by the authors of Refs [45, 49, 50] stems from the fact that the world line does not correspond to the trajectory of actual body motion, so that making use of the solid angle theorem is not valid in the case of considering body's rotation.

5. Sommerfeld's second study (1931)

After the discovery of electron spin by S A Goudsmit (1902–1979) and G E Uhlenbeck (1900–1974) in 1925 [52] the derivation of the expression for the relativistic precession of electron spin (TP) became of interest. The Borel studies [41–43] had been forgotten by that time. In 1926–1927, L H Thomas (1903–1992) obtained an expression for the TP [4, 5] with the help of Lorentz transformations, which, as became apparent later [9], was in error: he (in the same manner as Föppl and Daniell [45]) overestimated the angular frequency of the TP by a factor of γ.

In 1931, in the fifth edition of monograph [3] as well as in a talk at a conference in Rome [53] Sommerfeld suggested the derivation of the formula for TP using his own method [1, 2]. Let us consider expression (27) from his monograph [3, Chapter 12]:

$$\sin\theta = \frac{v_1 v_2}{c^2\left(1 + \sqrt{1 - v_1^2/c^2}\,\sqrt{1 - v_2^2/c^2}\,\right)}\,, \tag{1}$$

where θ is the angle of electron spin rotation pertaining to the TP, v_1 is the orbital electron velocity, and v_2 is the variation of the electron velocity over an infinitesimal time under the action of centripetal force (in this case, the Coulomb force due to the positively charged atomic nucleus). [3]

Consider the simplest but most interesting case of the circular motion of an electron at a particular orbital velocity ($v_1 \leqslant c$). Then, $v_1 = R\omega$ (where R is the radius of the electron orbit, and ω is the angular velocity of orbital motion), and $v_2 = R\omega^2\,\mathrm{d}t$ is the velocity increment for an infinitely small time interval $\mathrm{d}t$. The angular velocity of the TP in the IRF connected with a resting observer is defined as $\Omega_{\mathrm{T}} = \mathrm{d}\theta/\mathrm{d}t$ [9]. Taking the derivatives of both sides of expression (1) with respect to time, and noticing that $\mathrm{d}\sin\theta/\mathrm{d}t = \cos\theta\,\mathrm{d}\theta/\mathrm{d}t$, one obtains a rather cumbersome expression for Ω_{T} containing constant terms and terms proportional to $(\mathrm{d}t)^2$. The latter can be omitted as infinitesimally small quantities of second order. In this case, the expression for the TP takes the form

$$\Omega_{\mathrm{T}} = (1 - \gamma^{-1})\,\omega\,, \tag{2}$$

which is the correct result for the TP in the laboratory IRF [6–11]. As far as we know, the Sommerfeld method [1–3] had not been applied earlier to derive expression (2). In particular, from Eqn (2) it follows that for $v = c$ the spin of an electron exerting circular motion will accomplish a single turn for each orbital turn of the electron. Since expression (1) directly follows from the results of Refs [1, 2], formula (2) could have been obtained back in 1909. Admittedly, since Sommerfeld was interested in 1931 in the motion of electrons in an atom, which is characterized by fairly small velocities ($v \ll c$), he limited the analysis to respective expansion in v/c.

As shown in Ref. [9], researchers long used the erroneous Thomas expression for the TP [5]. In particular, C Møller (1904–1980) obtained an expression for the TP in his monograph [54] in 1952, which coincides with that of Thomas [5] up to the sign. Only in 1961 was the well-known work by Ritus [55] published, showing that for a massless particle moving along a curved trajectory with the speed c the direction of its spin will always coincide with that of the particle velocity, while the spin direction of a usual particle moving with the velocity $v < c$ will always lag behind that of its velocity. It follows from the results of Ref. [55] that the spin of a mass-less particle moving along a circular trajectory will make a rotation for each orbital turn of the particle relative to the laboratory IRF. Expression (2) can be obtained from the results of Ref. [55] after little manipulations.

Soon after that followed the monograph by J D Jackson (b. 1925) [56] and articles by Ya A Smorodinskii [47, 48] and A Chakrabarti (b. 1928) [57], where expression (2) was already written explicitly. Despite this, a considerable number of studies were published later where, as shown in Ref. [9], various incorrect expressions for the TP were used.

In 2007, V I Ritus in an interesting and important study [10] gave a conclusive solution to the question as to which of the expressions for the TP is the correct one, and, in particular, pointed to an error in the analysis made by Møller [54]. In the next study [11], Ritus considered the question of the relativistic addition law for noncollinear velocities in the most general case. References [10, 11] solve the problem of the Thomas precession for the relativistic rotation of a rigid body or the spin of a single material particle in a complete form.

