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On the invariant form of the wave and motion equations for a charged point mass[1]

V A Fock

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Abstract. **The Schrödinger wave equation and the equations of motion are written as an invariant Laplace equation and the equation for a geodesic in five-dimensional space, respectively. The superfluous fifth coordinate is closely related to the linear differential form of the electromagnetic potential.**

H Mandel in his yet unpublished work[2] uses the notion of five-dimensional space in order to consider gravity and the electromagnetic field from a unified standpoint. It seems to us that the introduction of a fifth coordinate facilitates representation of the Schrödinger wave equation and the equation of motion in an invariant form.

1. Special relativity

The Lagrange function corresponding to the motion of a charged point mass can be written, using the readily understandable notation, as

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} + \frac{e}{c}\,\mathfrak{A}\mathbf{v} - e\varphi\,, \tag{1}$$

and the respective Hamilton–Jacobi equation can be written as

$$(\mathrm{grad}\,W)^2 - \frac{1}{c^2}\left(\frac{\partial W}{\partial t}\right)^2 - \frac{2e}{c}\left(\mathfrak{A}\,\mathrm{grad}\,W + \frac{\varphi}{c}\frac{\partial W}{\partial t}\right) + m^2c^2 + \frac{e^2}{c^2}(\mathfrak{A}^2 - \varphi^2) = 0\,. \tag{2}$$

In analogy with the ansatz used in our earlier work,[3] we assume here that

$$\mathrm{grad}\,W = \frac{\mathrm{grad}\,\psi}{\partial\psi/\partial p}\,, \qquad \frac{\partial W}{\partial t} = \frac{\partial\psi/\partial t}{\partial\psi/\partial p}\,, \tag{3}$$

where p denotes some new parameter with the dimension of the quantum of action. Having multiplied by $(\partial\psi/\partial p)^2$, we obtain the quadratic form

$$Q = (\mathrm{grad}\,\psi)^2 - \frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2 - \frac{2e}{c}\frac{\partial\psi}{\partial p}\left(\mathfrak{A}\,\mathrm{grad}\,\psi + \frac{\varphi}{c}\frac{\partial\psi}{\partial t}\right) + \left[m^2c^2 + \frac{e^2}{c^2}(\mathfrak{A}^2 - \varphi^2)\right]\left(\frac{\partial\psi}{\partial p}\right)^2. \tag{4}$$

We note that the coefficients at the zeroth, first, and second powers of $\partial\psi/\partial p$ are four-dimensional invariants. Besides, the form Q remains invariant under the transformation

$$\begin{aligned} \mathfrak{A} &= \mathfrak{A}_1 + \mathrm{grad}\,f\,,\\ \varphi &= \varphi_1 - \frac{1}{c}\frac{\partial f}{\partial t}\,,\\ p &= p_1 - \frac{e}{c}\,f\,, \end{aligned} \tag{5}$$

where f is an arbitrary function of coordinates and time. The last transformation also leaves the linear differential form

$$\mathrm{d}'\Omega = \frac{e}{mc^2}(\mathfrak{A}_x\,\mathrm{d}x + \mathfrak{A}_y\,\mathrm{d}y + \mathfrak{A}_z\,\mathrm{d}z) - \frac{e}{mc}\,\varphi\,\mathrm{d}t + \frac{1}{mc}\,\mathrm{d}p \tag{6}$$

invariant.[4]

We now express the form Q as a squared gradient of ψ in a five-dimensional space (R_5) and find the respective interval. It readily follows that

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - c^2\,\mathrm{d}t^2 + (\mathrm{d}'\Omega)^2\,. \tag{7}$$

Laplace's equation in R_5 is written as

$$\Delta\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{2e}{c}\left(\mathfrak{A}\,\mathrm{grad}\,\frac{\partial\psi}{\partial p} + \frac{\varphi}{c}\frac{\partial^2\psi}{\partial t\,\partial p}\right) - \frac{e}{c}\frac{\partial\psi}{\partial p}\left(\mathrm{div}\,\mathfrak{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t}\right) + \left[m^2c^2 + \frac{e^2}{c^2}(\mathfrak{A}^2 - \varphi^2)\right]\frac{\partial^2\psi}{\partial p^2} = 0\,. \tag{8}$$

[1] The idea of this work grew from a discussion with Prof. V. Frederiks, to whom I am also indebted for some valuable suggestions.

A note in proofreading. When this note was already in press, an elegant work by Oskar Klein (*Z. Phys.* **37** 895 (1926)) [2] reached Leningrad. Its author had obtained results that are in principle identical to those of the current work. However, owing to the importance of the results, their derivation, carried out in a different way (the generalization of the substitution used in one of my previous studies), can also be of interest.

[2] The author has kindly provided me with the opportunity of reading the manuscript of his work.

[3] V Fock "Zur Schrödingerschen Wellenmechanik" *Z. Phys.* **37** 242 (1926) [4].

[4] The notation d' serves to emphasize that $\mathrm{d}'\Omega$ is not a full differential.

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Similarly to Eqns (7) and (4), it remains invariant under Lorentz transformations and the transformation given by Eqns (5).

Because the coefficients in Eqn (8) are independent of the parameter p, we can choose the dependence of ψ on p in an exponential form and write

$$\psi = \psi_0 \exp\left(2\pi\mathrm{i}\,\frac{p}{h}\right) \tag{9}$$

for consistency with observations.[5] The equation for ψ_0 is invariant under Lorentz transformations, but not transformation (5). The role of the additional fifth coordinate parameter p is precisely to ensure the invariance of the equations under the addition of an arbitrary gradient to the four-potential.

It is worth noting here that the coefficients of the equation for ψ_0 are complex-valued in the general case.

If we further assume that these coefficients are independent of t and use the representation

$$\psi_0 = \exp\left[-\frac{2\pi\mathrm{i}}{h}(E + mc^2)t\right]\psi_1\,, \tag{10}$$

then ψ_1 obeys an equation that does not contain time and is identical to the generalized Schrödinger equation proposed in our earlier work. Those values of E for which there exists a function ψ_1 satisfying certain conditions of boundedness and continuity are the Bohr energy levels. It follows from the foregoing that adding a gradient to the four-potential cannot influence the energy levels. The functions ψ_1 and $\bar{\psi}_1$ found for the four-potentials $\mathfrak{A}$ and $\bar{\mathfrak{A}} = \mathfrak{A} - \operatorname{grad} f$ differ only by the factor $\exp\left[2\pi\mathrm{i}ef/(ch)\right]$ with the absolute value 1, and consequently (under very general requirements on the function f) have the same continuity properties.

2. General relativity

A. Wave equation. For the squared interval in a five-dimensional space, we write

$$\left.\begin{aligned} \mathrm{d}s^2 &= \sum_{i,k=1}^{5} \gamma_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k \\ &= \sum_{i,k=1}^{4} g_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k + \frac{e^2}{m^2}\left(\sum_{i=1}^{5} q_i\,\mathrm{d}x_i\right)^2. \end{aligned}\right\} \tag{11}$$

Here, the quantities g_{ik} are the components of the Einstein fundamental tensor, the quantities q_1 $(i = 1, 2, 3, 4)$ are the components of the four-potential divided by c^2, i.e.,

$$\sum_{i=1}^{4} q_i\,\mathrm{d}x_i = \frac{1}{c^2}\left(\mathfrak{A}_x\,\mathrm{d}x + \mathfrak{A}_y\,\mathrm{d}y + \mathfrak{A}_z\,\mathrm{d}z - \varphi c\,\mathrm{d}t\right), \tag{12}$$

q_5 is a constant, and x_5 is the superfluous coordinate parameter. All coefficients are real-valued and independent of x_5.

[5] The appearance of the parameter p related to the linear form in the exponent is possibly linked to certain relations mentioned by E Schrödinger (*Z. Phys.* **12** 13 (1923)) [5].

The g_{ik} and q_i are dependent only on fields but not on the characteristics of the point mass; the latter are represented through the factor e^2/m^2. For brevity, however, we introduce the quantities dependent on e/m,

$$\frac{e}{m}\,q_i = a_i \qquad (i = 1, 2, 3, 4, 5)\,, \tag{13}$$

and use the convention that the summation sign is written when summing from 1 to 5 and omitted when summing from 1 to 4.