6. Sommerfeld's role in creating geometric phase methods

Since Sommerfeld's studies [1–3] occupy a prominent place in the development history of GP methods, which are now widely used in polarization optics, classical and quantum mechanics, and other branches of physics [14–23], we shall briefly highlight the milestones of the creation of the GP methods.

(1) 1853: Hamilton formulated the so-called solid angle theorem [24], which states that if an axis linked to a rigid body

[3] As demonstrated in Ref. [3], one has $\theta = \pi/2 - \alpha_1 - \alpha_2$, where $\alpha_{1,2}$ are the angles at two vertices of a spherical triangle composed by the great-circle arcs φ_1, φ_2, and φ_3. Since the velocities v_1 and v_2 are in this case orthogonal to each other, the arcs φ_1 and φ_2 are orthogonal too, and, consequently, $\alpha_3 = \pi/2$. Then, $\theta = \pi - \alpha_1 - \alpha_2 - \alpha_3$ is the spherical excess of the triangle, taken with the opposite sign [1–3, 31].

describes a closed trajectory in the process of conical motion, the body will turn around this axis over an angle numerically equal to the solid angle described by the axis. The proof of this theorem, based on the theory of quaternion multiplication, occupies 140 pages of his monograph [24] (for details, see Ref. [22]).

(2) 1909: Sommerfeld demonstrated that the angle between vectors of the sum of two mutually orthogonal velocities combined in a different sequence is numerically equal to the excess of a triangle on a sphere with unit imaginary radius, taken with the opposite sign [1, 2].

(3) 1913: Borel derived an approximate expression for the TP using the parallel translation method for a body velocity vector on the Lobachevsky plane [41]. In this same year, Föppl and Daniell attempted to obtain an expression for the TP by applying parallel translation of the axis connected with the body in the Minkowski space [45], but made an error.

(4) 1931: Sommerfeld obtained an approximate expression for the TP of electron spin by connecting it to the excess of a triangle on a sphere with unit imaginary radius [3, 53].

(5) 1938–1941: S M Rytov (1908–1996) [58] and V V Vladimirskii (b. 1915) [59] linked the rotation of a polarization plane of light propagating along a nonplanar trajectory with a solid angle described by the tangent to the ray trajectory (for details, see Refs [22, 23]).

(6) 1942: A D Galanin (1916–2000) obtained an approximate expression for the TP of the spins of an electron and meson by passing from the wave equation to equations of geometrical optics through the expansion in powers of a small parameter [60], which allowed him further to exploit the method employed in Refs [58, 59] (see Ref. [22] for details).

(7) 1944–1952: A Yu Ishlinskii (1913–2003) proved the solid angle theorem by making use of the parallel translation method in three-dimensional Euclidean space, and did it not only for a rigid body, but also for a gyrocompass [51, 61]. Ishlinskii's proof, based on Green's function technique, is very concise (for details, see Ref. [22]).

(8) 1956: S Pancharatnam (1934–1969) related the magnitude of an additional optical phase appearing in the process of cyclic evolution of the light polarization state to the area on the Poincaré sphere described by the point which maps the polarization state [62, 63] (for details, see Refs [14, 23]).

(9) 1984: M Berry (b. 1941) wrote an expression for the GP in quantum mechanics (Berry phase) [64] (for details, see Refs [14–18]).

The work by Berry [64] triggered the publication of an immense number of papers exploring various manifestations of the GP. In particular, the connection was established between the phase difference coming from polarization nonreciprocity [65] (and not linked to rotation) at the output of a fiber ring interferometer (FRI) and the area of a triangle on the Poincaré sphere, defined by the light polarization state at the entrance to and both exits from the FRI [19–21, 23]. It was also successfully shown that the angle of a rigid body rotation which occurs because of the TP as the body moves along a curvilinear trajectory is numerically equal to the solid angle, observed in a rest reference frame, which is described by an axis linked to the body as a consequence of relativistic aberration — the change in the tilt of the body image pertaining to the Lorentz length contraction and retardation of light emitted by various parts of the body [7–9].

7. Conclusions

Sommerfeld was the second, after Hamilton, to employ in 1909 the geometric phase method and the first to apply this method to computations of the relativistic sum of noncollinear velocities, and to demonstrate that this operation is noncommutative [1, 2]. In 1931, Sommerfeld used the GP method to compute the magnitude of the TP [3]. It is worth mentioning that the results of Refs [1, 2] allowed deriving the correct expression for the TP already in 1909. The technique of computation on a sphere with an imaginary radius, proposed by Sommerfeld, proved to be essentially simpler and physically more apparent than the method based on the Lobachevskian geometry [13].

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