In this notation, we find

$$\left.\begin{aligned} &\gamma_{ik} = g_{ik} + a_i a_k\,; \quad g_{i5} = 0 \qquad (14)\\ &\gamma = \|\gamma_{ik}\| = a_5^2 g \qquad (15) \end{aligned}\right\}\ (i, k = 1, 2, 3, 4, 5)\,,$$

$$\left.\begin{aligned} &\gamma^{lk} = g^{lk} \\ &\gamma^{5k} = -\frac{1}{a_5}\,g^{ik}a_i = -\frac{a^i}{a_5} \\ &\gamma^{55} = \frac{1}{a_5^2}(1 + a_i a^i) \end{aligned}\right\}\ (i, k, l = 1, 2, 3, 4)\,. \tag{16}$$

The wave equation corresponding to Eqn (8) takes the form

$$\sum_{i,k=1}^{5} \frac{\partial}{\partial x_i}\left(\sqrt{-\gamma}\,\gamma^{ik}\,\frac{\partial\psi}{\partial x_k}\right) = 0\,, \tag{17}$$

or, in more detail,

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x_i}\left(\sqrt{-g}\,g^{ik}\,\frac{\partial\psi}{\partial x_k}\right) - \frac{2}{a_5}\,a^i\,\frac{\partial^2\psi}{\partial x_i\,\partial x_5} \\ &\qquad + \frac{1}{a_5^2}(1 + a_i a^i)\,\frac{\partial^2\psi}{\partial x_5^2} = 0\,. \end{aligned} \tag{18}$$

Finally, by introducing the function ψ_0 and potentials q_i, this equation can be rewritten as

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\,\frac{\partial}{\partial x_i}\left(\sqrt{-g}\,g^{ik}\,\frac{\partial\psi_0}{\partial x_k}\right) - \frac{4\pi}{h}\sqrt{-1}\,ceq^i\,\frac{\partial\psi_0}{\partial x_i} \\ &\qquad - \frac{4\pi^2c^2}{h^2}(m^2 + e^2 q_i q^i)\,\psi_0 = 0\,. \end{aligned} \tag{19}$$

B. Equations of motion. Our intention here is to represent the equations of motion of a charged point mass as equations for a geodesic line in R_5.

For this, we first compute the Christoffel symbols. We let the five-dimensional symbols be denoted as

$$\begin{Bmatrix} k\ l \\ r \end{Bmatrix}_5\,,$$

and the four-dimensional ones as

$$\begin{Bmatrix} k\ l \\ r \end{Bmatrix}_4\,.$$

We also introduce the covariant derivatives of the four-potential,

$$A_{lk} = \frac{\partial a_l}{\partial x_k} - \begin{Bmatrix} k\ l \\ r \end{Bmatrix}_4 a_r \tag{20}$$

and decompose the tensor $2A_{ik}$ into its symmetric and antisymmetric parts:

$$B_{lk} = A_{lk} + A_{kl}\,, \qquad M_{lk} = A_{lk} - A_{kl} = \frac{\partial a_l}{\partial x_k} - \frac{\partial a_k}{\partial x_l}\,. \tag{21}$$

We then obtain

$$\begin{aligned}
\left\{ \begin{matrix} k\,l \\ r \end{matrix} \right\}_5 &= \left\{ \begin{matrix} k\,l \\ r \end{matrix} \right\}_4 + \frac{1}{2}\left(a_k g^{ir} M_{il} + a_l g^{ir} M_{ik}\right),\\
\left\{ \begin{matrix} k\,l \\ 5 \end{matrix} \right\}_5 &= \frac{1}{2a_5} B_{lk} - \frac{1}{2a_5}\left(a_k a^i M_{il} + a_l a^i M_{ik}\right),\\
\left\{ \begin{matrix} k\,5 \\ 5 \end{matrix} \right\}_5 &= -\frac{1}{2} a^i M_{ik}\,,\\
\left\{ \begin{matrix} 5\,5 \\ k \end{matrix} \right\}_5 &= 0\,,\\
\left\{ \begin{matrix} 5\,5 \\ 5 \end{matrix} \right\}_5 &= 0\,.
\end{aligned} \tag{22}$$

Accordingly, the equations for a geodesic line in R_5 take the form

$$\frac{\mathrm{d}^2 x_r}{\mathrm{d}s^2} + \left\{ \begin{matrix} k\,l \\ r \end{matrix} \right\}_4 \frac{\mathrm{d}x_k}{\mathrm{d}s}\frac{\mathrm{d}x_l}{\mathrm{d}s} + \frac{\mathrm{d}'\Omega}{\mathrm{d}s} g^{ir} M_{il} \frac{\mathrm{d}x_l}{\mathrm{d}s} = 0\,, \tag{23}$$

$$\frac{\mathrm{d}^2 x_5}{\mathrm{d}s^2} + \frac{1}{2a_5} B_{lk} \frac{\mathrm{d}x_k}{\mathrm{d}s}\frac{\mathrm{d}x_l}{\mathrm{d}s} - \frac{1}{a_5}\frac{\mathrm{d}'\Omega}{\mathrm{d}s} a^i M_{il} \frac{\mathrm{d}x_l}{\mathrm{d}s} = 0\,. \tag{24}$$

Here, as previously, $\mathrm{d}'\Omega$ denotes the linear form

$$\mathrm{d}'\Omega = a_i\,\mathrm{d}x_i + a_5\,\mathrm{d}x_5\,. \tag{25}$$

Multiplying four equations in (23) by a_r and the fifth equation (24) by a_5, and summing the results, we obtain an equation that can be written as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}'\Omega}{\mathrm{d}s}\right) = 0\,. \tag{26}$$

Hence, it follows that

$$\frac{\mathrm{d}'\Omega}{\mathrm{d}s} = \mathrm{const}\,. \tag{27}$$

If we multiply Eqn (23) by $g_{r\alpha}\,\mathrm{d}x_\alpha/\mathrm{d}s$ and sum over r and α, then because of the antisymmetry of M_{ik} we obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{r\alpha}\frac{\mathrm{d}x_r}{\mathrm{d}s}\frac{\mathrm{d}x_\alpha}{\mathrm{d}s}\right) = 0 \tag{28}$$

or, by introducing the proper time τ through the formula

$$g_{ik}\,\mathrm{d}x_i\,\mathrm{d}x_k = -c^2\,\mathrm{d}\tau^2\,, \tag{29}$$

the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2 = 0\,. \tag{30}$$

We note that because of the relation

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}\tau^2 + (\mathrm{d}'\Omega)^2\,, \tag{31}$$

Eqns (28) and (30) follow from Eqn (26).

It follows from the discussion above that Eqn (24) is a consequence of Eqn (23) and can be discarded. If the proper time is introduced in Eqn (23) as an independent variable, the fifth parameters disappears completely; we also drop the index 4 of the Christoffel symbols:

$$\frac{\mathrm{d}^2 x_r}{\mathrm{d}\tau^2} + \left\{ \begin{matrix} k\,l \\ r \end{matrix} \right\} \frac{\mathrm{d}x_k}{\mathrm{d}\tau}\frac{\mathrm{d}x_l}{\mathrm{d}\tau} + \frac{\mathrm{d}'\Omega}{\mathrm{d}\tau} g^{ir} M_{il} \frac{\mathrm{d}x_l}{\mathrm{d}\tau} = 0\,. \tag{32}$$

The last term in the left-hand side represents the Lorentz force. In the special relativity theory, the first of these equations can be written as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}\tau} + \frac{1}{c}\frac{\mathrm{d}'\Omega}{\mathrm{d}\tau}\left[\frac{e}{c}\left(\dot{z}H_y - \dot{y}H_z + \frac{\partial \mathfrak{A}_x}{\partial t}\right) + e\frac{\partial\varphi}{\partial x}\right] = 0\,. \tag{33}$$

In order to reach agreement with observations, the factor before the square bracket should be equal to 1. Hence,

$$\frac{\mathrm{d}'\Omega}{\mathrm{d}\tau} = c \tag{34}$$

and

$$\mathrm{d}s^2 = 0\,. \tag{35}$$

The trajectories of a point mass are geodesic null lines in the five-dimensional space.

To obtain the Hamilton–Jacobi equation, we equate the squared five-dimensional gradient of ψ to zero,

$$g^{ik}\frac{\partial\psi}{\partial x_i}\frac{\partial\psi}{\partial x_k} - \frac{2}{a_5}\frac{\partial\psi}{\partial x_5} a^i \frac{\partial\psi}{\partial x_i} + (1 + a_i a^i)\left(\frac{1}{a_5}\frac{\partial\psi}{\partial x_5}\right)^2 = 0\,. \tag{36}$$

If we set

$$mca_5\frac{\partial\psi/\partial x_i}{\partial\psi/\partial x_5} = \frac{\partial W}{\partial x_i} \tag{37}$$

and introduce potentials q_i instead of a_i, then we obtain the equation

$$g^{ik}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_k} - 2ecq^i\frac{\partial W}{\partial x_i} + c^2(m^2 + e^2 q_i q^i) = 0\,, \tag{38}$$

which can be considered a generalization of our Eqn (2), which served as a departing point.

Leningrad, Physics Department of the University,
24 July 1926

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